Online Unit Covering in Euclidean Space[*]

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Abstract

We revisit the online Unit Covering problem in higher dimensions: Given a set of n points in $\mathbb{R}^d$, that arrive one by one, cover the points by balls of unit radius, so as to minimize the number of balls used. In this paper, we work in $\mathbb{R}^d$ using Euclidean distance. The current best competitive ratio of an online algorithm, $O(2^d d \log d)$, is due to Charikar et al. (2004); their algorithm is deterministic.

(I) We give an online deterministic algorithm with competitive ratio $O(1.321^d)$, thereby sharply improving on the earlier record by a large exponential factor. In particular, the competitive ratios are 5 for the plane and 12 for 3-space (the previous ratios were 7 and 21, respectively). For $d = 3$, the ratio of our online algorithm matches the ratio of the current best offline algorithm for the same problem due to Biniaz et al. (2017), which is remarkable (and rather unusual).

(II) We show that the competitive ratio of every deterministic online algorithm (with an adaptive deterministic adversary) for Unit Covering in $\mathbb{R}^d$ under the L_2 norm is at least $d + 1$ for every $d \geq 1$. This greatly improves upon the previous best lower bound, $\Omega(\log d / \log \log \log d)$, due to Charikar et al. (2004).

(III) We obtain lower bounds of 4 and 5 for the competitive ratio of any deterministic algorithm for online Unit Covering in $\mathbb{R}^2$ and respectively $\mathbb{R}^3$; the previous best lower bounds were both 3.

(IV) When the input points are taken from the square or hexagonal lattices in $\mathbb{R}^2$, we give deterministic online algorithms for Unit Covering with an optimal competitive ratio of 3.

Keywords: online algorithm, unit covering, unit clustering, competitive ratio, lower bound, Newton number.

1 Introduction

Covering and clustering are fundamental problems in the theory of algorithms, computational geometry, optimization, and other areas. They arise in a wide range of applications, such as facility location, information retrieval, robotics, and wireless networks. While these problems have been studied in an offline setting for decades, they have been considered only recently in a more dynamic (and thereby practical) setting. Here we study such problems in a high-dimensional Euclidean space and mostly in the L_2 norm. We first formulate them in the classic *offline* setting.

Problem 1. k-CENTER. *Given a set of n points in $\mathbb{R}^d$ and a positive integer k, cover the set by k congruent balls centered at the points so that the diameter of the balls is minimized.*

The following two problems are dual to Problem 1.

Problem 2. UNIT COVERING. *Given a set of n points in $\mathbb{R}^d$, cover the set by balls of unit diameter so that the number of balls is minimized.*

Problem 3. UNIT CLUSTERING. *Given a set of n points in $\mathbb{R}^d$, partition the set into clusters of diameter at most one so that the number of clusters is minimized.*

Problems 1 and 2 are easily solved in polynomial time for points on the line, i.e., for $d = 1$; but both problems become NP-hard already in Euclidean plane [14, 22]. Factor 2 approximations are known for k-CENTER in any metric space (and so for any dimension) [13, 15]; see also [25, Ch. 5], [26, Ch. 2], while polynomial-time approximation schemes are known for UNIT COVERING for any fixed dimension [17]. However, these algorithms are notoriously inefficient and thereby impractical; see also [2] for a summary of such results and different time vs. ratio trade-offs.

Problems 2 and 3 are identical in the offline setting: indeed, one can go from clusters to balls in a straightforward way; and conversely, one can assign multiply covered points in an arbitrary fashion to unique balls. In regard to their *online* versions, it is worth emphasizing two common properties: (i) a point assigned to a cluster must remain in that cluster; and (ii) two distinct clusters cannot merge into one cluster, i.e., the clusters maintain their identities. In this paper we focus on the second problem, namely *online* UNIT COVERING; we however point out key differences between this problem and online UNIT CLUSTERING.

The performance of an online algorithm ALG is measured by comparing it to an optimal offline algorithm OPT using the standard notion of competitive ratio [3, Ch. 1]. The competitive ratio of ALG is defined as $\sup_\sigma \frac{\mathsf{ALG}(\sigma)}{\mathsf{OPT}(\sigma)}$, where σ is an input sequence of points, $\mathsf{OPT}(\sigma)$ is the cost of an optimal offline algorithm for σ and $\mathsf{ALG}(\sigma)$ denotes the cost of the solution produced by ALG for this input. For randomized algorithms, $\mathsf{ALG}(\sigma)$ is replaced by the expectation $E[\mathsf{ALG}(\sigma)]$, and the competitive ratio of ALG is $\sup_\sigma \frac{E[\mathsf{ALG}(\sigma)]}{\mathsf{OPT}(\sigma)}$. Whenever there is no danger of confusion, we use ALG to refer to an algorithm or the cost of its solution, as needed.

Charikar et al. [8, Sec. 6] studied the online version of UNIT COVERING (under the name of "Dual Clustering"). The points arrive one by one and each point needs to be assigned to a new or to an existing unit ball upon arrival; the L_2 norm is used in $\mathbb{R}^d$, $d \in \mathbb{N}$. The location of each new ball is fixed as soon as it is opened. The authors provided a deterministic algorithm of competitive ratio $O(2^d d \log d)$ and gave a lower bound of $\Omega(\log d / \log \log \log d)$ on the competitive ratio of any deterministic algorithm for this problem. For $d = 1$ a tight bound of 2 is folklore; for $d = 2$ the best known upper and lower bounds on the competitive ratio are 7 and 3, respectively, as implied by the results in [8][1].

The online UNIT CLUSTERING problem was introduced by Chan and Zarrabi-Zadeh [7] in 2006. While the input and the objective of this problem are identical to those for UNIT COVERING, this latter problem is more flexible in that the algorithm is not required to produce unit balls at any time, but rather the smallest enclosing ball of each cluster should have diameter *at most* 1; furthermore, a ball may change (grow or shift) in time. The authors showed that several standard approaches for UNIT CLUSTERING, namely the deterministic algorithms `Centered`, `Grid`, and `Greedy`, all have competitive ratio at most 2 for points on the line ($d = 1$). Moreover, the first two algorithms

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[1]Charikar et al. [8] claim (on p. 1435) that a lower bound of 4 for $d = 2$ under the L_2 norm follows from their Theorem 6.2; but this claim appears unjustified; only a lower bound of 3 is implied. Unfortunately, this misinformation has been carried over also by [7] and [9].

are applicable for UNIT COVERING, with a competitive ratio at most 2 for $d = 1$, as well. These algorithms naturally extend to any higher dimension (including `Grid` provided the L_∞ norm is used).

> **Algorithm Centered.** For each new point p, if p is covered by an existing unit ball, do nothing; otherwise open a new unit ball centered at p.

> **Algorithm Grid.** Build a uniform grid in $\mathbb{R}^d$ where cells are unit cubes of the form $\prod [i_j, i_j + 1)$, where $i_j \in \mathbb{Z}$ for $j = 1, \ldots, d$. For each new point p, if the grid cell containing p is nonempty, put p in the corresponding cluster; otherwise open a new cluster for the grid cell and put p in it.

Since in $\mathbb{R}^d$ each cluster of OPT can be split into at most 2^d grid-cell clusters created by the algorithm, its competitive ratio is at most 2^d, and this analysis is tight for the L_∞ norm. It is worth noting that there is no direct analogue of this algorithm under the L_2 norm.

Some (easy) remarks are in order. Any lower bound on the competitive ratio of an online algorithm for UNIT CLUSTERING applies to the competitive ratio of the same type of algorithm for UNIT COVERING. Conversely, any upper bound on the competitive ratio of an online algorithm for UNIT COVERING yields an upper bound on the competitive ratio of the same type of algorithm for UNIT CLUSTERING.

Related work. UNIT COVERING is a variant of SET COVER. Alon et al. [1] gave a deterministic online algorithm of competitive ratio $O(\log m \log n)$ for this problem, where n is the size of the ground set and m is the number of sets in the family. Buchbinder and Naor [6] obtained sharper results under the assumption that every element appears in at most Δ sets.

Chan and Zarrabi-Zadeh [7] showed that no online algorithm (deterministic or randomized) for UNIT COVERING can have a competitive ratio better than 2 in one dimension ($d = 1$). They also showed that it is possible to get better results for UNIT CLUSTERING than for UNIT COVERING. Specifically, they developed the first algorithm with competitive ratio below 2 for $d = 1$, namely a randomized algorithm with competitive ratio $15/8$. This fact has been confirmed by subsequent algorithms designed for this problem; the current best ratio $5/3$, for $d = 1$, is due to Ehmsen and Larsen [11], and this gives a ratio of $2^d \cdot \frac{5}{6}$ for every $d \geq 2$ (the L_∞ norm is used); their algorithm is deterministic. The appropriate "lifting" technique has been layed out in [7, 29]. From the other direction, the lower bound for deterministic algorithms has evolved from $3/2$ in [7] to $8/5$ in [12], and then to $13/8$ in [20].

Answering a question of Epstein and van Stee [12], Dumitrescu and Tóth [9] showed that the competitive ratio of any algorithm (deterministic or randomized) for UNIT CLUSTERING in $\mathbb{R}^d$ under the L_∞ norm must depend on the dimension d; in particular, it is $\Omega(d)$ for every $d \geq 2$.

Liao and Hu [21] gave a PTAS for a related disk cover problem (another variant of SET COVER): given a set of m disks of arbitrary radii and a set P of n points in $\mathbb{R}^2$, find a minimum-size subset of disks that jointly cover P; see also [23, Corollary 1.1].

Our results. (i) We show that the competitive ratio of `Algorithm Centered` for online UNIT COVERING in $\mathbb{R}^d$, $d \in \mathbb{N}$, under the L_2 norm is bounded by the Newton number of the Euclidean ball in the same dimension. In particular, it follows that this ratio is $O(1.321^d)$ (Theorem 1 in Section 2). This greatly improves on the ratio of the previous best algorithm due to Charikar et al. [8]. The competitive ratio of their algorithm is at most $f(d) = O(2^d d \log d)$, where $f(d)$ is the number of unit balls needed to cover a ball of radius 2 (i.e., the doubling constant). By a volume argument,

$f(d)$ is at least 2^d. In particular $f(2) = 7$ and $f(3) = 21$ [28]; see also [2]. The competitive ratios of our algorithm are 5 in the plane and 12 in 3-space, improving the earlier ratios of 7 and 21, respectively.

(ii) We show that the competitive ratio of every deterministic online algorithm (with an adaptive deterministic adversary) for UNIT COVERING in $\mathbb{R}^d$ under the L_2 norm is at least $d + 1$ for every $d \geq 1$ (Theorem 3 in Section 3). This greatly improves the previous best lower bound, $\Omega(\log d / \log \log \log d)$, due to Charikar et al. [8].

(iii) We obtain lower bounds of 4 and 5 for the competitive ratio of any deterministic algorithm (with an adaptive deterministic adversary) for UNIT COVERING in $\mathbb{R}^2$ and respectively $\mathbb{R}^3$ (Theorems 2 and 3 in Section 3). The previous best lower bounds were both 3.

(iv) For input point sequences that are subsets of the infinite square or hexagonal lattices, we give deterministic online algorithms for UNIT COVERING with an optimal competitive ratio of 3 (Theorems 4 and 5 in Section 4).

Notation and terminology.   For two points $p, q \in \mathbb{R}^d$, let $d(p, q)$ denote the Euclidean distance between them. Throughout this paper the L_2-norm is used. The closed ball of radius r in $\mathbb{R}^d$ centered at point $z = (z_1, \ldots, z_d)$ is

$$B_d(z, r) = \{x \in \mathbb{R}^d \mid d(z, x) \leq r\} = \{(x_1, \ldots, x_d) \mid \sum_{i=1}^{d} (x_i - z_i)^2 \leq r^2\}.$$

A *unit ball* is a ball of unit radius in $\mathbb{R}^d$. The UNIT COVERING problem is to cover a set of points in $\mathbb{R}^d$ by a minimum number of unit balls.

The *unit sphere* is the surface of the d-dimensional unit ball centered at the origin $\mathbf{0}$, namely, the set of points $S_d \subset B_d(\mathbf{0}, 1)$ for which equality holds: $\sum_{i=1}^{d} x_i^2 = 1$. A *spherical cap* $C(\alpha)$ of angular radius $\alpha \leq \pi$ and center P on S_d is the set of points Q of S_d for which $\angle P\mathbf{0}Q \leq \alpha$; see [24].

2   Analysis of `Algorithm Centered` for online unit covering in Euclidean d-space

For a convex body $C \subset \mathbb{R}^d$, the *Newton number* (a.k.a. *kissing number*) of C is the maximum number of nonoverlapping congruent copies of C that can be arranged around C so that each of them is touching C [5, Sec. 2.4]. Some values $N(B_d)$, where $B_d = B_d(\mathbf{0}, 1)$, are known exactly for small d, while for most dimensions d we only have estimates. For instance, it is easy to see that $N(B_2) = 6$, and it is known that $N(B_3) = 12$ and $N(B_4) = 24$. The problem of estimating $N(B_d)$ in higher dimensions is closely related to the problem of determining the densest sphere packing and the knowledge in this area is largely incomplete with large gaps between lower and upper bounds; see [5, Sec. 2.4] and the references therein; in particular, many upper and lower estimates up to $d = 128$ are given in [4] and [10]. In this section, we prove the following theorem.

Theorem 1. *Let $\varrho(d)$ be the competitive ratio of `Algorithm Centered` in $\mathbb{R}^d$ (when using the L_2 norm). Then $\varrho(2) = N(B_2) - 1 = 5$, $\varrho(3) = N(B_3) = 12$, and $\varrho(d) \leq N(B_d)$ for every $d \geq 4$. In particular, $\varrho(d) = O(1.321^d)$.*

A key fact for proving the theorem is the following easy lemma.

Lemma 1. *Let $B = B(o)$ be a unit ball centered at o, that is part of OPT. Let $p, q \in B$ be any two points in B presented to the online algorithm that forced the algorithm to open new balls centered at p and q; refer to Fig. 1. Then $\angle poq > \pi/3$.*
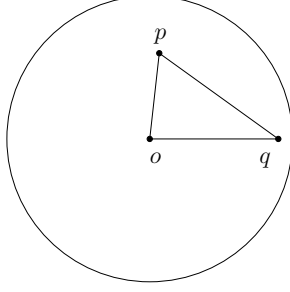
Figure 1: Lemma 1.

Proof. Assume for contradiction that $\alpha = \angle poq \leq \pi/3$. Assume also, as we may, that p arrives before q. Since $q \notin B(p)$, we have $|pq| > 1$. Consider the triangle Δpoq; we may further assume that $\angle opq \geq \angle oqp$ (if we have the opposite inequality, the argument is symmetric). In particular, we have $\angle opq \geq \pi/3$. Since $\angle poq \leq \pi/3$ and $\angle opq \geq \pi/3$, the law of sines yields that $|oq| \geq |pq| > 1$. However, this contradicts the fact that q is contained in B, and the proof is complete. □

Corollary 1. *Let $B = B(o)$ be a unit ball centered at o, that is part of* OPT*. Let $p \in B$ be any point in B presented to the online algorithm that forced the algorithm to open a new ball centered at p; consider the cone $\Psi(p)$ with apex at o, axis $\overrightarrow{op}$, and angle $\pi/6$ around $\overrightarrow{op}$. Then the cones $\Psi(p)$ are pairwise disjoint in B; hence the corresponding caps on the surface of B are also nonoverlapping.*

Proof of Theorem 1. For each unit ball B of OPT we bound from above the number of unit balls opened by `Algorithm Centered` whose center lies in B. Suppose this number is at most A (for each ball in OPT). Since the center of each unit ball opened by the algorithm is a point of the set and all points in the set are covered by balls in OPT, it follows that the competitive ratio of `Algorithm Centered` is at most A.

By Corollary 1 we are interested in the maximum number $A(\alpha)$ of nonoverlapping caps $C(\alpha)$ that can be placed on S_d, for $\alpha = \pi/6$. This is precisely the number of unit balls that can touch a given ball externally without overlapping, namely the Newton number $N(B_d)$ in the same dimension d.

For $d = 2$ we gain 1 in the bound due to the fact that the inequality in Lemma 1 is strict and we are dealing with the unit circle; the five vertices of a regular pentagon inscribed in a unit circle make a tight example with ratio 5; note that the minimum pairwise distance between points is $2\sin(\pi/5) > 1$, and so the algorithm places a new ball for each point. For $d = 3$ the twelve vertices of a regular icosahedron inscribed in a unit sphere make a tight example with ratio 12; note that the minimum pairwise distance between points is $(\sin(2\pi/5))^{-1} > 1$, and the same observation applies. □

Bounds on the Newton number of the ball. A classic formula established by Rankin [24] yields that

$$N(B_d) \leq \frac{\pi^{1/2}\,\Gamma\left(\frac{d-1}{2}\right)}{2\sqrt{2}\,\Gamma\left(\frac{d}{2}\right)\int_0^{\pi/4}(\sin\theta)^{d-2}\left(\cos\theta - \cos\frac{\pi}{4}\right)\mathrm{d}\theta} = A_d^*, \tag{1}$$

and

$$A_d^* \sim \sqrt{\frac{\pi}{8}}\,d^{3/2}\,2^{d/2}. \tag{2}$$

5

More recently, Kabatiansky and Levenshtein [19] have established a sharper upper bound

$$N(B_d) \leq 2^{0.401d(1+o(1))}. \tag{3}$$

In particular, $N(B_d) = O(1.321^d)$. It is worth noting that the best lower known on the Newton number, due to Jenssen et al. [18] is far apart; see also [27].

$$N(B_d) = \Omega\left(d^{3/2} \cdot \left(\frac{2}{\sqrt{3}}\right)^d\right). \tag{4}$$

In particular, $N(B_d) = \Omega(1.154^d)$.

Remark. From an earlier discussion in Section 1, it immediately follows that `Algorithm Centered` has a competitive ratio $O(1.321^d)$ for UNIT CLUSTERING in $\mathbb{R}^d$ under the L_2 norm. It is however worth noting that presently there is no online algorithm for UNIT CLUSTERING in $\mathbb{R}^d$ under the L_∞ norm with a competitive ratio $o(2^d)$. The best one known under this norm (for large d) has ratio $2^d \cdot \frac{5}{6}$ for every $d \geq 2$; note that this is only marginally better than the trivial 2^d ratio.

3   Lower bounds on the competitive ratio for online unit covering in Euclidean d-space

Theorem 3 that we prove in this section greatly improves the previous best lower bound on the competitive ratio of a deterministic algorithm, $\Omega(\log d / \log \log \log d)$, due to Charikar et al. [8].

Previous lower bounds for $d = 2, 3$. To clarify matters, we briefly summarize the calculation leading to the previous best lower bounds on the competitive ratio. Charikar et al. [8] claim (on p. 1435) that a lower bound of 4 for $d = 2$ under the L_2 norm follows from their Theorem 6.2; but this claim appears unjustified; only a lower bound of 3 is implied. The proof uses a volume argument. For a given d, the parameters R_t are iteratively computed for $t = 1, 2, \ldots$ by using the recurrence relation

$$R_{t+1} = \frac{R_t + t^{1/d}}{2}, \text{ where } R_1 = 0. \tag{5}$$

The lower bound on the competitive ratio of any deterministic algorithm given by the argument is the largest t for which $R_t \leq 1$. The values obtained for R_t, for $t = 1, 2, \ldots$ and $d = 2, 3$ are listed in Table 1; as such, both lower bounds are equal to 3.

d	R_1	R_2	R_3	R_4
2	0	0.5	$0.957\ldots$	$1.344\ldots$
3	0	0.5	$0.879\ldots$	$1.161\ldots$

Table 1: Values R_t, for $t = 1, 2, \ldots$

3.1   A new lower bound in the plane

In this section, we deduce an improved lower bound of 4 (an alternative proof will be provided by Theorem 3).

Theorem 2. *The competitive ratio of any deterministic online algorithm for* UNIT COVERING *in the plane (in the L_2 norm) is at least 4.*

Proof. Consider a deterministic online algorithm ALG. We present an input instance σ for ALG and show that the solution $\mathsf{ALG}(\sigma)$ is at least 4 times $\mathsf{OPT}(\sigma)$. Our proof works like a two player game, played by Alice and Bob. Here, Alice is presenting points to Bob, one at a time. Bob (who plays the role of the algorithm) takes the decision whether to place a new disk or not. If a new disk is required, Bob decides where to place it. Alice tries to force Bob to place as many new disks as possible by presenting the points in a smart way. Bob tries to place new disks in a way such that they may cover other points presented by Alice in the future, thereby reducing the need of placing new disks quite often.
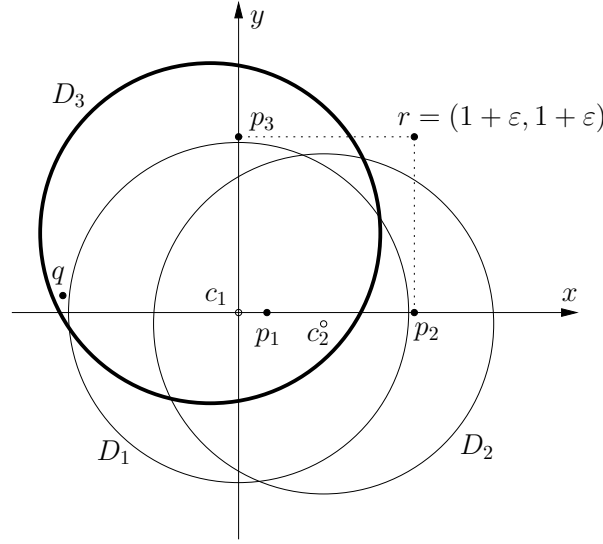

Figure 2: A lower bound of 4 on the competitive ratio in the plane. The figure illustrates the case $p_4 = r$.

The center of a disk D_i is denoted by c_i, $i = 1, 2, \ldots$. The point coordinates will depend on a parameter $\varepsilon > 0$; a sufficiently small $\varepsilon \leq 0.01$ is chosen so that the inequalities appearing in the proof hold. First, point p_1 arrives and the algorithm places disk D_1 to cover it. By symmetry, it can be assumed that $c_1 = (0, 0)$ and $p_1 = (x, 0)$, where $0 \leq x \leq 1$. The second point presented is $p_2 = (1 + \varepsilon^2, 0)$ and, since $p_2 \notin D_1$, a second disk D_2 is placed to cover it. By symmetry, it can be assumed that $y(c_2) \leq 0$. The third point presented is $p_3 = (0, 1 + \varepsilon)$, and neither D_1 nor D_2 covers it; thus a new disk, D_3, is placed to cover p_3.

Consider two other candidate points, $q = (-1 + \varepsilon, \sqrt{2\varepsilon})$ and $r = (1 + \varepsilon, 1 + \varepsilon)$. Since

$$|qc_1|^2 = (-1 + \varepsilon)^2 + 2\varepsilon = 1 + \varepsilon^2 - 2\varepsilon + 2\varepsilon = 1 + \varepsilon^2 > 1,$$

q is not covered by D_1; and clearly r is not covered by D_1. Since

$$|qc_2|^2 \geq (1 - \varepsilon + \varepsilon^2)^2 + 2\varepsilon = 1 + \varepsilon^2 + \varepsilon^4 - 2\varepsilon + 2\varepsilon^2 - 2\varepsilon^3 + 2\varepsilon$$
$$= 1 + 3\varepsilon^2 + O(\varepsilon^3) > 1,$$

q is not covered by D_2; and clearly r is not covered by D_2. Note also that the D_3 cannot cover both q and r, since their distance is close to $\sqrt{5} > 2$. We now specify p_4, the fourth point presented to the algorithm. If q is covered by D_3, let $p_4 = r$, otherwise let $p_4 = q$. In either case, a fourth disk, D_4, is required to cover p_4.

To conclude the proof, we verify that p_1, p_2, p_3, p_4 can be covered by a unit disk.

Case 1: $p_4 = r$. It is easily seen that p_1, p_2, p_3, p_4 can be covered by the unit disk D centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$; indeed, the four points are close to the boundary of the unit square $[0, 1]^2$.

Case 2: $p_4 = q$. Consider the unit disk D centered at the midpoint c of qp_2. We have

$$
\begin{aligned}
|qp_2|^2 &= (2 - \varepsilon + \varepsilon^2)^2 + 2\varepsilon = 4 + \varepsilon^2 + \varepsilon^4 - 4\varepsilon + 4\varepsilon^2 - 2\varepsilon^3 + 2\varepsilon \\
&= 4 - 2\varepsilon + O(\varepsilon^2) < 4.
\end{aligned}
$$

It follows that D covers p_2 and p_4. Note that

$$
c = \left(\frac{\varepsilon + \varepsilon^2}{2}, \sqrt{\frac{\varepsilon}{2}}\right).
$$

We next check the containment of p_1 and p_3.

$$
|cp_1|^2 \le \left(1 - \frac{\varepsilon + \varepsilon^2}{2}\right)^2 + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2} + O(\varepsilon^2) < 1,
$$

thus D also covers p_1. Finally, we have

$$
\begin{aligned}
|cp_3|^2 &\le \left(\frac{\varepsilon + \varepsilon^2}{2}\right)^2 + \left(1 + \varepsilon - \sqrt{\frac{\varepsilon}{2}}\right)^2 \le \varepsilon^2 + \left(1 + \varepsilon - \sqrt{\frac{\varepsilon}{2}}\right)^2 \\
&= \varepsilon^2 + 1 + \varepsilon^2 + \frac{\varepsilon}{2} + 2\varepsilon - \sqrt{2\varepsilon} - \sqrt{2\varepsilon^3} \\
&= 1 - \sqrt{2\varepsilon} + O(\varepsilon) < 1,
\end{aligned}
$$

thus D also covers p_3.

We have shown that $\mathsf{ALG}(\sigma)/\mathsf{OPT}(\sigma) \ge 4$, and the proof is complete. $\qquad\square$

3.2 A new lower bound in d-space

We introduce some additional terminology. For every integer k, $0 \le k < d$, a *k-sphere* of radius r centered at a point $c \in \mathbb{R}^d$ is the locus of points in $\mathbb{R}^d$ at distance r from a center c, and lying in a $(k+1)$-dimensional affine subspace that contains c. In particular, a $(d-1)$-sphere of radius r centered at c is the set of *all* points $p \in \mathbb{R}^d$ such that $|cp| = r$; a 1-sphere is a circle lying in a 2-dimensional affine plane; and a 0-sphere is a pair of points whose midpoint is c. A *k-hemisphere* is a k-dimensional manifold with boundary, defined as the intersection $S \cap H$, where S is a k-sphere centered at some point $c \in \mathbb{R}^d$ and H is a halfspace whose boundary ∂H contains c but does not contain S. For $k \ge 1$, the *relative boundary* of the k-hemisphere $S \cap H$ is the $(k-1)$-sphere $S \cap (\partial H)$ concentric with S; and the *pole* of $S \cap H$ is the unique point $p \in H$ such that $\overrightarrow{cp}$ is orthogonal to the k-dimensional affine subspace that contains $S \cap (\partial H)$. For $k = 0$, a 0-hemisphere consists of a single point, and we define the *pole* to be that point. We make use of the following observation.

Observation 1. *Let S be a k-sphere of radius $1 + \varepsilon$, where $0 \le k < d$ and $\varepsilon > 0$; and let B be a unit ball in $\mathbb{R}^d$. Then $S \setminus B$ contains a k-hemisphere.*

Proof. Without loss of generality, S is centered at the origin, and lies in the subspace spanned by the coordinate axes $x_1, \ldots, x_{k+1}$. By symmetry, we may also assume that the center of B is on the nonnegative x_1-axis, say, at $(b, 0, \ldots, 0)$ for some $b \ge 0$. If $b = 0$, then S and B are concentric and B lies in the interior of S, consequently, $S \setminus B = S$. Otherwise, $S \cap B$ lies in the open halfspace $x_1 > 0$, and $S \setminus B$ contains the k-hemisphere $S \cap \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 \le 0\}$. $\qquad\square$

Theorem 3. *The competitive ratio of every deterministic online algorithm (with an adaptive deterministic adversary) for* UNIT COVERING *in* $\mathbb{R}^d$ *under the* L_2 *norm is at least* $d + 1$ *for every* $d \geq 1$; *and at least* $d + 2$ *for* $d = 2, 3$.

Proof. Consider a deterministic online algorithm ALG. We present an input instance σ for ALG and show that the solution $\mathsf{ALG}(\sigma)$ is at least $d + 1$ times $\mathsf{OPT}(\sigma)$. In particular, σ consists of $d + 1$ points in $\mathbb{R}^d$ that fit in a unit ball, hence $\mathsf{OPT}(\sigma) = 1$, and we show that ALG is required to place a new unit ball for each point in σ. Similarly to the proof of Theorem 2, our proof works like a two player game between Alice and Bob.

Let the first point $p_0 = o$ be the origin in $\mathbb{R}^d$ (we will use either notation as convenient). For a constant $\varepsilon \in (0, \frac{1}{2d})$, let S_0 be the $(d-1)$-sphere of radius $1 + \varepsilon$ centered at the origin o. Refer to Fig. 3. Next, B_0 is placed to cover p_0. The remaining points $p_1, \ldots, p_d$ in σ are chosen adaptively, depending on Bob's moves. We maintain the following two invariants: For $i = 1, \ldots, d$, when Alice has placed points $p_0, \ldots, p_{i-1}$, and Bob placed unit balls $B_0, \ldots B_{i-1}$,

(I) the vectors $\overrightarrow{op_j}$, for $j = 1, \ldots, i - 1$, are pairwise orthogonal and have length $1 + \varepsilon$;

(II) there exists a $(d-i)$-hemisphere $H_i \subset S_0$ that lies in the $(d - i + 1)$-dimensional subspace orthogonal to $\langle \overrightarrow{op_j} : j = 1, \ldots, i - 1 \rangle$ and is disjoint from $\bigcup_{j=0}^{i-1} B_j$.

Both invariants hold for $i = 1$: (I) is vacuously true, and (II) holds by Observation 1 (the first condition of (II) is vacuous in this case).
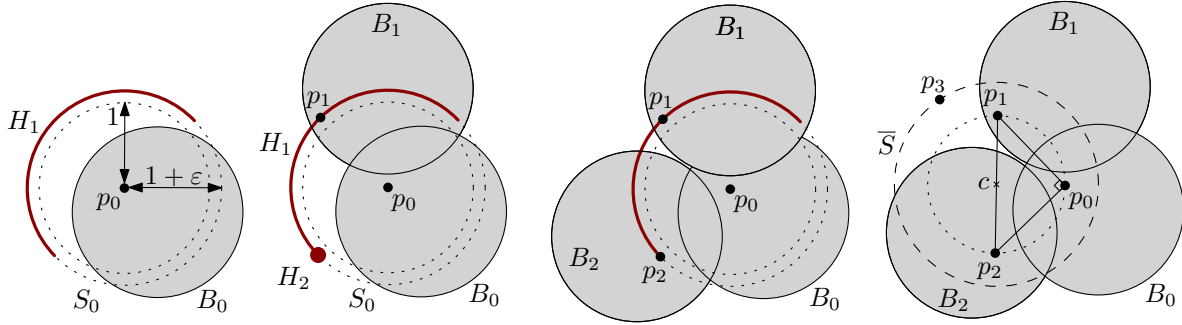

Figure 3: The first three steps of the game between Alice and Bob in the proof of Theorem 3 for $d = 2$. After the 3rd step, Alice can place a 4th point $p_3 \in \overline{S}$ which is not covered by the balls B_0, B_1, B_2.

At the beginning of step i (for $i = 1, \ldots, d$), assume that both invariants hold. Alice chooses p_i to be the pole of the $(d-i)$-hemisphere H_i. By Invariant (II), p_i is not covered by $B_0, \ldots, B_{i-1}$, and Bob has to choose a new unit ball B_i that contains p_i. By Invariant (I), $H_i \subset S_0$, so $|op_i| = 1 + \varepsilon$. By Invariant (II), $\overrightarrow{op_i}$ is orthogonal to the vectors $\overrightarrow{op_j}$, for $j = 1, \ldots, i - 1$. Hence Invariant (I) is maintained.

Let S_i be the relative boundary of H_i, which is a $(d - i - 1)$-sphere centered at the origin. Since p_i is the pole of H_i, $\overrightarrow{op_i}$ is orthogonal to the $(d - i)$-dimensional subspace spanned by S_i. By Observation 1, S_i contains a $(d - i - 1)$-hemisphere that is disjoint from B_i. Denote such a $(d - i - 1)$-hemisphere by $H_{i+1} \subset S_i$. Clearly, H_{i+1} is disjoint from the balls $B_0, \ldots, B_{i-1}, B_i$; so Invariant (II) is also maintained.

By construction, p_i $(i = 1, \ldots, d)$ is not covered by the balls $B_0, \ldots, B_{i-1}$, so Bob has to place a unit ball for each of the $d + 1$ points $p_0, p_1, \ldots, p_d$. By Invariant (I), the points $p_1, \ldots, p_d$ span a regular $(d-1)$-dimensional simplex of side length $(1 + \varepsilon)\sqrt{2}$. By Jung's Theorem [16, p. 46], the

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radius of the smallest enclosing ball of $p_1, \ldots, p_d$ is

$$R = (1+\varepsilon)\sqrt{2} \cdot \sqrt{\frac{d-1}{2d}} < \left(1 + \frac{1}{2d}\right)\sqrt{\frac{d-1}{d}} = \sqrt{\frac{(2d+1)^2(d-1)}{4d^3}} = \sqrt{\frac{4d^3 - 3d - 1}{4d^3}} < 1,$$

and this ball contains the origin p_0, as well.

We next show how to adjust the argument to derive a slightly better lower bound of $d+2$ for $d = 2, 3$. Let B be the smallest enclosing ball of the points $p_0, p_1, \ldots, p_d$, and let c be the center of B. As noted above, the radius of B is $R = (1+\varepsilon)\sqrt{(d-1)/d}$. Let $\overline{S}$ be the $(d-1)$-sphere of radius $2 - R = 2 - (1+\varepsilon)\sqrt{(d-1)/d}$ centered at c. Then the smallest enclosing ball of B and an arbitrary point $p_{d+1} \in \overline{S}$ has unit radius. That is, points $p_0, \ldots, p_d, p_{d+1}$ fit in a unit ball. This raises the question whether Alice can choose yet another point $p_{d+1} \in \overline{S}$ outside of the balls $B_0, \ldots, B_d$ placed by Bob.

For $d = 2$, $\overline{S}$ has radius $2 - (1+\varepsilon)\sqrt{1/2} = 2 - (1+\varepsilon)(\sqrt{2}/2) \geq 1.2928$ (provided that $\varepsilon > 0$ is sufficiently small). A unit disk can cover a circular arc in $\overline{S}$ of diameter at most 2. If 3 unit disks can cover $\overline{S}$, then $\overline{S}$ would be the smallest enclosing circle of a triangle of diameter at most 2, and its radius would be at most $\frac{2}{3}\sqrt{3} \leq 1.1548$ by Jung's Theorem. Consequently, Alice can place a 4th point $p_3 \in \overline{S}$ outside of B_0, B_1, B_2, and all four points $p_0, \ldots, p_3$ fit in a unit disk; see Fig. 3 (right) for an example. That is, $\mathsf{ALG}(\sigma) = 4$ and $\mathsf{OPT}(\sigma) = 1$; and we thereby obtain an alternative proof of Theorem 2.

For $d = 3$, $\overline{S}$ has radius $R_1 = 2 - (1+\varepsilon)(\sqrt{2/3}) \geq 1.1835$ (provided that $\varepsilon > 0$ is sufficiently small). Let c_i denote the center of B_i, for $i = 0, 1, 2, 3$; we may assume that at least one of the balls B_i, say B_0, is not concentric with $\overline{S}$, since otherwise $\bigcup_{i=0}^3 B_i$ would cover zero area of $\overline{S}$. We may also assume for concreteness that cc_0 is a vertical segment; let π_0 denote the horizontal plane incident to c. Then $C = \overline{S} \cap \pi_0$ is horizontal great circle (of radius R_1) centered at c. Note that $C \cap B_0 = \emptyset$, and so if $\bigcup_{i=0}^3 B_i$ covers $\overline{S}$, then $\bigcup_{i=1}^3 (B_i \cap \pi_0)$ covers C. However, the analysis of the planar case ($d = 2$) shows that this is impossible; indeed, we have $R_1 \geq 1.1835 > 1.1548$. Consequently, Alice can place a 5th point $p_4 \in \overline{S}$ outside of B_0, B_1, B_2, B_3, and all five points $p_0, \ldots, p_4$ fit in a unit ball. That is, $\mathsf{ALG}(\sigma) = 5$ and $\mathsf{OPT}(\sigma) = 1$ and a lower bound of 5 on the competitive ratio is implied. □

4 Unit covering for lattice points in the plane

In this section, we give optimal deterministic algorithms for online UNIT COVERING of points from the infinite unit square and hexagonal lattices. The input points always belong to the lattice under consideration. We start with the infinite unit square lattice $\mathbb{Z}^2$.

Theorem 4. *There exists a deterministic online algorithm for online UNIT COVERING of integer points (points in $\mathbb{Z}^2$) with competitive ratio 3. This result is tight: the competitive ratio of any deterministic online algorithm for this problem is at least 3.*

Proof. First, we prove the lower bound; refer to Fig. 4 (left). First, point p_1 arrives and disk D_1 covers it. Observe that D_1 misses at least one point from $\{p_2, p_3\}$, since $|p_2 p_3| = 2\sqrt{2} > 2$. We may assume that D_1 missed p_2; and this further implies that D_1 does not cover p_4 or p_5, since if it would, it would also cover p_2, a contradiction. Now, D_2 is placed to cover p_2. If D_2 covers p_4, next input point is p_5, otherwise it is p_4. In either case a third disk is needed. To finish the proof, observe that $\{p_1, p_2, p_4\}$ or $\{p_1, p_2, p_5\}$ can be covered optimally by a single unit disk; hence the competitive ratio of any deterministic algorithm is at least 3.
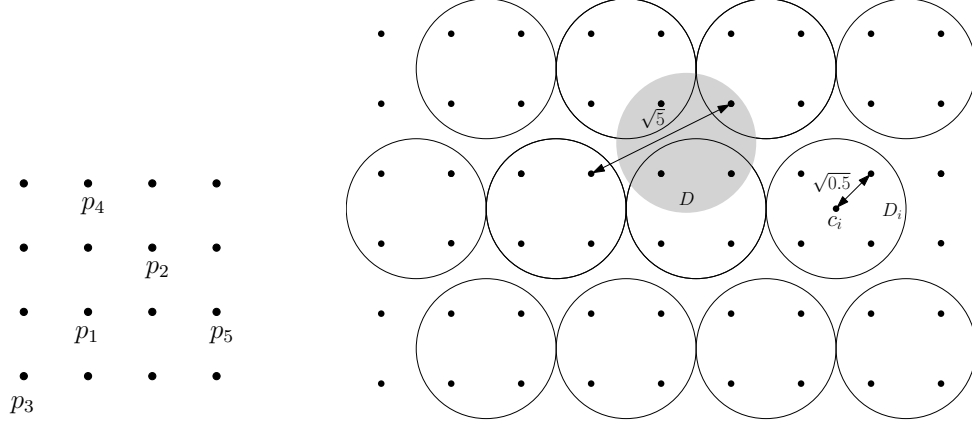
Figure 4: Left: Lower bound for $\mathbb{Z}^2$. Right: Illustration of the upper bound; the disk D is shaded.

Next, we present an algorithm which has competitive ratio 3. Refer to Fig. 4 (right). Partition the lattice points using unit disks as shown in the figure. When a point arrives, use the disk it belongs to in the partition. For the analysis, consider a disk D from an optimal cover. As seen in the figure, D can cover points which belong to at most three disks used for partitioning the lattice. Thus, we conclude that the algorithm has competitive ratio 3. □

In the following, we state our result for the infinite hexagonal lattice.

Theorem 5. *There exists a deterministic online algorithm for online* UNIT COVERING *of points of the hexagonal lattice with competitive ratio* 3. *This result is tight: the competitive ratio of any deterministic online algorithm for this problem is at least* 3.

Proof. The proof is similar to that of Theorem 4. We start by proving the lower bound of 3; refer to Fig. 5 (left). The first point, p_1, arrives and D_1 is used to cover it. D_1 misses at least one of
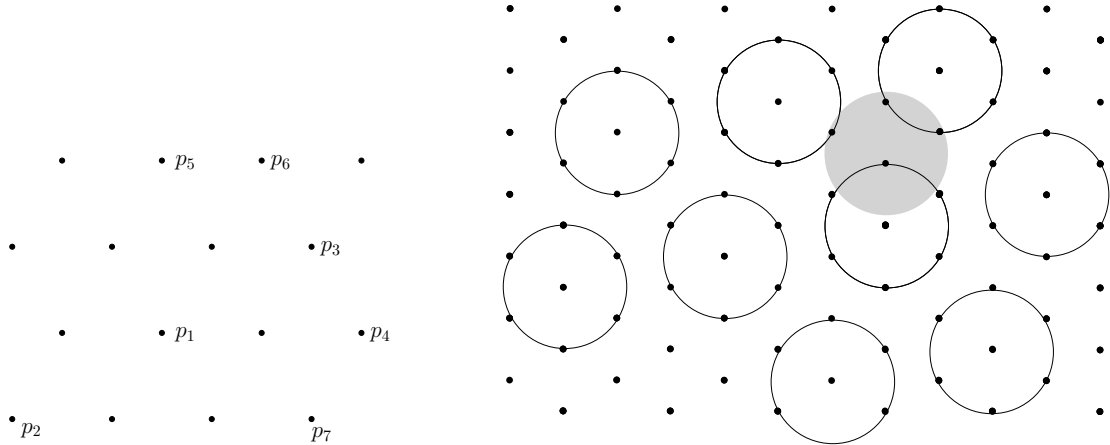

Figure 5: Left: Lower bound for the hexagonal lattice. Right: Illustration of the upper bound.

$\{p_2, p_3\}$, since $|p_2 p_3| = 2\sqrt{3} > 2$. By symmetry, we may assume that D_1 misses p_3. The algorithm uses D_2 to cover p_3. We distinguish two cases:

Case 1: D_2 misses p_4. Since D_1 misses p_3, D_1 also misses p_4. Otherwise, if D_1 covers p_4, then D_1 also covers p_3, a contradiction. The algorithm uses D_3 to cover p_4. Thus the ratio is 3 since

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p_1, p_3, p_4 can be covered by a single disk centered at $(p_1 + p_4)/2$, and the algorithm has used three disks, D_1, D_2, D_3.

Case 2: D_2 covers p_4. This means that D_2 misses p_5. Now, two things may happen:

1. *D_1 misses p_5 too.* Then p_5 is the next input point, and the algorithm uses D_3 to cover it. Here p_1, p_3, p_5 can be covered by a single disk centered at $(p_1 + p_3 + p_5)/3$, but the algorithm has used three disks, D_1, D_2, D_3.

2. *D_1 covers p_5.* Since D_1 does not cover p_3, D_1 cannot cover p_6. If D_2 misses p_6, let p_6 be the third point presented; the algorithm uses D_3 to cover p_6. Here p_1, p_3, p_6 can be covered by a single disk centered at $(p_1 + p_6)/2$, but the algorithm has used three disks, D_1, D_2, D_3. If D_2 covers p_6, let p_7 be the third point presented. Note that D_1 cannot cover p_7 since it covers p_5; also, D_2 cannot cover p_7 since it covers p_6. The algorithm uses D_3 to cover p_7. Here p_1, p_3, p_7 can be covered by a single disk centered at $(p_1 + p_3 + p_7)/3$, but the algorithm has used three disks, D_1, D_2, D_3.

In all cases a lower bound of 3 has been enforced by Alice, as required.

Now, we prove the upper bound of 3. As in the case of the unit square lattice, we partition the lattice points using disks as shown in Fig. 5 (right). Arguing similarly, it can be concluded that the same algorithm has competitive ratio 3 in this case. □

5   Conclusion

Our results suggest several directions for future study. For instance, the gap between the sublinear lower bound and the exponential upper bound in the competitive ratios for online UNIT COVERING is intriguing. We summarize a few specific questions of interest.

Problem 4. *Is there a lower bound on the competitive ratio for* UNIT COVERING *that is exponential in d? Is there a superlinear lower bound?*

Problem 5. *Can the online algorithm for integer points (with ratio 3 in the plane) be extended to higher dimensions, i.e., for covering points in $\mathbb{Z}^d$? What ratio can be obtained for this variant?*

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