

Drawing Clustered Graphs on Disk Arrangements^{*}

Tamara Mchedlidze¹, Marcel Radermacher¹, Ignaz Rutter², and Nina Zimbel¹

¹ Department of Computer Science, Karlsruhe Institute of Technology, Germany

² Department of Computer Science and Mathematics, University of Passau, Germany
mched@iti.uka.de, radermacher@kit.edu, rutter@fim.uni-passau.de

Abstract. Let $G = (V, E)$ be a planar graph and let \mathcal{V} be a partition of V . We refer to the graphs induced by the vertex sets in \mathcal{V} as *clusters*. Let $\mathcal{D}_{\mathcal{C}}$ be an arrangement of disks with a bijection between the disks and the clusters. Akitaya et al. [2] give an algorithm to test whether (G, \mathcal{V}) can be embedded onto $\mathcal{D}_{\mathcal{C}}$ with the additional constraint that edges are routed through a set of pipes between the disks. Based on such an embedding, we prove that every clustered graph and every disk arrangement without pipe-disk intersections has a planar straight-line drawing where every vertex is embedded in the disk corresponding to its cluster. This result can be seen as an extension of the result by Alam et al. [3] who solely consider biconnected clusters. Moreover, we prove that it is \mathcal{NP} -hard to decide whether a clustered graph has such a straight-line drawing, if we permit pipe-disk intersections.

1 Introduction

In practical applications, it often happens that a graph drawing produced by an algorithm has to be post processed by hand to comply with some particular requirements. Thus, the user moves vertices and modifies edges in order to fulfill these requirements. Interacting with large graphs is often time-consuming. It takes a lot of time to group and move the vertices or process them individually and to control the overall appearance of the produced drawing. The problem we study in this paper addresses this scenario. In particular, we assume that a user wants to modify a drawing of a large planar graph G . Instead we provide her with an abstraction of this graph. The user modifies the abstraction and thus providing some constraints on how the drawing of the initial graph should look like. Then our algorithm propagates the drawing of the abstraction to the initial graph so that the provided constraints are satisfied.

More formally, we model this scenario in terms of a (*flat*) *clustering* of a graph $G = (V, E)$, i.e., a partition $\mathcal{V} = \{V_1, \dots, V_k\}$ of the vertex set V . We refer to the pair $\mathcal{C} = (G, \mathcal{V})$ as a *clustered graph* and the graphs G_i induced by V_i as *clusters*. The set of edges E_i of a cluster G_i are *intra-cluster edges* and the set of

^{*} Work was partially supported by grant WA 654/21-1 of the German Research Foundation (DFG).

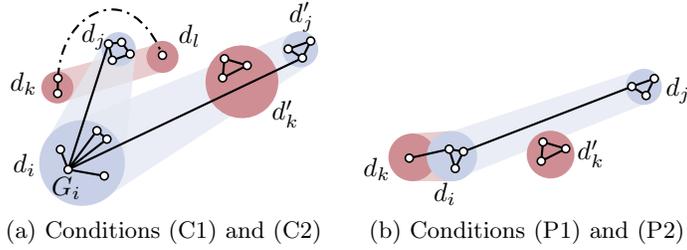


Fig. 1: (a) The blue disk arrangement satisfies the conditions (C1, C2) and (P1, P2). The disks d_k, d_l and d_i, d_j violate condition (C1). The disks d'_k and d_i, d'_j violate (C2). Note that the edge from disk d_i to d'_j has to cross the boundary of d'_k twice. (b) The disks d_k, d_i violate condition (P1) and d'_k and d_i, d_j violate condition (P2).

edges with endpoints in different clusters *inter-cluster edges*. A *disk arrangement* $\mathcal{D} = \{d_1, \dots, d_k\}$ is a set of pairwise disjoint disks in the plane together with a bijective mapping $\mu(V_i) = d_i$ between the clusters \mathcal{V} and the disks \mathcal{D} . We refer to a disk arrangement \mathcal{D} with a bijective mapping μ as a *disk arrangement of \mathcal{C}* , denoted by $\mathcal{D}_{\mathcal{C}}$. A $\mathcal{D}_{\mathcal{C}}$ -*framed drawing of a clustered graph $\mathcal{C} = (G, \mathcal{V})$* is a planar drawing of G where each cluster G_i is drawn within its corresponding disk d_i . We study the following problem: given a clustered planar graph $\mathcal{C} = (G, \mathcal{V})$, an embedding ψ of G and a disk arrangement $\mathcal{D}_{\mathcal{C}}$ of \mathcal{C} , does \mathcal{C} admit a $\mathcal{D}_{\mathcal{C}}$ -framed straight-line drawing homeomorphic to ψ ?

Related Work. Feng et al. [11] introduced the notion of *clustered graphs* and *c-planarity*. A graph G together with a recursive partitioning of the vertex set is considered to be a clustered graph. An embedding of G is *c-planar* if (i) each cluster c is drawn within a connected region R_c , (ii) two regions R_c, R_d intersect if and only if the cluster c contains the cluster d or vice versa, and (iii) every edge intersects the boundary of a region at most once. They prove that a c-planar embedding of a connected clustered graph can be computed in $O(n^2)$ time. It is an open question whether this result can be extended to disconnected clustered graphs. Many special cases of this problem have been considered [8].

Concerning drawings of c-planar clustered graphs, Eades et al. [10] prove that every c-planar graph has a c-planar straight-line drawing where each cluster is drawn in a convex region. Angelini et al. [5] strengthen this result by showing that every c-planar graph has a c-planar straight-line drawing in which every cluster is drawn in an axis-parallel rectangle. The result of Akitaya et al. [2] implies that in $O(n \log n)$ time one can decide whether an abstract graph with a flat clustering has an embedding where each vertex lies in a prescribed topological disk and every edge is routed through a prescribed topological pipe. In general they ask whether a simplicial map φ of G onto a 2-manifold M is a *weak embedding*, i.e., for every $\epsilon > 0$, φ can be perturbed into an embedding ψ_{ϵ} with $\|\varphi - \psi_{\epsilon}\| < \epsilon$.

Godau [12] showed that it is \mathcal{NP} -hard to decide whether an embedded graph has a \mathcal{D}_C -framed straight-line drawing. The proof relies on a disk arrangement \mathcal{D}_C of overlapping disks that have either radius zero or a large radius.

Banyassady et al. [6] study whether the intersection graph of unit disks has a straight-line drawing such that each vertex lies in its disk. They proved that this problem is \mathcal{NP} -hard regardless of whether the embedding of the intersection graph is prescribed or not. Angelini et al. [4] showed it is \mathcal{NP} -hard to decide whether an abstract graph G and an arrangement of unit disks have a \mathcal{D}_C -framed straight-line drawing. They leave the problem of finding a \mathcal{D}_C -framed straight-line drawing of G with a fixed embedding as an open question. Alam et al. [3] prove that it is \mathcal{NP} -hard to decide whether an embedded clustered graph has a c -planar straight-line drawing where every cluster is contained in a prescribed (thin) rectangle and edges have to pass through a defined part of the boundary of the rectangle. Further, they prove that all instances with biconnected clusters always admit a solution. Their result implies that graphs of this class have \mathcal{D}_C -framed straight-line drawings.

Ribó [13] shows that every embedded clustered graph where each cluster is a set of independent vertices has a straight-line drawing such that every cluster lies in a prescribed disk. In contrast to our setting Ribó allows an edge e to intersect a disk of a cluster G_i that does not contain an endpoint of e .

Contribution. A pipe p_{ij} of two clusters V_i, V_j is the convex hull of the disks d_i and d_j , i.e., the smallest convex set of points containing d_i and d_j ; see Fig. 1. We refer to a topological planar drawing of G as an *embedding of G* . A \mathcal{D}_C -framed embedding of G is a \mathcal{D}_C -framed topological drawing of G with the additional requirement that (i) each intra-cluster edge entirely lies in its disk (ii) each inter-cluster edge uv intersects with a pipe p_{ij} if and only if u and v are vertices of the clusters G_i and G_j , respectively, and (iii) each edge crosses the boundary of a disk at most once. This concept is also known as *c -planarity with embedded pipes* [9]. An embedding ψ of G is *compatible with \mathcal{D}_C* if ψ is homeomorphic to a \mathcal{D}_C -framed embedding of G . The result of Akitaya et al. can be used to decide whether an embedding ψ of G is compatible with \mathcal{D}_C .

The following two conditions are necessary, for \mathcal{C} to have a \mathcal{D}_C -framed embedding: (C1) if $(V_i \times V_j) \cap E \neq \emptyset$ and $(V_k \times V_l) \cap E \neq \emptyset$ (i, j, k, l pairwise distinct), then the intersection of the pipes p_{ij} and p_{kl} is empty, and (C2) the set $p_{ij} \setminus d_k$ is connected. Thus, in the following we assume that \mathcal{D}_C satisfies (C1) and (C2). A *planar* disk arrangement additionally satisfies the condition that (P1) the pairwise intersections of all disks are empty, and (P2) $(V_i \times V_j) \cap E \neq \emptyset$, the intersection of p_{ij} with all disks d_k (corresponding to V_k) is empty (i, j, k pairwise distinct). A planar disk arrangement can be seen as a thickening of a planar straight-line drawing of the graph obtained by contracting all clusters.

We prove that every clustered graph (G, \mathcal{V}) with planar disk arrangement \mathcal{D}_C and an \mathcal{D}_C -framed embedding ψ has a \mathcal{D}_C -framed planar straight-line drawing homeomorphic to ψ . Taking the result of Akitaya et al. [2] into account, our result can be used to test whether an abstract clustered graph with connected clusters has a \mathcal{D}_C -framed straight-line drawing. Cluster G_i in Fig. 1 shows that in general

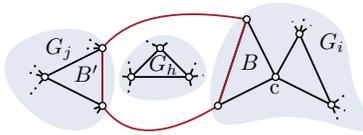


Fig. 2: A planar clustered graph \mathcal{C} that is not simple.

clusters cannot be augmented to be biconnected, if the embedding is fixed. Hence, our result is generalization of the result of Alam et al. [3]. In Section 3 we show that the problem is \mathcal{NP} -hard in the case that the disk arrangements does not satisfy condition (P2). From now on we refer to a planar straight-line drawing of G simply as a drawing of G .

2 Drawing on Planar Disk Arrangements

In this Section we prove that every *simple* clustered graph with a planar disk arrangement $\mathcal{D}_{\mathcal{C}}$ and $\mathcal{D}_{\mathcal{C}}$ -framed embedding has a $\mathcal{D}_{\mathcal{C}}$ -framed drawing. An embedded clustered graph \mathcal{C} is *simple* if for every i, j , there is no cluster $G_h (i, j \neq h)$ embedded in the interior of the subgraph induced by $V_i \cup V_j$; see Fig. 2. Note that this is a necessary condition for the corresponding disk arrangement to be planar. A clustered graph $\mathcal{C} = (G, \mathcal{V})$ is *connected* if each cluster G_i is connected.

We prove the statement by induction on the number of intra-cluster edges. In Lemma 1 we show that we can indeed reduce the number of intra-cluster edges by contracting intra-cluster edges. In Lemma 2, we prove that the statement is correct if the outer face is a triangle and \mathcal{C} is connected. In Theorem 3 we extend this result to clustered graphs whose clusters are not connected.

Let $\mathcal{C} = (G, \mathcal{V})$ with a disk arrangement $\mathcal{D}_{\mathcal{C}}$ and a $\mathcal{D}_{\mathcal{C}}$ -framed embedding ψ . Let uv be an intra-cluster edge of G that is not an edge of a separating triangle. We obtain a *contracted clustered graph* \mathcal{C}/e of \mathcal{C} by removing v from G and connecting the neighbors of v to u . We obtain a corresponding embedding ψ/e from ψ by routing the edges $vw \in E, w \neq u$ close to uv .

Lemma 1. *Let $\mathcal{C} = (G, \mathcal{V})$ be a connected simple clustered graph with a planar disk arrangement $\mathcal{D}_{\mathcal{C}}$ and a $\mathcal{D}_{\mathcal{C}}$ -framed embedding ψ . Let e be an intra-cluster edge that is not an edge of a separating triangle. Then \mathcal{C} has a $\mathcal{D}_{\mathcal{C}}$ -framed drawing that is homeomorphic to ψ if \mathcal{C}/e has a $\mathcal{D}_{\mathcal{C}}$ -framed drawing that is homeomorphic to ψ/e .*

Proof. Let $e = uv$ and denote by u_0, u_1, \dots, u_k the neighbors of u and v_0, v_1, \dots, v_l the neighbors of v in \mathcal{C} . Without loss of generality, we assume that $u_0 = v$ and $v_0 = u$. Since e is not an edge of a separating triangle the set $I := \{u_2, \dots, u_{k-1}\} \cap \{v_2, \dots, v_{l-1}\}$ is empty. Denote by u the vertex obtained by the contraction of e . Let G_i be the cluster of u and v , and let d_i be the corresponding disk in $\mathcal{D}_{\mathcal{C}}$.

Consider a \mathcal{D}_C -framed drawing Γ/e of \mathcal{C}/e homeomorphic to ψ/e . Then there is a small disk $d_u \subset d_i$ around u such that for every point p in d_u moving u to p yields a \mathcal{D}_C -framed drawing that is homeomorphic to ψ/e .

We obtain a straight-line drawing Γ of \mathcal{C} from Γ/e as follows. First, we remove the edges uv_i from Γ/e . The edges u_1, u_k partitions d_u into two regions r_u, r_v such that the intersection of r_v with uu_i is empty for all $i \in \{2, \dots, k-1\}$. We place v in r_v and connect it to u and the vertices v_1, \dots, v_l . Since r_v is a subset of d_u and $I = \emptyset$, we have that the new drawing Γ is planar. Since v is placed in r_v , the edge uv is in between u_1 and u_k in the rotational order of edges around u . Hence, Γ is homeomorphic to ψ . Finally, Γ is a \mathcal{D}_C -framed drawing since, d_u is entirely contained in d_i and thus are u and v . \square

Lemma 2. *Let \mathcal{C} be a connected simple clustered graph with a triangular outer face T , a planar disk arrangement \mathcal{D}_C , and a \mathcal{D}_C -framed embedding ψ . Moreover, let Γ_T be a \mathcal{D}_C -framed drawing of T . Then \mathcal{C} has a \mathcal{D}_C -framed drawing that is homeomorphic to ψ with the outer face drawn as Γ_T .*

Proof. We prove the theorem by induction on the number of intra-cluster edges.

First, assume that every intra-cluster edge of \mathcal{C} is an edge on the boundary of the outer face. Let Γ be the drawing obtained by placing every interior vertex on the center point of its corresponding disk and draw the outer face as prescribed by Γ_T . Since \mathcal{D}_C is a planar disk arrangement and Γ_T is convex, the resulting drawing is planar and thus a \mathcal{D}_C -framed drawing of \mathcal{C} that is homeomorphic to the embedding ψ .

Let S be a separating triangle of \mathcal{C} that splits \mathcal{C} into two subgraphs \mathcal{C}_{in} and \mathcal{C}_{out} so that $\mathcal{C}_{\text{in}} \cap \mathcal{C}_{\text{out}} = S$ and the outer face \mathcal{C}_{out} and \mathcal{C} coincide. Then by the induction hypothesis \mathcal{C}_{out} has the \mathcal{D}_C -framed drawing Γ_{out} with the outer face drawn as Γ_T and \mathcal{C}_{in} as a \mathcal{D}_C -framed drawing Γ_{in} with the outer face drawing as $\Gamma_{\text{out}}[S]$, where $\Gamma_{\text{out}}[S]$ is the drawing of S in Γ_{out} . Then we obtain a \mathcal{D}_C -framed drawing of \mathcal{C} by merging Γ_{in} and Γ_{out} .

Consider an intra-cluster edge e that does not lie on the boundary of the outer face and is not an edge of a separating triangle. Then by the induction hypothesis, \mathcal{C}/e has a \mathcal{D}_C -framed drawing with the outer face drawn as Γ_T . It follows by Lemma 1 that \mathcal{C} has a \mathcal{D}_C -framed drawing homeomorphic to ψ . \square

Theorem 3. *Every simple clustered graph \mathcal{C} with a \mathcal{D}_C -framed embedding ψ has a \mathcal{D}_C -framed drawing homeomorphic to ψ .*

Proof. We obtain a clustered graph \mathcal{C}' from \mathcal{C} by adding a new triangle T to the graph and assigning each vertex of T to its own cluster. Let Γ_T be a drawing of T that contains all disks in \mathcal{D}_C in its interior. We obtain a new disk arrangement \mathcal{D}'_C from \mathcal{D}_C by adding a sufficiently small disk for each vertex of Γ_T . The embedding ψ together with Γ_T is a \mathcal{D}'_C -framed embedding ψ' of \mathcal{C}' .

According to Feng, et al. [11] there is a simple connected clustered graph \mathcal{C}'' that contains \mathcal{C}' as a subgraph whose embedding ψ'' is \mathcal{D}_C -framed and contains ψ' . By Lemma 2 there is a \mathcal{D}_C -framed drawing Γ'' of \mathcal{C}'' homeomorphic to ψ'' with the outer face drawn as Γ_T . The drawing Γ'' contains a \mathcal{D}_C -framed drawing of \mathcal{C} . \square

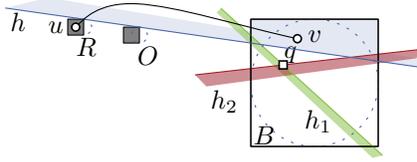


Fig. 3: Regulator

3 Drawing on General Disk Arrangements

We study the following problem referred to as \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS. Given a planar clustered graph $\mathcal{C} = (G, \mathcal{V})$, a disk arrangement \mathcal{D}_C that is not planar, i.e., \mathcal{D}_C satisfies condition (C1) and (C2) but not (P1) and (P2), and a \mathcal{D}_C -framed embedding ψ of G , is there a \mathcal{D}_C -framed straight-line drawing Γ that is homeomorphic to ψ and \mathcal{D}_C ? Note that if the disks \mathcal{D}_C are allowed to overlap (condition (P1)) and G is the intersection graph of \mathcal{D}_C , the problem is known to be \mathcal{NP} -hard [6]. Thus, in the following we require that the disks do not overlap, but there can be disk-pipe intersections, i.e., \mathcal{D}_C satisfies conditions (C1), (C1) and (P1) but not (P2). By Alam et al. [3] it follows that the problem restricted to thin touching rectangles instead of disks is \mathcal{NP} -hard. We strengthen this result and prove that in case that the rectangles are axis-aligned squares and are not allowed to touch the problem remains \mathcal{NP} -hard. Our illustrations contain blue dotted circles that indicate how the square in the proof can be replaced by disks. For the entire proof we refer to Appendix A.

To prove \mathcal{NP} -hardness we reduce from PLANAR MONOTONE 3-SAT [7]. For each literal and clause we construct a clustered graph \mathcal{C} with an arrangement of squares \mathcal{D}_C of \mathcal{C} such that each disk contains exactly one vertex. We refer to these instances as *literal* and *clause gadgets*. In order to transport information from the literals to the clauses, we construct a *copy* and *inverter gadget*. The design of the gadgets is inspired by Alam et al. [3], but due to the restriction to squares rather than rectangles, requires a more careful placement of the geometric objects. The green and red regions in the figures of the gadget correspond to *positive* and *negative* drawings of the literal gadget. The green and red line segments indicate that for each truth assignment of the variables our gadgets indeed have \mathcal{D}_C -framed straight-line drawings. Negative versions of the literal and clause gadget are obtained by mirroring vertically. Hence, we assume that variables and clauses are positive. Each gadget covers a set of checkerboard cells. This simplifies the assembly of the gadgets for the reduction.

An *obstacle of a pipe* p_{ij} is a disk d_k , $i, j \neq k$, that intersects p_{ij} . The *obstacle number of a pipe* p_{ij} is the number of obstacles of p_{ij} . Let $P = \{p_{ij} \mid V_i \times V_j \cap E \neq \emptyset\}$. The *obstacle number of a disk arrangement* \mathcal{D}_C is maximum obstacle number of all pipes p_{ij} with $V_i \times V_j \cap E \neq \emptyset$.

Regulator. The *regulator gadget* restricts the feasible placements of a vertex v that lies in the interior of a square B ; refer to Fig. 3. Let h_1, h_2 be two half planes such that the intersection q of their supporting lines lies in B . In a \mathcal{D}_C -framed

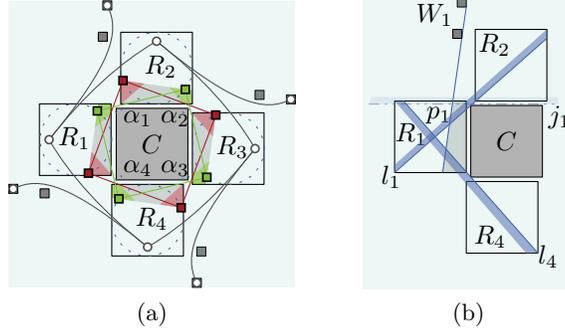


Fig. 4: Literal gadget

drawing of the regulator gadget the placement of v is restricted by a half plane h that excludes a placement of v in $h_1 \cap h_2$ but allows for a placement in $h_1 \cap B$ or $h_2 \cap B$. We refer to $h \cap B$ as the *regulated region* of B .

Literal Gadget. The *positive literal gadget* is depicted in Fig. 4. The *center block* is a unit square C with corners $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in clockwise order. For each corner α_i of C consider a line l_i that is tangent to C in α_i , i.e. $l_i \cap C = \{\alpha_i\}$. Let p_i be the intersection of lines l_{i-1} and l_i where $l_0 = l_4$; refer to Fig. 4b. Let R_1, \dots, R_4 be four pairwise non-intersecting squares that are disjoint from C such that R_i contains p_i in its interior. We add a cycle $v_1 v_2 v_3 v_4 v_1$ such that $v_i \in R_i$. We refer to the vertex v_i as the *cycle vertex* of the *cycle block* R_i . For each i , let j_i be a half plane that contains R_{i+1} but does not intersect C . We place a regulator W_i of v_i with respect to h_{i-1} and h_i and position it such that it lies in j_i , where h_i is the half plane spanned by l_i with $C \not\subseteq h_i$.

We now describe the two combinatorially different realizations of the literal gadgets. Consider R_1 and its two adjacent squares R_2 and R_4 . Let Q_i be the regulated region of R_i with respect to W_i . We refer to $\overline{h_2} \cap \overline{h_4} \cap Q_1$ as the *infeasible region* of R_1 , where $\overline{h_i}$ denotes the complement of h_i . The intersection $h_1 \cap Q_1$ is the *positive region* P_1 of R_1 . The region $\overline{h_4} \cap Q_1$ is the *negative region* N_1 of R_1 . All these regions are by construction not empty. The positive, negative and infeasible region of $R_i, i \neq 1$ are defined analogously.

Property 4. If Γ is a \mathcal{D}_C -framed drawing of a positive (negative) literal gadget, then no cycle vertex v_i lies in the infeasible region of R_i . Moreover, either each cycle vertex v_i lies in the positive region P_i or each vertex v_i lies in the negative region N_i .

Property 5. The positive and negative placements induce a \mathcal{D}_C -framed drawing of the literal gadget, respectively.

Copy and Inverter Gadget. The copy gadget in Fig. 5 connects two positive literal gadgets X and Y such that a drawing of X is positive if and only if the drawing of Y is positive. The inverter gadget connects a positive literal gadget

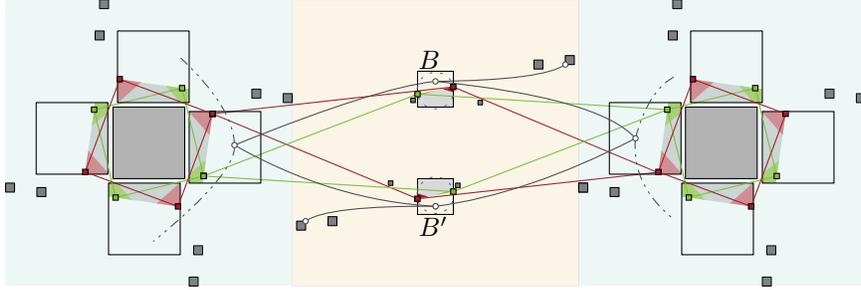


Fig. 5: Copy gadget

X to a negative literal gadget Y such that the drawing of X is positive if and only if the drawing of Y is negative. The construction of the gadgets uses ideas similar to the construction of the literal gadget. In contrast to the literal gadget, we replace the center block by four squares.

Property 6. Let Γ be a \mathcal{D}_C -framed drawing of two positive (negative) literal gadgets X and Y connected by a copy gadget. Then the \mathcal{D}_C -framed of X in Γ is positive if and only if the \mathcal{D}_C -framed drawing of Y is positive.

Property 7. The positive (negative) placement of two literal gadgets X, Y induces a \mathcal{D}_C -framed drawing of a copy [inverter] gadget that connects X and Y .

Clause Gadget. We construct a *clause gadget* with respect to three positive literal gadgets X, Y, Z arranged as depicted in Fig. 7. The negative clause gadget, i.e., a clause with three negative literal gadgets, is obtained by mirroring vertically.

We construct the clause gadget in two steps. First, we place a *transition block* T_A close to each literal gadget $A \in \{X, Y, Z\}$. In the second step, we connect the transition block to a vertex k in a *clause block* K such that for every placement of k in K at least one drawing of the literal gadgets has to be positive.

Consider the literal gadget X and let R_X be the right-most cycle block of X . Let h_X be a negative half plane of R_X , i.e., h_X contains the positive region P_X but not the negative region N_X , refer to Fig. 6. We now place a transition block T_X such that the intersection $T_X \cap h_X$ has small area. Further, let p_X^+ and p_X^- be the positive and negative placements of X , respectively. Let q_X^- be a point in $T_X \cap h_X$. Let i be the intersection point of the supporting line l_X of h_X and the line segment $p_X^- q_X^-$. We place an obstacle O_X^1 such that l_X is tangent to O_X^1 in point i . Finally, we place a *transition vertex* t_X in the interior of T_X and route the edge $v_X t_X$ through $h_X \cup T_X \cup R_X$, where $v_X \in R_X$.

Consider a half plane h'_X such that $O_X^1 \not\subseteq h'_X$ and $N_X \not\subseteq h'_X$ and such that the supporting line l'_X of h'_X contains p_X^+ and is tangent to O_X^1 . Let q_X^+ be a point $h'_X \cap R_X$. Observe that for q_X^+ and q_X^- there is a positive and negative drawing of X , respectively. Further, if X has a negative drawing, then t_X lies in

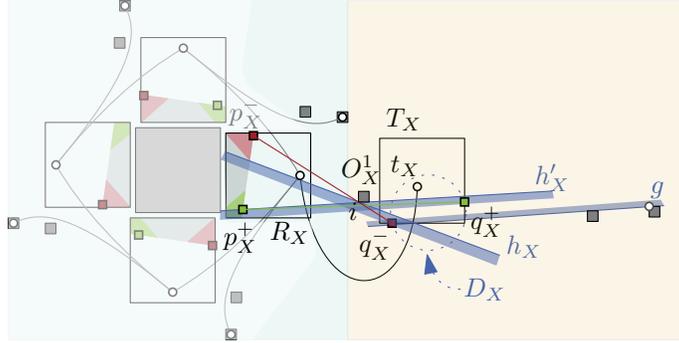


Fig. 6: Construction of the transition block.

the region $h_X \cap T_X$. In the following, we refer to $h_X \cap T_X$ as the *negative region* of T_X . The transition blocks of Y and Z are constructed analogously with only minor changes. The transition block T_Z of Z is constructed with respect to the top-most cycle block. Note that we can choose the points q_A^+ , $A \in \{X, Y, Z\}$ independent from each other as long as each of them induces a positive drawing the literal gadget A .

Denote by $\text{x-max } S$ the maximum x -coordinate of a point in a bounded set $S \subset \mathbb{R}^2$. Note that $\text{x-max } D_X \cap h_X > \text{x-max } T_X \cap h_X$, refer to Fig. 6. To ensure that our construction remains correct for disks we add a regulator R with respect a half plane g such that $\text{x-max } D_X \cap h_x \cap g = \text{x-max } T_X \cap h_x \cap g$ and g contains q_X^+ , q_X^- .

Given the placement of the transition block T_X, T_Y and T_Z as depicted in Fig. 7, we construct the *clause block* K as follows. We choose a point $q_{X,Y}$. Let l_X^- and l_Y^- be the lines through the points $q_X^-, q_{X,Y}$, and $q_Y^-, q_{X,Y}$, respectively. Further, consider a line l_Z^- with $q_Z^- \in l_Z^-$ such that the intersection point $q_{A,Z} := l_Z^- \cap l_A^-, A \in \{X, Y\}$ lies in between q_A^- and $q_{X,Y}$. Further, let l_X^+ be the line through $q_X^+, q_{Y,Z}$, l_Y^+ the line through $q_Y^+, q_{X,Z}$, and let l_Z^+ be the line through q_Z^+ and $q_{X,Y}$. Let h_A be a half plane that does not contain the negative region N_A and whose supporting line contains the intersection i_A of l_A^- and l_A^+ . We place obstacles O_A^2 such that $O_A^2 \not\subseteq h_A$ and the supporting line of h_A is tangent to O_A^2 in point i_A . We place the clause box K such that it contains $q_{X,Y}, q_{Y,Z}, q_{X,Z}$ and a new vertex k in its interior. We finish the construction by routing the edges kt_A through $K \cup h_A \cup T_A, A \in \{X, Y, Z\}$, where $t_A \in T_A$.

By construction we have that for each $y \in \{q_Y^-, q_Y^+\}$ and $z \in \{q_Z^-, q_Z^+\}$ the points y, z and $q_{Y,Z}$ induce a \mathcal{D}_C -framed drawing. The analog statement for the points $q_{X,Z}$ and $q_{X,Y}$ is also true. Further, if $h_X \cap h_Y \cap h_Z = \emptyset$, then there is no \mathcal{D}_C -framed drawing such that each vertex t_A lies on q_A^- . Fig. 7 shows that there is an arrangement of the clause block and the obstacles such that $h_X \cap h_Y \cap h_Z$ indeed is empty.

Property 8. There is no \mathcal{D}_C -framed drawing of the clause gadget such that the drawing of each literal gadget is negative. For all other combinations of positive

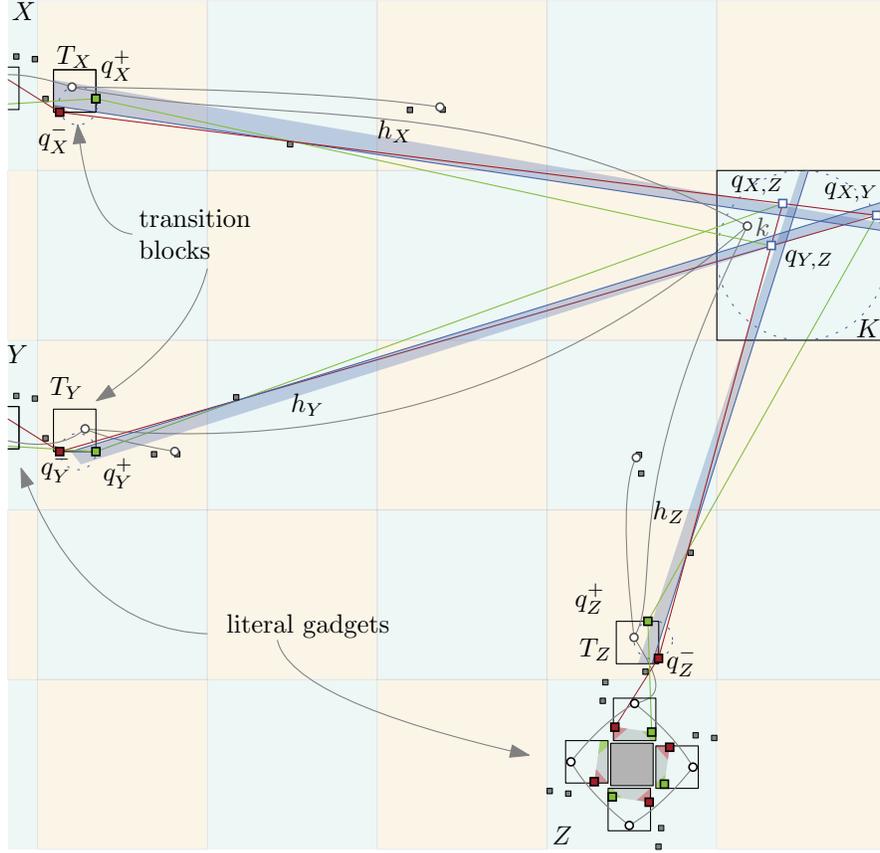


Fig. 7: Construction of the clause block.

and negative drawings of the literal gadgets there is a \mathcal{D}_C -framed drawing of the clause gadget.

Reduction. We reduce from a planar monotone 3-SAT instance (U, C) ; refer to Fig. 8. We modify its rectilinear representation such that each vertex and clause rectangle covers sufficiently many cells of a checkerboard and each edge covers the entire column between its two endpoints. We place positive literal gadgets in each blue cell of a rectangle corresponding to a variable. We place a clause gadget in each positive clause rectangle R_c such that it is aligned with the right-most edge of R_c . The literal gadget X of a variable x is connected to its corresponding literal gadget X' in R_c by placing a literal gadget in each blue cell that is covered by the Γ -shape that connects X to X' ; refer to Fig. 8b. Finally, we place a copy gadget in each orange cell between two literal gadgets of the same variable. The negative clauses are obtained by mirroring the modified rectilinear representation vertically and repeating the construction

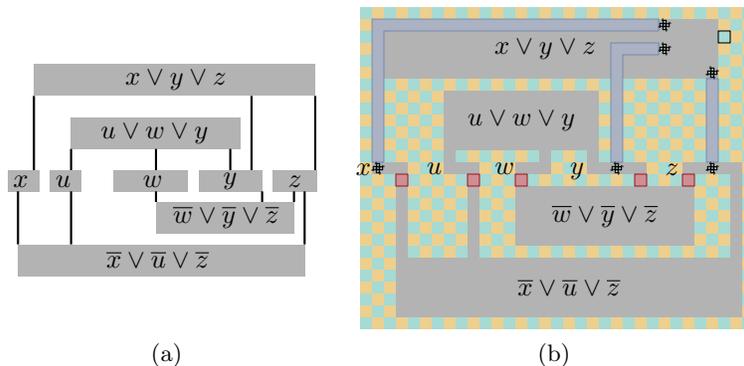


Fig. 8: (a) Planar monotone 3-SAT instance (U, C) with a rectilinear representation. (b) Modified rectilinear representation of (U, C) on a checkerboard.

for the positive clauses. To negate the state of the variable we place the inverter gadget immediately below a variable (red cells in Fig. 8b).

Correctness. Assume that (U, C) is satisfiable. Depending on whether a variable u is true or false, we place all vertices on a positive placement of a positive literal gadget and on the negative placement of negative literal gadget of the variable. By Property 5, the placement induces a \mathcal{D}_C -framed drawing of all literal gadgets and Property 7 ensures the copy and inverter gadgets have a \mathcal{D}_C -framed drawing. Since at least one variable of each clause is true, there is a \mathcal{D}_C -framed drawing of each clause gadget by Property 8.

Now consider that the clustered graph \mathcal{C} has a \mathcal{D}_C -framed drawing. Let X and Y be two positive (negative) literal gadgets connected with a copy gadget. By Property 6, a drawing of X is positive if and only if the drawing of Y is positive. Property 6 ensures that the drawing of a positive literal gadget X is positive if and only if the drawing of the negative literal gadget Y is negative, in case that both are joined with an inverter gadget. Further, Property 4 states that each cycle vertex lies either in a positive or a negative region. Thus, the truth value of a variable u can be consistently determined by any drawing of a literal gadget of u . By Property 8, the clause gadget K has no \mathcal{D}_C -framed drawing such that all drawings of the literal gadgets of K are negative. Thus, the truth assignment indeed satisfies C .

Theorem 9. *The problem \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS with axis-aligned squares is \mathcal{NP} -hard even when the clustered graph \mathcal{C} is restricted to vertex degree 5 and the obstacle number of \mathcal{D}_C is two.*

4 Conclusion

We proved that every clustered planar graph with a planar disk arrangement \mathcal{D}_C and a \mathcal{D}_C -framed embedding ψ has a \mathcal{D}_C -framed straight-line drawing home-

omorphic to ψ . If the requirement of the disk arrangement to satisfy condition (P2) is dropped, we proved that it is \mathcal{NP} -hard to decide whether \mathcal{C} has a $\mathcal{D}_{\mathcal{C}}$ -framed straight-line drawing. We are not aware whether our problem is known to be in \mathcal{NP} . We ask whether techniques developed by Abrahamsen et al. [1] can be used to prove $\exists\mathbb{R}$ -hardness of our problem.

Angelini et al. [4] showed that if \mathcal{C} is not embedded and all squares have the same size, it is \mathcal{NP} -hard to decide whether \mathcal{C} has a $\mathcal{D}_{\mathcal{C}}$ -framed drawing. They posed as an open problem whether the same is true for embedded graphs. In our construction, the squares have constant number of different side lengths and the side length of the largest square is 32 time longer than the side length of the smallest rectangle. We conjecture that our construction can be modified to show that it is indeed \mathcal{NP} -hard to decide whether a clustered graph \mathcal{C} with a non-planar arrangement of squares (disk) of unit size and a $\mathcal{D}_{\mathcal{C}}$ -framed embedding ψ has a $\mathcal{D}_{\mathcal{C}}$ -framed drawing that is homeomorphic to ψ . Further, we ask whether the obstacle number can be reduced to one.

References

1. Abrahamsen, M., Adamaszek, A., Miltzow, T.: The Art Gallery Problem is $\exists\mathbb{R}$ -complete. In: STOC'18. pp. 65–73. ACM (2018)
2. Akitaya, H.A., Fulek, R., Tóth, C.D.: Recognizing Weak Embeddings of Graphs. In: Artur Czumaj (ed.) SODA'18. pp. 274–292. SIAM (2018)
3. Alam, M., Kaufmann, M., Kobourov, S.G., Mchedlidze, T.: Fitting Planar Graphs on Planar Maps. *J. Graph Alg. Appl.* **19**(1), 413–440 (2015)
4. Angelini, P., Da Lozzo, G., Di Bartolomeo, M., Di Battista, G., Hong, S.H., Patrignani, M., Roselli, V.: Anchored Drawings of Planar Graphs. In: Duncan, C., Symvonis, A. (eds.) GD'14. pp. 404–415. Springer (2014)
5. Angelini, P., Frati, F., Kaufmann, M.: Straight-Line Rectangular Drawings of Clustered Graphs. *Disc. & Comput. Geom.* **45**(1), 88–140 (2011)
6. Banyassady, B., Hoffmann, M., Klemz, B., Löffler, M., Miltzow, T.: Obedient Plane Drawings for Disk Intersection Graphs. In: Ellen, F., Kolokolova, A., Sack, J.R. (eds.) WADS'17. pp. 73–84. Springer (2017)
7. de Berg, M., Khosravi, A.: Optimal Binary Space Partitions for Segments in the Plane. *Int. J. Comput. Geom. & Appl.* **22**(3), 187–206 (2012)
8. Bläsius, T., Rutter, I.: A New Perspective on Clustered Planarity as a Combinatorial Embedding Problem. *Theoretical Comput. Sci.* **609**(2), 306 – 315 (2016)
9. Cortese, P.F., Di Battista, G., Patrignani, M., Pizzonia, M.: On Embedding a Cycle in a Plane Graph. *Disc. Math.* **309**(7), 1856 – 1869 (2009)
10. Eades, P., Feng, Q., Lin, X., Nagamochi, H.: Straight-Line Drawing Algorithms for Hierarchical Graphs and Clustered Graphs. *Algorithmica* **44**(1), 1–32 (2006)
11. Feng, Q., Cohen, R.F., Eades, P.: Planarity for Clustered Graphs. In: Spirakis, P. (ed.) ESA'95. pp. 213–226. Springer (1995)
12. Godau, M.: On the Difficulty of Embedding Planar Graphs with Inaccuracies. In: Tamassia, R., Tollis, I.G. (eds.) GD'94. pp. 254–261. Springer (1995)
13. Ribó Mor, A.: Realization and Counting Problems for Planar Structures. Ph.D. thesis, FU Berlin (2006), <https://refubium.fu-berlin.de/handle/fub188/10243>

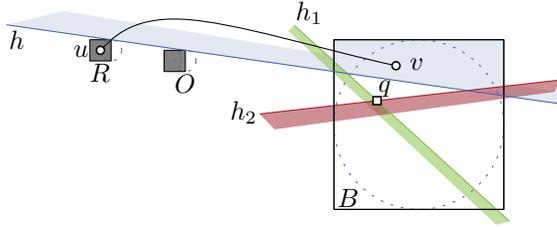


Fig. 9: Regulator

A Drawing on Non-Planar Disk Arrangements

We study the following problem referred to as \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS. Given a planar clustered graph \mathcal{C} with an embedding ψ and a non-planar disk arrangement \mathcal{D}_C , is there a \mathcal{D}_C -framed drawing Γ that is homeomorphic to ψ and \mathcal{D}_C . This requires that in the transition from ψ to Γ at any point in time an edge uv does not intersect a geometric object other than its own clusters. Note that if the disks \mathcal{D}_C are allowed to overlap and \mathcal{C} is the intersection graph of \mathcal{D}_C , the problem is known to be \mathcal{NP} -hard [6]. Thus, in the following we require that the disk may not overlap, but there can be disk-pipe intersection. By Alam et al. [3] it follows that the problem restricted to thin touching rectangles instead of disks is \mathcal{NP} -hard. We strengthen this result and prove that in case that the rectangles are axis-aligned squares and not allowed to touch the problem remains \mathcal{NP} -hard. The squares in the proof can be replaced in a straight-forward manner by disks.

To prove \mathcal{NP} -hardness for \mathcal{D}_C -framed Drawings of Non-planar Arrangements problem we reduce from PLANAR MONOTONE 3-SAT [7]. For each literal and clause we construct a graph \mathcal{C} with a disk arrangement \mathcal{D}_C of \mathcal{C} such that each disk contains exactly one vertex. We refer to these instances as *literal* and *clause gadget*. In order to transport information from the literals to the clauses, we construct a *copy* and *inverter gadget*. The design of the gadgets is inspired by Alam et al. [3], but due to the restriction to squares rather than rectangles, requires a more careful placement of the geometric objects.

Notation A line l separates the euclidean plane in two *half planes* h_1 and h_2 that are *spanned by* l . We say that l *supports* h_1 (h_2).

A.1 Regulator

Let B be an axis-aligned square that contains a vertex v in its interior and let h_1, h_2 be two half planes such that the intersection q of their supporting lines l_1, l_2 lies in the interior of B . We say that h_1 and h_2 are *proper half planes of* B . We describe the construction of a gadget that restricts the feasible placements of v in a \mathcal{D}_C -framed drawing by a half plane h that excludes a placement of v in $h_1 \cap h_2$ but allows for a placement in $h_1 \cap B$ or $h_2 \cap B$. Since q lies in the

interior of B , there is a half plane h that does not contain q and for each $i = 1, 2$, $h \cap h_i \cap B$ is non-empty.

We construct a *regulator gadget of v in B with respect to h_1 and h_2* as follows. Let l be the supporting line of h . We create two axis-aligned squares R and O such that R, O and B intersect l in this order and h neither intersects the interior of R nor the interior of O . Place a vertex u in R and route an edge uv through $h \cup R \cup B$.

Lemma 10. *Let W be a regulator gadget of v in B with respect to two proper half planes h_1 and h_2 . For every point $p_v \in h \cap B$ there is a \mathcal{D}_C -framed drawing Γ such that v lies on p_v . There is no \mathcal{D}_C -framed drawing of W such that v lies in $\bar{h} \cap B$.*

Proof. By construction of W , there is for every point $p_v \in h \cap B$ a \mathcal{D}_C -framed drawing Γ such that v lies on v .

The supporting line l of h intersects the boundary of R and does not intersect the interior of O . Let r and o be points in the intersection of l with R and O , respectively. Since Γ is homeomorphic to W the edge uv intersects l on the ray starting in o in the direction towards r . Therefore, u and v lie on different sides of l . Since $u \in R$, it follows that $v \in \bar{h}$. \square

We refer to the intersection $h \cap B$ as the *regulated region of v in B* . Thus, by the construction of W , the regulated region Q has a non-empty intersection with $h_1 \cap B$ and $h_2 \cap B$. Thus, by the lemma there is for each placement of v in $Q \cap h_i \cap B$, $i = 1, 2$, a \mathcal{D}_C -framed drawing. On the other hand, since $h \cap h_1 \cap h_2 \cap B = \emptyset$, there is no \mathcal{D}_C -framed drawing such that v lies in $h_1 \cap h_2 \cap B$.

A.2 Literal Gadget

In this section we construct a clustered graph \mathcal{C} with an arrangement of squares \mathcal{D}_C that models a literal u . The *positive literal gadget* is depicted in Fig. 10. We obtain the *negative literal gadget* by mirroring vertically.

The *center block* is a unit square C with corners $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in clockwise order. For each corner α_i of C consider a line l_i that is tangent to C in α_i , i.e., $l_i \cap C = \{\alpha_i\}$. Let p_i be the intersection of lines l_{i-1} and l_i where $l_0 = l_4$; refer to Fig. 10b. Let R_1, \dots, R_4 be four pairwise non-intersecting squares that are disjoint from C such that R_i contains p_i in its interior. We add a cycle $v_1v_2v_3v_4v_1$ such that $v_i \in R_i$. We refer to the vertices v_i as *cycle vertices* of the *cycle block* R_i . For each i , let j_i be a half plane that contains R_{i+1} but does not intersect C . We place a regulator W_i of v_i with respect to h_{i-1} and h_i and position it such that it lies in j_i , where h_i is the half plane spanned by l_i with $C \not\subseteq h_i$. This finishes the construction.

We now show that there exist two combinatorially different realizations. Consider R_1 and its two adjacent squares R_4 and R_2 . Let Q_i be the regulated region of R_i with respect to W_i . Then the intersection $I_1 := \bar{h}_4 \cap \bar{h}_2 \cap Q_1 \neq \emptyset$. We refer to I_1 as the *infeasible region of R_1* . The intersection $h_4 \cap Q_1$ is the *positive region*

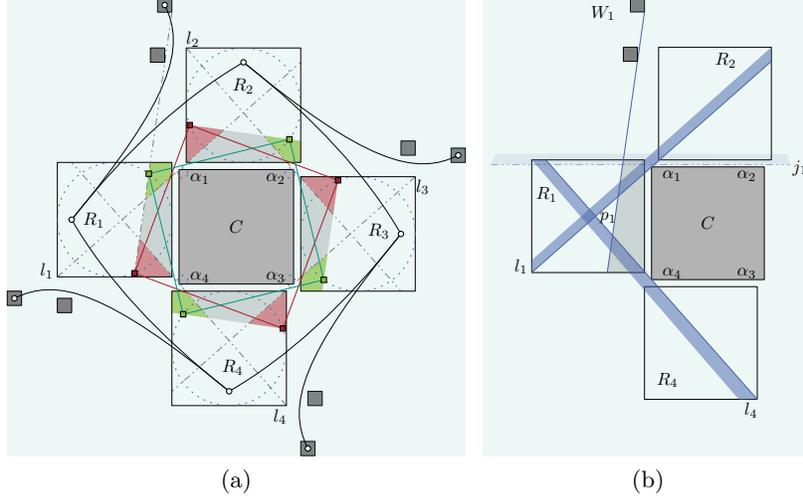


Fig. 10: Literal gadget

P_1 of R_1 . The region $h_2 \cap Q_1$ is the *negative region* N_1 of R_1 . All these regions are by construction not empty. The positive, negative and infeasible region of $R_i, i \neq 1$ are defined analogously.

Property 4. *If Γ is a \mathcal{D}_C -framed drawing of a positive (negative) literal gadget, then no cycle vertex v_i lies in the infeasible region of R_i . Moreover, either each cycle vertex v_i lies in the positive region P_i or each vertex v_i lies in the negative region N_i .*

Proof. Consider a \mathcal{D}_C -framed drawing Γ with an edge $v_i v_{i+1}$ such that v_i lies in P_i . We show that v_{i+1} lies in N_{i+1} . If v_{i+1} lies in h_i , then v_i and v_{i+1} lie on the same side of l_i , which is tangent to α_{i+1} . Thus, $v_i v_{i+1}$ intersects C . It follows that v_{i+1} lies in h_i and therefore in the negative region N_{i+1} .

Assume that v_1 lies in its infeasible region I_1 , then v_2 lies in N_2 by the above observation. Likewise, v_3, v_4, v_1 lie in N_3, N_4, N_1 , respectively. This contradicts $N_1 \cap I_1 = \emptyset$. This generalizes to all $v_i, i \neq 1$. Thus, each v_i either lies in P_i or in N_i . Moreover, if one v_i lies in N_i the above observation yields that all of them lie in their negative region. \square

In the following, we fix the placement of the cycle blocks for literal gadgets as depicted in Fig. 10. This allows us to show that the literal gadget has a \mathcal{D}_C -framed drawing where all cycle vertices lie in their positive region and one where all cycle vertices lie in their negative region. We refer to the former as *positive* and the latter as *negative* drawing. We construct two specific drawings. Let D_i be the circle inscribed in the square R_i . Since P_i and N_i are obtained from the intersection of two half planes with R_i , they are convex. The intersection p_i^+ of the boundary of P_i with D_i that does not lie on the boundary of Q_i is the

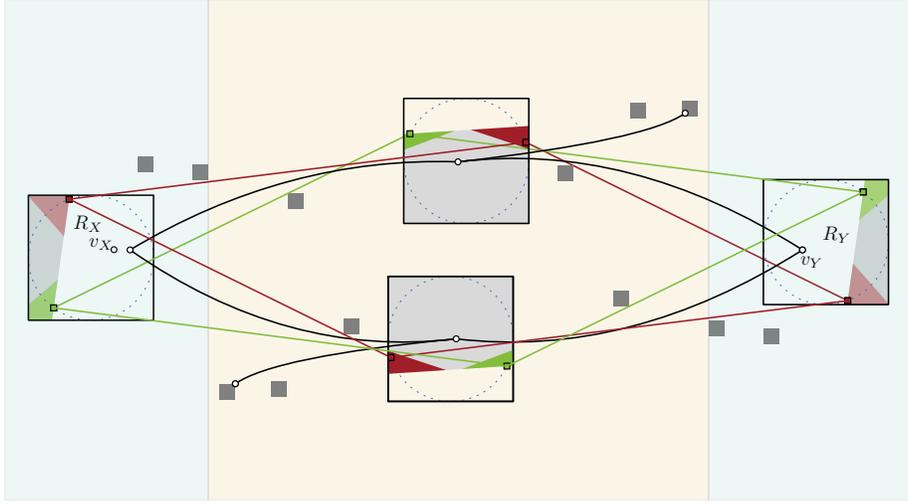


Fig. 11: Copy gadget. Green and red regions depict positive and negative regions, respectively.

positive placement of v_i . Analogously, we obtain the negative placement p_i^- of v_i . The positive and negative placement induce two straight-line drawings of the graph induced by cycle vertices. By Lemma 10 we can extend these drawings to \mathcal{D}_C -framed drawings of the whole literal gadget, including the regulators.

Property 5. *The positive and negative placements induce a \mathcal{D}_C -framed drawing of the literal gadget, respectively.*

A.3 Copy and Inverter Gadget

In this section, we describe the copy and inverter gadget; see Fig. 11. The copy gadget connects two positive or two negative literal gadgets X and Y such that a drawing of X is positive if and only if the drawing of Y is positive. Correspondingly, the inverter gadget connects a positive literal gadget X to a negative literal gadget Y such that the drawing of X is positive if and only if the drawing of Y is negative. The construction of the inverter and the copy gadget are symmetric.

A copy gadget of two negative literal gadgets is obtained by vertically mirroring the copy gadget that connects two positive literals. Correspondingly, we obtain an inverter gadget that connects a negative literal with a positive literal by mirroring the inverter gadget that connects a positive literal with a negative literal. Thus, in the following we describe only the construction of the copy gadget with two positive literals. Whenever necessary we emphasize differences for the inverter gadget.

Let X and Y be two positive literal gadgets whose center blocks are aligned on the x -axis with a sufficiently large distance. We construct the copy gadget that connects X and Y as follows. Let R_X and R_Y be the two cycle blocks of the

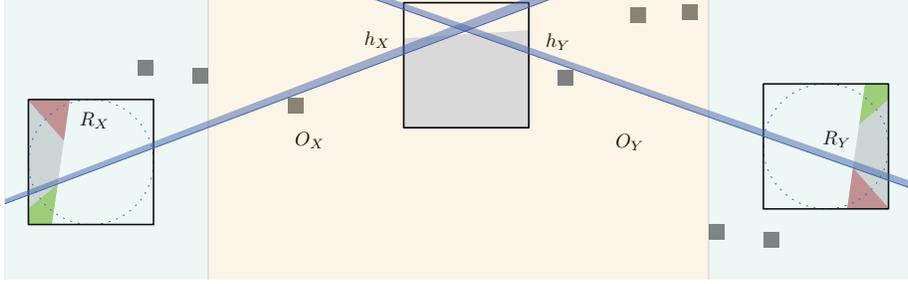


Fig. 12: Construction of the square B of the copy gadget.

literal gadgets X and Y , respectively, with minimal distance on the x -axis. For $Z \in \{X, Y\}$, let P_Z and N_Z be the positive and negative regions of R_Z . Since N_Z and P_Z are convex and their intersection is empty, there exist a half plane h_Z that contains N_Z but not P_Z , and vice versa. We refer to h_Z as *positive half plane of Z* if it contains the positive region P_Z , otherwise it is *negative*.

Consider a negative half plane h_X of X and a positive half plane h_Y of Y ; refer to Fig. 12. We create two non-intersecting squares O_X and O_Y that are contained in the intersection of $\overline{h_X}$ and $\overline{h_Y}$ such that a corner of O_Z lies on the supporting line of h_Z , $Z \in \{X, Y\}$. Let I be the intersection of the supporting lines of h_X and h_Y . We place a square B with a vertex b and the intersection I in its interior. Additionally, we add a regulator of b with respect to h_X and h_Y to exclude the intersection $h_X \cap h_Y$ as feasible placement of b . We route the edges bv_X and bv_Y through $R_X \cup h_X \cup B$ and $R_Y \cup h_Y \cup B$ respectively. This construction ensures that in a \mathcal{D}_C -framed drawing a placement of the vertex v_X in the negative region N_X excludes the possibility that the vertex v_Y lies in the positive region P_Y . In order to ensure that v_X cannot lie at the same time in P_X as v_Y in N_Y , we construct a square B' with respect to a positive half plane of X and a negative half plane of Y analogously to B . If the distance between X and Y is sufficiently large, we can ensure the intersection of B and B' is empty. In the construction of the inverter gadget one has to consider two positive half planes and two negative half planes, respectively. We refer to the corresponding gadgets as *copy* and *inverter gadget*. We say that the copy and inverter gadget *connect* two literals.

Property 6. *Let Γ be a \mathcal{D}_C -framed drawing of two positive (negative) literals gadgets X and Y connected by a copy gadget. Then the \mathcal{D}_C -framed of X in Γ is positive if and only if the \mathcal{D}_C -framed drawing of Y is positive.*

Proof. By Property 5 the vertices v_X and v_Y of X and Y cannot lie in the infeasible regions of X and Y , respectively. Thus, similar to the proof of Lemma 4 we can assume for the sake of contradiction that the vertex b of block B lies in the intersection of $\overline{h_X}$ and $\overline{h_Y}$. Thus, vertex v_X lies in the negative region of R_X and v_Y in the positive region of R_Y . But then vertex b' of the block B' lies in h'_X and h'_Y . This contradicts that $Q' \cap h'_X \cap h'_Y \cap B'$ is empty, where Q' is the

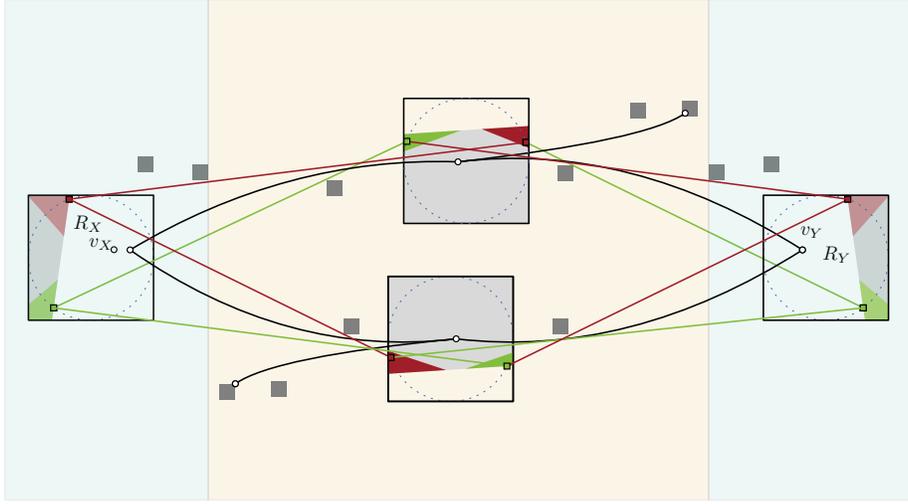


Fig. 13: Inverter gadget. Green and red regions depict positive and negative regions, respectively.

regulated region of B' , and h'_X and h'_Y are the negative and positive half planes of R_X and R_Y , respectively. \square

The same argumentation is applicable to the inverter gadget.

Property 11. Let Γ be a \mathcal{D}_C -framed drawing of a positive literal X and a negative literal Y connected by an inverter gadget. Then the \mathcal{D}_C -framed drawing of X in Γ is positive if and only if the \mathcal{D}_C -framed drawing of Y is negative.

From now on we refer to the exact placement of the squares as depicted in Fig. 11 and Fig. 13 as the copy and inverter gadget, respectively. The positive and negative placement of the literal gadgets induce a \mathcal{D}_C -framed drawing of the copy gadgets as indicated by the green and red straight-line segments, respectively. By Lemma 10 we can extend these drawings to drawings of the entire gadget.

Property 7. *The positive (negative) placement of two literals gadgets X, Y induces a \mathcal{D}_C -framed drawing of a copy [inverter] gadget that connects X and Y .*

A.4 Clause Gadget

We construct a *clause gadget* with respect to three positive literal gadgets X, Y, Z arranged as depicted in Fig. 14. The negative clause gadget, i.e., a clause with three negative literal gadgets, is obtained by mirroring vertically.

We construct the clause gadget in two steps. First, we place a *transition block* T_A close to each literal gadget $A \in \{X, Y, Z\}$. In the second step, we connect the

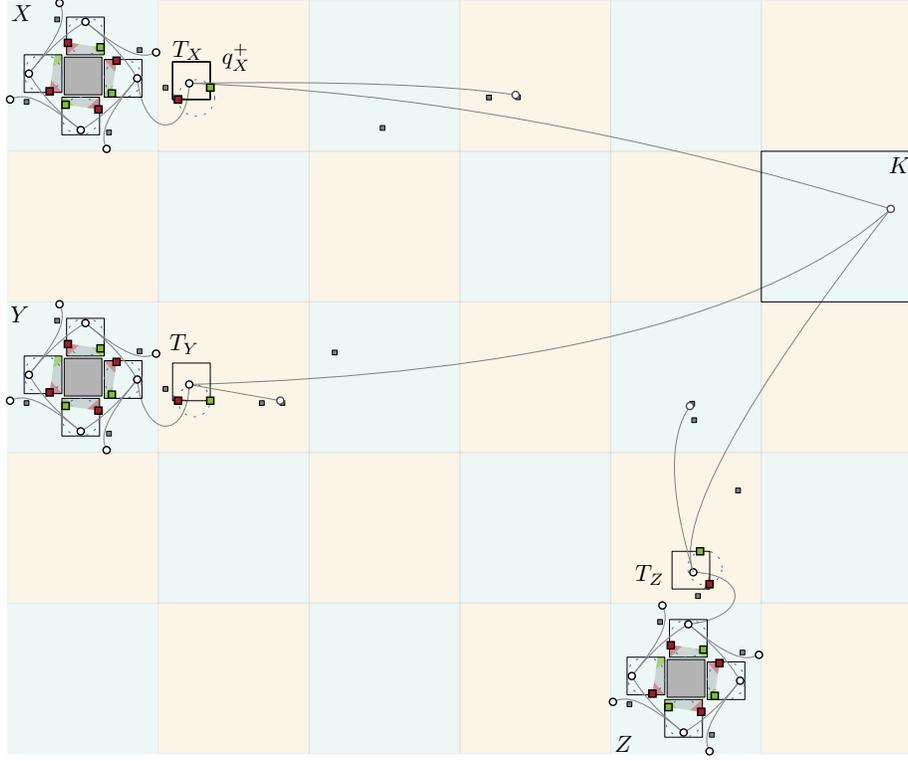


Fig. 14: Clause gadget.

transition block to a vertex k in a *clause block* K such that for every placement of k in K at least one drawing of the literal gadgets has to be positive.

Consider the literal gadget X and let R_X be the right-most cycle block of X . Let h_X be a negative half plane of R_X , i.e., h_X contains the positive region P_X but not the negative region N_X , refer to Fig. 15. We now place a transition block T_X such that the intersection $T_X \cap h_X$ has small area. Further, let p_X^+ and p_X^- be the positive and negative placements of X , respectively. Let q_X^- be a point in $T_X \cap h_X$. Let i be the intersection point of the supporting line l_X of h_X and the line segment $p_X^- q_X^-$. We place an obstacle O_X^1 such that l_X is tangent to O_X^1 in point i . Finally, we place a *transition vertex* t_X in the interior of T_X and route the edge $v_X t_X$ through $h_X \cup T_X \cup R_X$, where $v_X \in R_X$.

Consider a half plane h'_X such that $O_X^1 \not\subseteq h'_X$ and $N_X \not\subseteq h'_X$ and such that the supporting line l'_X of h'_X contains p_X^+ and is tangent to O_X^1 . Let q_X^+ be a point $h'_X \cap R_X$. Observe that q_X^+ and q_X^- induce a positive and negative drawing of X , respectively. Further, if X has a negative drawing, then t_X lies in the region $h_X \cap T_X$. In the following, we refer to $h_X \cap T_X$ as the *negative region* of T_X . The transition blocks of Y and Z are constructed analogously with only minor changes. The transition block T_Z of Z is constructed with respect to the

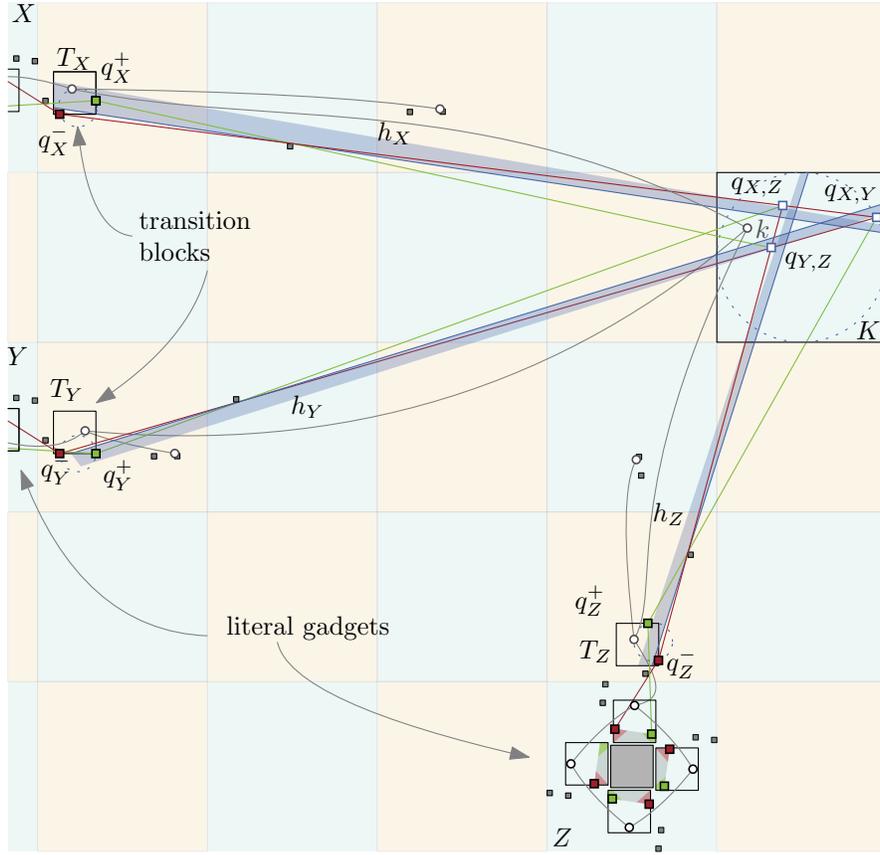


Fig. 16: Construction of the clause block.

and negative drawings of the literal gadgets there is a \mathcal{D}_C -framed drawing of the clause gadget.

A.5 Reduction

A 3-SAT instance (U, C) on a set U of n boolean variables and m clauses C is *monotone* if each clause either contains only positive or only negative literals. It is *planar* if the bipartite graph $G_{U,C} = (U \cup C, \{uc \mid u \in c \text{ or } \bar{u} \in c \text{ with } u \in U \text{ and } c \in C\})$ is planar. A *rectilinear representation* of a monotone planar 3-SAT instance is a drawing of $G_{U,C}$ where each vertex is represented as an axis-aligned rectangle and the edges are vertical line segments touching their endpoints; see Fig. 17. Further, all vertices corresponding to a variable lie on common line l , the positive and negative clauses are separated by l . The problem **MONOTONE PLANAR 3-SAT** asks whether a monotone planar 3-SAT instance with a given rectilinear representation is satisfiable. De Berg and Khosravi [7]

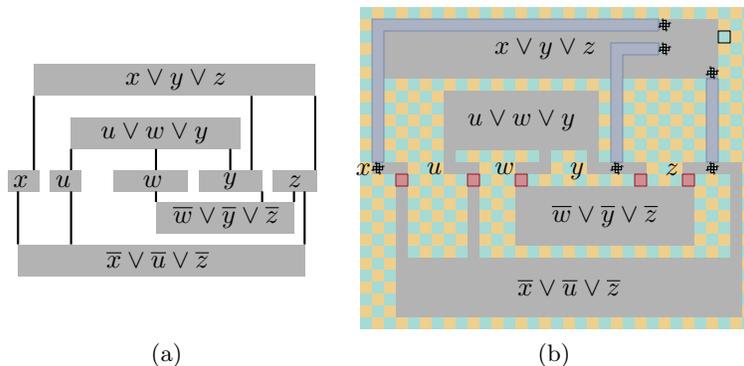


Fig. 17: Example of planar monotone 3-SAT instance with a corresponding rectilinear representation.

proved that MONOTONE PLANAR 3-SAT is \mathcal{NP} -complete. We use this problem to show that the \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS problem is \mathcal{NP} -hard.

Theorem 12. *The problem \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS with axis-aligned squares is \mathcal{NP} -hard even when the clustered graph \mathcal{C} is restricted to vertex degree 5 and the obstacle number is 2.*

Proof. Let (U, C) be a planar monotone 3-SAT instance with a rectilinear representation Π . Let l be a horizontal or vertical line that intersects Π . Further, let l_ϵ be a tunnel of width ϵ around l . We obtain from Π a new rectilinear representation by increasing the width of l_ϵ by an arbitrary positive factor x . This operations allows us to do the necessary modifications.

In the following we modify Π to fit on a checkerboard of $O(|C|)$ rows and columns where each column has width d and every row has height d . A row or column is *odd* if its index is an odd number, otherwise it is *even*. The pair (i, j) refers to the cell in column i and row j . We align all vertices corresponding to variables in the rectilinear representation in a common row and such that the left-most variable vertex is in column 1; refer to Fig. 18. The width of each rectangle r_u of variable u is increased to cover $2 \cdot n_u - 1$ columns where n_u is the number of occurrences of u and \bar{u} in C . To ensure that each r_u starts in an odd column, we increase the distance between two consecutive variables such that the number of columns between the variables is odd and is at least three. Since we are able to add an arbitrary number of columns between two consecutive variables, we can assume without loss of generality that no two edges of the rectilinear representation share a column and that their columns are odd. We adapt the rectangle of a clause such that it covers five rows and at least six columns, and such that its left and right sides are aligned with the left-most and right-most incoming edges, respectively. Note that the positive clauses lie

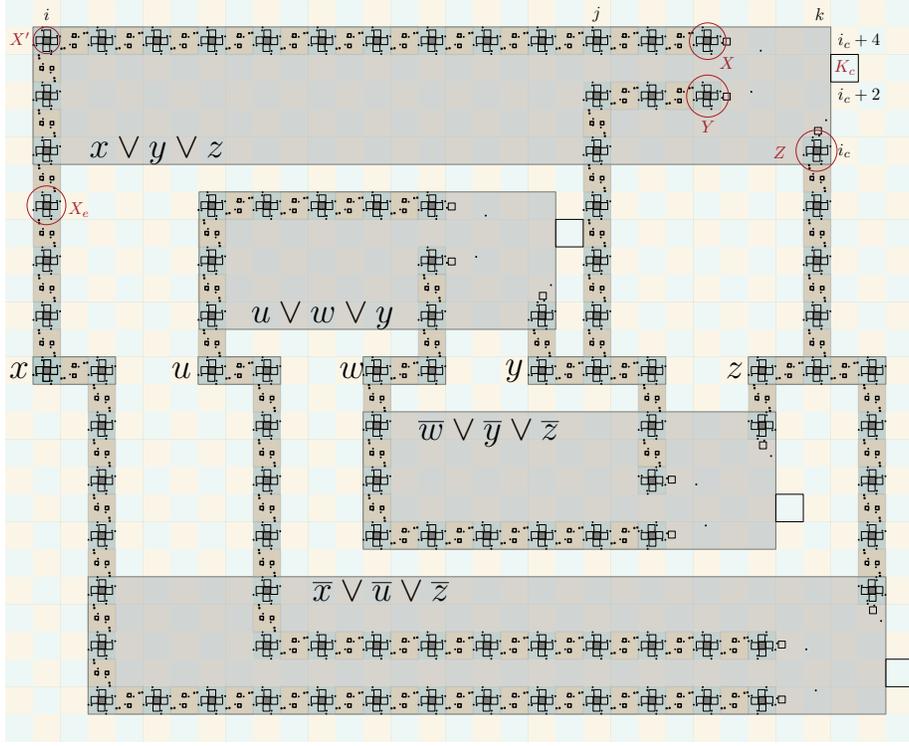


Fig. 18: Sketch of the final arrangement of squares. Edges are omitted from the drawing.

in rows with a positive index and the negative clauses in rows with a negative index. Each operation adds at most a constant number of columns and rows per vertex and per edge to the layout. Thus, the width and height of the final layout is in $O(|C|)$. Further, it can be computed in time polynomial in $|C|$.

In the following we construct a planar embedded graph \mathcal{C} and an arrangement of squares $\mathcal{D}_{\mathcal{C}}$ of \mathcal{C} . We use the modified rectilinear layout to locally replace the variable by a sequence of positive and negative literals connected by either a copy or an inverter gadget. Clauses are replaced with the clause gadget and then connected with a sequence of literals and copy gadget to the respective literal in the variable.

Observe that the literal gadget is constructed such that all its squares fit in a larger square S . The copy and inverter gadget together with two literals is constructed such that they fit in rectangle three times the size of S . The clause gadget fits in a rectangle of width six times the size of the square S and its height is five times the height of S .

Thus, we assume that the size of the square S and the size of the squares of the checkerboard coincide. Let r be the row that contains the variable vertices.

Every column contains at most one edge of the rectilinear representation. Thus, we place a positive literal gadget in cell (i, r) if the edge in column i connects a variable u to a positive clause. Otherwise, if the edge connects u to a negative clause, we place a negative literal gadget in cell (i, r) . Since every edge of the rectilinear representation lies in an odd column, we can connect two literals of the same variable by either a copy or inverter gadget depending on whether both literals are positive or negative, or one is positive and the other negative.

We substitute an edge e of the rectilinear representation that connects a variable to a positive clause as follows. Let i be the column of e . For every odd row r that is covered by e we place a positive literal gadget in cell (i, r) . The copy gadget can be rotated in order to connect a literal gadget in cell (i, r) to a literal gadget in a cell $(i, r + 2)$.

Let R_c be the clause rectangle in the modified rectilinear representation of a positive clause c . Let $i < j < k$ be the columns of the edges in the rectilinear representation that connect R_c to the variables x, y, z . We place the clause gadget in the clause rectangle such that literal gadget Z , refer to Fig. 18, lies in column k . Let i_c denote the lowest row of c . Then the literal gadget of X and Y lie in column $k - 4$ with X in row $i_c + 4$ and Y in column $i_c + 2$. The clause block K_c lies in cell $(i_c + 3, k + 1)$. We place a positive literal gadget X' in cell $(i, i_c + 4)$ and connect it with an alternating sequence of literal and copy gadgets to X . Let X_e be the literal gadget in cell $(i, i_c - 2)$. Since between two variables there are at least three empty columns we can connect X' to X_e with an alternating sequence of literal and copy gadgets. Analogously, we connect Y to the edge in column j by placing a positive literal gadget in cell $(j, i_c + 2)$. A negative clause is obtained by vertically mirroring the construction of a positive clause.

We now argue that the embedding of the graph \mathcal{C} is planar and that the pairwise intersections of squares in the arrangement $\mathcal{D}_{\mathcal{C}}$ are empty. Observe that, with the exception of the clause blocks K_c every gadget is entirely embedded in the modified rectilinear representation. The column of K_c is even, and therefore it cannot intersect with an edge of the rectilinear representation. Recall that the rectilinear representation is planar and all gadget are placed in disjoint cells. Therefore, the pairwise intersection of squares in $\mathcal{D}_{\mathcal{C}}$ is empty. Moreover, each literal gadget is planar embedded in a single cell, each clause is embedded in a rectangle that covers five rows and six columns, and finally each copy and inverter gadget together with its two literal gadget is embedded in either a single row and 3 columns or in 3 rows and a single column. Thus, since the modified rectilinear representation is planar and the pairwise intersections of squares in $\mathcal{D}_{\mathcal{C}}$ are empty, the graph \mathcal{C} has a planar embedding. Finally, the maximal vertex degree of the literal gadget is three, the maximal degree a clause gadget is four. Connecting two literal gadgets by copy or inverter gadget increases the maximum vertex degree of \mathcal{C} to five. Further, the obstacle number of the literal gadget and clause gadget is one and the obstacle number of the copy and inverter gadget is two.

It is left to show that the layout can be computed in polynomial time. As already argued the modified rectilinear representation Π of the monotone planar

3-SAT instance can be computed polynomial time. Moreover, the height and width of Π is linear in $|C|$. Thus, we inserted a number of gadgets linear in $|C|$. Further, the coordinates of each gadget are independent of the instance (U, C) , thus overall the representation of the final arrangement \mathcal{D}_C is polynomial in $|U|$ and $|C|$. Placing a single gadget requires polynomial time, thus overall the clustered graph \mathcal{C} and the arrangement \mathcal{D}_C of squares is can be computed in polynomial time.

Correctness Assume that (U, C) is satisfiable. Depending on whether a variable u is true or false, we place all cycle vertices on a positive placement of a positive literal gadget and on the negative placement of negative literal gadget of the variable. Correspondingly, if u is false, we place the vertices on the negative and positive placements, respectively. By Property 5, the placement induces a \mathcal{D}_C -framed drawing of all literal gadgets and Property 7 ensures the copy and inverter gadgets have a \mathcal{D}_C -framed drawing. Since at least one variable of each clause is true, there is a \mathcal{D}_C -framed drawing of each clause gadget by Lemma 8.

Now consider the clustered graph \mathcal{C} has a \mathcal{D}_C -framed drawing. Let X and Y be two positive literal gadgets or two negative literal gadgets connected with a copy gadget. By Lemma 6, a drawing of X is positive if and only if the drawing of Y is positive. Property 11 ensures that the drawing of a positive literal gadget X is positive if and only if the drawing of the negative literal gadget Y is negative, in case that both are joined with an inverter gadget. Further, Lemma 4 states that each cycle vertex lies either in a positive or negative region. Thus, the truth value of a variable u can be consistently determined by any drawing of a positive or negative literal gadget of u . By Lemma 8, the clause gadget has no \mathcal{D}_C -framed drawing of the clause gadget such that all literal gadgets have a negative drawing. Thus, the truth assignment indeed satisfies C . \square

Observe that we placed blue dotted circles in the figures of the gadgets. The squares can be replaced by these circles without changing the essential properties of the gadgets. More precisely, the disks can be replaced such that they contain the positive and negative placements of the corresponding squares. In case of obstacles, O is tangent to some line l in a point p , thus, O is replaced by a disk that is tangent to l in p . In the clause gadget we placed the obstacles O_A^2 with respect to two intersecting lines l_1, l_2 . In this case we place a disk with small radius such that it is tangent to l_1 and l_2 . Therefore, the proof for squares can be adapted to proof that the \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS problem with disks is \mathcal{NP} -hard.

Corollary 13. *The problem \mathcal{D}_C -FRAMED DRAWINGS OF NON-PLANAR ARRANGEMENTS with disks is \mathcal{NP} -hard, even when the clustered graph is restricted to vertex degree 5.*