

Greedy Rectilinear Drawings*

P. Angelini¹, M. A. Bekos¹, W. Didimo², L. Grilli², P. Kindermann³, T. Mchedlidze⁴,
R. Prutkin⁴, A. Symvonis⁵, and A. Tappini²

¹Institut für Informatik, Universität Tübingen, Germany,
{angelini,bekos}@informatik.uni-tuebingen.de

²Università degli Studi di Perugia, Italy,
{walter.didimo,luca.grilli}@unipg.it, alessandra.tappini@studenti.unipg.it

³Lehrstuhl für Informatik I, Universität Würzburg, Germany,
philipp.kindermann@uni-wuerzburg.de

⁴Institute of Theoretical Informatics, Karlsruhe Institute of Technology, Germany,
mched@iti.uka.de, roman.prutkin@kit.edu

⁵School of Applied Mathematical & Physical Sciences, NTUA, Greece,
symvonis@math.ntua.gr

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Abstract

A drawing of a graph is *greedy* if for each ordered pair of vertices u and v , there is a path from u to v such that the Euclidean distance to v decreases monotonically at every vertex of the path. From an application perspective, greedy drawings are especially relevant to support routing schemes in ad hoc wireless networks. The existence of greedy drawings has been widely studied under different topological and geometric constraints, such as planarity, face convexity, and drawing succinctness. We introduce *greedy rectilinear drawings*, where edges are horizontal or vertical segments. These drawings have several properties that improve human readability and support network routing.

We address the problem of testing whether a planar *rectilinear representation*, i.e., a plane graph with prescribed vertex angles, admits a greedy rectilinear drawing. We give a characterization, a linear-time testing algorithm, and a full generative scheme for *universal* greedy rectilinear representations, i.e., those for which *every* drawing is greedy. For general greedy rectilinear representations, we give a combinatorial characterization and, based on it, a polynomial-time testing and drawing algorithm for a meaningful subset of instances.

1 Introduction

In a *greedy drawing* of a graph in the plane every vertex is mapped to a distinct point and, for each ordered pair of vertices u and v , there is a *distance-decreasing* path from u to v , i.e., a path such that the Euclidean distance to v decreases monotonically at every vertex of the path. Greedy drawings have been originally proposed to support *greedy routing schemes* for ad hoc wireless networks [23, 24]. In such schemes, a node that has to send a packet to a destination v just forwards the packet to one of its neighbors that is closer to v than itself. In their seminal work, Papadimitriou and Ratajczak [23] showed that 3-connected planar graphs

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form the largest class of graphs for which every instance may admit a greedy drawing, and they formulated two conjectures:

Weak conjecture: Every 3-connected planar graph admits a greedy drawing.

Strong conjecture: Every 3-connected planar graph admits a *convex* greedy drawing, i.e., a planar greedy drawing with convex faces.

Concerning the weak conjecture, Dhandapani [9] provided an existential proof for maximal planar graphs. Later on, Leighton and Moitra [19] and Angelini et al. [3] independently settled the weak conjecture positively, by also describing constructive algorithms. Da Lozzo et al. [8] strengthened these results, showing that in fact every 3-connected planar graph admits a *planar* greedy drawing, which may however contain non-convex faces. As such, this result sits in between the two conjectures, leaving the strong conjecture still open. For graphs that are not 3-connected, Nöllenburg and Prutkin [21] characterized the trees that admit a greedy drawing. Note that every greedy drawing of a tree is planar [2].

Greedy drawings have also been investigated in terms of *succinctness*, an important property that helps to make greedy routing schemes work in practice. A drawing is *succinct* if the vertex coordinates are represented by a polylogarithmic number of bits. Since there exist greedy-drawable graphs in the Euclidean sense that do not admit a succinct greedy drawing [2], several papers studied succinct greedy drawings in spaces different from the Euclidean one or according to a metric different from the Euclidean distance [14, 16, 17, 20, 29].

A model related to greedy drawings is the one of *self-approaching drawings* [1, 22]. A straight-line drawing is self-approaching if for any ordered pair of vertices u and v , there is a path P from u to v in the drawing such that, for any point q on P , as a point p continuously moves along P from u to q , the Euclidean distance from p to q always decreases. Clearly, every self-approaching drawing is greedy, but not vice versa. Hence, self-approaching drawings are greedy drawings with stronger properties. In particular, their *dilation* is bounded by a constant [18], while for greedy drawings it may be unbounded [1]. The dilation (or “stretch-factor”) of a straight-line drawing is the maximum value of the ratio between the length of the shortest path between two vertices in the drawing and their Euclidean distance.

Motivation and Contribution. The rich literature on greedy drawings described above witnesses the relevance of these kind of drawings both from a practical and from a theoretical perspective. In particular, our work enhances the research on greedy drawings that satisfy some interesting topological or geometric requirements, such as planarity [8] and face convexity [17, 23, 29].

We initiate the study of greedy drawings in the popular *orthogonal drawing* convention [11, 12, 26]: Vertices are mapped to points and edges are sequences of horizontal and vertical segments (consequently, each vertex has degree at most 4). More precisely, we introduce *planar greedy rectilinear drawings*, i.e., crossing-free greedy drawings where each edge is either a horizontal or a vertical segment. We address the following general question: “Let H be a *planar rectilinear representation*, i.e., a plane graph with given values (90, 180, 270 degrees) for the geometric angles around each vertex; is it possible to assign coordinates to the vertices of H so that the resulting drawing is greedy rectilinear?”.

Figure 1a shows a rectilinear drawing that is not greedy; nonetheless, the corresponding rectilinear representation has a greedy drawing, as shown in Fig. 1b. Our question fits into the effective *topology-shape-metrics* approach [4, 26], which first computes a planar embedding of the graph, then finds an embedding-preserving orthogonal representation, and finally assigns coordinates to vertices and bends to complete the drawing. The topology-shape-metrics approach is successfully used to compute graph layouts in several application domains, including information systems [27, 10], software design [13], and computer networks [6]; see also [12]. We consider orthogonal drawings without bends and we address the last step of the topology-shape-metrics approach. Our contribution is as follows.

- We discuss basic properties of greedy rectilinear drawings (Section 3). In particular, we prove that the faces are always convex and the dilation is bounded by a small constant. This makes these representations useful to improve human readability and support network routing. In contrast, we provide convex (non-rectilinear) greedy drawings in which every distance-decreasing path between two vertices is arbitrarily longer than the Euclidean distance.

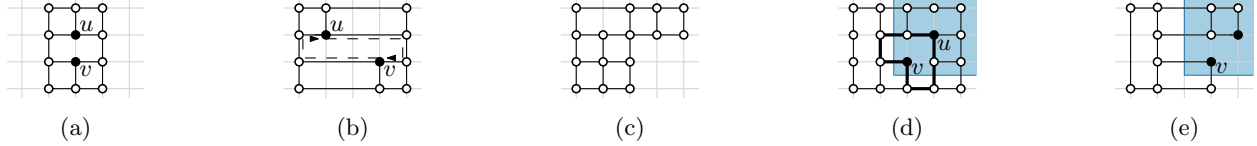


Figure 1: (a) A rectilinear drawing that is not greedy. (b) A greedy rectilinear drawing of the same representation (the distance-decreasing paths between u and v are dashed). (c) Drawing of a universal greedy rectilinear representation. (d)–(e) H is not greedy realizable if an internal face is not a rectangle or the external face is not orthoconvex.

- We investigate planar *universal greedy* rectilinear representations, i.e., representations for which *every* drawing is greedy (Section 4); see for example Fig. 1c). We give a characterization and a linear-time recognition algorithm which, in the positive case, computes a greedy drawing of minimum area on an integer grid. Our characterization may help in the design of algorithms that compute orthogonal representations in the second step of the topology-shape-metrics approach. We also describe a generative scheme for constructing any possible universal greedy rectilinear representation starting from a rectangle.
- We extend our study to general rectilinear greedy representations (Section 5). We give a non-geometric characterization, which leads to a linear-time testing algorithm for a meaningful subset of instances. If the condition of the characterization is satisfied, a greedy drawing of minimum area within that condition can be computed in quadratic time. However, we show that greedy rectilinear representations may require exponential area in general. Our non-geometric characterization opens up the way to intriguing theoretical problems, as discussed in Section 6.

We introduce basic concepts of graph drawing and the terminology used in the paper in Section 2. Conclusions and open problems are reported in Section 6.

Methodological tools and strategy. This paper mainly concentrates on 2-connected graphs, because, as it will be shown in Section 3 (Theorem 2), the set of greedy rectilinear representations for 1-connected graphs may be very limited. The main results of Section 4 and Section 5 make use of two auxiliary planar DAGs (directed acyclic graphs) D_x and D_y associated with the input rectilinear representation H , which allow us to capture and summarize the relative x - and y -positions of pairs of vertices in a drawing of H . We prove that H is universal greedy if and only if both D_x and D_y are Hamiltonian, which leads to an efficient linear-time testing algorithm for this family of representations (Theorem 5). More in general, the existence of a greedy drawing for H depends on the existence of an st -ordering for each of D_x and D_y that guarantees specific connectivity properties for the subgraphs induced by consecutive nodes in that ordering (Theorem 7). This provides an interesting non-geometric characterization of greedy rectilinear representations and makes it possible to easily design a polynomial-time testing algorithm of greedy realizability for a large subclass of rectilinear representations, namely those for which D_x and D_y are series-parallel graphs. If the test is positive, then a greedy drawing of minimal area for the input representation can be found by solving a linear program (Theorem 8).

Concerning our generative scheme to create any universal greedy representation (Section 4.2), this is based on incrementally augmenting a rectilinear representation, starting from a rectangle and by using a small set of primitive operations that either subdivide edges of the external face or attach to the external face a simple path of reflex vertices. Path additions resemble those of an *open ear decomposition* for 2-connected graphs [25], but they are tailored to plane graphs and of course enhanced with information concerned with the structure of a greedy rectilinear representation.

2 Background

Drawings and Planarity. Let $G = (V, E)$ be a graph. A *drawing* of G is a geometric representation Γ of G in the plane such that each vertex $v \in V$ is placed at a distinct point p_v and each edge $e = (u, v) \in E$ is drawn as a simple curve connecting p_u and p_v . We denote by $x(v)$ and $y(v)$ the x - and the y -coordinate of a vertex $v \in V$ in Γ , respectively. For two vertices $u, v \in V$, we denote by $d(u, v)$ the Euclidean distance between u and v in Γ , and by $d_M(u, v)$ the Manhattan distance between them. Also, we say that a path from u to v in Γ is a *u-v-path*. The degree of v is denoted as $\deg(v)$.

A drawing Γ of a graph G is *planar* if no two edges intersect except at their common end-vertices (when they are adjacent). Graph G is *planar* if it admits a planar drawing Γ . Such a drawing divides the plane into topologically connected regions, called *faces*. Exactly one face of Γ is an unbounded region and it is called the *external* face of Γ ; the other faces are called *internal*. Each internal face is described by the counterclockwise sequence of vertices and edges that form its boundary; while for the external face we use the clockwise sequence. The description of the set of (internal and external) faces determined by a planar drawing of G is called a *planar embedding* of G . Recall that a planar embedding uniquely determines, for each vertex v , a clockwise ordering of the edges incident to v . A planar graph G together with one of its planar embeddings is a *plane graph*: If Γ is a planar drawing of G whose set of faces coincides with that described by the planar embedding of G , then Γ *preserves* this embedding.

Graph Connectivity. A graph G is *k-connected* if every two vertices are connected by at least k disjoint paths. If a graph is k -connected, for $k = 1, 2, 3$, we also say that it is *connected*, *biconnected*, and *triconnected*, respectively. Let G be a connected graph that is not biconnected. Then, G contains at least a *cutvertex*, namely a vertex whose removal disconnects G . Also, any maximal subgraph of G that is biconnected is called a *block* of G . Finally, the *block-cutvertex tree* \mathcal{T} of G is a tree whose *C-nodes* are the cutvertices of G , and whose *B-nodes* are the blocks of G ; then, \mathcal{T} contains an edge between a B-node b and a C-node c if and only if the cutvertex c belongs to block b .

Directed Graphs and Series-Parallel Compositions. A *DAG* (directed acyclic graph) is a directed graph without directed cycles. A node of a DAG with only outgoing (incoming) edges is a *source* (*sink*). A DAG D is an *st-digraph* if it has a single source s and a single sink t . An *st-ordering* of D is a linear order $S = v_1, \dots, v_n$ of its nodes such that $i < j$ for any directed edge $(v_i, v_j) \in D$; observe that $v_1 = s$ and $v_n = t$ always holds. Every *st-digraph* D admits an *st-ordering*, which can be computed in $O(n)$ time [15]. Finally, D is *series-parallel* if one of the following holds:

- (i) D is a single edge (s, t) connecting a source to a sink;
- (ii) D is obtained from a set of series-parallel *st-digraphs* D_1, \dots, D_k with sources s_1, \dots, s_k and sinks t_1, \dots, t_k , by identifying s_1, \dots, s_k into a single node s , which becomes the source of D , and t_1, \dots, t_k into a single node t , which becomes the sink of D . This operation is a *parallel composition*;
- (iii) D is obtained from a set of series-parallel *st-digraphs* D_1, \dots, D_k with sources s_1, \dots, s_k and sinks t_1, \dots, t_k , by identifying node t_i with s_{i+1} , for each $i = 1, \dots, k - 1$. Here, $s = s_1$ and $t = t_k$ are the source and the sink of the resulting graph D . This operation is called *series composition*.

Orthogonal Drawings and Representations. The concept of (rectilinear) orthogonal drawing has been already defined in the introduction. We now give a more formal definition of (rectilinear) orthogonal representations. Let G be a plane graph, v be a vertex of G , and e_1, e_2 be two edges incident to v that are consecutive in the clockwise order around v (note that $e_1 = e_2$, if v has degree 1). We say that $a = \langle e_1, v, e_2 \rangle$ is an *angle at v* of G , or simply an *angle* of G . Let Γ and Γ' be two rectilinear orthogonal drawings of G that preserve its planar embedding. We say that Γ and Γ' are *shape equivalent* if for any angle a of G , the geometric angle corresponding to a is the same in Γ and Γ' . In other words, two shape equivalent rectilinear orthogonal drawings Γ and Γ' may only differ for the coordinates of their vertices, while the angles around any vertex are the same in the two drawings. Clearly, the shape equivalence relationship partitions the

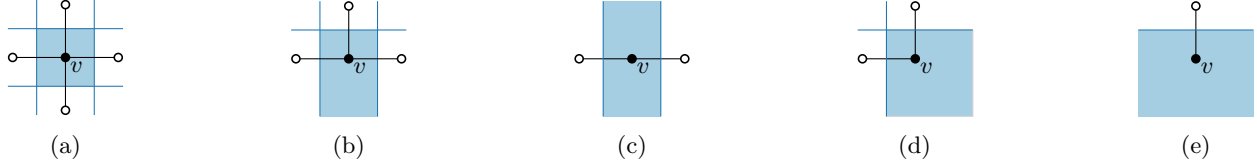


Figure 2: Different types of cells (shaded regions) of a vertex v in a rectilinear drawing of a graph: (a) $\deg(v) = 4$; (b) $\deg(v) = 3$; (c)-(d) $\deg(v) = 2$, (e) $\deg(v) = 1$.

infinite set of rectilinear orthogonal drawings of a plane graph into a finite number of equivalence classes. Each of these classes is called a *rectilinear orthogonal representation* H of G . One can regard H as a partial description of a drawing Γ that only specifies the angles at each vertex but that does not fix the vertex coordinates. Hence, H can be described by the embedding of G together with the geometric value of each angle of G (90, 180, 270 degrees)¹. If Γ is a rectilinear orthogonal drawing within class H , we also say that Γ is a rectilinear orthogonal drawing of H . For example, Fig. 1a and Fig. 1b are two shape equivalent drawings, i.e., they are drawings of the same rectilinear orthogonal representation.

For the sake of simplicity, we will use the term *rectilinear drawing* in place of rectilinear orthogonal drawing and the term *rectilinear representation* in place of rectilinear orthogonal representation.

Consider a rectilinear drawing Γ of a rectilinear representation H . Since H just fixes the angles around the vertices of Γ , rotating Γ by a multiple of 90° does not change H . Due to this observation, we can assume, without loss of generality, that H always comes with a specific orientation of its edges, i.e., we shall assume that for every edge (u, v) of H , it is fixed whether u is to the left, to the right, above, or below v in every rectilinear drawing Γ of H . A *flat vertex* of H (or of Γ) is a vertex with a flat angle (180 degrees). A flat angle formed by two horizontal segments is *north-oriented* (*south-oriented*) if it is above (below) the two segments. A flat angle between two vertical segments is either *east-oriented* or *west-oriented*. Finally, a *staircase path* between two vertices u and v of H (resp. of Γ) is either an edge or an x, y -monotone (zigzag) path that connects u and v .

Greedy Drawings. Let Γ be a drawing of G . A path (v_0, v_1, \dots, v_k) of G is *distance-decreasing* if $d(v_{i+1}, v_k) < d(v_i, v_k)$, for $i = 0, \dots, k-1$. Drawing Γ is *greedy* if for any ordered pair of vertices u, v , there exists a distance-decreasing u - v -path. If a rectilinear representation H admits a greedy rectilinear drawing, H is *greedy realizable* or, equivalently, it is a *greedy rectilinear representation*.

3 Basic Properties of Greedy Rectilinear Representations

In this section, we discuss some properties of rectilinear representations with respect to their possible greedy rectilinear drawings. We start with an additional definition concerning general (not necessarily rectilinear) greedy drawings. Let G be a graph, and let v be a vertex of G with neighbors u_1, u_2, \dots, u_h . The *cell* of v in a drawing Γ of G , denoted by $\text{cell}(v)$, is the (possibly unbounded) region of all points of the plane that are closer to v than to any u_i . The following geometric characterization is proven in [23].

Theorem 1 (Papadimitriou and Ratajczak [23]). *A drawing of a graph is greedy if and only if for every vertex v , $\text{cell}(v)$ contains no vertex other than v .*

For a rectilinear representation H , the cell of each vertex v has a specific shape in any rectilinear drawing of H , which depends on $\deg(v)$ and on the angles at v . Fig. 2 shows all the possible shapes. Note that, if $\deg(v) \leq 3$, then $\text{cell}(v)$ is always unbounded.

We restrict our study to biconnected graphs, because if a graph is not biconnected, the set of its greedy rectilinear drawings may be very limited, as shown by the following result for trees.

¹Every degree-1 vertex has a single angle of 360 degrees, thus one can avoid to specify it.



Figure 3: (a) A tree T with four leaves. (b) A greedy rectilinear representation of T

Theorem 2. *A tree T of vertex degree at most four admits a greedy rectilinear drawing if and only if it has at most four leaves.*

Proof. Given a leaf v of T and a rectilinear drawing Γ of T , we say that v is *north-oriented* (*south-oriented*) if v is above (below) its neighbor in Γ . Similarly, v is *east-oriented* (*west-oriented*) if v is to the right (left) of its neighbor in Γ . If T has at least five leaves, then there are at least two leaves u and v in Γ that are equally oriented, say north-oriented. This implies that $\text{cell}(u)$ contains v or $\text{cell}(v)$ contains u (or both). By Theorem 1, Γ is not greedy.

Suppose vice versa that T has at most four leaves u, v, w, z . A greedy drawing Γ of T is constructed as follows (see Figs. 3a and 3b). All vertices of the path π between u and v in T are horizontally aligned, so that u is west-oriented and v is east-oriented. Let π' be the path connecting π to w and π'' be the path connecting π to z ; note that π' and π'' may be attached to the same vertex of π . Then, all vertices of π' are vertically aligned in such a way that w is north-oriented, while all vertices of π'' are vertically aligned in such a way that z is south-oriented. It is immediate to see that Γ is a greedy drawing. \square

Observe that, with the same argument as in the proof of Theorem 2, it is possible to prove that any graph with more than four degree-1 vertices does not admit a greedy rectilinear drawing.

3.1 Convexity of greedy rectilinear representations

We now show that we can further restrict our study to *convex* rectilinear representations, i.e., those whose every internal face is rectangular and whose external face is bounded by an *orthoconvex* polygon. Recall that a simple polygon P is orthoconvex if for any vertical or horizontal line ℓ , the intersection between P and ℓ is either empty or a single segment. An illustration of a convex rectilinear representation is given in Fig. 4a.

Lemma 1. *A rectilinear representation is greedy realizable only if it is convex.*

Proof. Suppose first that a rectilinear representation H has an internal face f that is not a rectangle. This means that there is a vertex v with an angle of 270 degrees inside f . Let Γ be any rectilinear drawing of H . Suppose that, when moving counterclockwise along the boundary of f , we enter v horizontally from west and leave v vertically towards south; the other cases are symmetric. Since Γ has no bend, there exists a vertex u to the right of v and above v (see Fig. 4b). Therefore, $\text{cell}(v)$ contains u , which means that Γ is not greedy by Theorem 1.

Suppose now that the polygon P defined by the external boundary of H is not orthoconvex. Let Γ be any rectilinear drawing of H and let P be the polygon defined by the boundary of the external face of Γ . Since P is not orthoconvex, we can assume without loss of generality that there exists a vertical line ℓ whose intersection with P consists of at least two segments s' and s'' . Suppose that s' is above s'' . Clearly, ℓ cuts P into at least three distinct polygons, two of which lie on the same side of ℓ , say to the right of ℓ : The polygon having s' as a leftmost side is denoted by P' while the polygon having s'' as a leftmost side is denoted by P'' . Refer to Fig. 4c for an illustration. Let r' be a rightmost vertical side of P' and let r'' be a

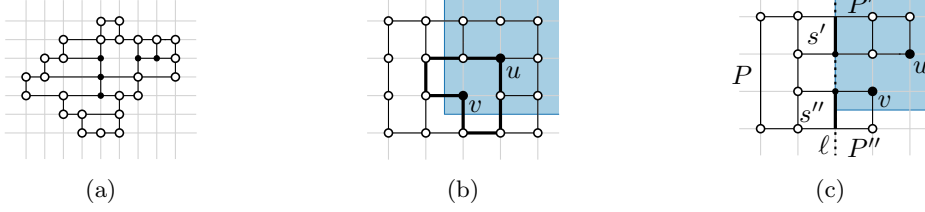


Figure 4: (a) A convex rectilinear representation, where the vertices on the orthoconvex outer face are filled white. (b-c) Illustration for the proof that a rectilinear representation is not greedy realizable if (b) an internal face is not a rectangle, or (c) the external face is not orthoconvex.

rightmost vertical side of P'' ; also, denote by $x(r')$ and $x(r'')$ the x -coordinate of r' and of r'' , respectively. Assume first that $x(r') \geq x(r'')$ and let p be the topmost point on r'' . Clearly, Γ has a vertex v at point p , and v is a degree-2 vertex that forms an angle of 270 degrees on the external face of Γ . Also, since P' and P'' cannot intersect, there must be at least a vertex u of Γ on P' that is above v and not to the left of v . It follows that $\text{cell}(v)$ contains u , and hence Γ is not greedy. The case in which $x(r') < x(r'')$ is handled symmetrically, choosing p as the bottommost point on r' . \square

For a rectilinear drawing of a convex rectilinear representation H , and for any two vertices u and v of H , let $R(u, v)$ denote the minimum bounding box (rectangle or segment) including u and v . The next property immediately follows from the convexity of H .

Property 1. *Let f be a face of H and w be any vertex of H with an angle of 90 degrees inside f . Denote by u and v the two neighbors of w along the boundary of f . In any rectilinear drawing of H , there is no vertex properly inside $R(u, v)$.*

3.2 Dilation of greedy rectilinear representations

We now exploit Property 1 to show that the dilation of greedy rectilinear representations is always bounded by a small constant.

Theorem 3. *In a rectilinear greedy drawing on an integer grid, for every two vertices s, t there is a distance-decreasing s - t -path of length at most $3\sqrt{2} \cdot d(s, t)$.*

Proof. Let Γ be a rectilinear greedy drawing of a rectilinear representation H . We prove that for every two vertices $s, t \in H$ there exists a distance-decreasing s - t -path in Γ of length at most $3d_M(s, t)$. Then, the statement follows by the fact that the Manhattan distance between two points is at most $\sqrt{2}$ times the Euclidean distance between them, that is, $d_M(s, t) \leq \sqrt{2}d(s, t)$.

We use induction on $d_M(s, t)$, which is always an integer number, since Γ has integer vertex coordinates. First note that every vertex u of H is connected to every vertex that is closest to u in Γ with respect to the Euclidean distance [23].

In the base case $d_M(s, t) = 1$, we have that $x(s) = x(t)$ or $y(s) = y(t)$. Since t is the closest vertex to s , we have that s and t are adjacent in Γ , and the statement trivially holds.

Suppose now that $d_M(s, t) > 1$ and that the statement holds for every pair of vertices whose Manhattan distance is less than $d_M(s, t)$. If $x(s) = x(t)$ or $y(s) = y(t)$, then there must be in Γ a distance-decreasing straight s - t -path (horizontal or vertical), as otherwise Γ would not be greedy. In this case, the length of this path equals $d_M(s, t) = d(s, t)$. Suppose now that s and t are not horizontally or vertically aligned. Without loss of generality, let t lie to the right of s and below it. Recall that $R(s, t)$ denotes the bounding box of s and t . We distinguish between the following cases:

Case 1. There is a vertex $v \neq s$ on the top or left boundary of $R(s, t)$. Then, s is connected to v by a straight path. Since $d_M(v, t) < d_M(s, t)$, by induction there exists a distance-decreasing v - t -path of length at

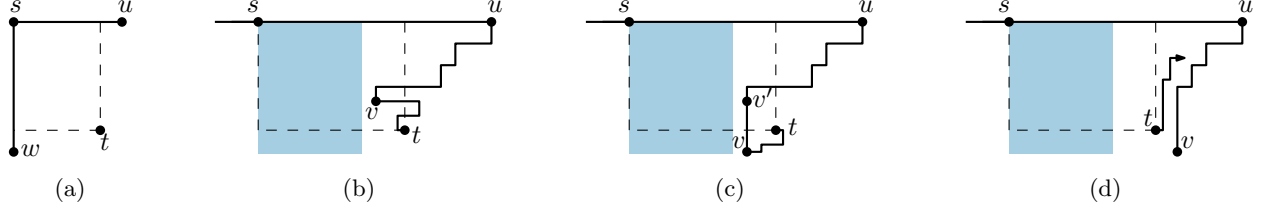


Figure 5: Illustration for the proof of Theorem 3. (a) There is no edge (s, w) with $y(w) < y(s)$, and (b)–(d) the path from u to v is the longest staircase on ρ .

most $3d_M(v, t)$. Concatenating this v - t -path and the straight s - v -path creates a distance-decreasing s - t -path of length at most

$$d_M(s, v) + 3d_M(v, t) < 3(d_M(s, v) + d_M(v, t)) = 3d_M(s, t).$$

Case 2. There is no vertex $v \neq s$ on the top or left boundary of $R(s, t)$. Consider the shortest distance-decreasing s - t -path ρ . Let (s, u) be the first edge of this path, and assume, without loss of generality, that this edge is horizontal (the other case is symmetric). Then, we have

$$x(t) < x(u) < x(s) + 2(x(t) - x(s))$$

and hence $d_M(u, t) < d_M(s, t)$. Note that, in this case, s cannot have a neighbor w below it, as this would imply $y(w) < y(t)$, thus violating Property 1; see Fig. 5a. Then, the region delimited by the vertical lines $x = x(s)$, $x = \frac{1}{2}(x(s) + x(u))$, and by the upper horizontal line $y = y(s)$ is a subset of $\text{cell}(s)$, and thus it contains no vertex other than s , since Γ is greedy; see Fig. 5b. Let v be the latest successor of u along ρ , such that the u - v -subpath of ρ is a staircase. The following subcases are possible:

Case 2.1. $v = t$. Then, the length of ρ is

$$x(u) - x(s) + d_M(u, t) \leq 2d_M(s, t) + d_M(u, t) \leq 3d_M(s, t).$$

Case 2.2. $v \neq t$ and $y(v) > y(t)$. Then, $x(v) > \frac{1}{2}(x(s) + x(u))$. Furthermore, $x(v) < x(t)$, as otherwise the edge following v would not be distance decreasing; see Fig. 5b. The s - v -subpath of ρ has length

$$x(u) - x(s) + x(u) - x(v) + y(s) - y(v) < 3(x(v) - x(s)) + y(s) - y(v) < 3d_M(s, v).$$

Since $d_M(v, t) < d_M(s, t)$ by induction, the v - t -subpath of ρ has length at most $3d_M(v, t)$. Therefore, the length of ρ is at most $3d_M(s, v) + 3d_M(v, t) = 3d_M(s, t)$.

Case 2.3. $v \neq t$ and $y(v) \leq y(t)$. If $x(v) < x(t)$, then for the predecessor v' of v on ρ , we have $y(v') > y(t)$; see Fig. 5c. Then, Case 2.2 can be applied to v' instead of v . Conversely, assume that $x(v) \geq x(t)$; see Fig. 5d. In this case, starting from t , we repeatedly go upwards or to the right in Γ . We cannot get stuck, since otherwise we have a vertex with no edge to the right and no edge upwards, a contradiction to face convexity. This implies that at some point we reach (intersect) the s - v -subpath of ρ by going upwards or to the right, and thus we construct a distance-decreasing s - t -path shorter than ρ , a contradiction. \square

Observe that the property we proved in Theorem 3 cannot be guaranteed for general greedy drawings, even if they are convex. For instance, Fig. 6 depicts a greedy (non-orthogonal) convex drawing of a biconnected planar graph in which every distance-decreasing path from vertex s to vertex t can be made arbitrarily longer than the Euclidean distance between s and t . This construction shows the existence of a family of greedy convex drawings with unbounded dilation.

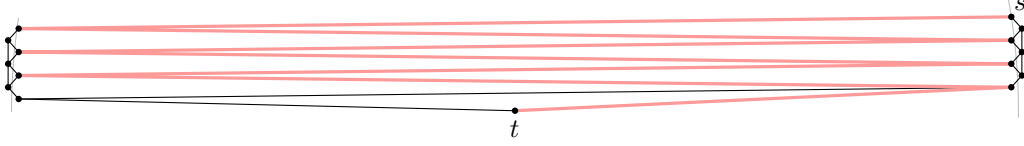


Figure 6: The bold zigzag path is the shortest distance-decreasing s - t -path.

3.3 Conflicts in rectilinear representations.

In this subsection we define the concept of “conflict” between two vertices of a rectilinear representation H . Intuitively, two vertices are in conflict if they can have different relative positions (either right/left or top/bottom) in different drawings of H . Studying the pairs of conflicting vertices of H will be fundamental for our results. To formalize this concept, we first define two directed acyclic graphs (DAGs) D_x and D_y associated with H , which have been already used in previous works on orthogonal compaction [5, 26]; refer to Fig. 7 for an illustration.

The DAG D_x is obtained from H by orienting the horizontal edges from left to right and by contracting each maximal path of vertical edges into a node; D_y is defined symmetrically on the maximal paths of horizontal edges. Note that D_x and D_y may have multiple edges: also, since the external face of H is orthoconvex, each of D_x and D_y has a single source and a single sink, that is, it is an st -digraph. For any vertex u of H , we denote by $c_x(u)$ (by $c_y(u)$) the node of D_x (of D_y) corresponding to the maximal vertical (horizontal, respectively) path containing u in H . If $c_x(u) \neq c_x(v)$, the notation $u \prec_x v$ ($u \not\prec_x v$) denotes the existence (absence) of a directed path from $c_x(u)$ to $c_x(v)$ in D_x . The notation $u \sim_x v$ means that either $u \prec_x v$ or $v \prec_x u$ holds, while $u \not\sim_x v$ means that none of them holds. The notations $u \prec_y v$, $u \not\prec_y v$, $u \sim_y v$, and $u \not\sim_y v$ are symmetric for D_y . Clearly, \prec_x and \prec_y are transitive relations.

Lemma 3 shows that there is a directed path between any two vertices of H in at least one of D_x and D_y . For this, we first give an auxiliary lemma.

Lemma 2. *Let Γ be a drawing of a convex rectilinear representation H . For two distinct nodes $c_x(u)$ and $c_x(v)$ of D_x such that there exists a horizontal line crossing both the vertical paths corresponding to $c_x(u)$ and to $c_x(v)$ in Γ , we have that $u \sim_x v$. A symmetric property holds for the nodes of D_y .*

Proof. We give the proof for the first part of the statement; the argument for the second part is symmetric. Consider a horizontal line crossing both the vertical paths corresponding to $c_x(u)$ and $c_x(v)$ in Γ . Let s_x be the portion of this line between the two vertical paths. If s_x does not traverse any face, then it overlaps with a set of horizontal edges in Γ . Thus, there is a path in D_x between $c_x(u)$ and $c_x(v)$. Otherwise, since every face of H is rectangular, there exists a path in D_x between $c_x(u)$ and $c_x(v)$ whose internal vertices are the vertical paths containing the vertical edges of the faces traversed by s_x . \square

Lemma 3. *For any two vertices u and v of a convex rectilinear representation H , at least one of the following holds: (i) $u \sim_x v$ or (ii) $u \sim_y v$.*

Proof. If $c_x(u) = c_x(v)$, then u and v belong to the same vertical path, and thus $u \sim_y v$. The case $c_y(u) = c_y(v)$ is symmetric. Suppose now that $c_x(u) \neq c_x(v)$ and $c_y(u) \neq c_y(v)$. Let Γ be any rectilinear drawing of H , and assume that u is below and to the left of v ; the other cases are symmetric. Consider a maximal path $\pi = (u_1, \dots, u_k)$ in H , with $u_1 = u$, such that for each edge (u_i, u_{i+1}) , u_i lies either below or to the left of u_{i+1} . Since all internal faces of H are rectangles, u_k is a top-right corner of H on the external face.

Note that, if there is a node u_i in π such that $c_x(u_i) = c_x(v)$, then $u \prec_x v$ and $u \prec_y v$. Otherwise, π crosses either the horizontal line ℓ_h through v (to the left of v) or the vertical line through v (below v). In the former case, let u_i be a node of π such that the vertical path containing u_i is crossed by ℓ_h . Note that, by the construction of π , either $c_x(u) = c_x(u_i)$ or $u \prec_x u_i$. Also, by Lemma 2, $u_i \prec_x v$. By transitivity, $u \prec_x v$. In the latter case, a symmetric argument is used to prove that $u \prec_y v$. This concludes the proof of the lemma. \square

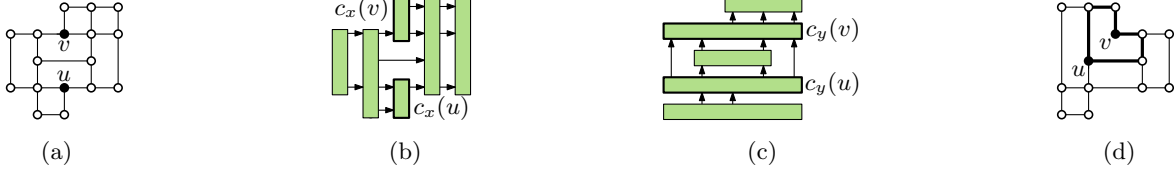


Figure 7: (a) A greedy realizable convex rectilinear representation H that is not universal greedy due to conflict $\{u, v\}$; (b)–(c) The DAGs D_x and D_y for H . (d) A non-convex representation such that $u \in \text{cell}(v)$ for any drawing, even though $u \prec_x v$ and $u \prec_y v$.

Let u and v be two vertices of H such that $c_x(u) \neq c_x(v)$ and $c_y(u) \neq c_y(v)$. By Lemma 3, at least one of $u \sim_x v$ and $u \sim_y v$ holds. If both such conditions hold, then the relative positions (left/right/top/bottom) of u and v are fixed (they are the same in any drawing of H); in this case, we prove that none of the two vertices lies in the cell of the other in any drawing of H (refer to Lemma 5). Conversely, this is not guaranteed when the two vertices are not comparable in one of the two DAGS, that is, when either $u \not\prec_x v$ or $u \not\prec_y v$ (refer to Theorem 4). In this case, we say that u and v form a *conflict*, denoted by $\{u, v\}$; in particular, if $u \not\prec_x v$, then $\{u, v\}$ is an x -conflict, otherwise it is a y -conflict.

Let $\{u, v\}$ be an x -conflict. By Lemma 3, we have that either $u \prec_y v$ or $v \prec_y u$; suppose that $u \prec_y v$, the other case is symmetric. Consider the topmost vertex u' of the vertical path corresponding to $c_x(u)$ and the bottommost vertex v' of the vertical path corresponding to $c_x(v)$. We say that u' and v' are *responsible* for the conflict $\{u, v\}$; e.g., in Fig. 7a, $\{u, v\}$ is an x -conflict with $u' = u$ and $v' = v$. The next lemma proves that both u' and v' are flat vertices.

Lemma 4. *Let u' and v' be the responsible vertices for a conflict $\{u, v\}$. Then, both u' and v' are flat vertices.*

Proof. Assume, without loss of generality, that the conflict is an x -conflict and that $u \prec_y v$. The convexity of H implies that also $u' \prec_y v'$. We show that u' is a north-oriented flat vertex, the argument for showing that v' is a south-oriented flat vertex is symmetric. Since by definition u' is the topmost vertex of a vertical path, it must be either north-oriented flat or reflex. In the latter case, u' lies on the outer face of H ; in particular, it lies on the top boundary of the orthoconvex polygon delimiting the outer face of H ; assume that it also lies on the left side, the other case is symmetric. Since $u' \prec_y v'$, we have that v' lies in the part of the orthoconvex polygon that is above u' . Since every point of this part also lies to the right of u' , we have that $u' \prec_x v'$, a contradiction. Therefore, u' is a north-oriented flat vertex. \square

A conflict is *resolved* in a drawing Γ of H if none of the two vertices that are responsible for it lies in the cell of the other. Finally, a convex rectilinear representation H is *conflict-free* if it has no conflict.

Lemma 5. *Let H be a convex rectilinear representation of a biconnected graph. A rectilinear drawing Γ of H is greedy if and only if every conflict is resolved in Γ .*

Proof. By Theorem 1, drawing Γ is greedy if and only if for any vertex v of H , we have that $\text{cell}(v)$ contains no vertex distinct from v . This already proves the necessity, since a conflict that is not resolved implies that a vertex lies in the cell of another vertex, by definition.

We now prove the sufficiency. First note that, if v is a vertex on the external face, then the portion of $\text{cell}(v)$ that belongs to the external face is empty, since the external boundary defines an orthoconvex polygon. Also, since all internal faces of H are rectangles, there is no internal angle of 270 degrees. Further, by Property 1, if two edges incident to a vertex v create an angle of 90 degrees, then the portion of $\text{cell}(v)$ delimited by these two edges is always empty. Thus, the only possible vertices whose cells may be non-empty in Γ are the flat vertices. Let v be a flat vertex, and assume that the flat angle at v is south-oriented (the other cases are symmetric). Consider any other vertex u . If u and v are not in an x -conflict, then either $u \prec_x v$ or $v \prec_x u$, say the former. Then, u lies to the left of v in Γ ; also, u does not lie to the right of the left neighbor of v , which implies that it lies to the left of $\text{cell}(v)$. Finally, if u and v are in an x -conflict,

then this is resolved by assumption. Hence, $u \notin \text{cell}(v)$ by definition. Repeating this argument for every flat vertex proves the statement. \square

4 Universal Greedy Rectilinear Representations

We recall that a rectilinear representation H is universal greedy if every rectilinear drawing of H is greedy. In this section, we first provide a linear-time algorithm to test whether a rectilinear representation is universal greedy (refer to Section 4.1) and then we describe a full generative scheme for this family of representations (refer to Section 4.2).

4.1 Testing algorithm

Our algorithm to test whether a convex rectilinear representation is universal greedy is based on the following concise characterization.

Theorem 4. *Let H be a convex rectilinear representation of a biconnected plane graph. Then, H is universal greedy if and only if it is conflict-free.*

Proof. By Lemma 5, if H is conflict-free, every rectilinear drawing of H is greedy (note that a rectilinear representation may be conflict-free without being convex, which would imply that it is not universal greedy; see Fig. 7d).

We now prove the other direction. Suppose, for a contradiction, that H is universal greedy but not conflict-free. Let Γ be any rectilinear drawing of H . Consider two vertices u and v that are responsible for a conflict in H ; assume without loss of generality that $\{u, v\}$ is an x -conflict, that is, $u \not\prec_x v$. We can further assume that u and v are consecutive along the x -axis in Γ , that is, there is no vertex w such that $x(u) < x(w) < x(v)$. Indeed, if such a vertex w exists (which implies $w \not\prec_x u$ and $v \not\prec_x w$), at least one of $u \not\prec_x w$ and $w \not\prec_x v$ holds, as otherwise $u \prec_x v$. Hence, we could have selected either u and w or w and v instead of u and v .

First observe that, if $x(u) = x(v)$, then Γ is not greedy, since $u \in \text{cell}(v)$ and $v \in \text{cell}(u)$. On the other hand, if $x(u) < x(v)$, then we can transform Γ into a drawing Γ' of H by moving u and all the vertices in its vertical path to the right of a quantity $x(v) - x(u)$, so that $x(u) = x(v)$. Since u and v are consecutive along the x -axis in Γ and since H is convex, Γ' is still planar but not greedy, which contradicts the fact that H is universal greedy. \square

Before giving our testing algorithm, we observe that it is possible to state an alternative characterization of universal greedy representations as a corollary of Theorem 4. Namely, suppose that H is a convex rectilinear representation of a biconnected plane graph and suppose that there exists a staircase path from any two vertices u and v . This immediately implies that either u and v belong to the same horizontal or vertical path in H (i.e., $c_x(u) = c_x(v) \vee c_y(u) = c_y(v)$), or there exist two directed paths connecting $c_x(u)$ and $c_x(v)$ in D_x and $c_y(u)$ and $c_y(v)$ in D_y . Thus, in this case, H is conflict-free. It is not difficult to prove that the reverse is also true, which thanks to Theorem 4 implies the following:

Corollary 1. *H is universal greedy if and only if there exists a staircase path between any two vertices of H .*

We now present our efficient testing algorithm.

Theorem 5. *Let H be a rectilinear representation of an n -vertex biconnected plane graph. It can be tested in $O(n)$ time whether H is universal greedy.*

Proof. The algorithm first checks in linear time whether H is convex. If not, the instance is rejected. Otherwise, it checks whether both D_x and D_y contain a (directed) Hamiltonian path, which can be done in linear time in the size of D_x and D_y , which is $O(n)$. Namely, since each of D_x and D_y is an st -digraph, computing a longest path from s to t is done in $O(n)$ time from a topological sorting. We claim that H is

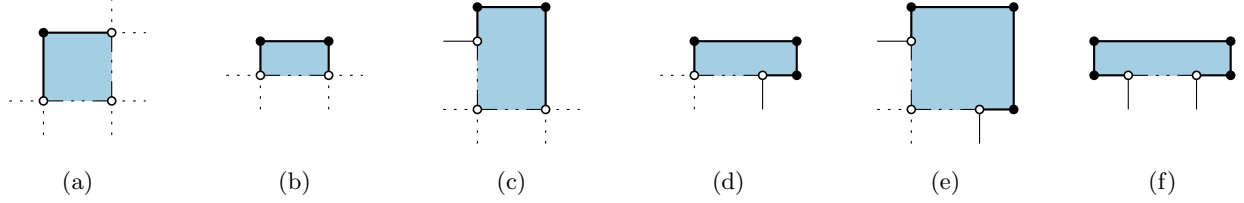


Figure 8: Schematic illustration of k -reflex vertex additions. The new face introduced by the operation is shaded; the new k -reflex vertices are in black. (a) $k = 1$; (b)-(c) $k = 2$; (d)-(e) $k = 3$; (f) $k = 4$.

universal greedy if and only if this test succeeds. By Theorem 4, to prove this claim, it is enough to show that a DAG D contains a Hamiltonian path if and only if for any two vertices u and v of D , there is a directed path either from u to v or from v to u . If D has a Hamiltonian path π , a directed path between any two vertices of D is a subpath of π . Conversely, suppose that there exists a directed path between any two vertices of D . Then, we can construct a topological sorting of D , which determines a total order of its nodes, and hence a Hamiltonian path. \square

Since conflict-free rectilinear representations form a subclass of the *turn-regular* orthogonal representations [5], for which a minimum-area drawing can be found in linear time, we can also state the following as a corollary of Theorem 5.

Corollary 2. *Let H be a universal greedy rectilinear representation. There is a linear-time algorithm to compute a (greedy) drawing of H with minimum area.*

4.2 Generative scheme

We now describe a generative scheme to obtain any possible universal greedy rectilinear representation, starting from a rectangle, by applying a suitable sequence of primitive operations, which incrementally add simple paths on the external face, possibly subdividing external edges.

Let H be a biconnected universal greedy rectilinear representation. Each of the following operations on H produces a new biconnected universal greedy rectilinear representation, as Lemma 6 proves.

- *k -reflex vertex addition.* Attach to the external face of H a path of $1 \leq k \leq 4$ reflex vertices (corners) that forms a new rectangular internal face, provided that the resulting representation is convex (see Fig. 8).
- *flat vertex addition.* Subdivide an external edge (u, v) of H with a flat vertex of degree two, provided that the open strip of the plane between the two lines orthogonal to (u, v) and passing through u and v , respectively, has no vertices in its interior.

Figure 9 shows an example of universal greedy representation generated through a sequence of k -reflex and flat-vertex additions.

Lemma 6. *Let H be a universal greedy rectilinear representation of a biconnected plane graph. Let H' be the rectilinear representation obtained from H by applying either a k -reflex vertex addition or a flat vertex addition. Then, H' is biconnected and it is a universal greedy rectilinear representation.*

Proof. To prove that H' is biconnected, note that subdividing an edge or attaching a simple path between two vertices of a biconnected graph cannot create cutvertices. We claim that H' is also convex. In fact, a flat vertex addition does not change the shape of any face of H , while a k -reflex vertex addition creates a new rectangular face and maintains the property that the outer face is delimited by an orthoconvex polygon.

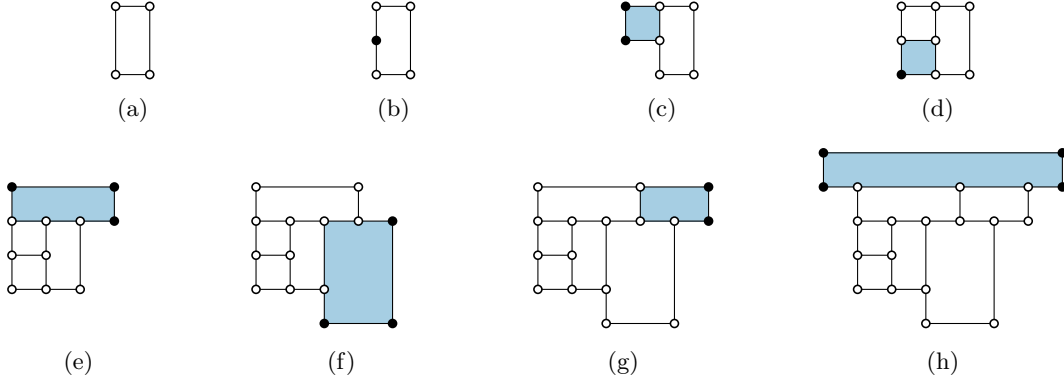


Figure 9: A sequence of primitive operations that generates a universal greedy rectilinear representation. (a) A single rectangular face; (b) flat vertex addition; (c) 2-reflex vertex addition; (d) 1-reflex vertex addition; (e)-(f) 3-reflex vertex additions; (g) 2-reflex vertex addition; (h) 4-reflex vertex addition.

We now show that H' is universal greedy. By Theorem 4, this is equivalent to proving that H' is conflict-free. For this, we show that none of the two operations introduces conflicts to H , which is universal greedy and thus conflict-free.

Consider first a flat vertex addition that subdivides an edge (u, v) into two edges (u, w) and (w, v) . Without loss of generality, assume that (u, v) is horizontal, with u to the left of v . Since $c_y(w) = c_y(u) = c_y(v)$, vertex w cannot form a y -conflict with any other vertex in H' . Also, by hypothesis, the open strip between the two lines orthogonal to (u, v) and passing through u and v , respectively, does not contain vertices in its interior. Hence, there cannot be vertices in H' that form an x -conflict with w . It follows that H' is still conflict-free.

Assume now that H' is obtained by applying a k -reflex vertex addition to H . Let u and v be the two vertices of H that are joined by a path u, w_1, \dots, w_k, v , with $1 \leq k \leq 4$, in order to obtain H' . Suppose, for a contradiction that there exists a conflict in H' , and let a and b be the vertices responsible for it. First observe that $a, b \notin \{w_1, \dots, w_k\}$, since w_i is a reflex vertex, for each $i = 1, \dots, k$, while a and b are flat vertices by Lemma 4.

We now consider the case in which one of a and b coincides with one of u and v , say $a = u$, and the flat angle at u involved in the conflict $\{a, b\}$, call it ϕ , is delimited by edge (u, w_1) ; see Figs. 8c–8f. We claim that, in any drawing Γ' of H' , the part of the cell of u that is determined by ϕ does not contain b , which implies that u is not in conflict with b , a contradiction. The claim follows from the fact that this part of the cell of u is a subset of the cell of u in the drawing of H obtained by removing w_1, \dots, w_k from Γ' , which is empty as H is universal, and from the fact that $b \in H$. Symmetrically, we can prove that v is not responsible for any conflict due to a flat angle delimited by edge (v, w_k) . Hence, the conflict $\{a, b\}$ is determined by two flat angles that also exist in H , contradicting the fact that H is conflict-free. Therefore, we have that H' is conflict-free, and thus universal by Theorem 4. \square

The next lemma is used to prove Theorem 6.

Lemma 7. *Let H be a rectilinear representation of a biconnected plane graph G , such that all internal faces of H are rectangles. If G is not a simple cycle, then there exists an internal face f of G such that:*

- (i) f is adjacent to the external face of G ;
- (ii) f has a degree-2 vertex that is a reflex vertex in the external face of H ;
- (iii) G remains biconnected if we remove from it all the external degree-2 vertices of f and their incident edges.

Proof. Let G^* be the *weak dual* of G , i.e., the node set of G^* is the set of the internal faces of G , and for each edge e of G shared by two internal faces f and g , there is a dual edge of e in G^* that connects the two nodes corresponding to f and g . Note that G^* can be obtained from the (non-weak) dual of G by deleting the node corresponding to the external face of G ; hence, G^* is (at least) connected, since the dual of G is biconnected (see, e.g., [28]). Also, as soon as G becomes non-biconnected due to the removal of some edges of the external face, then G^* becomes disconnected. Indeed, in this case, G would have a cutvertex c on the external face, which means that in G^* there would be no path between any two nodes corresponding to faces that belong to different biconnected components of G with respect to c . Therefore, it is sufficient to prove that there exists a face f in G that verifies properties (i) and (ii), and such that G^* remains connected after the removal from G of all the external degree-2 vertices of f . To this aim, we distinguish between two cases, based on whether G^* is biconnected or simply connected.

Case 1. G^* is biconnected. Since the external boundary of H is a rectilinear polygon, the external face of H has at least four reflex vertices. Let v be one of them and f be the internal face containing v . Removing from G all the external degree-2 vertices of f (included v) causes the removal of the node corresponding to f in G^* . Since G^* was biconnected, it remains connected after such a removal.

Case 2. G^* is connected but not biconnected. Let \mathcal{T} be the block-cutvertex tree of G^* , and let B^* be a block of G^* that is a leaf B-node of \mathcal{T} . Hence, B^* contains only one cutvertex of G^* , which corresponds to an internal face f_c of G . Denote by F_c the set of internal faces of G distinct from f_c and whose corresponding nodes of G^* are in B^* . It can be seen that there is a face $f \in F_c$ that contains a reflex vertex in the external face of H . More precisely, let s be the number of sides of f_c in H that are incident to some face of F_c , and let r be the number of reflex vertices in the external face of H that belong to some faces of F_c . Since the boundary of the rectilinear representation H restricted to F_c is a rectilinear polygon, we have that: (a) if $s = 1$, then $r \geq 2$; (b) if $s = 2$, then $r \geq 3$; (c) if $s \in \{3, 4\}$, then $r \geq 4$. Hence, removing from G all the external degree-2 vertices of f causes the removal of the node corresponding to f in B^* . Since B^* was biconnected, it remains connected after such a removal, and G^* remains connected as well. \square

We are now ready to prove Theorem 6.

Theorem 6. *Let H be a universal greedy rectilinear representation of a biconnected planar graph. Then, H can be obtained by a suitable sequence of k -reflex vertex and flat vertex additions, starting from a rectangle.*

Proof. We prove that there exists a sequence H_0, H_1, \dots, H_r ($r \in \mathbb{N}$) of universal greedy rectilinear representations such that H_0 is a rectangle, $H_r = H$, and H_{i+1} is obtained by applying either a k -reflex vertex addition or a flat vertex addition on H_i ($i = 0, \dots, r-1$). To this aim, it suffices to show that from each H_{i+1} we can derive H_i by applying a reverse operation of either a k -reflex vertex addition or a flat vertex addition. We distinguish between two cases:

Case 1. H_{i+1} has a flat degree-2 vertex w on the external face. Let u and v be the neighbors of w . Let H_i be the rectilinear representation obtained from H_{i+1} by deleting the edges (u, w) and (w, v) , and by adding the edge (u, v) (as a single segment). Clearly, H_i remains biconnected, convex, and greedy universal. Also, H_{i+1} is obtained from H_i by applying a flat vertex addition that subdivides (u, v) .

Case 2. Every degree-2 vertex on the external face of H_{i+1} is a reflex vertex. Note that the external face contains at least four reflex vertices. Let f be an internal face having the Properties (i)–(iii) in the statement of Lemma 7 (this lemma guarantees that such a face exists). By the proof of Lemma 7, the external degree-2 vertices of f form a path π , and their removal preserves biconnectivity. Since by hypothesis there is no external flat vertex of degree two in H_{i+1} , all vertices of π are reflex vertices in the external face of H_{i+1} . Also, since f is rectangular, π is formed by at most k vertices, with $k \in \{1, 2, 3, 4\}$. Now, let u and v be the two vertices of f to which π is attached, and let π' be the path from u to v containing all the internal edges of f (the boundary of f is the union of π and π'). Since H_{i+1} is universal greedy, π' cannot contain two vertices with an angle of 90 degrees inside f (i.e., π' is either a straight-line path or it is an L -shaped path). Indeed, in such a case, u and v would be two flat vertices on opposite sides of f , which, as already observed, contradicts the fact that H_{i+1} is universal greedy. Let H_i be the rectilinear representation derived

from H_{i+1} by removing π . For the above properties, H_i remains convex. Also, H_{i+1} is universal greedy, because π' is a staircase path from u to v and thus every staircase path of H_{i+1} that contains π can be replaced with a staircase path in which π is substituted with π' . This proves that H_{i+1} is obtained from H_i by applying a k -reflex vertex addition. \square

5 General Greedy Rectilinear Representations

In this section, we consider convex rectilinear representations of biconnected plane graphs that may contain conflicts. In particular, we investigate conditions under which a biconnected plane graph H is greedy realizable. We present a characterization (refer to Theorem 7), which yields a polynomial-time testing algorithm for a meaningful subclass of instances, namely when D_x and D_y are series-parallel (refer to Theorem 10).

Let D be one of the two DAGs D_x and D_y associated with H . Since D is an st -digraph, it has an st -ordering $\mathcal{S} = v_1, \dots, v_m$. For two indices i and j , with $1 \leq i < j \leq m$, $D\langle i, j \rangle$ denotes the subgraph of D induced by v_i, \dots, v_j . We say that \mathcal{S} is *good* if:

- S.1 For any two indices i and j , with $1 \leq i < j \leq m$, $D\langle i, j \rangle$ has at most two connected components, and
- S.2 if $D\langle i, j \rangle$ has exactly two components, then all nodes of one component precede those of the other in \mathcal{S} .

Further, we say that a drawing of H *respects* an st -ordering \mathcal{S}_x of D_x (\mathcal{S}_y of D_y) if for any two vertices u and w of H , we have that u lies to the left of w (below w) in the drawing if and only if $c_x(u)$ precedes $c_x(w)$ in \mathcal{S}_x ($c_y(u)$ precedes $c_y(w)$ in \mathcal{S}_y). Finally, when we refer to the x -coordinate (y -coordinate) of a node v_i of D_x (of D_y), we mean the one of all the vertices $w \in H$ with $c_x(w) = v_i$ (with $c_y(w) = v_i$), as these vertices belong to the same vertical (horizontal) path. We prove the following characterization.

Theorem 7. *A convex rectilinear representation H of a biconnected plane graph is greedy realizable if and only if both DAGs D_x and D_y admit good st -orderings.*

The following three subsections are devoted to the proof of Theorem 7. In particular, we prove the necessity of the existence of good st -orderings in Section 5.1; then, for a proof of the sufficiency, we first discuss in Section 5.2 some properties of greedy rectilinear representations with respect to their conflicts, and then we use these properties in Section 5.3 to derive a drawing algorithm, assuming a good st -ordering.

5.1 Necessity of the condition in Theorem 7

In this section, we prove that the existence of good st -orderings for both DAGs is a necessary condition for H to be greedy realizable.

Lemma 8. *If D_x or D_y admits no good st -ordering, H is not greedy realizable.*

Proof. Let \mathcal{S}_x be any st -ordering of D_x that is not good. We prove that H does not admit any greedy drawing respecting \mathcal{S}_x . Suppose, for a contradiction, that there exists such a greedy drawing Γ of H . Since \mathcal{S}_x is not good, there exist two indices i and j , with $1 \leq i < j \leq m$, such that $D_x\langle i, j \rangle$ consists of at least two connected components.

Let ℓ_A be a vertical line with x -coordinate between $x(v_{i-1})$ and $x(v_i)$, and let ℓ_B be a vertical line with x -coordinate between $x(v_j)$ and $x(v_{j+1})$ in Γ . Observe that, for a connected component C of $D_x\langle i, j \rangle$, the following property holds. Consider the smallest rectangle $R(C)$ having its vertical sides along ℓ_A and ℓ_B and containing all the vertices of H corresponding to nodes of C in its interior; then, every horizontal segment connecting two points on the two vertical sides of $R(C)$ intersects at least a vertical edge between two vertices u and w of H such that $c_x(u) = c_x(w) \in C$. In fact, if this was not the case, then there would be two vertices of C that are not joined by any path in C , contradicting the fact that C is a connected component.

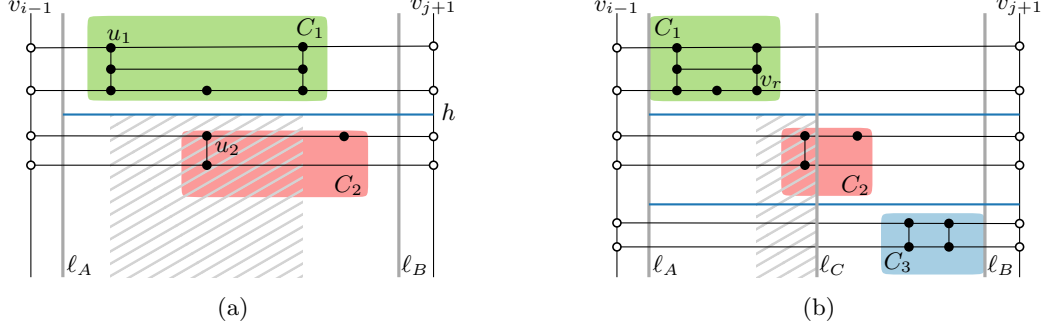


Figure 10: Illustration for the proof of Lemma 8. (a) There are two components in $D_x\langle i, j \rangle$. C_1 and C_2 are separated by h . The cells of the south-oriented flat vertices of C_1 span at least the tiled area. (b) There are at least three components. There are vertices of C_2 to the left of ℓ_C . The cell of v_r spans at least the tiled area.

Let C_1 and C_2 be two components of $D_x\langle i, j \rangle$, and consider two vertices u_1 and u_2 of H such that $c_x(u_1) \in C_1$ and $c_x(u_2) \in C_2$; see Fig. 10a. This implies that $u_1 \approx_x u_2$. Thus, by Lemma 3, either $u_1 \prec_y u_2$ or $u_2 \prec_y u_1$ holds; assume the latter. Consider another pair of vertices u'_1 and u'_2 of H such that $c_x(u'_1) \in C_1$ and $c_x(u'_2) \in C_2$. By the same argument, either $u'_1 \prec_y u'_2$ or $u'_2 \prec_y u'_1$ holds; we claim that $u'_2 \prec_y u'_1$. Suppose for a contradiction that $u'_1 \prec_y u'_2$. Consider the two rectangles $R(C_1)$ and $R(C_2)$ as defined above. Note that, since $u_2 \prec_y u_1$ and $u'_1 \prec_y u'_2$, we have $R(C_1) \cap R(C_2) \neq \emptyset$, as the rectangle $R(C_1)$ contains u_1 and u'_1 , and thus it contains also u_2 and u'_2 . Therefore, there exist a vertical path corresponding to a node of C_1 and a vertical path corresponding to a node of C_2 that are crossed by the same horizontal line. By Lemma 2, there exists a directed path in D_x between the two nodes of C_1 and C_2 , contradicting the fact that C_1 and C_2 are different connected components. Repeating this argument for any pair of vertices, we conclude that there exists a horizontal line-segment h from ℓ_A to ℓ_B such that all the vertices of H corresponding to nodes of C_1 lie above h and all those corresponding to nodes of C_2 lie below h in Γ .

Further, since for each node of C_1 there is a flat vertex of H that is south-oriented (the bottommost vertex of the vertical path corresponding to the node of C_1), we have that the union of the cells of these flat vertices, restricted to the region below h , consists of a rectangle of infinite height spanning at least all the x -coordinates between those of the leftmost and of the rightmost node of C_1 (see the tiled region in Fig. 10a). Since the same holds for the cells of the north-oriented flat vertices that are the topmost points of the vertical paths corresponding to nodes of C_2 , we have that all the nodes of C_1 are to the left of all the nodes of C_2 in Γ , or vice versa. Therefore, $D_x\langle i, j \rangle$ contains at least another connected component C_3 , as otherwise the st -ordering \mathcal{S}_x would be good. By the same argument as before, we can claim that C_1 , C_2 , and C_3 are separated by horizontal line-segments; we further assume that C_1 , C_2 , and C_3 appear in this order from top to bottom in Γ . Also, either all the nodes of C_3 lie to the left of all the nodes of C_2 in Γ , or vice versa, and the same holds for the nodes of C_3 and of C_1 ; see Fig. 10b.

Assume that all the nodes of C_1 are to the left of all those of C_2 , which are to the left of those of C_3 ; the other cases are analogous. Let ℓ_C be the vertical line that is equidistant from ℓ_A and ℓ_B . We claim that all the nodes of C_2 are required to lie to the right of ℓ_C . Namely, if there is at least a node of C_1 to the right of ℓ_C , this is trivially true since the nodes of C_2 are to the right of those of C_1 , by assumption. Further, if all the nodes of C_1 lie to the left of ℓ_C , let v_r be the bottommost vertex of the vertical path corresponding to the rightmost node of C_1 . Let x_r , x_A , x_B , and x_C be the x -coordinates of v_r , ℓ_A , ℓ_B , and ℓ_C , respectively. Assuming all positive x -coordinates, we have that $x_C - x_r < x_C - x_A = x_B - x_C$; thus, $x_r + (x_B - x_r)/2 > x_C$. This implies that the right boundary of cell(v_r) lies to the right of ℓ_C , since the neighbor of v_r in H with its same y -coordinate and with larger x -coordinate lies to the right of ℓ_B , as otherwise $c_x(v_r)$ would not be the rightmost node of C_1 . Hence, the claim follows, since the nodes of C_2 must lie to the right of cell(v_r). With analogous arguments we can prove that the topmost vertex of the vertical path corresponding to the leftmost node of C_3 enforces all the nodes of C_2 to lie to the left of ℓ_C . This results in a contradiction and concludes the proof. \square

We now prove that the necessary condition of Theorem 7 is also sufficient for the existence of a greedy rectilinear drawing. Our proof is constructive, as we provide a polynomial-time algorithm that, given a good st -ordering for each of the two DAGs of a rectilinear representation H , constructs a greedy drawing Γ of H . Our algorithm is based on some properties concerning the conflicts of H , which we discuss in the following subsection.

5.2 Properties of conflicts in greedy rectilinear representations

We start by proving a lemma that allows us to assign the x - and y -coordinates of the vertices in Γ in two independent steps.

Lemma 9. *Let H be a convex rectilinear representation of a biconnected embedded planar graph G . Let Γ_1 and Γ_2 be two drawings of H such that all x -conflicts are resolved in Γ_1 and all y -conflicts are resolved in Γ_2 . Then, the drawing Γ_3 of H in which the x -coordinate of each vertex is the same as in Γ_1 and the y -coordinate of each vertex is the same as in Γ_2 is greedy.*

Proof. By Theorem 1, in order to prove that Γ_3 is greedy, it is enough to prove that for any vertex v of H , we have that $\text{cell}(v)$ contains no vertex distinct from v . Since H is convex, there is no internal angle of 270 degrees. Also, by Property 1, if two edges incident to a vertex v create an angle of 90 degrees, then the portion of $\text{cell}(v)$ delimited by these two edges is always empty. Thus, if, for a vertex v , the cell $\text{cell}(v)$ is non-empty in Γ_3 , then v forms one or two flat angles. However, the fact that a vertex lies inside a cell determined by a north-oriented or by a south-oriented flat angle only depends on the x -coordinates of the vertices in the drawing; thus, all these cells are empty in Γ_3 , since they are empty in Γ_1 . Analogously, all the cells determined by east-oriented or by west-oriented flat angles are empty in Γ_3 , since they are empty in Γ_2 . This concludes the proof of the lemma. \square

In view of Lemma 9, we only focus on the assignment of the x -coordinates based on the good st -ordering $\mathcal{S}_x = v_1, \dots, v_m$ of D_x , which implies that the only conflicts that we have to consider are the x -conflicts. The assignment of the y -coordinates based on the good st -ordering of D_y works symmetrically.

We now give an overview of our strategy. We first prove in Lemma 10 that, to guarantee that every x -conflict is resolved, it suffices to resolve a specific subset of them, called *minimal*. Namely, we say that an x -conflict $\{u, v\}$ *dominates* an x -conflict $\{w, z\}$, with $c_x(u) = v_i$, $c_x(v) = v_j$, $c_x(w) = v_k$, and $c_x(z) = v_\ell$, if $k \leq i < j \leq \ell$. A *minimal* x -conflict is not dominated by any x -conflict. In Fig. 11b, the x -conflict $\{z, r\}$ is minimal and dominates the x -conflict $\{u, w\}$.

By Lemmas 5 and 10, we conclude that a greedy rectilinear drawing can be obtained by resolving all the minimal conflicts. In our algorithm, described in Section 5.3, we encode that a minimal x -conflict is resolved with a single inequality on the horizontal distances between the vertices in the x -conflict. Then, in Lemma 11, we prove that, for a minimal x -conflict $\{u, v\}$, the nodes $c_x(u)$ and $c_x(v)$ of D_x are consecutive in \mathcal{S}_x . We use this property to show that the system of inequalities describing the conditions for the minimal x -conflicts to be resolved always admits a solution.

Lemma 10. *Let Γ be a rectilinear drawing of H respecting \mathcal{S}_x . If every minimal x -conflict dominating an x -conflict $\{u, w\}$ is resolved in Γ , $\{u, w\}$ is resolved.*

Proof. We may assume without loss of generality that u and w are responsible for $\{u, w\}$. Let $v_i = c_x(u)$ and $v_j = c_x(w)$, with $i < j$. Consider the graph $D_x \langle i, j \rangle$. Since \mathcal{S}_x is good, this graph has at most two connected components C_1 and C_2 . Assume that $v_i \in C_1$.

Suppose first that also $v_j \in C_1$. Consider the right neighbor u' of u in H , which exists since u is a flat vertex; see Fig. 11a. Note that node $c_x(u')$ precedes $c_x(w)$ in \mathcal{S}_x , that is, $c_x(u') \in D_x \langle i, j \rangle$; in fact, if this were not the case, then v_j would not belong to C_1 . Thus, u' lies to the left of w in any rectilinear drawing of H respecting \mathcal{S}_x . Hence, the mid-point of edge (u, u') , which defines the right boundary of $\text{cell}(u)$, lies to the left of w , which implies that $w \notin \text{cell}(u)$. Symmetrically, we can show that $u \notin \text{cell}(w)$.

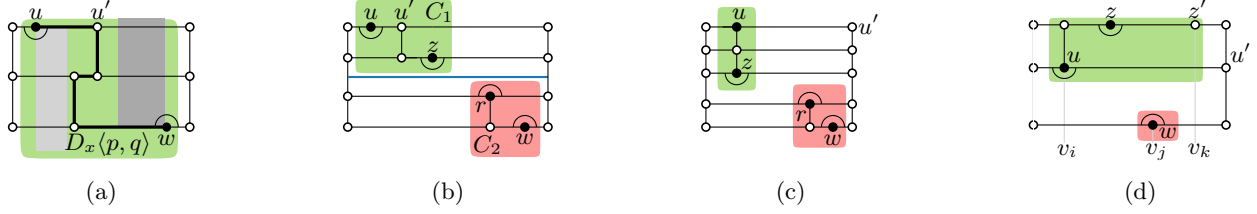


Figure 11: Illustration for the proof of Lemma 10. (a) $w \notin \text{cell}(u)$ and $u \notin \text{cell}(w)$; (b) the x -conflict $\{z, r\}$ is resolved; (c) $c_x(u)$ and $c_x(z)$ are sinks in $D_x(i, j)$; and (d) $c_x(u)$ is not a sink in C_1 .

Suppose now that $v_j \in C_2$. As in the proof of Lemma 8, we can assume that all the vertices corresponding to nodes of C_1 lie above those corresponding to nodes of C_2 . Further, we can assume that all the nodes of C_2 follow all those of C_1 in the good st -ordering \mathcal{S}_x . The other cases are symmetric. Let z be the bottommost vertex of the vertical path corresponding to the last node $c_x(z)$ of C_1 in \mathcal{S}_x ; see Fig. 11b. Also, let r be the topmost vertex of the vertical path corresponding to the first node $c_x(r)$ of C_2 in \mathcal{S}_x . Note that vertices z and r are responsible for a minimal x -conflict $\{z, r\}$, which is resolved by assumption. We now show that also $\{u, w\}$ is resolved. In particular, we show that $w \notin \text{cell}(u)$; the argument for $u \notin \text{cell}(w)$ is symmetric.

First observe that, if the right neighbor u' of u in H belongs to C_1 , $\text{cell}(u)$ does not extend beyond $c_x(z)$; since every node of C_2 is completely to the right of $c_x(z)$, we have $w \notin \text{cell}(u)$; see Fig. 11b. Thus, we assume that u lies on the right boundary of C_1 , i.e., its right neighbor u' does not belong to C_1 . Note that, if $c_x(u)$ is also a sink of C_1 , then C_1 does not contain any other node other than $c_x(u)$, since $c_x(u)$ is the first node of C_1 ; see Fig. 11c. Thus, either v_i is not a sink of C_1 , or $c_x(u) = c_x(z)$. In the latter case, $r \notin \text{cell}(u)$, since the minimal x -conflict $\{z, r\}$ is resolved, which implies $w \notin \text{cell}(u)$. Hence, it remains to consider the case that $c_x(u) \neq c_x(z)$ and $c_x(u)$ is not a sink of C_1 ; see Fig. 11d. This implies that there is a directed path from $c_x(u)$ to $c_x(z)$ in C_1 .

Since $u' \notin C_1$ and since u is a south-oriented flat vertex, u lies below z . Consider the right neighbor z' of z , which lies to the right of w because z is the sink of C_1 . Recall that also u' lies to the right of w . Assume first that z' lies to the left of u' , and let $c_x(z') = v_k$. Consider now the graph $D_x(i, k)$ from v_i to v_k . This graph contains two connected components, one containing $c_x(u) = v_i$ and $c_x(z') = v_k$, and another one containing $c_x(w) = v_j$, due to the presence of the edge (u, u') , which cannot be crossed. However, this implies a contradiction to Condition S.2 of a good st -ordering, since $i < j < k$. Thus, z' must lie to the right of u' ; since z is to the right of u , the right boundary of $\text{cell}(z)$ is to the right of the right boundary of $\text{cell}(u)$. Hence, the fact that $r \notin \text{cell}(z)$ implies that $r \notin \text{cell}(u)$, and thus $w \notin \text{cell}(u)$. \square

Lemma 11. *For any two vertices u and w of H such that $\{u, w\}$ is a minimal x -conflict, we have that $c_x(u)$ and $c_x(w)$ are consecutive in a good st -ordering \mathcal{S}_x .*

Proof. Suppose that there is a vertex $z \in H$ such that $c_x(z) = v_j$ lies between $c_x(u) = v_i$ and $c_x(w) = v_k$ in \mathcal{S}_x , i.e., $i < j < k$. First, suppose that $c_x(u)$ and $c_x(w)$ belong to the same connected component C of $D_x(i, k)$. Then, by definition, v_i and v_k are a source and a sink of C , respectively. Since $\{u, w\}$ is an x -conflict, we have $u \not\prec_x w$; hence, there is another source $c_x(s)$ in C , for some vertex $s \in H$, such that $s \prec_x w$. Since $c_x(u)$ and $c_x(s)$ are different sources of C , we have $u \not\prec_x s$, and thus $\{u, s\}$ is an x -conflict dominating the minimal x -conflict $\{u, w\}$; a contradiction. Suppose now that $c_x(u)$ and $c_x(w)$ belong to different components. Then, $c_x(z)$ does not belong to the same component as one of them, say $c_x(u)$. Thus, $u \not\prec_x z$, i.e., $\{u, z\}$ is an x -conflict dominating $\{u, w\}$; a contradiction. \square

We are now ready to present our algorithm to assign x -coordinates to the vertices of H so that all minimal x -conflicts are resolved.

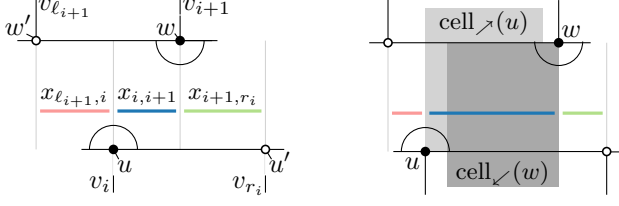


Figure 12: Solving the inequalities of $x_{i,i+1}$ implies $u \notin \text{cell}(w)$ and $w \notin \text{cell}(u)$.

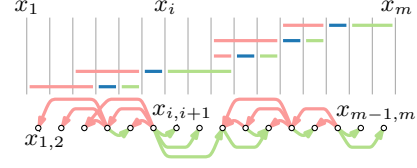


Figure 13: The relation graph defined by the left and right inequalities

5.3 A greedy drawing algorithm when D_x and D_y admit good st -orderings

We extend some definitions from vertices of H to nodes of D_x . Namely, we say $v_i \prec_x v_j$, if there is a directed path in D_x from v_i and v_j . Also, we say that there is a (minimal) x -conflict $\{v_i, v_j\}$ in D_x , if there is a (minimal) x -conflict $\{u, w\}$ in H such that $c_x(u) = v_i$ and $c_x(w) = v_j$.

For $0 < i, j \leq m$, let $x_{i,j} := x_j - x_i$ be the x -distance between v_i and v_j . To prove that a good st -ordering \mathcal{S}_x allows for a greedy realization, we develop a system of inequalities describing the geometric requirements for the x -distance of consecutive nodes in \mathcal{S}_x in a greedy drawing, and then prove that this system always admits a solution since \mathcal{S}_x is good. First note that, for every $0 < i < m$ such that there is no minimal x -conflict $\{v_i, v_{i+1}\}$, we only require the x -distance to be positive, so we define the following *trivial inequality*.

$$x_{i,i+1} > 0. \quad (\text{trivial inequality})$$

For every $0 < i < m$ such that there is a minimal x -conflict $\{v_i, v_{i+1}\}$, we define two inequalities that describe the necessary conditions for the x -conflict to be resolved. Let u and w , with $c_x(u) = v_i$ and $c_x(w) = v_{i+1}$, be responsible for $\{v_i, v_{i+1}\}$. We assume that $u \prec_y w$; the other case is symmetric.

By assumption, v_i lies to the bottom left of v_{i+1} , so we only have to consider the part of $\text{cell}(w)$ to the bottom left of w , which we denote by $\text{cell}_{\swarrow}(w)$ (dark region in Fig. 12). Let (w', w) be the bottommost incoming edge of v_{i+1} with $c_x(w') = v_{\ell_{i+1}}$. Then, the left boundary of $\text{cell}_{\swarrow}(w)$ is delimited by the vertical line through the mid-point of (w', w) . Thus, we require

$$x_{i,i+1} > x_{\ell_{i+1},i+1}/2 \Leftrightarrow x_{i,i+1} > x_{\ell_{i+1},i}$$

Symmetrically, we only consider the part $\text{cell}_{\nearrow}(u)$ of $\text{cell}(u)$ to the top right of u (light region in Fig. 12), which is bounded by the vertical line through the mid-point of the topmost outgoing edge (u, u') of v_i with $c_x(u') = v_{r_i}$. Thus, we require

$$x_{i,i+1} > x_{i,r_i}/2 \Leftrightarrow x_{i,i+1} > x_{i+1,r_i}$$

Since $v_{\ell_{i+1}}$ and v_i (and v_{i+1} and v_{r_i}) are not necessarily consecutive in the st -ordering, we express the x -distance $x_{\ell_{i+1},i}$ (and x_{i,r_i}) as the sum of the x -distances between the consecutive nodes between them in the st -ordering. This gives the *left* and the *right inequality*.

$$x_{i,i+1} > \sum_{j=\ell_{i+1}}^{i-1} x_{j,j+1} \quad (\text{left inequality})$$

$$x_{i,i+1} > \sum_{j=i+1}^{r_i-1} x_{j,j+1} \quad (\text{right inequality})$$

Note that for every variable $x_{i,i+1}$ there exists either a trivial inequality or a left and right inequality. Consider the following triangulated matrices, where $c_{i,j} = -1$ if $i > j \geq \ell_{i+1}$ or $i < j \leq r_i$, where $c_{i,j} = 1$ if $i = j$, and $c_{i,j} = 0$ otherwise.

$$\begin{aligned}
A &= \begin{pmatrix} c_{1,1} & & 0 \\ \vdots & \ddots & \\ c_{m-1,1} & \cdots & c_{m-1,m-1} \end{pmatrix} \\
B &= \begin{pmatrix} c_{1,1} & \cdots & c_{1,m-1} \\ & \ddots & \vdots \\ 0 & & c_{m-1,m-1} \end{pmatrix} \\
x &= \begin{pmatrix} x_{1,2} \\ \vdots \\ x_{m-1,m} \end{pmatrix}
\end{aligned}$$

We express the left and trivial (right and trivial) inequalities as $Ax > 0$ (as $Bx > 0$). Any vector $x > 0$ determines a unique rectilinear drawing: we assign to each vertex the y -coordinate defined by S_y , we assign to v_1 the x -coordinate $x_1 = 0$ and to every other v_i the x -coordinate $x_i = x_{i-1} + x_{i-1,i}$. Since $x > 0$, the x -coordinates preserve the good st -ordering and resolve all x -conflicts.

Lemma 12. *A vector $x = (x_{1,2}, \dots, x_{m-1,m})^\top > 0$ solves both $Ax > 0$ and $Bx > 0$ if and only if it determines a drawing where all x -conflicts are resolved.*

Proof. First, suppose that x solves both $Ax > 0$ and $Bx > 0$. Let $\{v_i, v_j\}$ be a minimal x -conflict. By Lemma 11, we have either $j = i - 1$ or $j = i + 1$; without loss of generality, assume $j = i + 1$. Consider the i -th row a_i in matrix A . By definition of A , we have that $a_i \cdot x > 0$ is equivalent to the left inequality of $\{v_i, v_j\}$. If the left inequality of $\{v_i, v_j\}$ is resolved, then v_i lies outside the cell of v_j ; see Fig. 12. Analogously, the i -th row b_i in matrix B gives the right inequality $b_i \cdot x > 0$ of $\{v_i, v_j\}$, which implies that v_j lies outside the cell of v_i . Hence, the minimal x -conflict $\{v_i, v_j\}$ is resolved and, by Lemmas 5 and 10, all x -conflicts are resolved in Γ .

Now, suppose that x does not solve both $Ax > 0$ and $Bx > 0$; without loss of generality, assume that some row a_i of A is not resolved, that is, we have

$$x_{i,i+1} \leq \sum_{j=\ell_{i+1}}^{i-1} x_{j,j+1} = x_{\ell_{i+1},i} = x_{\ell_{i+1},i+1} - x_{i,i+1}.$$

Then, v_i lies in $\text{cell}(v_{i+1})$, so the drawing determined by x is not greedy. \square

Note that we can always solve $Ax > 0$ and $Bx > 0$ independently by solving the linear equation systems $Ax = 1$ and $Bx = 1$ via forward substitution, since A and B are triangular. We prove that there is always a vector x solving $Ax > 0$ and $Bx > 0$ simultaneously. Let $C = A + B - I_{m-1}$ be the matrix defined by the values of $c_{i,j}$. We claim that any solution to the linear inequality system $Cx > 0$ is also a solution to both $Ax > 0$ and $Bx > 0$. To see this, consider the inequalities described by the i -th row a_i , b_i , and c_i of A , B , and C , respectively. We have

$$\begin{aligned}
a_i \cdot x > 0 &\Leftrightarrow x_{i,i+1} > \sum_{j=1}^{i-1} (c_{i,j} x_{j,j+1}), \\
b_i \cdot x > 0 &\Leftrightarrow x_{i,i+1} > \sum_{j=i+1}^{m-1} (c_{i,j} x_{j,j+1}), \\
c_i \cdot x > 0 &\Leftrightarrow x_{i,i+1} > \sum_{j=1}^{i-1} (c_{i,j} x_{j,j+1}) + \sum_{j=i+1}^{m-1} (c_{i,j} x_{j,j+1}).
\end{aligned}$$

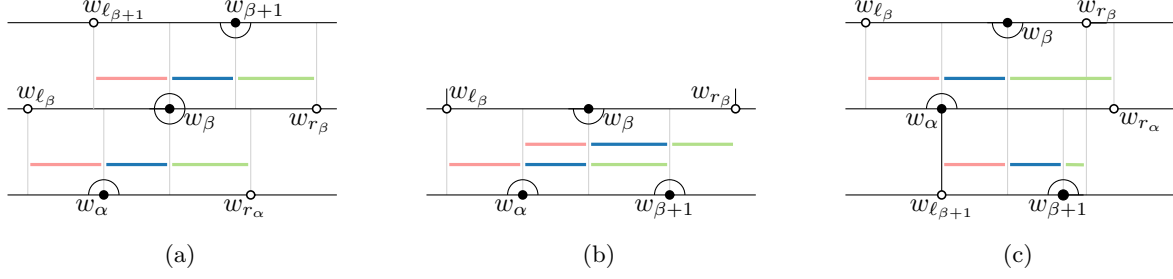


Figure 14: Illustration for the proof of Property 3: $\beta > \alpha + 1$ by assuming that $\beta = \alpha + 1$. (a) $v_{\alpha} \not\prec_x v_{\beta+1}$; (b) $v_{\alpha} \prec_x v_{\beta+1}$; (c) $v_{\alpha} \prec_x v_{\beta+1}$.

So $c_i \cdot x > 0$ implies both $a_i \cdot x > 0$ and $b_i \cdot x > 0$ and our claim follows. We now show that C can be triangulated. For this, we define the *relation graph* corresponding to the adjacency matrix $I_{m-1} - C$ that contains a vertex u_i for each interval $x_{i,i+1}$, $1 \leq i < m$, and a directed edge from a vertex u_i to a vertex u_j if and only if $c_{i,j} = -1$; see Fig. 13.

Lemma 13. *The relation graph of a good st-ordering is acyclic.*

Proof. Let $S_x = v_1, \dots, v_m$ be a good st-ordering of D_x , let A, B, C be the matrices as defined above, and let u_1, \dots, u_m be the vertices of its relation graph. We call a directed edge (u_i, u_j) a *left edge* if $i < j$ and a *right edge* otherwise. Note that a left (right) edge corresponds to a part of a left (right) inequality. We first have to prove the following property for the values $c_{i,j}$ of the matrices A, B, C .

Property 2. *For any $0 < i < j < k < m$, we have $c_{k,j} \leq c_{k,i}$ and $c_{i,j} \leq c_{i,k}$.*

Proof. Assume that $0 = c_{k,j} > c_{k,i} = -1$. By definition of $c_{k,i}$, we have that $i \geq \ell_{k+1}$. But then $j > i \geq \ell_{k+1}$ implies $c_{k,j} = -1$. Furthermore, assume that $0 = c_{i,k} > c_{i,j} = -1$. By definition of $c_{i,j}$, we have that $k \leq r_i$. But then $j < k \leq r_i$ implies $c_{i,k} = -1$. \square

Consider the shortest cycle $u_{\lambda_1}, \dots, u_{\lambda_k}, u_{\lambda_1}$ in the relation graph. Obviously, there is at least one left edge and at least one right edge in the cycle. We will first show that this shortest cycle has length 2. Without loss of generality, assume that $(u_{\lambda_1}, u_{\lambda_2})$ is a right edge; the other case is symmetric. Let i be the smallest number such that $(u_{\lambda_i}, u_{\lambda_{i+1}})$ is a left edge. Then $\lambda_1 < \dots < \lambda_i$, and we consider three cases.

Case I. $\lambda_{i+1} = \lambda_{i-1}$. Then there is a cycle $u_{\lambda_{i-1}}, u_{\lambda_i}, u_{\lambda_{i+1}} = u_{\lambda_{i-1}}$ of length 2.

Case II. $\lambda_{i+1} < \lambda_{i-1} < \lambda_i$. The edge $(u_{\lambda_i}, u_{\lambda_{i+1}})$ is a left edge, so $c_{\lambda_i, \lambda_{i+1}} = -1$. However, by Property 2, $c_{\lambda_i, \lambda_{i-1}} \leq c_{\lambda_i, \lambda_{i+1}} = -1$, so there must be a left edge $(u_{\lambda_i}, u_{\lambda_{i-1}})$. Then there is a cycle $u_{\lambda_{i-1}}, u_{\lambda_i}, u_{\lambda_{i-1}}$ of length 2.

Case III. $\lambda_{i-1} < \lambda_{i+1} < \lambda_i$. The edge $(u_{\lambda_{i-1}}, u_{\lambda_i})$ is a right edge, so $c_{\lambda_{i-1}, \lambda_i} = -1$. However, by Property 2, $c_{\lambda_{i-1}, \lambda_{i+1}} \leq c_{\lambda_{i-1}, \lambda_i} = -1$, so there must be a right edge $(u_{\lambda_{i-1}}, u_{\lambda_{i+1}})$. Hence, there is a shorter cycle $u_{\lambda_1}, \dots, u_{\lambda_{i-1}}, u_{\lambda_{i+1}}, \dots, u_{\lambda_k}$.

From our case analysis, it follows that $k = 2$. Let $\alpha = \lambda_1$ and $\beta = \lambda_2$. Then there are two minimal x -conflicts $\{\alpha, \alpha + 1\}$ and $\{\beta, \beta + 1\}$ with $\alpha < \beta$, $\alpha \geq \ell_{\beta+1}$, and $\beta + 1 \leq r_{\alpha}$. Let $w_{\alpha}, w_{\alpha+1}, w_{\beta}, w_{\beta+1}$ be the responsible vertices for these two x -conflicts with $c_x(w_{\alpha}) = v_{\alpha}$, $c_x(w_{\alpha+1}) = v_{\alpha+1}$, $c_x(w_{\beta}) = v_{\beta}$, and $c_x(w_{\beta+1}) = v_{\beta+1}$. Assume that $w_{\alpha} \prec_y w_{\alpha+1}$; the other case is symmetric. By the definition of x -conflicts, we have $v_{\alpha} \not\prec_x v_{\beta}$ and $v_{\beta} \not\prec_x v_{\beta+1}$. We first show that we cannot have $\beta = \alpha + 1$.

Property 3. $\beta > \alpha + 1$.

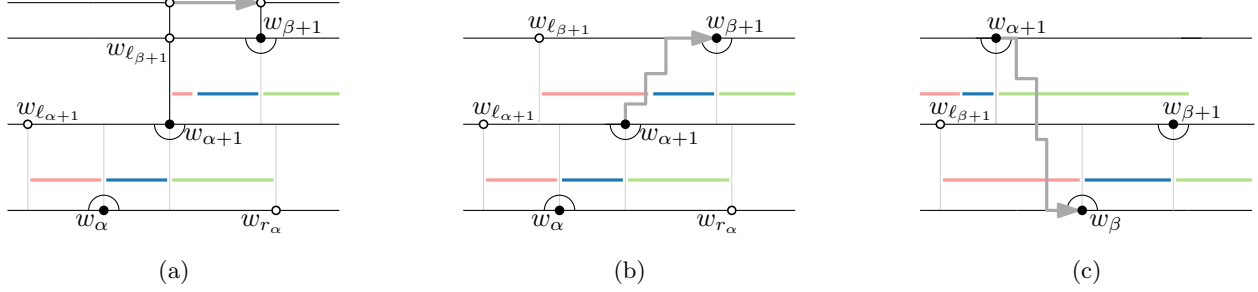


Figure 15: Illustration for Case 1: $\psi_{\beta} < \psi_{\beta+1}$; (a)–(b) The two cases for the path from $v_{\alpha+1}$ to $v_{\beta+1}$ in Case 1.1: $\psi_{\beta+1} > \psi_{\alpha+1}$. In (a) the path starts above $\psi_{\beta+1}$, while in (b) the path starts below $\psi_{\beta+1}$. (c) Case 1.3: $\psi_{\beta} < \psi_{\beta+1} < \psi_{\alpha+1}$. The path from $v_{\alpha+1}$ to v_{β} .

Proof. Assume that $\beta = \alpha + 1$; see Fig. 14. If there is no directed path between v_{α} and $v_{\beta+1}$, then the graph $D_x\langle\alpha, \beta + 1\rangle$ contains three connected components, which contradicts Condition S.1. On the other hand, if there is a directed path between v_{α} and $v_{\beta+1}$, then the graph $D_x\langle\alpha, \beta + 1\rangle$ contains two connected components; one component that contains exactly v_{α} and $v_{\beta+1}$, and one component that contains only v_{β} . However, since $\alpha < \beta < \beta + 1$, this contradicts Condition S.2. \square

We now show some properties on the existence of directed paths between vertices $v_{\alpha}, v_{\alpha+1}, v_{\beta}, v_{\beta+1}$.

Property 4. $v_{\alpha} \prec_x v_{\beta}$.

Proof. Assume that $v_{\alpha} \not\prec_x v_{\beta}$. Consider the graph $D_x\langle\alpha, \beta + 1\rangle$. If $v_{\alpha} \not\prec_x v_{\beta+1}$, then the graph has three connected components, which contradicts Condition S.1. Otherwise, there is a connected component that contains v_{β} and a connected component that contains v_{α} and $v_{\beta+1}$, which contradicts Condition S.2. \square

Property 5. $v_{\alpha+1} \prec_x v_{\beta+1}$.

Proof. Assume that $v_{\alpha+1} \not\prec_x v_{\beta+1}$. Consider the graph $D_x\langle\alpha, \beta + 1\rangle$. If $v_{\alpha} \not\prec_x v_{\beta+1}$, then the graph has three connected components, which contradicts Condition S.1. Otherwise, there is a connected component that contains $v_{\alpha+1}$ and one that contains v_{α} and $v_{\beta+1}$, which contradicts Condition S.2. \square

Property 6. $v_{\alpha+1} \prec_x v_{\beta}$.

Proof. Assume that $v_{\alpha+1} \not\prec_x v_{\beta}$. Since $v_{\alpha} \prec_x v_{\beta}$, by Property 4, it follows that the graph $D_x\langle\alpha, \beta\rangle$ has a connected component that contains $v_{\alpha+1}$ and one that contains v_{α} and v_{β} ; a contradiction to Condition S.2. \square

Assume that we know the exact y -coordinates of every vertex. We now have to analyze the relative positions of the vertices $w_{\alpha}, w_{\alpha+1}, w_{\beta}, w_{\beta+1}$ with y -coordinates $\psi_{\alpha}, \psi_{\alpha+1}, \psi_{\beta}, \psi_{\beta+1}$. We will show that any choice of y -coordinates gives a contradiction. Note that $\psi_{\alpha} < \psi_{\alpha+1}$ by assumption and $\psi_{\beta} \neq \psi_{\beta+1}$ by the x -conflict $\{\beta, \beta + 1\}$. Recall that $\alpha \geq \ell_{\beta+1}$ and $\beta + 1 \leq r_{\alpha}$. Further, let $(w_{\alpha}, w_{r_{\alpha}})$ with $c_x(w_{r_{\alpha}}) = v_{r_{\alpha}}$ be the right horizontal edge of w_{α} , let $(w_{\ell_{\alpha+1}}, w_{\alpha+1})$ with $c_x(w_{\ell_{\alpha+1}}) = v_{\ell_{\alpha+1}}$ be the left horizontal edge of $w_{\alpha+1}$, let $(w_{\beta}, w_{r_{\beta}})$ with $c_x(w_{r_{\beta}}) = v_{r_{\beta}}$ be the right horizontal edge of w_{β} , and let $(w_{\ell_{\beta+1}}, w_{\beta+1})$ with $c_x(w_{\ell_{\beta+1}}) = v_{\ell_{\beta+1}}$ be the left horizontal edge of $w_{\beta+1}$; refer to the definition of the left and right inequalities. We distinguish between the following cases.

Case 1. $\psi_{\beta} < \psi_{\beta+1}$. This implies that $w_{\beta} \prec_y w_{\beta+1}$ and thus w_{β} has a north-oriented flat angle and $w_{\beta+1}$ has a south-oriented flat angle; see Fig. 15.

Case 1.1. $\psi_{\beta+1} > \psi_{\alpha+1}$. By Property 5, $v_{\alpha+1} \prec_x v_{\beta+1}$. If the corresponding path starts in $v_{\alpha+1}$ at a y -coordinate $\geq \psi_{\beta+1}$, then we have that $\alpha \geq \ell_{\beta+1} \geq \alpha + 1$, a contradiction; see Fig. 15a. Otherwise,

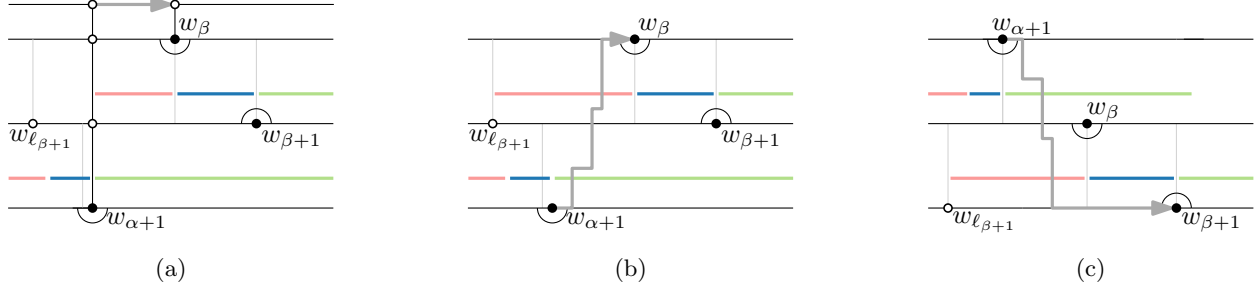


Figure 16: Illustration for Case 2: $\psi_{\beta+1} < \psi_{\beta}$. (a)–(b) The two cases for the path from $v_{\alpha+1}$ to v_{β} in Case 2.1: $\psi_{\beta} > \psi_{\beta+1} > \psi_{\alpha+1}$; in (a) the path starts above $\psi_{\beta+1}$, while in (b) the path starts below $\psi_{\beta+1}$. (c) Case 2.3: $\psi_{\beta+1} < \psi_{\alpha+1}$. The path from $v_{\alpha+1}$ to $v_{\beta+1}$.

since $v_{\beta+1}$ has a south-oriented flat angle, this path has to end at a y -coordinate $\geq \psi_{\beta+1}$ and its last segment is a horizontal segment. Hence, the path has to traverse some point with y -coordinate $\psi_{\beta+1}$ and with x -coordinate between $\alpha + 1$ and $\beta + 1$; see Fig. 15b. However, all of these points lie on the edge $(w_{\ell_{\beta+1}}, w_{\beta+1})$, due to $\ell_{\beta+1} \leq \alpha$, which contradicts planarity.

Case 1.2. $\psi_{\beta+1} = \psi_{\alpha+1}$. Then we have $\ell_{\beta+1} = \alpha + 1 > \ell_{\beta+1}$; a contradiction.

Case 1.3. $\psi_{\beta} < \psi_{\beta+1} < \psi_{\alpha+1}$; see Fig. 15c. By Property 6, $v_{\alpha+1} \prec_x v_{\beta}$. Since $w_{\alpha+1}$ has a south-oriented flat angle and w_{β} has a north-oriented flat angle, the corresponding path has to traverse some point with y -coordinate $\psi_{\beta+1}$ and with x -coordinate between $v_{\alpha+1}$ and v_{β} . However, all of these points lie on the edge $(w_{\ell_{\beta+1}}, w_{\beta+1})$, due to $\ell_{\beta+1} \leq \alpha$, which contradicts planarity.

Case 2. $\psi_{\beta+1} < \psi_{\beta}$. This implies that $w_{\beta+1} \prec_y w_{\beta}$ and thus $w_{\beta+1}$ has a north-oriented flat angle and v_{β} has a south-oriented flat angle; see Fig. 16.

Case 2.1. $\psi_{\beta} > \psi_{\beta+1} > \psi_{\alpha+1}$. By Property 6, $v_{\alpha+1} \prec_x v_{\beta}$. If the corresponding path starts in $v_{\alpha+1}$ at a y -coordinate $> \psi_{\beta+1}$, then we have that $\alpha \geq \ell_{\beta+1} \geq \alpha + 1$, a contradiction; see Fig. 16a. Otherwise, since w_{β} has a south-oriented flat angle, this path has to end at a y -coordinate $\geq \psi_{\beta} > \psi_{\beta+1}$ and its last segment is a horizontal segment. Hence, the path has to traverse some point with y -coordinate $\psi_{\beta+1}$ and with x -coordinate between $\alpha + 1$ and β ; see Fig. 16b. However, all of these points lie on the edge $(w_{\ell_{\beta+1}}, w_{\beta+1})$, due to $\ell_{\beta+1} \leq \alpha$, which contradicts planarity.

Case 2.2. $\psi_{\beta+1} = \psi_{\alpha+1}$. Then we have $\ell_{\beta+1} = \alpha + 1 > \ell_{\beta+1}$; a contradiction.

Case 2.3. $\psi_{\beta+1} < \psi_{\alpha+1}$; see Fig. 16c. By Property 5, $v_{\alpha+1} \prec_x v_{\beta+1}$. Since $w_{\alpha+1}$ has a south-oriented flat angle and $w_{\beta+1}$ has a north-oriented flat angle, the corresponding path has to traverse some point with y -coordinate $\psi_{\beta+1}$ and with x -coordinate between $v_{\alpha+1}$ and $v_{\beta+1}$. However, all of these points lie on the edge $(w_{\ell_{\beta+1}}, w_{\beta+1})$, due to $\ell_{\beta+1} \leq \alpha$, which contradicts planarity.

From the above case analysis, it follows that there is no valid y -coordinate $\psi_{\beta+1}$ in any rectilinear drawing. Thus, there cannot be any cycle in the relation graph and the proof of the lemma follows. \square

From the acyclicity of the relation graph, we show in the following lemma that C is triangularizable.

Lemma 14. *The matrix C is triangularizable.*

Proof. By Lemma 13, the relation graph described by the matrix $I_{m-1} - C$ is acyclic. Hence, there is a permutation matrix P (corresponding to a topological sort) such that $P(I_{m-1} - C)P^{-1}$ is triangulated with only 0's on the diagonal. Thus, $PI_{m-1}P^{-1} - PCP^{-1} = I_{m-1} - PCP^{-1}$ is triangulated with only 0's on the diagonal, so PCP^{-1} is triangulated with only 1's on the diagonal. \square

Since C is triangularizable by Lemma 14, the system of linear equations $Cx = 1$ always has a solution, which solves $Ax > 0$ and $Bx > 0$ simultaneously. This concludes the sufficiency proof for Theorem 7.

5.4 Area requirements of greedy rectilinear drawings

In this subsection we consider the area requirements of greedy drawings of rectilinear representations. The first observation in this direction is that the construction presented in the previous subsection ensures that all the coordinates are integer. However, the area of the produced drawing is in general not minimum, since we just require an interval to be larger than the sum of the length of all intervals of the left and right inequality.

In the following, we strengthen the algorithmic part of the characterization by showing that, given good st -orderings of the two DAGs D_x and D_y , we can construct in polynomial time a rectilinear greedy drawing with minimum area respecting the given st -orderings.

Theorem 8. *Let H be a convex rectilinear representation of a biconnected plane graph and let \mathcal{S}_x and \mathcal{S}_y be good st -orderings of D_x and D_y . We can compute a greedy drawing of H that respects \mathcal{S}_x and \mathcal{S}_y with minimum area in $O(n^2)$ time.*

Proof. By Theorem 7, there is always a rectilinear greedy drawing of H , and we can construct one by solving the linear equality system $Cx = 1$ as described above for both D_x and D_y . Since all inequalities are necessary and sufficient, by Lemma 12, a solution of minimum area will have the form $x_{i,i+1} = 1$, if there is no minimal x -conflict $\{v_i, v_{i+1}\}$, while otherwise we have:

$$x_{i,i+1} = \max\left\{\sum_{j=i+1}^{r_i-1} x_{j,j+1}, \sum_{j=\ell_{i+1}}^{i-1} x_{j,j+1}\right\} + 1$$

We can find such a solution in quadratic time by solving the following linear program.

$$\text{minimize } \sum_{i=1}^{m-1} x_{i,i+1} \quad \text{subject to } Ax \geq 1 \quad \text{and} \quad Bx \geq 1.$$

Note that the inequalities already imply $x \geq 1$. By the acyclicity of the constraints, there is always a solution that satisfies $a_i x_i = 1$ or $b_i x_i = 1$ for each $1 \leq i < m$, where a_i and b_i correspond to the i -th row of the matrices A and B , respectively; hence, the linear program will assign to each $x_{i,i+1}$ the value $x_{i,i+1} = \max\left\{\sum_{j=i+1}^{r_i-1} x_{j,j+1}, \sum_{j=\ell_{i+1}}^{i-1} x_{j,j+1}\right\} + 1$, which is an integer².

For the running time, we first have to find all minimal x -conflicts. To this end, we only have to check whether two consecutive nodes in the st -orderings have an x -conflict; this can clearly be done in linear time per node pair, so in $O(n^2)$ time in total. Then, we have to create the matrices A , B , and C , which have at most $m - 1$ rows and columns each (since D_x and D_y might have fewer nodes than H). This takes $O(n^2)$ time each. In order to triangularize C , we have to compute a topological order on the DAG defined by the adjacency matrix $I_m - C$; this can be done in $O(n^2)$ time using, e.g., depth-first search. Finally, we can solve the linear program in polynomial time.

In general, it is not known whether the linear program can be solved in $O(n^2)$ time; the best-known bound is $O^*(n^\omega)$ where $\omega \approx 2.372$ is the current matrix multiplication time [7]. However, we can reduce the runtime for finding a rectilinear greedy drawing of H with minimum area by solving the inequalities “by hand”. Let $A^* = (a_1^*, \dots, a_m^*)^\top = PAP^{-1}$ and $B^* = (b_1^*, \dots, b_m^*)^\top = PBP^{-1}$. Let $C^* = (a_1^*, b_1^*, \dots, a_m^*, b_m^*)^\top$ and $x^* = xP^{-1}$. Obviously, the following linear program is equivalent to the one above and since both A^* and B^* are upper triangulated, we can solve it bottom-up two rows at a time in $O(n^2)$ time.

$$\text{minimize } \sum_{i=1}^{m-1} x_{i,i+1} \quad \text{subject to } C^* x^* \geq 1$$

We can also use a more algorithmical approach. We can assign the values to each $x_{i,i+1}$ already while using the topological sort to triangulate $I_m - C$; according to this topological sort, we can assign $x_{i,i+1} = 1$

²Formally, one would have to prove that the constraint matrix is totally unimodular, from which we refrain here since the fact that we obtain an integral solution should be clear.

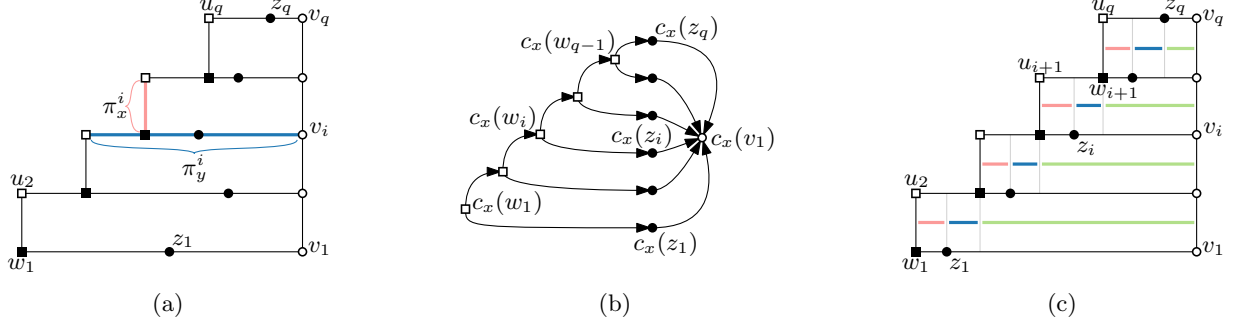


Figure 17: Illustration for the proof of Theorem 9. (a) Rectilinear representation H ; (b) The DAG D_x ; (c) A good st -ordering and its constraints.

to all sources of the DAG $I_m - C$ and the maximum of $\sum_{j=i+1}^{r_i-1} x_{j,j+1} + 1$ and $\sum_{j=\ell_{i+1}}^{i-1} x_{j,j+1} + 1$ to all non-sources. By this, all linear inequalities are resolved and the minimality follows by the necessity of the constraints. Since the DAG $I_m - C$ has at most $O(m^2)$ edges, this algorithm works in $O(n^2)$ time. \square

In the following we show that, although minimum, the area of the drawings produced by our algorithm may be non-polynomial in some cases; namely, Theorem 9 states that there exist convex rectilinear representations whose DAGs admit good st -orderings, but there is no combination of them resulting in a succinct greedy drawing, since the solutions of the corresponding system of inequalities are always exponential in the input size. Observe that, on the contrary, every universal greedy rectilinear representation of an n -vertex graph is succinct, since by Corollary 2 it has a (greedy) drawing of minimum area on an integer grid of size $O(n^2)$ [5, 26].

Theorem 9. *There exist rectilinear representations whose every greedy rectilinear drawing has exponential area, even if D_x and D_y are series-parallel.*

Proof. We first describe a rectilinear representation H , and then we show that it satisfies the properties of the statement; see Fig. 17a. The vertex set of H consists of four sets v_1, \dots, v_q , w_1, \dots, w_{q-1} , z_1, \dots, z_q , and u_2, \dots, u_q , connected as follows. Vertices v_1, \dots, v_q belong to a vertical path $\pi_x(v)$, so that they appear in this order from bottom to top. Then, for each $i = 2, \dots, q-1$, we add a horizontal path $\pi_y^i = u_i, w_i, z_i, v_i$ such that these vertices appear in this left-to-right order. Also, we add a horizontal path $\pi_y^1 = w_1, z_1, v_1$ and a horizontal path $\pi_y^q = u_q, z_q, v_q$ such that these vertices appear in this left-to-right order. Finally, for each $i = 1, \dots, q-1$, we add a vertical path π_x^i composed of a single edge (w_i, u_{i+1}) . Observe that H is convex.

We now consider DAGs D_x and D_y , and their possible good st -orderings. The first observation is that D_y is a directed path from the vertex $c_y(v_1)$ corresponding to π_y^1 to the vertex $c_y(v_q)$ corresponding to π_y^q . Thus, D_y admits a unique st -ordering, which is trivially good. As for D_x , it consists of the series-parallel graph depicted in Fig. 17b, whose unique source is the vertex $c_x(w_1)$ corresponding to π_x^1 and whose unique sink is the vertex $c_x(v_1)$ corresponding to $\pi_x(v)$. Although D_x admits several st -orderings, we claim that only two of them are good.

Observe that D_x contains a directed path $c_x(w_1), \dots, c_x(w_{q-1}), c_x(z_q), c_x(v_1)$, and thus these vertices appear in this order in any st -ordering. Thus, the only possible st -orderings differ by the placement of vertices $c_x(z_1), \dots, c_x(z_{q-1})$. Note that $c_x(z_{q-1})$ must appear after $c_x(w_{q-1})$ in any st -ordering. Thus, if we consider the subgraph of D_x induced by the vertices from the one following $c_x(w_{q-1})$ to the one preceding $c_x(v_1)$ in any st -ordering, we always have a connected component consisting only of $c_x(z_{q-1})$, and another connected component consisting only of $c_x(z_q)$. This implies that no other vertex can be placed after $c_x(w_{q-1})$, as otherwise the resulting st -ordering would not be good. Consider now the subgraph of D_x induced by the vertices from the one following $c_x(w_{q-2})$ to the one preceding $c_x(v_1)$ in any st -ordering. Again, we have already two connected components, namely one consisting only of $c_x(z_{q-2})$ and one consisting of $c_x(w_{q-1}), c_x(z_{q-1})$, and $c_x(z_q)$. This implies that no other vertex can be placed after $c_x(w_{q-2})$, as otherwise the

resulting *st*-ordering would not be good. In particular, this implies that $c_x(w_{q-2})$, $c_x(z_{q-2})$, and $c_x(w_{q-1})$ are consecutive in any good *st*-ordering of D_x . Repeating this argument for every $i \leq q-2$, we obtain that $c_x(w_1), c_x(z_1), c_x(w_2), c_x(z_2), \dots, c_x(w_{q-2}), c_x(z_{q-2})$, and $c_x(w_{q-1})$ are consecutive in any good *st*-ordering of D_x . Thus, there exist only two good *st*-orderings, which only differ for the position of $c_x(z_{q-1})$ with respect to the positions of $c_x(w_q)$ and of $c_x(z_q)$; recall that, in a good *st*-ordering, $c_x(z_{q-1})$ must appear either before or after both of $c_x(w_q)$ and $c_x(z_q)$.

Assume that $c_x(z_{q-1})$ appears before $c_x(z_q)$ in the good *st*-ordering; see Fig. 17c. The other case is analogous. For ease of notation, we say $w_q := z_q$. By the good *st*-ordering, we have:

$$x(z_i) > x(u_i)$$

for every $1 \leq i < q$. Recall the definitions of minimal *x*-conflict and right inequality from Section 5. For each $1 \leq i < q$, we have a minimal *x*-conflict between $c_x(z_i)$ and $c_x(w_{i+1})$ with $r_{c_x(z_i)} = c_x(v_1)$. This gives us the right inequality:

$$\begin{aligned} x(w_{i+1}) - x(z_i) &> x(v_1) - x(w_{i+1}) \\ \Leftrightarrow x(v_1) - x(z_i) &> 2(x(v_1) - x(w_{i+1})) >_{(5.4)} 2(x(v_1) - x(z_{i+1})) \end{aligned} \quad (1)$$

Hence, we obtain:

$$x(v_1) - x(z_1) >_{(1)} 2(x(v_1) - x(z_2)) > \dots >_{(1)} 2^{q-1}(x(v_1) - x(z_q)). \quad (2)$$

Since a solution to the right inequalities is necessary for a rectilinear greedy drawing by Lemma 12, any greedy drawing of H must satisfy Equation 2. However, by $q \in \Omega(n)$, this implies a lower bound on the area of $2^{\Omega(n)}$. \square

5.5 A linear-time algorithm for a special family of instances

We conclude the section by presenting an algorithm to efficiently test, for a meaningful subset of instances, whether a rectilinear representation is greedy realizable. In particular we show that, when an *st*-graph is series-parallel, it is possible to test efficiently whether it admits a good *st*-ordering, and thus satisfies the condition of the characterization presented in Theorem 7.

Theorem 10. *Let H be a convex rectilinear representation of a biconnected plane graph. If D_x and D_y are series-parallel, we can test in $O(n)$ time if H is greedy realizable. If the test succeeds, a greedy drawing of H is computed in $O(n^2)$ time.*

Proof. By Theorem 7, we need to check whether both D_x and D_y admit a good *st*-ordering. We show how to check this for D_x in linear time, the algorithm for D_y is the same.

Consider the recursive construction of D_x through series and parallel compositions. For the base case, notice that a graph consisting of a single edge trivially has a good *st*-ordering. Let D_x be composed of a set of subgraphs D_1, \dots, D_k , forming a parallel or a series composition. If we assume that if D_x is composed by a parallel (resp. series) composition, then each of D_i was composed by series (resp. parallel) composition. A construction with this property can be obtained by considering each composition to be maximal.

First note that, if D_1, \dots, D_k form a parallel composition, then either $k = 2$ or $k = 3$ and one of D_1, D_2, D_3 is a single edge. In fact, let s and t be the source and sink of D_1, \dots, D_k . Thus, for any *st*-ordering $\mathcal{S}_x = v_1, \dots, v_m$ of D_x , it holds that $s = v_1$, $t = v_m$, and for each internal vertex u of a component in D_1, \dots, D_k , we have $u = v_q$, for some $1 < q < m$. This implies that, for each component in D_1, \dots, D_k that is not a single edge, there exists a connected component in $D_x \setminus \{1, m\}$. Hence, both $k > 3$ and $k = 3$ where none of D_1, D_2, D_3 is a single edge would violate Condition S.1 of a good *st*-ordering.

Consider now a parallel composition between two vertices s and t consisting of exactly two components D_1 and D_2 that are not a single edge. Let D° denote the subgraph of D_x induced by the nodes $V(D_x) \setminus \{s, t\}$.

Recall that, by Condition S.2, all nodes of D_1° must precede all nodes of D_2° in a good *st*-ordering, or vice versa. Consider the case in which all nodes of D_1° precede those of D_2° , the other one is analogous. We claim that this results in a good *st*-ordering only if D_1° has a single sink and D_2° has a single source. This follows from the observation that, for any set of sinks of D_1° and sources of D_2° , it is possible to find a pair of nodes v_p and v_q in any *st*-ordering $\mathcal{S}_x = v_1, \dots, v_m$ such that each of these sources/sinks define a connected component in $D_x \langle p, q \rangle$; thus, if there exist more than two sources/sinks, then there exists no good *st*-ordering. On the other hand, if D_1° has only one sink and D_2° only one source, none of the conditions for a good *st*-ordering are violated. From the above discussion, it follows that the only two checks to perform are whether either D_1° has only one sink and D_2° only one source, and vice versa. If one of the checks succeeds, we compute a good *st*-ordering of D_1° and of D_2° , and we merge them according to the result of the check; otherwise, we reject the instance.

When D_1, \dots, D_k form a series composition, the number of components D_1, \dots, D_k and their structure can be arbitrary. We construct good *st*-orderings of D_1, \dots, D_k recursively and merge them in a good *st*-ordering of D_x .

To conclude, the necessary and sufficient condition for D_x to have a good *st*-ordering is that at every parallel composition either exactly two components are merged or exactly three components are merged and additionally one of them is a single edge, one has a single source and one has a single sink. This condition can be checked in time linear to the number of nodes of D_x . The time complexity for the construction of a rectilinear greedy drawing follows from Theorem 8. \square

6 Conclusions and Open Problems

In this work, we introduced rectilinear greedy drawings, i.e., planar greedy drawings in the orthogonal drawing style with no bends. Our work reveals several interesting open problems.

1. The main problem raised by our work is whether we can test in polynomial time whether whether a rectilinear representation is greedy realizable. Due to our characterization, this is equivalent to asking whether a planar DAG admits a good *st*-ordering.
2. For the aforementioned open problem, we provided a linear-time testing algorithm when the DAG is series-parallel. As a further step towards an answer to our main open problem, it is worth studying the special case in which the DAG has only one source and one sink.
3. It is known that not all degree-4 plane graphs admit a rectilinear representation, while all of them have an orthogonal representation with bends [26]. This motivates to extend the study to greedy orthogonal drawings with bends along the edges.
4. Given a biconnected plane graph G (that is, without prescribed values for the geometric angles around each vertex), what is the complexity of deciding whether G admits a (universal) greedy rectilinear representation? This question pertains the intermediate step of the topology-shape-metrics approach [26].

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