Timing Matters: Online Dynamics in Broadcast Games^{*}

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Abstract

A central question in algorithmic game theory is to measure the inefficiency (ratio of costs) of Nash equilibria (NE) with respect to socially optimal solutions. The two established metrics used for this purpose are price of anarchy (POA) and price of stability (POS), which respectively provide upper and lower bounds on this ratio. A deficiency of these metrics, however, is that they are purely existential and shed no light on which of the equilibrium states are reachable in an actual game, i.e., via natural game dynamics. This is particularly striking if these metrics differ significantly in value, such as in network design games where the exponential gap between the best and worst NE states originally prompted the notion of POS in game theory (Anshelevich *et al.*, FOCS 2002). In this paper, we make progress toward bridging this gap by studying network design games under natural game dynamics.

First we show that in a completely decentralized setting, where agents arrive, depart, and make improving moves in an arbitrary order, the inefficiency of NE attained can be polynomially large. To the best of our knowledge, this is the first demonstration of an NE with polynomial inefficiency that can actually be attained starting at an empty state. This negative result implies that the game designer must have some control over the interleaving of these events (arrivals, departures, and moves) in order to force the game to attain efficient NE. We complement our negative result by showing that if the game designer is allowed to execute a sequence of improving moves to create an equilibrium state after every batch of agent arrivals or departures, then the resulting equilibrium states attained by the game are exponentially more efficient, i.e., the ratio of costs compared to the optimum is only logarithmic. This result is obtained by a careful dual charging argument where the goal of the improving moves executed by the game designer is to dissipate the excessive charge accumulated on any region of the network by the previous phase of agent arrivals or departures. Overall, our two results establish that in network games, the efficiency of equilibrium states is dictated by whether agents are allowed to join or leave the game in arbitrary states, an observation that might be useful in analyzing the dynamics of other classes of games with divergent POS and POA bounds.

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1 Introduction

In multi-agent systems where different agents have competing objectives, it is well-known that selfish behavior can lead to suboptimal system performance. A natural question is to quantify how the system performs at a stable state or equilibrium that is consistent with selfish interests of users, relative to an optimal solution designed by a central authority. The *Price of Anarchy* (POA), that captures the relative performance of the worst possible equilibrium, and the *Price of Stability* (POS), that captures the relative performance of the best possible equilibrium, are two successful and widely applied concepts developed to address this question. In a scenario where these two measures are close to each other, they provide a satisfactory resolution to understanding the quality of stable states the system is expected to reach. On the other hand, when these two measures differ significantly, the system exhibits multiple equilibria with highly varying performance, and they do not adequately address whether or not good performance would be achieved. A natural direction is to understand the quality of equilibria that can be reached organically by the agents via some dynamics. More generally, what is the minimal guidance put in place by a central authority so as to guarantee that the relative quality of the equilibrium reached is close to the best possible, that is, the price of stability?

We study these questions on *broadcast games* which form a subclass of a more general class of congestion games called *network design games* that were introduced by Anshelevich *et al.* [1]. In a broadcast game, we are given a rooted undirected graph with costs on the edges. Every vertex has an agent residing on it and the goal of the agent is to select a path to connect to the root. The cost of the edges thus selected must be paid collectively by the agents using those edges. The Shapley cost sharing scheme stipulates that the cost of each edge is divided equally among all the agents using that edge; an agent's total *shared cost* is the sum of her cost share on the edges along her selected path. A state of the system, i.e., a *routing solution*, is defined by a collection of paths, one for each agent. The state is in Nash equilibrium (or NE) if no agent can lower her shared cost by unilaterally changing her routing path. The existence of NE in broadcast games is proved through a potential function argument, originally given by Rosenthal [17, 15]. The social cost of a solution is the sum of the costs of all edges contained in it. Observe that the socially optimal solution is a minimum spanning tree (MST) of the graph, and thus both the POA and POS are defined relative to its cost.

Anshelevich *et al.* [1] observed that there exist instances of the broadcast game with equilibria whose cost differs from the optimum by a factor of $\Omega(n)$, where *n* denotes the size of the graph. In other words, the POA of the game can be as large as $\Omega(n)$. However, examples achieving this lower bound are somewhat artificial for two reasons. First, there exist alternative NE states in every example that are much more efficient: following a long line of work [9, 13, 14], Bilo *et al.* [5] showed that every instance of the game contains an equilibrium whose cost is within a constant factor of the optimum, i.e., the POS is $O(1)^1$. This implies that broadcast games display multiple equilibria of widely varying quality. Second, there are no known natural (e.g., best response) dynamics that lead to these inefficient NE states. Given that we know of the existence of both efficient and inefficient equilibria, an intriguing question is which of these would be attained via natural game dynamics. For example, if starting from an empty graph agents are allowed to enter the game and choose their strategies sequentially, what can we say about the quality of equilibria that emerge in such situations?

Our Results. We consider the evolution of the state in a broadcast game under the following dynamics. Starting with an empty graph, we allow the following events to modify the routing solution.

• Arrival: A set of new agents joins the game; each new agent chooses her best response (least shared cost) path given the current solution.

¹Appendix D provides examples illustrating these bounds.

- Departure: A subset of existing agents leaves the game.
- Move: An agent changes her routing path to decrease her shared cost. We will call this an *improving move* (or if there is no scope of confusion, simply a move).

Our goal is to understand whether, and under what assumptions, the system can reach good quality (i.e., low social cost) equilibrium states. We assume that edge costs satisfy the triangle inequality and compare the cost of the equilibrium reached to OPT, defined to be the MST of all the vertices in the graph².

We first show that if the central authority does not place any restrictions on the dynamics by which the game evolves, then the equilibrium reached can be significantly worse off than the social optimum.

Theorem 1. For any large enough integer n, there exists an instance of the broadcast game with n vertices and a sequence of arrivals and departures that terminates in an NE of cost $\Omega(n^{1/3})$ times that of the minimum spanning tree on all the vertices.

A crucial feature of the instances we construct for the proof of Theorem 1 is that the dynamics consists only of arrivals and departures, with no improving moves in between. Although intermediate states are far from being in equilibrium (i.e., many agents want to change their paths), no improving moves are allowed until the sequence of arrivals and departures ends. When all of the arrivals and departures are done, the resulting final state is in equilibrium with significantly higher cost than the social optimum.

This lower bound shows that the central authority must have some control over the arrival/departure events in order to ensure good quality equilibrium states. In fact, we show a sharp dependence of the efficiency of NE states reachable via the above dynamics on the *timing* of the arrival and departure events. In particular, we consider the following dynamics: if an arrival or departure event moves the system out of equilibrium, the central authority is allowed to restore equilibrium through a sequence of improving moves before the next batch of arrivals/departures happens. The sequence of arrivals and departures is otherwise allowed to be arbitrary, indeed adversarial, as in the lower bound instance. We call this dynamics *equilibrium-preserving* (EQ-P) and show the following result.

Theorem 2. For every instance of the broadcast game using EQ-P dynamics, the system converges to NE of $cost O(\log n)$ times that of the minimum spanning tree on all the vertices.

1.1 Our Techniques

We first outline our techniques for the upper bound (Theorem 2), which is our main technical result. At a very high level, our argument relies on structural properties of states reachable in EQ-P dynamics that we prove via induction. We then employ a charging argument for the cost of any such state against a family of dual solutions for the minimum spanning tree over the underlying graph.

Our main construct for charging the cost of a solution is a family of $O(\log n)$ dual solutions³ for the minimum spanning tree. The *j*th dual in the family is a partition of the vertices into subsets of diameter (roughly) 2^j whose "centers" are at a distance of (roughly) 2^j from one another; each such subset is called a

²Observe that because we allow agents to leave the game, at the end of a sequence of moves, many vertices in the graph may not have any active agents residing at them. A natural goal then is to argue that the cost of the equilibrium reached at the end of the sequence is comparable to the cost of the minimum Steiner tree over all vertices with an active agent. However, even in extremely simple graphs, the gap between the cost of the equilibrium and the minimum Steiner tree can be as large as $\Omega(k)$, where k is the number of terminals active in the end. See Appendix D for an example. A more reasonable comparison, then, is against the minimum cost tree spanning all vertices that ever contained an active agent.

³Each dual solution serves as a lower bound on the MST.

level j cut. For any state of the system where the routing paths form a tree, we make every vertex in the tree "responsible" for the first edge on its unique tree path to the root. We charge the cost of an edge of length ℓ to a cut of diameter roughly $\ell/4$, namely the cut in level $\lfloor \log(\ell/4) \rfloor$ that contains the vertex responsible for that edge. Our main goal is to show that for any equilibrium reachable via EQ-P dynamics, each cut in the dual family is charged at most once; we call such an equilibrium a "balanced equilibrium". A simple accounting then bounds the cost of such an equilibrium within $O(\log n)$ of the cost of the MST.

To get a feel for how we might maintain such a balanced charging, consider a simple case where we start from a balanced equilibrium and an agent arrives at a new vertex v. The agent picks a path to connect to the root that, without loss of generality, consists of a new edge from v to a vertex, say w, in the existing routing tree, and the tree path from w to the root. The new edge potentially places an extra charge on a cut C that was already being charged by a different vertex, say u, previously. Thus, our charging is out of balance. We show that it must be the case that u benefits from changing its current routing path to the path obtained by using edge (u, v) followed by v's new path to the root: while u must now pay the cost of unilaterally using edge (v, w). Vertex u's descendants in the routing tree likewise follow u to the new path; we collectively call this sequence of improving moves a *tree-follow* move. This fixes the issue of overcharging of the cut C. Of course, if u moves to this new path, it creates a new edge (u, v), which in turn potentially overcharges a different cut C'. We then repeat the above argument for C', and so on, till a balanced state is restored.

The key invariant in the above argument is that only one cut is overcharged at a time. If multiple new arrivals happen all at once, however, this invariant no longer holds. Nevertheless, we can argue that any overcharging is done by leaf nodes, i.e. nodes that do not have any descendants in the routing tree because they are freshly arrived. This observation allows us to argue as before that for any cut that is overcharged, one of the vertices charging it has an improving move that removes this extra charge. We can now distinguish between states of the routing tree that are "balanced", and those that are "leaf-unbalanced", meaning in the latter case that every cut is charged by at most one non-leaf, but potentially many leaves. From leaf-unbalanced states we carry out a carefully ordered sequence of improving moves with the goal of eventually reaching a balanced state. Unfortunately, some of these moves apply to non-leaves, and can lead to a state where a cut is charged simultaneously by two non-leaf vertices. We show, however, that in any state reachable within EQ-P dynamics, at most one cut can be charged by multiple non-leaf vertices, and never by more than two non-leaf vertices. We call such states "non-leaf-unbalanced".

The departure of one or more agents does not affect the charging of cuts, but may lead to the introduction of Steiner vertices. We continue to hold Steiner vertices responsible for the first edge on their tree path to the root. This does not create a problem, as our family of dual solutions covers the entire graph and its total cost is bounded against the cost of the MST rather than a minimum Steiner tree.

Putting everything together, we argue that EQ-P dynamics cycles through four types of states – non-leafunbalanced, leaf-unbalanced, balanced, and balanced-equilibrium. In states of the first three types, we can always find an improving move leading to one of the four types of states. Each improving move leads to a decrease in the standard potential function for the game [17] (discussed in more detail in the following section). Therefore, the sequence of moves terminates within a finite number of steps at a balanced-equilibrium state. The cost of this equilibrium can then be charged against the family of dual solutions described above.

For our lower bound for NON-EQ-P dynamics (Theorem 1), the high level idea is to create a POA type of instance in which multiple different agents, that are located much closer to each other relative to the root, nevertheless follow independent paths to the root in the final solution. Such a solution can be made stable by ensuring that each agent is co-located with a large group of other agents. Call this entire set of agents the primary agents in the game, and the corresponding vertices the primary vertices. One challenge with

creating such a state dynamically is that after we have placed an agent at one of the primary vertices in the graph, when we place an agent at another close-by primary vertex, the best path for this agent is to take the short-cut to the first agent, and follow the latter's path to the root, thereby saving on cost via sharing. In order to get around this, and force every agent to take an independent path to the root, we introduce many new "auxiliary" agents at intermediate vertices along the desired path so as to make this path look cheap. We then remove these auxiliary agents so as to continue the process of introducing new primary agents at close-by vertices. The second challenge that arises is to guarantee that the introduction of auxiliary agents does not increase the cost of the MST by too much. In other words, while the collection of independent paths for the primary agents should have a large total cost, there exists an MST covering all intermediate vertices on all of the independent paths at a much lower cost. We achieve this by interconnecting the independent paths in such a manner that these paths successively converge and diverge from each other in a zig-zag fashion from the root to the primary vertices. We present this construction in Appendix A.

1.2 Related Work

Broadcast games form a subclass of a more general class of congestion games called *network design games* that were introduced by Anshelevich *et al.* [1]. The existence of NE in such games is guaranteed by the fact that all congestion games are also potential games [17, 15]. Substantial research effort in the last decade has been spent on bounding the *price of stability* (POS) of broadcast games. The potential function of Rosenthal [17], originally used to show the existence of NE, was also used to prove that the POS is at most $O(\log n)$ [1], which is already an exponential improvement over the POA bound. The POS bound was subsequently improved to $O(\log \log n)$ by Fiat *et al.* [9], further to $O(\log \log n)$ by Lee and Liggett [13], and eventually to a (large) constant by Bilo *et al.* [5] (see also Li [14]).

Chekuri *et al.* [7] initiated the line of inquiry of whether natural game dynamics can lead to efficient NE states. They considered the following two-phase dynamics: in the first phase, agents arrive in sequence and choose their best response path upon arrival, and in the second phase, agents can change their routing path to lower their shared cost (called "moves"). Chekuri *et al.* [7] showed that the resulting equilibrium costs at most $O(\sqrt{n} \log^2 n)$ times the cost of the MST, a factor that was later improved to $O(\log^3 n)$ by Charikar *et al.* [6]⁴. A lower bound of $\Omega(\log n)$ for this ratio was also given by [6], building on a known lower bound for the online Steiner tree problem [12]. A different approach was taken by Balcan *et al.* [3], who considered the problem of influencing game dynamics in network design games with Shapley sharing, in order to achieve a socially efficient equilibrium. In their model, players use expert learning, choosing between a best response expert and a central authority expert suggesting (near-)optimal global behavior. At a high level, our results are also for these two distinct approaches – we show a lower bound for a natural game dynamics with arbitrary arrivals and departures of agents, and then show an exponentially better upper bound if the central authority can suggest moves between successive arrival/departure phases.

The analysis of game dynamics in this paper crucially relies on the construction of a hierarchial family of multiple dual solutions. This method of analysis has been highly influential in designing online algorithms for network design problems. Implicit use of this method dates back to the work of Imase and Waxman [12] on online Steiner trees, and a subsequent line of work of [2, 4, 16]. More recently, this method has been explicitly employed in solving a range of node and edge-weighted Steiner network design problems in the online setting [10, 11, 8]. In terms of the exact techniques, perhaps the closest to our work is that of Umboh [18], who uses hierarchical tree embeddings to analyze greedy-like algorithms for network design problems. In contrast to these applications in competitive analysis where decisions are irrevocable, our

⁴This result applies to a more general setting called *multicast games*, where agents reside at any subset of vertices in the graph.

application in game dynamics allows temporary overcharging of dual solutions, of which we take advantage in this work.

2 Model and EQ-P Dynamics

In this section and the next we describe and analyze EQ-P dynamics for the broadcast game. We first set up our notation and terminology, and prove some basic structural properties that are used in the rest of the paper. Let G = (V, E) be a complete graph, |V| = n, with metric costs $c : V \times V \to \mathbb{R}_+$ defined on the edges. We assume without loss of generality that every vertex has a unique agent (a.k.a. terminal) residing at it. The graph G is revealed via an online process that is divided into *epochs* (indexed by time t). At the start of epoch t, the set of vertices in G that have already appeared is denoted by V_t . We denote the set of active terminals among them by $A_t \subseteq V_t$, i.e., those vertices whose agents are present in the game at the current epochs. Each terminal $v \in A_t$ has a current routing path p_v connecting it to the common root r. The cost share of v along this routing path is the sum of v's cost share over the edges in the path, where the cost of an edge is equally shared between all terminals currently using the edge. In the EQ-P scenario, we further enforce the invariant that the set of paths p_v are in NE, i.e., no terminal has an incentive to unilaterally deviate to a different routing path.

The routing at any time t is defined to be the set of routing paths $(p_v)_{v \in A_t}$. A best response path of a terminal v with respect to a routing, denoted p_v^* , is a path from v to r with the minimum shared cost if v were to move to this path. If there are multiple such paths, we break ties in favor of paths having fewer edges with no terminal other than v using them. Note that this may not break all ties, in which case, any of these paths can be designated as the best response path. A terminal $v \in A$ is said to have an *improving move* with respect to a routing if by moving from its current path p_v to a new path q_v strictly decreases v's cost share. Given a routing, its potential [17] is defined to be $\Phi = \sum_{e \in E} \sum_{i=1}^{N_e} c_e/i$, where N_e is the number of agents using e. A standard argument shows that any improving move decreases the potential by bound that is uniformly bounded away from zero resulting in a finite convergence of our dynamics. The following well-known lemma states that in equilibrium the routing paths always form a tree.

Lemma 3. In equilibrium, the routing paths of a broadcast game form a tree.

Each epoch t is divided into several phases. The first phase consists of an arrival or departure event. In the former case, a new set of terminals $U_t \subseteq V \setminus V_t$ arrive, and the cost of all edges incident on terminals in U_t is revealed. Each new terminal $u \in U_t$ chooses a *best response* routing path p_u . In the latter case, a set of terminals leave, thereby removing the corresponding vertices from the set of terminals A_t . (Note that the corresponding vertices remain in V_t .) Lemmas 9 and 10 establish that the structure of the set of routing paths after arrivals or departures remains a tree.

Both arrival and departure events lead to changes in the cost shares of edges. In the EQ-P scenario, this might lead to a violation of the equilibrium state that was being previously maintained. In this case, the system performs a sequence of *improving moves*, in each of which a terminal changes its routing path in order to reduce its cost share.

Improving moves may temporarily create cycles in the collection of routing paths $\{p_v\}_{v \in A_t}$. We order and group improving moves into contiguous blocks or phases such that every phase ends with the routing paths forming a tree. Furthermore, the trees at the beginning and end of the phase differ in a single pair of edges. The collection of moves in each such phase is called a *tree-follow* move.

Definition 4 (Tree-follow moves). A tree-follow move from u to v in T is a collection of improving moves that start with routing tree T and end with routing tree $T' = T \setminus (u, parent(u)) \cup (u, v)$, where parent(u)

is the parent vertex of u in T. Observe that each terminal in the subtree rooted at u in T reroutes its path to the root in the new rerouting T'. (See Figure 3 and Figure 4 in Appendix E for an example.)

A priori, it is not clear whether improving moves can always be grouped into tree-follow moves. In Lemma 11, we show that in every routing tree T which is not in equilibrium, there exists a sequence of improving moves that collectively form the tree-follow move from u to v for some vertices u and v. In algorithm SELECT TREE MOVE, we use a careful charging scheme to identify the order in which tree-follow moves should be implemented.

Since every vertex in a tree has a unique path to the root, it suffices to specify the tree itself in lieu of all of the routing paths. Henceforth, we will use T_t to denote the tree induced by $\{p_v\}_{v \in A_t}$ without explicitly specifying the paths themselves.

EQ-P Dynamics

1. Initialization.
$$t = 1, V_0 = \{r\}, T_0 = \{r\}, A_0 = \emptyset$$
.

- 2. For $t = 1, 2, \ldots$
 - (Arrivals.) Let U_t be the set of terminals arriving. Let $A_t \leftarrow A_{t-1} \cup U_t$. For each $v \in U_t$, let $p_v = p_v^*$ where p_v^* the best response path with respect to T_{t-1} . Let $T_t = T_{t-1} \cup_{v \in U_t} p_v^*$.
 - (Departures.) Let D_t be the set of terminals arriving. Let $A_t = A_t \setminus D_t$. Let $T_t = \bigcup_{v \in A_t} p_v$.
 - (Tree Follow Moves.) While T_t is not in equilibrium: Use algorithm SELECT TREE MOVE to determine a tree-follow move to implement in T_t ; let this be a move from u to v, and let parent(u) denote the parent of u in T_t . Implement the sequence of improving moves for this tree-follow move to obtain the new routing tree $T_t \leftarrow T_t \setminus (u, parent(u)) \cup (u, v)$.

Because of departure events, the routing tree may contain non-terminal vertices as Steiner vertices. It is convenient to extend the notion of an improving move to vertices that are not terminals. Let $w \notin A$ be a non-terminal vertex. We say that w has an improving move if the following properties hold: (1) There exists a terminal v whose routing path p_v includes w; let p_w denote the segment of p_v between w and r; (2) There exists a path q_w between w and r such that if v were to retain its current routing path from v to w but move from p_w to q_w , then the cost share of v would strictly decrease.

2.1 Charging Scheme and Classification of Tree Solutions

In proving the upper bound for EQ-P dynamics, we use a dual charging scheme to bound the cost of the routing tree. We define the dual and the corresponding lower bound on the optimal cost next. We call a partition $P = (S_1, \dots, S_m)$ of the vertex set V a *level-j* dual for an integer j if it satisfies the following properties:

- P is a partition: $\bigcup_{S \in P} S = V$, and for any $S_a, S_b \in P, S_a \cap S_b = \emptyset$.
- The components have bounded diameter: for any $S \in P$, and any vertices $x, y \in S$, $c(x, y) < 2^{j}$.
- The components are far from each other: there exists a "center" s_i in each component S_i, such that for all S_a, S_b ∈ P, c(s_a, s_b) ≥ 2^{j-1}.

We use the term *cuts* to denote the components S of the partition. The lemma below follows immediately from the observation that any spanning tree over V must connect the centers of all of the cuts in a level-j dual P.

Lemma 5. For any level-j dual P, the cost of the minimum spanning tree OPT is at least $2^{j-1}(|P|-1)$.

In order to bound the cost of an equilibrium resulting from EQ-P, we relate the cost of the edges used in the solution to a family of duals. Let $\Pi = \{P_j\}_{j \in \mathbb{Z}}$ denote a family of partitions, where P_j is a level-j dual.

Our charging scheme for routing solutions that form a tree proceeds as follows. Every vertex in the routing tree is responsible for the cost of its parent edge. Consider an edge e = (v, parent(v)) with length in $[2^{j+2}, 2^{j+3})$ for some $j \in \mathbb{Z}$. We charge the cost of this edge to the cut in the level-j dual that contains v: $S \in P_j$ such that $v \in S$. Our goal is to show that every cut gets charged a small number of times.

Lemma 6. Suppose that our charging scheme charges each cut in the family Π at most once. Then the cost of the solution is at most $O(\log n)$ OPT.

For much of our analysis, we will assume that the dual family Π is provided to us. In Section C we discuss how to construct this family algorithmically as terminals arrive online.

Classification of a Tree Routing. We classify the tree routings reachable via EQ-P dynamics into one of four states depending on the charging structure defined by the solution. We remark that not all tree routings are reachable via EQ-P dynamics, indeed even the set of equilibria obtained is smaller than the set of all equilibria. Let T be a routing tree for some set of active terminals A. We say a vertex u is a leaf (non-leaf) if it is a leaf (non-leaf) in T. Note that all leaves must be terminals, but a non-leaf vertex may or may not be a terminal.

- 1. BALANCED-EQUILIBRIUM: In this state, no terminal (and therefore, no non-terminal vertex in T) has an improving move. Furthermore, every cut is charged at most once. (Note that not every NE is a BALANCED-EQUILIBRIUM state.)
- 2. BALANCED: In this state, some terminals (and potentially non-terminals) may have improving moves, but every cut is charged at most once.
- 3. LEAF-UNBALANCED: In this state, every cut is charged by at most one non-leaf vertex (and any number of leaf terminals). (Recall that leaf vertices in the routing tree are necessarily terminals.)
- 4. NON-LEAF-UNBALANCED: In this state, all but one of the cuts are charged by at most one non-leaf vertex (and any number of leaf terminals). The exceptional cut, that we denote by S^* , is charged by at most two non-leaf vertices, say u and v (and any number of leaf terminals). One of these, u or v, must be the last vertex to have made a (tree-follow) move.

Note that: BALANCED-EQUILIBRIUM \subseteq BALANCED \subseteq LEAF-UNBALANCED \subseteq NON-LEAF-UNBALANCED, where $A \subseteq B$ implies that a routing tree in state A is also in state B.

2.2 Selecting a Tree-Follow Move

To define the tree-follow move performed in a non-equilibrium tree state T, we establish a system of priorities among the improving tree moves based on the current state of the routing tree. A tree follow move of uto v is said to be a *leaf move* if v is a leaf in T, and a *non-leaf move* otherwise. Algorithm SELECT TREE MOVE

- 1. BALANCED-EQUILIBRIUM: No terminal has an improving move. The system can deviate from an equilibrium state only via arrivals or departure events.
- 2. BALANCED: In this state, for any vertex u that has an improving tree move, move u to the closest vertex to which it has an improving move.
- 3. LEAF-UNBALANCED:
 - (a) If there exists a leaf terminal u with a non-leaf move, then make any such move for u.
 - (b) Else, if there exists a non-leaf vertex u with a non-leaf move then move u to the closest such non-leaf v.
 - (c) Else, if there exists a non-leaf vertex u and a leaf terminal v such that u and v are charging the same dual cut, then move u to v. (Claim 7 shows that either u to v or v to u is an improving move; The latter is covered by case (3a).) If there are multiple such leaf terminals v, then make any such move.
 - (d) Else, make any improving move. (This will necessarily be a leaf-to-leaf move by exclusion of the previous three cases.)
- 4. NON-LEAF-UNBALANCED: Let u and v be the non-leaf vertices that are charging the special cut S^* . If u has an improving move to v then move u, else move v, in either case to the closest vertex to which they have an improving move. (We show in Claim 8 that one of u and v has an improving move.)

The validity of the algorithm depends on the following two claims that we prove in Appendix B. The first claim shows that whenever a cut is being charged by a leaf and a non-leaf, at least one of these two vertices has an improving move to the other. In this case, we can find a valid tree-move for Step (3c) of SELECT TREE MOVE. The second claim shows that in a NON-LEAF-UNBALANCED state, whenever a cut is being charged by two non-leaves, at least one of these two vertices has an improving move to the other; we can then find a valid tree-move for Step (4) of SELECT TREE MOVE.

Claim 7. Let the routing tree T be in NON-LEAF-UNBALANCED state but not in BALANCED state. Let u, v be a pair of vertices in T charging the same dual cut D at some level i, where at most one of u or v is a non-leaf vertex in T. Then, at least one of u moving to v and v moving to u is an improving move.

Claim 8. Let T be a routing tree in a NON-LEAF-UNBALANCED but not a LEAF-UNBALANCED state, arrived at by implementing one of the steps (2)–(4) in algorithm SELECT TREE MOVE. Let u and v be the non-leaf vertices charging the special cut S^* . Then either u to v or v to u is an improving move.

3 Analysis of the EQ-P Dynamics

In this section we prove Theorem 2. Some proofs from this section are deferred to Appendix B.

Our argument hinges on a closure property: the epoch starts with the routing tree being in the BALANCED-EQUILIBRIUM state; Lemma 11 argues that whenever the current routing tree is not in equilibrium, at least one improving move exists, and we can use algorithm SELECT TREE MOVE to make a move; Lemma 12 then shows that for the moves made by algorithm SELECT TREE MOVE, the routing tree remains in one of the four states defined above, in particular, it is always in a NON-LEAF-UNBALANCED state. The epoch ends when the routing tree re-enters a BALANCED-EQUILIBRIUM state. At this point, by definition, each dual cut is charged at most once, and therefore, by Lemma 6 the cost of the routing tree is bounded, and Theorem 2 follows. We must also argue termination of the sequence of moves, but this follows directly from a standard potential argument based on the fact that all our moves are improving moves.

The arrival or departure phase. We first consider the situation where the epoch begins with an arrival event. Every arriving terminal u chooses its best response path p_u^* as its current routing path p_u . We claim that the new routing paths $\{p_u\}_{u \in U_t}$ along with the current routing tree T continue to form a tree solution and we end up in a LEAF-UNBALANCED state.

Lemma 9. Suppose a set of new terminals U_t arrive in epoch t when the routing paths of the existing terminals are in an equilibrium state. Then, for each new terminal $u \in U_t$, the chosen routing path p_u comprises a single edge (u, v) connecting u to an existing vertex v in the current routing tree T, and then following the unique path in T from v to r.

We therefore obtain the following lemma.

Lemma 10. After the arrival or departure of a set of terminals in an BALANCED-EQUILIBRIUM state, the routing tree T remains in a LEAF-UNBALANCED state.

Sequence of tree-follow moves. We now consider improving moves made by the algorithm SELECT TREE MOVE. We first observe that for any routing tree that is not at equilibrium, there must exist an improving tree-follow move; see Appendix B for a proof.

Lemma 11. If the routing tree is not in equilibrium, then at least one improving tree-follow move exists.

We therefore have the following simple observation.

Observation 1. In EQ-P dynamics the routing paths at the end of a phase always form a tree.

We are now ready to establish our main technical result of this section, namely that the four states defined in Section 2.1 are closed under EQ-P dynamics.

Lemma 12. Let *T* be the routing tree for which we make an improving tree-move in Step (3) of algorithm EQ-P.

- (*i*) If T is in a BALANCED state but not in a BALANCED-EQUILIBRIUM state, then after the move selected in Step (2) of SELECT TREE MOVE, the resulting routing tree is in a NON-LEAF-UNBALANCED state.
- (*ii*) If T is in a LEAF-UNBALANCED state, then after the move selected in Step (3) of SELECT TREE MOVE, the resulting routing tree is in a NON-LEAF-UNBALANCED state.
- (iii) If T is in a NON-LEAF-UNBALANCED state, then after the move selected in Step (4) of SELECT TREE MOVE, the resulting routing tree is in a NON-LEAF-UNBALANCED state.

Proof. We complete the proof by a detailed case analysis.

- (i) Let T be in a BALANCED state but not a BALANCED-EQUILIBRIUM. Therefore, every dual is being charged at most once in T. After the move, the new tree T' contains exactly one edge not in T. The charging for this edge can introduce at most one dual cut which is charged more than once. Thus the new routing tree is in a NON-LEAF-UNBALANCED state.
- (ii) Let T be a tree in a LEAF-UNBALANCED state. We now consider different cases depending on the move chosen by algorithm SELECT TREE MOVE.
- Step (3a): Suppose the algorithm makes a leaf to non-leaf move in Step (3a), then the only new edge introduced is u's parent edge. Since u remains a leaf in T and its new parent was already a non-leaf vertex, no new non-leaf vertex is introduced. Thus, in the new routing tree, every dual is charged by at most one non-leaf vertex. This implies that the LEAF-UNBALANCED state is preserved.
- **Step** (3b): Suppose there is no improving move in Step (3a), and algorithm makes a non-leaf to non-leaf move in Step (3b) for vertex u. Let the new edge introduced be (u, v) where v is a non-leaf vertex. After this move, the only cut that has an additional non-leaf vertex charging to it is the cut being charged by u (call it S). Prior to this move, S had at most one non-leaf vertex charging it. After u's move, it has at most two non-leaves charging it, with one of them (u) having made the last move. Therefore, T is in a LEAF-UNBALANCED or NON-LEAF-UNBALANCED state.
- Step (3c): Suppose there are no improving moves in Steps (3a) and (3b), and the algorithm performs a nonleaf to leaf move in Step (3c) from u to v. Prior to the move, u and v were charging the same cut, say S at level i. After u's move, u's parent edge, (u, v), is of length < 2ⁱ, and therefore, u charges a cut different from S (call it S'), whereas v continues to charge S.
 Therefore, the only cuts that get charged by new non-leaves after u's move are S and S'. S was previously being charged by a single non-leaf, namely u; Now it is charged by only one non-leaf, namely v. S' was previously being charged by at most one non-leaf (by virtue of the routing tree being in a LEAF-UNBALANCED state), so now it is being charged by at most two non-leaves, one
- **Step** (3d): Finally, consider the scenario where there are no improving moves in any of Steps (3a), (3b), and (3c). In this case, the algorithm makes a leaf to leaf move in Step (3d) from u to v. The only new non-leaf vertex created by the move is v. We first argue that the dual charged by v (say S) does not have a second non-leaf vertex charging it. Since Step (3c) was not executed, it follows that no cut was being charged by both a non-leaf and a leaf vertex before the move (although multiple leaf terminals might be charging the same cut). In particular, v was a leaf vertex charging S before the move, and so, no other non-leaf vertex was charging S before or after the move.

of which is u. Therefore, T is in a LEAF-UNBALANCED or NON-LEAF-UNBALANCED state.

The only other cut that gets a new charge after the move is the cut charged by u's new edge. Since u is a leaf, this cut continues to have at most one non-leaf charging it. Therefore, the routing tree is in a LEAF-UNBALANCED state.

(iii) Let T be in a NON-LEAF-UNBALANCED state. Recall from the definition of Step (4) that u and v are the two non-leaf vertices charging the special cut S^* at some level i, and u has an improving move to v. Then we have $c_{uv} < 2^i$. Suppose that u made the improving move to w, where w can be v. Since u chooses the improving move to the closest vertex, we have $c_{uw} \le c_{uv} < 2^i$. After the move, u must charge a cut, say S whose level is strictly less than i. Thus $S \ne S^*$, and S^* now has only a single non-leaf charging it. Moreover, before the move S had at most one non-leaf charging it. Now, along with u, it can have two non-leaves charging it but one of them, u, has made the last move. Thus, the new tree is in a NON-LEAF-UNBALANCED state.

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A Lower Bound for NON-EQ-P Dynamics

In this section, we will show that if arrivals and departures are allowed at non equilibrium states, then no dynamics can lead to a good equilibrium.

Theorem 1. (Restatement) There exists a graph on n vertices with a sequence σ of $N = \Theta(n^2)$ arrivals and departures such that if every terminal chooses its best path at its time of arrival, then the state of the network after the final arrival is an equilibrium and its cost is $\Omega(N^{1/6})$ times the cost of the minimum spanning tree over all vertices of the underlying graph.

The rest of this section is devoted to proving Theorem 1. In the previous sections, it was convenient to assume that players arrive at distinct vertices. For this section, we will assume that multiple players can arrive at the same vertex. Recall that multiple players at a vertex can be modeled in the single player case using zero cost edges; hence, there is no technical distinction between the two cases.

We construct a family of lower bound instances parameterized by an integer $m \ge 1$. The *m*th instance uses the metric induced by weighted graph G_m (see Fig 1). The vertex set of this graph consists of a root rand m + 1 layers V^0, \ldots, V^m . For $1 \le i \le m$, layer V^i consists of m clusters C_1^i, \ldots, C_m^i , each of which is a clique over m vertices. We use $v_{j,k}^i$ to denote the k-th vertex of C_j^i ; recall that each of i, j, and k take on integral values in [m]. Layer V^0 also consists of m^2 vertices, which are labeled $v_{j,k}^0$ for $j, k \in [m]$, but there are no edges between these vertices. The vertices of V^m are called *end* vertices, and those of V^0 are called *auxiliary* vertices. Observe that the graph G_m has $n = m^2(m+1) + 1$ vertices in all.

Next, we describe the edges. Each pair of vertices within the same cluster C_j^i is connected by an edge of length 1/m for all layers except V^0 . The remaining edges in the graph connect vertices in neighboring layers and are all of length 1. Each auxiliary vertex $v_{j,k}^0$ in V_0 is connected to the root and to its corresponding vertex $v_{j,k}^1$ in layer 1. For $1 \le i \le m-1$, we have an edge $(v_{j,k}^i, v_{k,j}^{i+1})$ for each $j, k \in [m]$. In other words, the vertices of the j-th cluster in layer i are connected to the k-th vertices of the clusters in layer i + 1; in particular, the k-th vertex of the j-th cluster in layer i is connected to the j-th vertex of the k-th cluster in layer i are exactly $m^2(m+1)$ inter-layer edges, and exactly $m^3(m-1)/2$ intra-cluster edges.

Observe that each end vertex $v_{j,k}^m$ has a unique path $P_{j,k}$ to the root that consists of only inter-layer edges (see Figure 1). We call these paths *canonical paths*. Note that each inter-layer edge belongs to exactly one canonical path. In other words, the set of inter-layer edges is a disjoint union of all the canonical paths $P_{j,k}$.



Figure 1: Example for m = 4. Auxiliary vertices are in red, end vertices are in blue. Ovals represent clusters. Intra-cluster edges are shown as dashed edges. The two bold paths starting from the same cluster successively diverge into different clusters and converge into the same cluster on their way to the root.

The cost of the final equilibrium. Our lower bound instance consists of a sequence of arrivals and departures of terminals. Each terminal/player, upon arrival, chooses its best response path to the root at the time of its arrival. Terminals are not allowed to change their chosen path until after the sequence ends. We show that: (1) each terminal chooses its canonical path as its best response path to the root; (2) at the end of the sequence, these paths collectively form an equilibrium, and therefore, no terminal wants to move; (3) this equilibrium state is far from optimal in cost.

Before we describe the sequence of arrivals and departures in full detail, we will analyze the final equilibrium state and its cost relative to the optimal cost. Let OPT denote the cost of the minimum spanning tree over all vertices in G_m . Observe that this is an upper bound on the cost of any optimal solution at the end. The final state following our sequence of arrivals and departures, denoted \mathcal{F} , consists of m players situated at every end vertex $v_{j,k}^m$ in layer m; each player uses the canonical path $P_{j,k}$ to route to the root. The following lemma shows that this is an equilibrium state with a polynomially larger cost relative to OPT.

Lemma 13. State \mathcal{F} is an equilibrium and the cost of \mathcal{F} is $\Omega(m)$ OPT.

Proof. First, we prove that \mathcal{F} is an equilibrium. Consider a player at end vertex $v_{j,k}^m$ with path $P_{j,k}$ and an alternative path p'. For every intra-cluster edge e, we have $N_e(\mathcal{F}) = 0$, and for every inter-layer edge e, we have $N_e(\mathcal{F}) = m$, where N_e denotes the number of terminals using edge e. So the player's current cost

share is $\frac{m+1}{m}$. Path p' contains at least one intra-cluster edge and at least m + 1 inter-layer edges. Thus, the player's cost share when it switches to p' is at least $\frac{1}{m} + \frac{m+1}{m+1} = \frac{m+1}{m}$. Therefore, \mathcal{F} is an equilibrium state. Next, we prove that $c(\mathcal{F}) = \Omega(m)$ OPT. \mathcal{F} consists of all unit-length inter-layer edges, and therefore

Next, we prove that $c(\mathcal{F}) = \Omega(m)$ OPT. \mathcal{F} consists of all unit-length inter-layer edges, and therefore its cost is $m^2(m+1)$. On the other hand, one way of constructing a spanning tree for G_m is to select an arbitrary spanning tree within each of the m^2 cliques, all of the edges from layer 0 to the root, as well as one inter-layer edge per clique connecting it to its preceding layer, say the edge $(v_{j,j}^i, v_{j,j}^{i+1})$ to connect C_j^{i+1} to C_i^i for all $i, j \in [m]$. The total cost of this solution is

$$\frac{1}{m}m^3 + m^2 + m^2 = 3m^2.$$

This concludes the proof.

Sequence of arrivals and departures. The sequence is constructed in *m* phases, each phase consisting of m^2 rounds, one per end vertex $v_{j,k}^m$, and indexed by (j,k). Informally, the objective of each phase is to add one more terminal at each of the end vertices $v_{j,k}^m$. Within round (j,k) in a phase, we use a set of "temporary" terminals whose sole aim is to force the terminal at $v_{j,k}^m$ that arrives at the end of the round to choose the canonical path as its best response. The temporary terminals are introduced at intermediate vertices along the canonical path during the round, and removed at the end of the round.

Formally, let \prec be an arbitrary total order on the pairs (j, k). The sequence σ is constructed to maintain the following invariant: at the end of round (j, k) of phase ℓ , there will be ℓ players on $v_{j',k'}^m$ for $(j', k') \prec (j, k)$, and $\ell - 1$ players on the remaining end vertices. Furthermore, each player on $v_{j,k}^m$ uses the path $P_{j,k}$.

We now specify the subsequence for each round. Consider round (j,k) of phase ℓ . For simplicity of notation, we use v^i to denote the vertex of V^i on $P_{j,k}$. We also use P^i to denote the segment of $P_{j,k}$ starting at v^i and ending at the root. The round consists of m + 1 *iterations*. In iteration $0 \le i \le m - 1$, m^2 players arrive at v^i . In iteration i = m, one player arrives at v^m . Finally, the players on v^0, \ldots, v^{m-1} depart.

Using induction over the terminal arrivals, we now show that for every terminal, the best-response path on arrival is the segment of the canonical path connecting it to the root.

Lemma 14. Consider a terminal arriving at vertex v^i in iteration *i* of round (j,k) in phase ℓ . The bestresponse path of the terminal to the root is the segment of its canonical path P^i .

Proof. We prove the invariant by induction over the sequence of terminal arrivals, ie, over (ℓ, j, k, i) . Moreover, we will only prove the that the inductive hypothesis holds for the first terminal arriving in an iteration; every subsequent terminal will clearly choose the same path as the first terminal since they are arriving at the same vertex. The invariant trivially holds prior to the start of round 1 of phase 1.

Now consider the start of iteration i of round (j, k) in phase ℓ and assume the invariant held in previous iterations, rounds and phases. Let us count the number of terminals on each edge. So far, on each end vertex, there are either ℓ or $\ell - 1$ terminals and each of them chose their canonical path on arrival, by the inductive hypothesis. The inductive hypothesis also tells us that for each i' < i, there are at least m^2 terminals on $v^{i'}$ using the path $P^{i'}$. Thus, each inter-layer edge belonging to P^{i-1} has at least m^2 terminals, and each inter-layer edge that does not belong to P^{i-1} has at most ℓ terminals. Moreover, none of the intra-cluster edges are used by any terminal.

Note that P^i consists of the inter-layer edge (v^i, v^{i-1}) followed by P^{i-1} . Since $|P^{i-1}| \le m$, the cost share of the new terminal at v^i (call this terminal a) on P^i is at most $1/\ell + m/(m^2 + 1) < 1/\ell + 1/m$. Any other path Q for a contains at least two inter-layer edges that do not belong to P^{i-1} and at least one intra-cluster edge, so a's cost share on Q is at least $2/(\ell + 1) + 1/m \ge 1/\ell + 1/m$. Thus, player a's unique best-response path is P^i .

Lemma 14 shows that the sequence of arrivals and departures described above terminates in the final state \mathcal{F} , which costs $\Omega(m)$ OPT by Lemma 13. This completes the proof of Theorem 1.

B Deferred Proofs

We will now present proofs that were skipped in Sections 2 and 3.

Lemma 3. In equilibrium, the routing paths of a broadcast game form a tree.

Proof. The lemma is a direct consequence of the following downward closure property that holds in an equilibrium state. Suppose w is vertex (not necessarily a terminal) that appears on the routing paths p_u and p_v of two terminals u and v respectively. Then, the segment of p_u and p_v between w and r must be identical. For the sake of contradiction, suppose this claim is false, and let q_u and q_v denote the non-identical segments of p_u and p_v between w and r. Assume wlog that the current shared cost of q_u is at most that of q_v . If v now moves to a new routing path that follows p_v until w and then uses q_u to reach r, then the change in cost of v will be the difference of the new shared cost of q_u and the current shared cost of q_v . This is clearly non-positive by our assumption on the relative order of current shared costs of q_u and q_v . In fact, we argue that this difference is negative. First, note that there is at least one edge that is in q_u but not in q_v since the two paths are non-identical. Now, let (x, y) be the closest edge to u in q_u that does not appear in q_v , where x is closer to u than y. Then, x must also appear on q_v , and therefore, cannot appear in the segment of p_v between v and w. It follows that edge (x, y) does not appear in the segment between v and w in p_v , and hence is absent from the entire path p_v . When v moves to its new path, the shared cost on (x, y) decreases below its current value, and therefore, the shared cost of q_u decreases as a whole. This implies that v has an improving move, which contradicts the premise that the terminals are in an equilibrium state. \square

Lemma 6. Suppose that our charging scheme charges each cut in the family Π at most once. Then the cost of the solution is at most $O(\log n)$ OPT.

Proof. Let D denote the largest edge length in the graph. Then, we first note that we may ignore edges in our solution of length at most D/n. This is because there are at most n such edges, and OPT is at least D by the metric property of edge lengths. For the remainder, we charge duals at levels j for $j \in (\log(D/n) - 3, \log D - 2]$. There are at most $\log n + 1$ such duals. The total cost of edges charged to a dual at level-j is at most $2^{j+3}|P| < 32 2^{j-1}(|P| - 1)$. Lemma 5 then implies the result.

Claim 7. Let the routing tree T be in NON-LEAF-UNBALANCED state but not in BALANCED state. Let u, v be a pair of vertices in T charging the same dual cut D at some level i, where at most one of u or v is a non-leaf vertex in T. Then, at least one of u moving to v and v moving to u is an improving move.

Proof. Let p_u and p_v be the routing paths of u and v respectively, where u's parent edge is (u, x) and v's parent edge is (v, y). Then, we have $c_{ux}, c_{vy} \in [2^{i+2}, 2^{i+3})$. Moreover, since u and v belong to the same cut at level $i, c_{uv} < 2^i$. Therefore,

$$c_{ux}, c_{vy} > 4c_{uv}.\tag{1}$$

Let N_e be the number of vertices using edge e. Since at most one of u or v is a non-leaf in T, we can assume wlog that u is a leaf in T (otherwise, the contradiction that we derive below for edge (u, x) can be derived instead for edge (v, y)). In other words, $N_{ux} = 1$.

For the sake of contradiction, let us assume that neither u moving to v, nor v moving to u is an improving move. Since u moving to v is not an improving move, we have the shared cost on path p_u is at most the shared cost for u if it shifts to the path $(u, v) \cup p_v$. Formally, we have

$$c_{ux} + \sum_{e \in p_u \setminus (u,x)} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e}$$

i.e.,
$$c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} + \sum_{e \in p_u \cap p_v} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e}$$

i.e.,
$$c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1}.$$
 (2)

Similarly, since v moving to u is not an improving move, we have

$$\sum_{e \in p_v} \frac{c_e}{N_e} \leq c_{uv} + \frac{c_{ux}}{N_{ux} + 1} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e + 1} + \sum_{e \in p_u \cap p_v} \frac{c_e}{N_e}$$

i.e.,
$$\sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e} \leq c_{uv} + \frac{c_{ux}}{N_{ux} + 1} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e + 1} + \sum_{e \in p_u \cap p_v} \frac{c_e}{N_e}$$

i.e.,
$$\sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} \leq c_{uv} + \frac{c_{ux}}{N_{ux} + 1} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e + 1}.$$
 (3)

Adding inequalities (2) and (3), and replacing $N_{ux} = 1$, we get

$$\begin{split} c_{ux} + \sum_{e \in (p_u \backslash (u,x)) \backslash p_v} \frac{c_e}{N_e} + \sum_{e \in p_v \backslash p_u} \frac{c_e}{N_e} &\leq c_{uv} + \sum_{e \in p_v \backslash p_u} \frac{c_e}{N_e + 1} + c_{uv} + \frac{c_{ux}}{2} + \sum_{e \in (p_u \backslash (u,x)) \backslash p_v} \frac{c_e}{N_e + 1} \\ &\text{i.e.,} \quad c_{ux} \quad \leq \quad 4c_{uv}, \end{split}$$

which contradicts inequality (1). Thus, it must be the case that at least one of u moving to v or v moving to u is an improving move.

We will prove Claim 8 next. Before we restate and prove that claim, we show a stability property of the improving moves made by algorithm SELECT TREE MOVE, namely that moving a terminal u does not create any new improving moves for it.

Claim 15. Suppose that in the current routing tree, vertex u moving to vertex v is **not** an improving move for u. Then, after u moves to some other vertex x (using a tree-follow move), u moving to v is still not an improving move.

Proof. Let p_u and p_v denote the paths connecting u and v respectively to r in the routing tree T after u's move. Let N_e and N'_e be the number of terminals, except u, that are routing through edge e before and after u's tree move respectively. For all edges in p_u , we have $N'_e \ge N_e$; for all other edges in T, $N'_e \le N_e$. Since u moving to v was not an improving move but moving to x was, the shared cost of u if it moved to v would

have been more than its shared cost after the tree move to p_u . Hence,

$$c_{uv} + \sum_{e \in p_v} \frac{c_e}{N_e + 1} > \sum_{e \in p_u} \frac{c_e}{N_e + 1}$$

i.e., $c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e + 1} > \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1} + \sum_{e \in p_u \cap p_v} \frac{c_e}{N_e + 1}$
i.e., $c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} > \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1}$
i.e., $c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e' + 1} > \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e' + 1}$
i.e., $c_{uv} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e' + 1} > \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e' + 1}$
i.e., $c_{uv} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e' + 1} > \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e' + 1} + \sum_{e \in p_u \cap p_v} \frac{c_e}{N_e' + 1}$
i.e., $c_{uv} + \sum_{e \in p_v} \frac{c_e}{N_e' + 1} > \sum_{e \in p_u} \frac{c_e}{N_e' + 1}$.

Hence, u moving to v is not an improving move after u's move to x.

Claim 8. Let T be a routing tree in a NON-LEAF-UNBALANCED but not a LEAF-UNBALANCED state, arrived at by implementing one of the steps (2)–(4) in algorithm SELECT TREE MOVE. Let u and v be the non-leaf vertices charging the special cut S^{*}. Then either u to v or v to u is an improving move.

Proof. By the definition of the NON-LEAF-UNBALANCED state, it must be the case that either u or v was the last to start a tree-follow move. Without loss of generality, say that u was the last vertex to move. After u's move, let p_u and p_v denote the routing paths of u and v, where (u, x) and (v, y) are respectively the parent edges of u and v. Then, we have $c_{ux}, c_{vy} \in [2^{i+2}, 2^{i+3})$. Moreover, $c_{uv} < 2^i$ since u and v both belong to the cut S^* at level i. Then, we have

$$c_{ux}, c_{vy} > 4c_{uv}.\tag{4}$$

We first claim that prior to u's move to x, u did not have an improving move to v. Suppose, for contradiction, that u did have an improving move to v. Then, since (u, v) is shorter than (u, x), and u and v are non-leaves, u's potential move would have triggered Steps (2), (3b), or (4). In each of these cases, u would have preferred the move to the closer vertex v over the move to x. It follows that u to v was not an improving move to x either. We therefore have,

$$c_{ux} + \sum_{e \in p_u \setminus (u,x)} \frac{c_e}{N_e} \le c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e}$$

Here N_e denotes the number of terminals using edge e after u's tree-move.

$$i.e., \quad c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e} \le c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e}$$
$$i.e., \quad c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} \le c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1}.$$
(5)

The rest of the proof is devoted to showing that v moving to u is an improving move. Let us assume for the sake of contradiction that v moving to u is not an improving move either. Then, we have

$$\sum_{e \in p_v} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1} + \sum_{e \in p_u \cap_v} \frac{c_e}{N_e}$$

i.e.,
$$\sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} + \sum_{e \in p_v \cap p_u} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1} + \sum_{e \in p_u \cap_v} \frac{c_e}{N_e}$$

i.e.,
$$\sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} \leq c_{uv} + \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1}$$
(6)

Adding the inequalities (5) and (6), and observing that $N_{ux} \ge 1$, we get

$$\begin{aligned} c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} &\leq c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + c_{uv} + \sum_{e \in p_u \setminus p_v} \frac{c_e}{N_e + 1} \\ \text{i.e.,} \quad c_{ux} + \sum_{e \in (p_u \setminus (u,x)) \setminus p_v} \frac{c_e}{N_e} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e} &\leq 2c_{uv} + \sum_{e \in p_v \setminus p_u} \frac{c_e}{N_e + 1} + \frac{c_{ux}}{N_{ux} + 1} + \sum_{e \in (p_u \setminus p_v) \setminus (u,x)} \frac{c_e}{N_e + 1} \\ &\text{i.e.,} \quad c_{ux} \leq 2c_{uv} + \frac{c_{ux}}{2} \\ &\text{i.e.,} \quad c_{ux} \leq 4c_{uv}, \end{aligned}$$

which contradicts inequality (4). Thus, it must be the case that v moving to u is an improving move.

Lemma 9. Suppose a set of new terminals U_t arrive in epoch t when the routing paths of the existing terminals are in an equilibrium state. Then, for each new terminal $u \in U_t$, the chosen routing path p_u comprises a single edge (u, v) connecting u to an existing vertex v in the current routing tree T, and then following the unique path in T from v to r.

Proof. The lemma has two parts:

- p_u has a single edge (u, v) connecting u to v, and
- the path p_u does not deviate from T between v and r.

The first statement is a direct consequence of our tie-breaking rule for best response paths: u is the only terminal using the portion of p_u from u to T; if this segment consists of a multi-hop path from u to some vertex $v \in T$, short-cutting this segment and using the direct (u, v) edge instead is potentially cheaper and has fewer edges with only u using them.

For the sake of contradiction, suppose the second statement is false. Then we claim that the vertex v has an improving move. This contradicts the fact that the routing paths are in equilibrium when terminal u arrives. To prove the claim, suppose first that v is a terminal. Let p_v denote the path along T from v to the root, and let q_v denote the segment of p_u from v to the root. Let E_v be the set of edges in $p_v \setminus q_v$, and E_u the set of edges in $q_v \setminus p_v$. Let N_e denote the number of terminals using edge e prior to any arrivals in this epoch. Since u follows its best response path, we have that

$$\sum_{e \in E_u} \frac{c_e}{N_e + 1} \leq \sum_{e \in E_v} \frac{c_e}{N_e + 1}.$$

The cost share of v over edges in E_v is $\sum_{e \in E_v} \frac{c_e}{N_e}$, whereas over edges in E_u if v were to switch to taking the path q_v would be $\sum_{e \in E_u} \frac{c_e}{N_e+1}$. From the above inequality, it follows that

$$\sum_{e \in E_u} \frac{c_e}{N_e + 1} < \sum_{e \in E_v} \frac{c_e}{N_e},$$

which implies that v's cost share would improve strictly by switching from p_v to q_v . When v is not a terminal, but is on the current path of some other terminal w, an identical argument shows that w (and therefore v) has an improving move.

Lemma 10. After the arrival or departure of a set of terminals in an BALANCED-EQUILIBRIUM state, the routing tree T remains in a LEAF-UNBALANCED state.

Proof. For an arrival event, this is a direct consequence of Lemma 9. Since every new terminal is a leaf, there is at most one non-leaf vertex charging every cut. After a set of terminal departures, the routing tree clearly remains in a BALANCED state since charges to dual cuts can only decrease. \Box

Lemma 11. If the routing tree is not in equilibrium, then at least one improving tree-follow move exists.

Proof. Let the current routing path of every terminal x be denoted p_x and the union of these paths be tree T. For notational convenience, we also denote the path in the routing tree from a non-terminal vertex x to r by p_x . We first identify a vertex w which has an improving path q_w that contains exactly one arc not in T. Since T is not an equilibrium, there exists a terminal x whose best response path p_x^* differs from p_x . Let (u, v) be the edge that is closest to r on p_x^* and is not in T, with v closer to r than u. Let w be any terminal in the subtree of T rooted at u. We claim the path q_w formed by taking the subpath in T from w to u and then the subpath of p_x^* from u to r is an improving path for w. Indeed, if its shared cost is at least the shared cost of p_w , we can find a path for x whose shared cost is at most the shared cost of p_x^* . Consider the path q_x form x to u and then taking the subpath from u to r. The difference in shared cost for x between q_x and p_x^* equals the difference in shared cost for w in p_w and q_w . Due to our tie-breaking rule, the best response path p_x^* must be strictly better than q_x since q_x has fewer new edges. Thus q_w must be strictly better than p_w as desired.

Now let $T' = T \setminus (u, \operatorname{parent}(u)) \cup (u, v)$ where $\operatorname{parent}(u)$ is the parent of u in T. We now give a series of improving moves which give us the routing tree T'. Order the terminals in subtree of T rooted at u. For each such terminal w change its path from p_w to q_w as defined above. It is clear that the union of all routing paths is exactly T'. It remains to show that they are all improving paths for their respective terminals. From the above argument, for the first terminal w_1 , q_{w_1} is clearly an improving path over p_{w_1} in T. For any latter terminal w in the sequence, the shared cost of q_w has only reduced as compared to its shared cost in the solution T since more terminals are using the edges on the subpath from u to x. Moreover, the shared cost of p_w has only increased as compared to its shared in T as fewer terminals are using the subpath of wfrom u to r. The subpath from w to u remains identical and same number of terminals keep using it and therefore, q_w remains an improving path in the intermediate solution as well for each of the terminals in the sequence.

C Constructing the Dual in an Online Fashion

The classification of tree routings described in Section 2.1 as well as the description of algorithm SELECT TREE MOVE relies on the knowledge of the dual family Π defined in Section 2.1 to which we charge the



Figure 2: Bad Examples for Price of Anarchy and comparison with Minimum Steiner Tree

cost of our solution. If the underlying graph G is known in advance, there are standard techniques for constructing a family of duals with the desired properties. In our setting, the underlying graph may not be known in advance, and may instead be revealed over time as terminals arrive. We now describe a simple greedy procedure for constructing the dual family in an online fashion.

Recall that the graph G is complete and edge lengths form a metric, i.e. they satisfy the triangle inequality. We use c(u, v) to denote the length of the edge between vertices u and v. The graph is revealed vertex by vertex, and every time a vertex is added, all edges between that vertex and previously added vertices are revealed. The level j dual for any integer j is constructed as follows. At any point of time, we have some number of components S_1, \dots, S_m in the dual, with centers s_1, \dots, s_m , respectively. At the beginning when the graph contains a single vertex, we have a single component with that vertex as its center. When the next vertex, say v, arrives, if there exists a center s_i with $c(s_i, v) < 2^{j-1}$, we add v to the *i*th component S_i . Otherwise, we create a new component S_{m+1} with center $s_{m+1} = v$. By construction, it holds that every vertex in component S_i is at a distance less than 2^{j-1} from its center s_i , and therefore, the diameter of the component is less than 2^j . Also, by construction, the distance between any two centers is at least 2^{j-1} . Therefore, the constructed dual satisfies the properties listed in Section 2.1.

D Bad Examples

In Figure 2(a) we give an instance from [1] where the price of anarchy is arbitrary large. In the above example, there are n + 1 agents at u. If the solution picks the edge of weight n instead of the edge of weight 1, a simple check shows that it is still in equilibrium. Since the optimal solution picks the edge of weight 1 instead of weight n, the price of anarchy is n.

In Figure 2(b), we give an instance where the natural dynamics leads to solution that is much more expensive than the minimum Steiner tree. Consider the following sequence of arrivals where each agent picks the best response path on arrival. First n agents arrive on v_{n-1} , then n agents arrive on v_{n-2} , and so on. Finally n agents arrive on u. Observe that in each phase of n arrivals, the best response dynamics

introduces the edge (v_i, v_{i+1}) and thus the solution at the end is the long path from u to r. Now, all the agents, except the n agents at u, depart. Observe that the solution is still in equilibrium since it is identical to the price of anarchy solution in Figure 2. But the weight of minimum Steiner tree is 1 for the agents which survive. This shows that the dynamics can lead to a much costlier solution as compared to the minimum Steiner tree. A more apt comparison is to the cost of the minimum spanning tree over all of the arriving clients. In this case, the minimum spanning tree costs n which is exactly our solution.

E Illustration of a Tree-Follow Move



(a) Solution A (b) Solution B Figure 3: Tree Move from u to v changes the solution from A to B.



Figure 4: Implementing the tree-follow move by individual moves. In Solution A, the current paths for u and its children w_1 and w_2 are shown. In Solution Int1, u changes its path and introduces a new edge (u, v). Vertices w_1 and w_2 follow u's path in Solution Int2 and D, respectively.