# Mechanically Proving Determinacy of Hierarchical Block Diagram Translations 

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#### Abstract

Hierarchical block diagrams (HBDs) are at the heart of embedded system design tools, including Simulink. Numerous translations exist from HBDs into languages with formal semantics, amenable to formal verification. However, none of these translations has been proven correct, to our knowledge.

We present in this paper the first mechanically proven HBD translation algorithm. The algorithm translates HBDs into an algebra of terms with three basic composition operations (serial, parallel, and feedback). In order to capture various translation strategies resulting in different terms achieving different tradeoffs, the algorithm is nondeterministic. Despite this, we prove its semantic determinacy: for every input HBD, all possible terms that can be generated by the algorithm are semantically equivalent. We apply this result to show how three Simulink translation strategies introduced previously can be formalized as determinizations of the algorithm, and derive that these strategies yield semantically equivalent results (a question left open in previous work). All results are formalized and proved in the Isabelle theorem-prover.


## 1 Introduction

Dozens of tools, including Simulink ${ }^{1}$, the most widespread embedded system design environment, are based on hierarchical block diagrams (HBDs). Being a graphical notation (and in the case of Simulink a "closed" one in the sense that the tool is not open-source), such diagrams need to be translated into other formalisms more amenable to formal analysis. Several such translations exist, e.g., see [3, 39, 27, 42, 35, 10, 43, 44, 45, 29] and related discussion in $\S 2$. To our knowledge, none of these translations has been formally verified. This paper aims to remedy this fact.

This work is part of a larger project, the Refinement Calculus of Reactive Systems (RCRS) - see [16, 31, $33,41]$ and http://rcrs.cs.aalto.fi/. RCRS is a compositional framework for modeling and reasoning about reactive systems. It allows to capture systems which can be both non-deterministic and non-inputreceptive, and offers compositional refinement and other features for modular specification and verification. RCRS comes with a toolset [16] which includes a full implementation of the RCRS theory and related analysis tools on top of the Isabelle theorem prover [30], and a Translator of Simulink diagrams to RCRS theories.

The Translator, first described in [14], implements three translation strategies from HBDs to an algebra of components with three basic composition operators: serial, parallel, and feedback. The several translation strategies are motivated by the fact that each strategy has its own pros and cons. For instance, one strategy may result in shorter and/or easier to understand algebra terms, while another strategy may result in terms that are easier to simplify by manipulating formulas in a theorem prover. But a fundamental question is left open in [14]: are these translation strategies semantically equivalent, meaning, do they produce semantically equivalent terms? This is the question we study and answer (positively) in this paper.

[^0]The question is non-trivial, as we seek to prove the equivalence of three complex algorithms which manipulate a graphical notation (hierarchical block diagrams) and transform models in this notation into a different textual language, namely, the algebra mentioned above. Terms in this algebra have intricate formal semantics, and formally proving that two given specific terms are equivalent is already a non-trivial exercise. Here, the problem is to prove that a number of translation strategies $T_{1}, T_{2}, \ldots, T_{k}$ are equivalent, meaning that for any given graphical diagram $D$, the terms resulting from translating $D$ by applying these strategies, $T_{1}(D), T_{2}(D), \ldots, T_{k}(D)$, are all semantically equivalent.

This equivalence question is important for many reasons. Just like a compiler has many choices when generating code, a HBD translator has many choices when generating algebraic expressions. Just like a correct compiler must guarantee that all possible results are equivalent (independently of optimization or other flags/options), the translator must also guarantee that all possible algebraic expressions are equivalent. Moreover, the algebraic expressions constitute the formal semantics of HBDs, and hence also those of tools like Simulink. Therefore, this determinacy principle is also necessary in order for the formal Simulink semantics to be well-defined.

In order to formulate the equivalence question precisely, we introduce an abstract and nondeterministic algorithm for translating HBDs into an abstract algebra of components with three composition operations (serial, parallel, feedback) and three constants (split, switch, and sink). By abstract algorithm we understand an algorithm that produces terms in this abstract algebra. Concrete versions for this algorithm are obtained when using it for concrete models of the algebra (e.g., constructive functions). The algorithm is nondeterministic in the sense that it consists of a set of basic operations (transformations) that can be applied in any order. This allows to capture various deterministic translation strategies as determinizations (refinements [5]) of the abstract algorithm.

The main contributions of the paper are the following:

1. We formally and mechanically define a translation algorithm for HBDs.
2. We prove that despite its internal nondeterminism, the algorithm achieves deterministic results in the sense that all possible algebra terms that can be generated by the different nondeterministic choices are semantically equivalent.
3. We formalize two translation strategies introduced in [14] as refinements of the abstract algorithm.
4. We formalize also the third strategy (feedbackless) introduced in [14] as an independent algorithm.
5. We prove the equivalence of these three translation strategies.

To our knowledge, our work constitutes the first and only mechanically proven hierarchical block diagram translator.

We remark that our translation is compositional [14]. We also remark that our abstract algorithm can be instantiated in many different ways, encompassing not just the three translation strategies of [14], but also any other HBD translation strategy that can be devised by combining the basic composition operations defined in the abstract algorithm. As a consequence, our results imply not just the equivalence of the translation strategies of [14], but also the equivalence of any other translation strategy that could be devised as stated above. More generally, any translation of a graphical formalism into expressions in some language would have to deal with problems similar to those tackled in this paper, and our work offers an example of how to address these problems in a formal manner.

The entire RCRS framework, including all results in this paper, have been formalized and proved in the Isabelle theorem prover [30] and are part of the RCRS toolset which is publicly available in a figshare repository [17]. The theories relevant to this paper are under RCRS/Isabelle/TranslateHBD. The RCRS toolset can be downloaded also from the RCRS web page: http://rcrs.cs.aalto.fi/.

## 2 Related Work

Model transformation and the verification of its correctness is a long standing line of research, which includes classification of model transformations [4] and the properties they must satisfy with respect to their intent [25], verification techniques [1], frameworks for specifying model transformations (e.g., ATL [18]), and various implementations for specific source and target meta-models. Extensive surveys of the above can be found in $[4,9,1]$.

Several translations from Simulink have been proposed in the literature, including to Hybrid Automata [3], BIP [39], NuSMV [27], Lustre [42], Boogie [35], Timed Interval Calculus [10], Function Blocks [43], I/O Extended Finite Automata [44], Hybrid CSP [45], and SpaceEx [29]. It is unclear to what extent these approaches provide formal guarantees on the determinism of the translation. For example, the order in which blocks in the Simulink diagram are processed might a-priori influence the result. Some works fix this order, e.g., [35] computes the control flow graph and translates the model according to this computed order. In contrast, we prove that the results of our algorithm are equivalent for any order. To the best of our knowledge, the abstract translation proposed hereafter for Simulink is the only one formally defined and mechanically proven correct.

The focus of several works is to validate the preservation of the semantics of the original diagram by the resulting translation (e.g., see $[43,36,8,37]$ ). In contrast, our goal is to prove equivalence of all possible translations. Given that Simulink semantics is informal ("what the simulator does"), ultimately the only way to gain confidence that the translation conforms to the original Simulink model is by simulation (e.g., as in [14]).

The work of [2] presents a correspondence between formulas and proofs in linear logic [20] and types and computations in process calculi [23, 28]. A sequent in the logic $\left(\vdash A, C^{\perp}, B\right)$ is interpreted as an interface specification for a concurrent process and how this process is connected to the environment. In this example $A$ and $B$ are inputs to the process and $C^{\perp}$ is output. In our approach we connect components by naming their inputs and outputs, and an output is connected to an input if they have the same name.

With respect to the target algebra of our translation, the most relevant related works are the algebra of flownomials [40] and the relational model for non-deterministic dataflow [21]. A comparison with these works is presented in Section 5.

In [11], graphs and graph operations are represented by algebraic expressions and operations, and a complete equational axiomatization of the equivalence of the graph expressions is given. This is then applied to flow-charts as investigated in [38].

To our knowledge, none of the theoretical works on flownomials, nor graph represented as expressions, nor the more practical works on translating HBDs/Simulink, are mechanically formalized or verified.

## 3 Preliminaries

For a type or set $X, X^{*}$ is the type of finite lists with elements from $X$. We denote the empty list by $\epsilon$, $\left(x_{1}, \ldots, x_{n}\right)$ denotes the list with elements $x_{1}, \ldots, x_{n}$, and for lists $x$ and $y, x \cdot y$ denotes their concatenation. The length of a list $x$ is denoted by $|x|$. The list of common elements of $x$ and $y$ in the order occurring in $x$ is denoted by $x \otimes y$. The list of elements from $x$ that do not occur in $y$ is denoted by $x \ominus y$. We define $x \oplus y=x \cdot(y \ominus x)$, the list of $x$ concatenated with the elements of $y$ not occurring in $x$. A list $x$ is a permutation of a list $y$, denoted perm $(x, y)$, if $x$ contains all elements of $y$ (including multiplicities) possibly in a different order. For a list $x, \operatorname{set}(x)$ denotes the set of all elements of $x$.

### 3.1 Constructive Functions

We introduce in this section the constructive functions as used in the constructive semantics literature [26, 7, 19]. They will provide a concrete model for the abstract algebra of HBDs, introduced in Section 5. These functions are also used in the example from Section 4.

We assume that $\perp$ is a new element called unknown, and that $\perp$ is not an element of other sets that we use. For a set $A$ we define $A^{\perp}=A \cup\{\perp\}$, and on $A^{\perp}$ we introduce the pointed complete partial order (cpo) [12] by $(a \leq b) \Longleftrightarrow(a=\perp \vee a=b)$. We extend the order on $A^{\perp}$ to the Cartesian product $A_{1}^{\perp} \times \cdots A_{n}^{\perp}$ by $\left(x_{1}, \ldots, x_{n}\right) \leq\left(y_{1}, \ldots, y_{n}\right) \Longleftrightarrow\left(\forall 1 \leq i \leq n: x_{i} \leq y_{i}\right)$.

Constructive functions are the monotonic functions $f: A_{1}^{\perp} \times \ldots \times A_{n}^{\perp} \rightarrow B_{1}^{\perp} \times \ldots \times B_{m}^{\perp}$, i.e., $(\forall x, y: x \leq$ $y \Rightarrow f(x) \leq f(y))$. We denote these functions by $A_{1} \cdots A_{n} \xrightarrow{c} B_{1} \cdots B_{m}\left(f: A_{1} \cdot \ldots \cdot A_{n} \xrightarrow{c} B_{1} \cdot \ldots \cdot B_{m}\right.$ for $\left.f: A_{1}^{\perp} \times \ldots \times A_{n}^{\perp} \rightarrow B_{1}^{\perp} \times \ldots \times B_{m}^{\perp}\right)$. Id $: A \xrightarrow{c} A$ denotes the identity function on $A: \forall x: \operatorname{ld}(x)=x$.

For constructive functions $f: A \xrightarrow{c} B$ and $g: B \xrightarrow{c} C$, their serial composition $g \circ f$ is the normal function composition $(g \circ f)(x)=g(f(x))$. The parallel composition of $f: A \xrightarrow{c} B$ and $g: A^{\prime} \xrightarrow{c} B^{\prime}$ is denoted $f \| g: A \cdot A^{\prime} \xrightarrow{c} B \cdot B^{\prime}$ and is defined by $(f \| g)(x, y)=(f(x), g(y))$. We assume that parallel composition operator binds stronger than serial composition, i.e. $f \| g \circ h$ is the same as $(f \| g) \circ h$.

For a constructive function $f: A \xrightarrow{c} A$, its least fixpoint always exists [12], and we use it to define a feedback composition. If $f: A \cdot B \xrightarrow{c} A \cdot B^{\prime}$ is a constructive function, then its feedback (on $A$ ), denoted feedback $(f): B \xrightarrow{c} B^{\prime}$, is defined by

$$
\text { feedback }(f)(y)=f\left(\mu x: f_{1}(x, y), y\right)
$$

where $f_{1}: A \cdot B \xrightarrow{c} A$ is the first component of $f$ and $\left(\mu x: f_{1}(x, y)\right)$ is the least fixpoint of the function that, for fixed $y$, maps $x$ into $f_{1}(x, y)$.

Let $x_{1}, \ldots, x_{n}$ be variables ranging over types $A_{1}, \ldots, A_{n}$, and $e_{1}, \ldots, e_{m}$ expressions using basic operations $(+,-, \ldots)$ on these variables, ranging over types $B_{1}, \ldots, B_{m}$. We define the constructive function

$$
\left[x_{1}, \ldots, x_{n} \leadsto e_{1}, \ldots, e_{m}\right]: A_{1} \cdots A_{n} \xrightarrow{c} B_{1} \cdots B_{n}
$$

as the function that maps $\left(x_{1}, \ldots, x_{n}\right) \in A_{1}^{\perp} \times \ldots \times A_{n}^{\perp}$ into $\left(e_{1}, \ldots, e_{m}\right)$, where the basic operations are extended to unknown values in a standard way (e.g. $3+\perp=\perp, \perp \cdot 0=0$ ).

### 3.2 Refinement Calculus and Hoare Total Correctness Triples

We model the (nondeterministic) algorithms using monotonic predicate transformers [13] within the refinement calculus [5].

We assume a state space $\Sigma$. A state $\sigma \in \Sigma$ gives values to all program variables. Programs are modeled as monotonic predicate transformers on $\Sigma$, that is monotonic functions from predicates to predicates $((\Sigma \rightarrow$ bool $) \rightarrow(\Sigma \rightarrow$ bool $)$ ) with a weakest precondition interpretation. For $P:(\Sigma \rightarrow$ bool $) \rightarrow(\Sigma \rightarrow$ bool $)$ and a post condition $q: \Sigma \rightarrow$ bool, $P(q)$ is the predicate that is true for the initial states from which the execution of the program modeled by $P$ always terminates, and it terminates in a state from $q$. In the rest of the paper we refer to monotonic elements of $(\Sigma \rightarrow$ bool $) \rightarrow(\Sigma \rightarrow$ bool $)$ as programs. The program statements are modeled as operations on monotonic predicate transformers.

For predicates ( $\Sigma \rightarrow$ bool), we use the notations $\cup, \cap$, $\neg$, and $\subseteq$ for the union, intersection, complement, and subset operations, respectively.

The nondeterministic assignment statement, denoted $\left[x:=x^{\prime} \mid p\left(y, x^{\prime}\right)\right]$, assigns a new value $x^{\prime}$ to variable $x$ such that the property $p\left(y, x^{\prime}\right)$ is true. In $p\left(y, x^{\prime}\right)$, variable $y$ stands for the current value (before the assignement) of $y$ used for updating varible $x$. We can choose $y=x$, to refer to the current value of variable $x$. For example $\left[x:=x^{\prime} \mid x^{\prime}>x+1\right]$ assigns to $x$ a new value greater that the current value of $x+1$.

Formally, the nondeterministic assignment statement is defined by:

$$
\left[x:=x^{\prime} \mid p\left(y, x^{\prime}\right)\right](q)(\sigma)=\left(\forall x^{\prime}: p\left(\sigma(y), x^{\prime}\right) \Rightarrow q\left(\sigma\left[x:=x^{\prime}\right]\right)\right)
$$

where $\sigma(y)$ is the value of variable $y$ in state $\sigma$, and $\sigma\left[x:=x^{\prime}\right]$ is a new state obtained from $\sigma$ by changing the value of $x$ to $x^{\prime}$.

The standard assignment statement $x:=e$ is defined as $\left[x:=x^{\prime} \mid x^{\prime}=e\right]$, where $x^{\prime}$ is a new name.

For a predicate $p: \Sigma \rightarrow$ bool, the assert statement, denoted $\{p\}$, starting from a state $\sigma$ behaves as skip if $p(\sigma)$ is true, and it fails otherwise. By fail we mean a program that runs forever.

$$
\{p\}(q)=p \cap q
$$

The sequential composition of programs $P, P^{\prime}$, denoted $P ; P^{\prime}$ is the function composition of predicate transformers:

$$
\left(P ; P^{\prime}\right)(q)=P\left(P^{\prime}(q)\right) .
$$

The nondeterministic choice of $P$ and $P^{\prime}$, denoted $P \sqcap P^{\prime}$, is the pointwise extension of the intersection on predicates to predicate transformers:

$$
\left(P \sqcap P^{\prime}\right)(q)=P(q) \cap P^{\prime}(Q)
$$

For a predicate $b$ and programs $P$ and $P^{\prime}$, the if statement, denoted if $b$ then $P$ else $P^{\prime}$ is defined by

$$
\text { if } b \text { then } P \text { else } P^{\prime}=(\{b\} ; P) \sqcup\left(\{\neg b\} ; P^{\prime}\right)
$$

where $\sqcup$ is the pointwise extension of union on predicates to predicate transformers $\left(\left(P \sqcup P^{\prime}\right)(q)=P(q) \cup\right.$ $\left.P^{\prime}(q)\right)$.

For predicate $b$ and program $P$, the while statement, denoted while $b$ do $P$, is defined as a least fixpoint:

$$
\text { while } b \text { do } P=(\mu X: \text { if } b \text { then } P ; X \text { else skip })
$$

where skip is the program that does not change the state, modeled as the identity predicate transformer, and ( $\mu X$ : if $b$ then $P ; X$ else skip) is the least fixpoint of the function mapping $X$ into if $b$ then $P ; X$ else skip. The fixpoint always exists because of the monotonicity assumption.

The refinement relation of programs, denoted $P \sqsubseteq P^{\prime}$, is again the pointwise extension of the inclusion order on predicates to predicate transformers:

$$
\left(P \sqsubseteq P^{\prime}\right)=\left(\forall q: P(q) \subseteq P^{\prime}(q)\right)
$$

If a program $P^{\prime}$ is a refinement of another program $P,\left(P \sqsubseteq P^{\prime}\right)$, then we can use $P^{\prime}$ in every context where we can use $P$. In a refinement $P \sqsubseteq P^{\prime}$, the program $P^{\prime}$ is more deterministic than $P$, and it fails for less input states. For example we have the following refinement:

$$
\{x>10\} ;\left[x:=x^{\prime} \mid x^{\prime}=1 \vee x^{\prime}=2 \vee x^{\prime}=3\right] \sqsubseteq\{x>0\} ;\left[x:=x^{\prime} \mid x^{\prime}=1 \vee x^{\prime}=3\right]
$$

In this example, the second program fails for less states $x>0$ as opposed to $x>10$, and it is more deterministic. The second program can assign to $x$ only the values 1 and 3 , while the first program can assign also value 2 .

Finally we introduce Hoare [22] total correctness triples for programs. If $p$ is a precondition predicate on states, $q$ is a postcondition predicate on states, and $P$ is a program, then the Hoare total correctness triple $p\{\mid P\} q$ is defined by

$$
(p\{|P|\} q)=(p \subseteq P(q))
$$

The interpretation of the triple $p\{|P|\} q$ is the following. If the program $P$ starts from a state $\sigma$ satisfying the precondition $p$, then $P$ always terminates, and it terminates into a state satisfying the postcondition $q$.

In general, the correctness of a program is stated as a Hoare triple, and it is proved by reducing this correctness problem to smaller and smaller programs using Hoare rules. As examples we give here two Hoare rules for the correctness of the nondeterministic assignment and while statements. ${ }^{2}$

[^1]

Figure 1: Running example: diagram for summation.

Lemma 1 (Hoare rule for the nondeterministic assignment). If $p, q$ are predicates on state and $b$ is a predicate on $y, x^{\prime}$ such that

$$
\left(\forall \sigma, x^{\prime}: p(\sigma) \wedge b\left(\sigma(y), x^{\prime}\right) \Rightarrow q\left(\sigma\left[x:=x^{\prime}\right]\right)\right)
$$

then

$$
\left(p\left\{\left[x:=x^{\prime} \mid b\left(y, x^{\prime}\right)\right] \mid\right\} q\right)
$$

Lemma 2 (Hoare rule for while). If $p, q, b, I$ are predicates on state, $t: \Sigma \rightarrow$ nat is a function from state to natural numbers, and $P$ is a program such that

$$
(\forall n:(I \wedge t=n)\{|P|\}(I \wedge t<n)) \quad \text { and } \quad(p \subseteq I) \quad \text { and } \quad(\neg b \cap I \subseteq q)
$$

then

$$
(p\{\mid \text { while } b \text { do } P \mid\} q)
$$

In this lemma $I$ is called the invariant and its role is to ensure the correctness of the while program based on the correctness of the body $(P)$. The function (term) $t$ is called the variant and it is used to ensure the termination of the while program for all possible input states satisfying $p$.

There is an important relationship between Hoare rules and refinement, expressed by the next lemma.
Lemma 3. If $P, P^{\prime}$ are programs, then

$$
P \sqsubseteq P^{\prime} \quad \Leftrightarrow \quad\left(\forall p, q:(p\{|P|\} q) \Rightarrow\left(p\left\{\left|P^{\prime}\right|\right\} q\right)\right)
$$

## 4 Overview of the Translation Algorithm

A block diagram $N$ is a network of interconnected blocks. A block may be a basic (atomic) block, or a composite block that corresponds to a sub-diagram. If $N$ contains composite blocks then it is called a hierarchical block diagram (HBD); otherwise it is called flat. An example of a flat diagram is shown in Figure 1a. The connections between blocks are called wires, and they have a source block and a target block. For simplicity, we will assume that every wire has a single source and a single target. This can be achieved by adding extra blocks. For instance, the diagram of Figure 1a can be transformed as in Figure 1b by adding an explicit block called Split.

Let us explain the idea of the translation algorithm. We first explain the idea for flat diagrams, and then we extend it recursively for hierarchical diagrams.

A diagram is represented in the algorithm as a list of elements corresponding to the basic blocks. One element of this list is a triple containing a list of input variables, a list of output variables, and a function.

The function computes the values of the outputs based on the values of the inputs, and for now it can be thought of as a constructive function. Later this function will be an element of an abstract algebra modeling HBDs. Wires are represented by matching input/output variables from the block representations.

A block diagram may contain stateful blocks such as delays or integrators. We model these blocks using additional state variables (wires). In Figure 1, the only stateful block is the block Delay. We model this block as an element with two inputs $(x, s)$, two outputs ( $y, s^{\prime}$ ) and function $\left(y, s^{\prime}\right):=(s, x)$ (Figure 1c). More details about this representation can be found in [14].

In summary, the list representation of the example of Figure 1 is the following:

$$
\begin{aligned}
& \text { (Add, Delay, Split) where } \\
& \text { Add }=((z, u), x,[z, u \leadsto z+u]) \\
& \text { Delay }=\left((x, s), \quad\left(y, s^{\prime}\right),[x, s \leadsto s, x]\right) \\
& \text { Split }=(y,(z, v), \quad[y \leadsto y, y])
\end{aligned}
$$

The algorithm works by choosing nondeterministically some elements from the list and replacing them with their appropriate composition (serial, parallel, or feedback). The composition must connect all the matching variables. Let us illustrate how the algorithm may proceed on the example of Figure 1; for the full description of the algorithm see Section 6.

Suppose the algorithm first chooses to compose Add and Delay. The only matching variable in this case is $x$, between the output of Add and the first input of Delay. The appropriate composition to use here is serial composition. Because Delay also has $s$ as input, Add and Delay cannot be directly connected in series. This is due to the number of outputs of Add that need to match the number of inputs of Delay. To compute the serial composition, Add must first be composed in parallel with the identity block Id, as shown in Figure 1d. Doing so, a new element $A$ is created:

$$
A=\left((z, u, s),\left(y, s^{\prime}\right), \text { Delay } \circ(\text { Add } \| \text { Id })\right) .
$$

Next, $A$ is composed with Split. In this case we need to connect variable $y$ (using serial composition), as well as $z$ (using feedback composition). The resulting element is

$$
\left((u, s),\left(v, s^{\prime}\right), \text { feedback }((\text { Split } \| \text { Id }) \circ \text { Delay } \circ(\text { Add } \| \text { Id }))\right)
$$

where we need again to add the Id component for variable $s^{\prime}$.
Suppose now that the algorithm chooses to compose first the blocks Split and Add (Fig. 2a) into B.

$$
B=((y, u),(x, v),(\text { Add } \| \text { Id }) \circ(\operatorname{Id} \|[v, u \sim u, v]) \circ(\text { Split } \| \text { Id }))
$$

In this composition, in addition to the Id components, we need now also a switch ( $[v, u \sim u, v]$ ) for wires $v$ and $u$. Next the algorithm composes $B$ and Delay (Fig. 2b):

$$
\left((u, s),\left(s^{\prime}, v\right), \text { feedback }((\text { Delay } \| \text { Id }) \circ(\text { Id } \|[v, s \sim s, v]) \circ(B \| \text { Id }))\right)
$$

As we can see from this example, by considering the blocks in the diagram in different orders, we obtain different expressions. On this example, the first expression is simpler (it has less connectors) than the second one. In general, a diagram, being a graph, does not have a predefined canonical order, and we need to show that the result of the algorithm is the same regardless of the order in which the blocks are considered.

We make two remarks here. First, the final result of the algorithm is a triple with the same structure as all elements on the original list: (input variables, output variables, function), where the function represents the computation performed by the entire diagram. Therefore, the algorithm can be applied recursively on HBDs.

Second, the variables in the representation occur at most twice, once as input, and once as output. The variables occurring only as inputs are the inputs of the resulting final element, and variables occurring only as outputs are the outputs of the resulting final element. This is true in general for all diagrams, due to the representation of splitting of wires. This fact is essential for the correctness of the algorithm as we will see in Section 6.


Figure 2: A different composition order for the example from Fig. 1.

## 5 An Abstract Algebra for Hierarchical Block Diagrams

We assume that we have a set of Types. We also assume a set of diagrams Dgr. Every element $S \in \operatorname{Dgr}$ has input type $t \in$ Types* and output type $t^{\prime} \in$ Types*. If $t=t_{1} \cdots t_{n}$ and $t^{\prime}=t_{1}^{\prime} \cdots t_{m}^{\prime}$, then $S$ takes as input a tuple of the type $t_{1} \times \ldots \times t_{n}$ and produces as output a tuple of the type $t_{1}^{\prime} \times \ldots \times t_{m}^{\prime}$. We denote the fact that $S$ has input type $t \in$ Types* and output type $t^{\prime} \in$ Types* by $S: t \xrightarrow{\circ} t^{\prime}$. The elements of Dgr are abstract.

### 5.1 Operations of the Algebra of HBDs

Constants. Basic blocks are modeled as constants on Dgr. For types $t, t^{\prime} \in$ Types* we assume the following constants:

$$
\begin{aligned}
& \operatorname{Id}(t): t \xrightarrow{\circ} t \\
& \operatorname{Split}(t): t \xrightarrow{\circ} t \cdot t \\
& \operatorname{Sink}(t): t \stackrel{\circ}{\longrightarrow} \epsilon \\
& \operatorname{Switch}\left(t, t^{\prime}\right): t \cdot t^{\prime} \xrightarrow{\circ} t^{\prime} \cdot t
\end{aligned}
$$

Id corresponds to the identity block. It copies the input into the output. In the model of constructive functions $\operatorname{ld}(t)$ is the identity function. Split $(t)$ takes an input $x$ of type $t$ and outputs $x \cdot x$ of type $t \cdot t$. $\operatorname{Sink}(t)$ returns the empty tuple $\epsilon$, for any input $x$ of type $t$. Switch $\left(t, t^{\prime}\right)$ takes an input $x \cdot x^{\prime}$ with $x$ of type $t$ and $x^{\prime}$ of type $t^{\prime}$ and returns $x^{\prime} \cdot x$. In the model of constructive functions these diagrams are total functions and they are defined as explained above. In the abstract model, the behaviors of these constants is defined with a set of axioms (see below).

Composition operators. For two diagrams $S: t \xrightarrow{\circ} t^{\prime}$ and $S^{\prime}: t^{\prime} \xrightarrow{\circ} t^{\prime \prime}$, their serial composition, denoted $S ; S^{\prime}: t \xrightarrow{\circ} t^{\prime \prime}$ is a diagram that takes inputs of type $t$ and produces outputs of type $t^{\prime \prime}$. In the model of constructive functions, the serial composition corresponds to function composition ( $S ; S^{\prime}=S^{\prime} \circ S$ ). Please note that in the abstract model we write the serial composition as $S ; S^{\prime}$, while in the model of constructive functions the first diagram that is applied to the input occurs second in the composition.

The parallel composition of two diagrams $S: t \xrightarrow{\circ} t^{\prime}$ and $S^{\prime}: r \xrightarrow{\circ} r^{\prime}$, denoted $S \| S^{\prime}: t \cdot r \xrightarrow{\circ} t^{\prime} \cdot r^{\prime}$, is a diagram that takes as input tuples of type $t \cdot r$ and produces as output tuples of type $t^{\prime} \cdot r^{\prime}$. This parallel composition corresponds to the parallel composition of constructive functions.

Finally we introduce a feedback composition. For $S: a \cdot t \xrightarrow{\circ} a \cdot t^{\prime}$, where $a \in$ Types is a single type, the feedback of $S$, denoted feedback $(S): t \xrightarrow{\circ} t^{\prime}$, is the result of connecting in feedback the first output of $S$ to its first input. Again this feedback operation corresponds to the feedback of constructive functions.

We assume that parallel composition operator binds stronger than serial composition, i.e. $S \| T ; R$ is the same as $(S \| T) ; R$.

Graphical diagrams can be represented as terms in the abstract algebra, as illustrated in Figure 3. This figure depicts two diagrams, and their corresponding algebra terms. As it turns out, these two diagrams are equivalent, in the sense that their corresponding algebra terms can be shown to be equal using the axioms presented below.

feedback $(\operatorname{Id}(a)\|S ; \operatorname{Switch}(a, a)\| \operatorname{Id}(t) ; \operatorname{Id}(a) \| T)=$

$S ; T$

Figure 3: Two flat diagrams and their corresponding terms in the abstract algebra.

### 5.2 Axioms of the Algebra of HBDs

In the abstract algebra, the behavior of the constants and composition operators is defined by a set of axioms, listed below:

1. $S: t \xrightarrow{\circ} t^{\prime} \Longrightarrow \operatorname{ld}(t) ; S=S ; \operatorname{ld}\left(t^{\prime}\right)=S$
2. $S: t_{1} \xrightarrow{\circ} t_{2} \wedge T: t_{2} \xrightarrow{\circ} t_{3} \wedge R: t_{3} \xrightarrow{\circ} t_{4} \Longrightarrow S ;(T ; R)=(S ; T) ; R$
3. $\operatorname{ld}(\epsilon)\|S=S\| \operatorname{Id}(\epsilon)=S$
4. $S\|(T \| R)=(S \| T)\| R$
5. $S: s \xrightarrow{\circ} s^{\prime} \wedge S^{\prime}: s^{\prime} \xrightarrow{\circ} s^{\prime \prime} \wedge T: t \xrightarrow{\circ} t^{\prime} \wedge T^{\prime}: t^{\prime} \xrightarrow{\circ} t^{\prime \prime}$ $\Longrightarrow(S \| T) ;\left(S^{\prime} \| T^{\prime}\right)=\left(S ; S^{\prime}\right) \|\left(T ; T^{\prime}\right)$
6. $\operatorname{Split}(t) ; \operatorname{Sink}(t) \| \operatorname{Id}(t)=\operatorname{Id}(t)$
7. Split $(t)$; Switch $(t, t)=\operatorname{Split}(t)$
8. Split $(t) ; \operatorname{Id}(t)\|\operatorname{Split}(t)=\operatorname{Split}(t) ; \operatorname{Split}(t)\| \operatorname{Id}(t)$
9. Switch $\left(t, t^{\prime} \cdot t^{\prime \prime}\right)=\operatorname{Switch}\left(t, t^{\prime}\right)\left\|\operatorname{Id}\left(t^{\prime \prime}\right) ; \operatorname{Id}\left(t^{\prime}\right)\right\| \operatorname{Switch}\left(t, t^{\prime \prime}\right)$
10. $\operatorname{Sink}\left(t \cdot t^{\prime}\right)=\operatorname{Sink}(t) \| \operatorname{Sink}\left(t^{\prime}\right)$
11. $\operatorname{Split}\left(t \cdot t^{\prime}\right)=\operatorname{Split}(t)\left\|\operatorname{Split}\left(t^{\prime}\right) ; \operatorname{Id}(t)\right\| \operatorname{Switch}\left(t, t^{\prime}\right) \| \operatorname{Id}\left(t^{\prime}\right)$
12. $S: s \xrightarrow{\circ} s^{\prime} \wedge T: t \xrightarrow{\circ} t^{\prime} \Longrightarrow \operatorname{Switch}(s, t) ; T\left\|S ; \operatorname{Switch}\left(t^{\prime}, s^{\prime}\right)=S\right\| T$
13. feedback $(\operatorname{Switch}(a, a))=\operatorname{Id}(a)$
14. $S: a \cdot s \xrightarrow{\circ} a \cdot t \Longrightarrow$ feedback $(S \| T)=$ feedback $(S) \| T$
15. $S: a \cdot s \xrightarrow{\circ} a \cdot t \wedge A: s^{\prime} \xrightarrow{\circ} s \wedge B: t \xrightarrow{\circ} t^{\prime}$

$$
\Longrightarrow \text { feedback }(\operatorname{ld}(a)\|A ; S ; \operatorname{ld}(a)\| B)=A ; \text { feedback }(S) ; B
$$

16. $S: a \cdot b \cdot s \xrightarrow{\circ} a \cdot b \cdot t$

$$
\Longrightarrow \text { feedback }^{2}(\operatorname{Switch}(b, a)\|\operatorname{Id}(s) ; S ; \operatorname{Switch}(a, b)\| \operatorname{Id}(t))=\text { feedback }^{2}(S)
$$



Figure 4: Axiom (5) Distributivity of serial and parallel compositions.


Figure 5: Axiom (11) Split switch.

Axioms (1) and (2) express the fact that the identity is the neutral element for the serial composition, and the serial composition is associative. Similarly, axioms (3) and (4) state that the identity of the empty type is the neutral element for the parallel composition, and that parallel composition is associative.

Axiom (5) introduces a distributivity property of serial and parallel compositions. Figure 4 represents graphically this axiom.

Axioms (6) - (11) express the properties of Split, Sink, and Switch. For example Axiom (11), represented in Figure 5, says that if we duplicate $x \cdot y$ of type $t \cdot t^{\prime}$, then this is equivalent to duplicate $x$ and $y$ in parallel, and then switch the middle $x$ and $y$.

Axiom (12) says that switching the inputs and outputs of $T \| S$ is equal to $S \| T$.
Axioms (13) - (16) are about the feedback operator. Axiom (13), represented in Figure 6, states that feedback of switch is the identity. Axiom (14), represented in Figure 7, states that feedback of the parallel composition of $S$ and $T$ is the same as the parallel composition of the feedback of $S$ and $T$. Axiom (15), Figure 8, states that components $A$ and $B$ can be taken out of the feedback operation. Finally, Axiom (16) represented in Figure 9, states that the order in which we apply the feedback operations does not change the result.

These axioms are equivalent to a subset of the axioms of algebra of flownomials [40], which implies that all models of flownomials are also models of our algebra. In [21], a relational model for dataflow is introduced. This model is also based on a set of axioms on feedback, serial and parallel compositions, but [21] does not use the split constant. Our axioms that are not involving split are equivalent to the axioms used in [21]. The focus of [21] is the construction of a relational model for the axioms.

The following theorem provides a concrete semantic domain for HBDs.
Theorem 1. Constructive functions with the operations defined in Section 3 are a model for axioms (1) (16).

We remark that constructive functions are only one example of a model for axioms (1) - (16), and by no means the only model. As mentioned above, all models of flownomials are also models of our algebra. In particular, relations are a model of flownomials and therefore also a model for axioms (1) - (16) [40].


Figure 6: Axiom (13) Feedback of switch.


Figure 7: Axiom (14) Feedback of parallel.


Figure 8: Axiom (15) Feedback of serial.

## 6 The Abstract Translation Algorithm and its Determinacy

### 6.1 Diagrams with Named Inputs and Outputs

The algorithm works by first transforming the graph of a HBD into a list of basic components with named inputs and outputs as explained in Section 4. For this purpose we assume a set of names or variables Var and a function T: Var $\rightarrow$ Types. For $v \in \operatorname{Var}, \mathrm{~T}(v)$ is the type of variable $v$. We extend T to lists of variables by $\mathrm{T}\left(v_{1}, \ldots, v_{n}\right)=\left(T\left(v_{1}\right), \ldots, T\left(v_{n}\right)\right)$.

Definition 1. A diagram with named inputs and outputs or io-diagram for short is a tuple (in, out, $S$ ) such that in, out $\in \mathrm{Var}^{*}$ are lists of distinct variables, and $S: \mathrm{T}($ in $) \xrightarrow{\circ} \mathrm{T}($ out $)$.

In what follows we use the symbols $A, A^{\prime}, B, \ldots$ to denote io-diagrams, and $\mathrm{I}(A), \mathrm{O}(A)$, and $\mathrm{D}(A)$ to denote the input variables, the output variables, and the diagram of $A$, respectively.

Definition 2. For io-diagrams $A$ and $B$, we define $\mathrm{V}(A, B)=\mathrm{O}(A) \otimes \mathrm{I}(B) \in \mathrm{Var}^{*}$.
$\mathrm{V}(A, B)$ is the list of common variables that are output of $A$ and input of $B$, in the order occurring in $\mathrm{O}(A)$. We use $\mathrm{V}(A, B)$ later to connect for example in series $A$ and $B$ on these common variables, as we did for constructing $A$ from Add and Delay in Section 4.

### 6.2 General Switch Diagrams

We compose diagrams when their types are matching, and we compose io-diagrams based on matching names of input output variables. For example if we have two io-diagrams $A$ and $B$ with $\mathrm{O}(A)=u \cdot v$ and $\mathrm{I}(B)=v \cdot u$, then we can compose in series $A$ and $B$ by switching the output of $A$ and feeding it into $B$, i.e., $(A ; \operatorname{Switch}(\mathrm{T}(u), \mathrm{T}(v)) ; B)$.

In general, for two lists of variables $x=\left(x_{1} \cdots x_{n}\right)$ and $y=\left(y_{1} \cdots y_{k}\right)$ we define a general switch diagram $\left[x_{1} \cdots x_{n} \leadsto y_{1} \cdots y_{k}\right]: \mathbf{\top}\left(x_{1} \cdots x_{n}\right) \xrightarrow{\circ} \mathbf{T}\left(y_{1} \cdots y_{k}\right)$. Intuitively this diagram takes as input a list of values of type $\mathrm{T}\left(x_{1} \cdots x_{n}\right)$ and outputs a list of values of type $\mathrm{T}\left(y_{1} \cdots y_{k}\right)$, where the output value corresponding


Figure 9: Axiom (16) Feedback of switched inputs/outputs.


Figure 10: The diagram Arb.
to variable $y_{j}$ is equal to the value corresponding to the first $x_{i}$ with $x_{i}=y_{j}$ and it is arbitrary (unknown) if there is no such $x_{i}$. For example in the constructive functions model $[u, v \leadsto v, u, w, u]$ for input $(a, b)$ outputs $(b, a, \perp, a)$.

To define [_~_] we use Split, Sink, and Switch, but we need also an additional diagram that outputs an arbitrary (or unknown) value for an empty input. For $a \in \operatorname{Types}$, we define $\operatorname{Arb}(a): \epsilon \xrightarrow{\circ} a$ by

$$
\operatorname{Arb}(a)=\text { feedback }(\operatorname{Split}(a))
$$

The diagram Arb is represented in Figure 10.
We define now $[x \leadsto y]: \mathrm{T}(x) \xrightarrow{\circ} \mathrm{T}(y)$ in two steps. First for $x \in \operatorname{Var}{ }^{*}$ and $u \in \operatorname{Var}$, the diagram $[x \sim u]: \mathrm{T}(x) \xrightarrow{\circ} \mathrm{T}(u)$, for input $a_{1}, \ldots, a_{n}$ it outputs the value $a_{i}$ where $i$ is the first index such that $x_{i}=u$. Otherwise it outputs an arbitrary (unknown) value.

$$
\begin{array}{ll}
{[\epsilon \sim u]} & =\operatorname{Arb}(\mathrm{T}(u)) \\
{[u \cdot x \sim u]} & =\operatorname{Id}(\mathrm{T}(u)) \| \operatorname{Sink}(\mathrm{T}(x)) \\
{[v \cdot x \sim u]} & =\operatorname{Sink}(\mathrm{T}(v)) \|[x \sim u] \quad(\text { if } u \neq v)
\end{array}
$$

and

$$
\begin{array}{ll}
{[x \sim \epsilon]} & =\operatorname{Sink}(\mathrm{T}(x)) \\
{[x \sim u \cdot y]} & =\operatorname{Split}(\mathrm{T}(x)) ;([x \sim u] \|[x \sim y])
\end{array}
$$

### 6.3 Basic Operations of the Abstract Translation Algorithm

The algorithm starts with a list of io-diagrams and repeatedly applies operations until it reduces the list to only one io-diagram. These operations are the extensions of serial, parallel and feedback from diagrams to io-diagrams.
Definition 3. The named serial composition of two io-diagrams $A$ and $B$, denoted $A ; ; B$ is defined by $A ; ; B=($ in, out, $S)$, where $x=\mathrm{I}(B) \ominus \mathrm{V}(A, B), y=\mathrm{O}(A) \ominus \mathrm{V}(A, B)$, in $=\mathrm{I}(A) \oplus x$, out $=y \cdot \mathrm{O}(B)$ and

$$
S=[\text { in } \leadsto \mathrm{I}(A) \cdot x] ; \mathrm{D}(A)\|[x \sim x] ;[\mathrm{O}(A) \cdot x \sim y \cdot \mathrm{I}(B)] ;[y \sim y]\| \mathrm{D}(B)
$$

The construction of $A$ from Section 4 can be obtained by applying the named serial composition to Add and Delay.

Figure 11 illustrates an example of the named serial composition. In this case we have $\mathrm{V}(A, B)=u$, $x=(a, b), y=(v, w)$, in $=(a, c, b)$, and out $=(v, w, d, e)$. The component $A$ has outputs $u, v, w$, and $u$ is also input of $B$. Variable $u$ is the only variable that is output of $A$ and input of $B$. Because the outputs $v, w$ of $A$ are not inputs of $B$ they become outputs of $A ; ; B$. Variable $a$ is input for both $A$ and $B$, so in $A ; ; B$ the value of $a$ is split and fed into both $A$ and $B$. The diagram for this example is:

$$
[a, c, b \leadsto a, c, a, b] ; A\|\operatorname{Id}(\mathbf{T}(a, b)) ;[u, v, w, a, b \leadsto v, w, a, u, b] ; \operatorname{ld}(\mathbf{T}(v, w))\| B
$$

The result of the named serial composition of two io-diagrams is not always an io-diagram. The problem is that the outputs of $A ; ; B$ are not distinct in general. The next lemma gives sufficient conditions for $A ; ; B$ to be an io-diagram.

Lemma 4. If $A, B$ are io-diagrams and $(\mathrm{O}(A) \ominus \mathrm{I}(B)) \otimes \mathrm{O}(B)=\epsilon$ then $A ; ; B$ is an io-diagram. In particular if $\mathrm{O}(A) \otimes \mathrm{O}(B)=\epsilon$ then $A ; ; B$ is an io-diagram.


Figure 11: Example of named serial composition.


Figure 12: Example of named parallel composition.

The named serial composition is associative, expressed by the next lemma.
Lemma 5. If $A, B, C$ are io-diagrams such that $(\mathrm{O}(A) \ominus \mathrm{I}(B)) \otimes \mathrm{O}(B)=\epsilon$ and $(\mathrm{O}(A) \otimes \mathrm{I}(B)) \otimes \mathrm{I}(C)=\epsilon$ then

$$
(A ; ; B) ; ; C=A ; ;(B ; ; C)
$$

Next we introduce the corresponding operation on io-diagrams for the parallel composition.
Definition 4. If $A, B$ are io-diagrams, then the named parallel composition of $A$ and $B$, denoted $A \| B$ is defined by

$$
A \| B=(\mathrm{I}(A) \oplus \mathrm{I}(B), \mathrm{O}(A) \cdot \mathrm{O}(B), S)
$$

where

$$
S=[\mathrm{I}(A) \oplus \mathrm{I}(B) \sim \mathrm{I}(A) \cdot \mathrm{I}(B)] ;(A \| B)
$$

Figure 12 presents an example of a named parallel composition. The named parallel composition is meaningful only if the outputs of the two diagrams have different names. However, the inputs may not necessarily be distinct as shown in Figure 12.

As in the case of named serial composition, the parallel composition of two io-diagrams is not always an io-diagram. Next lemma gives conditions for the parallel composition to be io-diagram and also states that the named parallel composition is associative.

Lemma 6. Let $A, B$, and $C$ be io-diagrams, then

1. $\mathrm{O}(A) \otimes \mathrm{O}(B)=\epsilon \Rightarrow A \| B$ is an io-diagram.
2. $(A \| B)\|C=A\| \|(B \| C)$

Next definition introduces the feedback operator for io-diagrams.
Definition 5. If $A$ is an io-diagram, then the named feedback of $A$, denoted $\mathrm{FB}(A)$ is defined by $($ in, out, $S$ ), where in $=\mathrm{I}(A) \ominus \mathrm{V}(A, A)$, out $=\mathrm{O}(A) \ominus \mathrm{V}(A, A)$ and

$$
S=\text { feedback }^{|\mathrm{V}(A, A)|}([\mathrm{V}(A, A) \cdot \text { in } \leadsto \mathrm{I}(A)] ; S ;[\mathrm{O}(A) \sim \mathrm{V}(A, A) \cdot \text { out }])
$$



Figure 13: Example of named feedback composition.

The named feedback operation of $A$ connects all inputs and outputs of $A$ with the same name in feedback. Figure 13 illustrates an example of named feedback composition. The named feedback applied to an iodiagram is always an io-diagram.
Lemma 7. If $A$ is an io-diagram then $\mathrm{FB}(A)$ is an io-diagram.

### 6.4 The Abstract Translation Algorithm

We have now all elements for introducing the abstract translation algorithm. The algorithm starts with a list $\mathcal{A}=\left(A_{1}, A_{2}, \ldots, A_{n}\right)$ of io-diagrams, such that for all $i \neq j$, the inputs and outputs of $A_{i}$ and $A_{j}$ are disjoint respectively $\left(\mathrm{I}\left(A_{i}\right) \otimes \mathrm{I}\left(A_{j}\right)=\epsilon\right.$ and $\left.\mathrm{O}\left(A_{i}\right) \otimes \mathrm{O}\left(A_{j}\right)=\epsilon\right)$. We denote this property by io- $\operatorname{distinct}(\mathcal{A})$. The algorithm is given in Alg. 1. Formally the algorithm is represented as a monotonic predicate transformer [13], within the framework of refinement calculus [5].

$$
\begin{aligned}
& \text { input: } \mathcal{A}=\left(A_{1}, A_{2}, \ldots, A_{n}\right) \quad \text { (list of io-diagrams) } \\
& \text { while }|\mathcal{A}|>1: \\
& \text { choose: } \\
& \begin{aligned}
\text { (a) } & {\left[\mathcal{A}:=\mathcal{A}^{\prime} \mid \exists k, B_{1}, \ldots, B_{k}, \mathcal{C}: k>1 \wedge \operatorname{perm}\left(\mathcal{A},\left(B_{1}, \ldots, B_{k}\right) \cdot \mathcal{C}\right)\right.} \\
& \left.\wedge \mathcal{A}^{\prime}=\mathrm{FB}\left(B_{1}| ||\ldots|| | B_{k}\right) \cdot \mathcal{C}\right] \\
\text { (b) }[\mathcal{A} & \left.:=\mathcal{A}^{\prime} \mid \exists A, B, \mathcal{C}: \operatorname{perm}(\mathcal{A},(A, B) \cdot \mathcal{C}) \wedge \mathcal{A}^{\prime}=\mathrm{FB}(\mathrm{FB}(A) ; ; \mathrm{FB}(B)) \cdot \mathcal{C}\right]
\end{aligned} \\
&
\end{aligned}
$$

$A:=\mathrm{FB}\left(A^{\prime}\right) \quad$ (where $A^{\prime}$ is the only remaining element of $\left.\mathcal{A}\right)$

Alg. 1: Nondeterministic algorithm for translating HBDs.
Computing $\mathrm{FB}(A)$ in the last step of the algorithm is necessary only if $\mathcal{A}$ contains initially only one element. However, computing $\mathrm{FB}(A)$ always at the end does not change the result since, as we will see later in Theorem 2, FB operation is idempotent, i.e. $\mathrm{FB}(\mathrm{FB}(A))=\mathrm{FB}(A)$. In the presentation of the algorithm, we have used the keyword choose for the nondeterministic choice $\Pi$, to emphasize the two alternatives.

Note that, semantically, choice (b) of the algorithm is a special case of choice (a), as shown later in Theorem 2. But syntactically, choices (a) and (b) result in different expressions that achieve different performance tradeoffs as observed in Section 4 and as further discussed in [14]. The point of our translator is to be indeed able to generate semantically equivalent but syntactically different expressions, which achieve different performance tradeoffs [14].

The result for the running example from Section 4 can be obtained by applying the second choice of the algorithm twice for the initial list of io-diagrams ([Add, Delay, Split]), first to Add and Delay to obtain $A$, and next to $A$ and Split to obtain

$$
\left((u, s),\left(v, s^{\prime}\right), \text { feedback }((\mathrm{D}(\text { Add }) \| \mathrm{Id}) ; \mathrm{D}(\text { Delay }) ;((\text { Split }) \| \text { Id }))\right) .
$$

As opposed to the example from Section 4, the elements are composed serially in the order occurring in the diagram.

### 6.5 Determinacy of the Abstract Translation Algorithm

The result of the algorithm depends on how the nondeterministic choices are resolved. However, in all cases the final io-diagrams are equivalent modulo a permutation of the inputs and outputs. To prove this, we introduce the concept io-equivalence for two io-diagrams.

Definition 6. Two io-diagrams $A, B$ are io-equivalent, denoted $A \sim B$ if they are equal modulo a permutation of the inputs and outputs, i.e., $\mathrm{I}(B)$ is a permutation of $\mathrm{I}(A), \mathrm{O}(B)$ is a permutation of $\mathrm{O}(A)$ and

$$
\mathrm{D}(A)=[\mathrm{I}(A) \sim \mathrm{I}(B)] ; \mathrm{D}(B) ;[\mathrm{O}(B) \sim \mathrm{O}(A)]
$$

Lemma 8. The relation io-equivalent is a congruence relation, i.e, for all $A, B, C$ io-diagrams we have:

1. $A \sim A$
2. $A \sim B \Rightarrow B \sim A$
3. $A \sim B \wedge B \sim C \Rightarrow A \sim C$.
4. $A \sim B \Rightarrow \mathrm{FB}(A) \sim \mathrm{FB}(B)$.
5. $\mathrm{O}(A) \otimes \mathrm{O}(B)=\epsilon \Rightarrow A\|B \sim B\| A$.
6. If io - $\operatorname{distinct}\left(A_{1}, \ldots, A_{n}\right)$ and $\operatorname{perm}\left(\left(A_{1}, \ldots, A_{n}\right),\left(B_{1}, \ldots, B_{n}\right)\right)$ then

$$
A_{1}\left\|\ldots A_{n} \sim B_{1}\right\| \ldots B_{n}
$$

To prove correctness of the algorithm we also need the following results:
Theorem 2. If $A, B$ are io-diagrams such that $\mathrm{I}(A) \otimes \mathrm{I}(B)=\epsilon$ and $\mathrm{O}(A) \otimes \mathrm{O}(B)=\epsilon$ then

$$
\mathrm{FB}(A\|\|)=\mathrm{FB}(\mathrm{FB}(A) ; ; \mathrm{FB}(B))
$$

and

$$
\mathrm{FB}(\mathrm{FB}(A))=\mathrm{FB}(A)
$$

The proof of Theorem 2 is quite involved and requires several properties of diagrams (see the RCRS formalization [17] for details).

We can now state and prove one of the main results of this paper, namely, determinacy of Algorithm 1.
Theorem 3. If $\mathcal{A}=\left(A_{1}, A_{2}, \ldots, A_{n}\right)$ is the initial list of io-diagrams satisfying io-distinct $(\mathcal{A})$, then Algorithm 1 terminates, and if $A$ is the io-diagram computed by the algorithm, then

$$
A \sim \mathrm{FB}\left(A_{1}\| \| \ldots| | A_{n}\right)
$$

Proof. It is easy to see that the algorithm terminates because at each step, the size of the list $\mathcal{A}$ decreases. The termination variant in Lemma 2 is $|\mathcal{A}|$, the length of list $\mathcal{A}$.

To prove the correctness of the algorithm we use the Hoare rule for the while statement (Lemma 2), which requires an invariant. The invariant must be true at the beginning of the while loop, it must be preserved by the body of the while loop, and it must establish the final post-condition $\left(A \sim \operatorname{FB}\left(A_{1}\| \| \ldots \| A_{n}\right)\right.$ ).

If $\mathcal{A}_{0}=\left(A_{1}, \ldots, A_{n}\right)$ is the initial list of the io-diagrams, and $\mathcal{A}=\left(C_{1}, \ldots, C_{m}\right)$ is the current list of io-diagrams, then the invariant is

$$
\operatorname{inv}(\mathcal{A})=\text { io }-\operatorname{distinct}(\mathcal{A}) \wedge \mathrm{FB}\left(C_{1}\| \| \ldots \| C_{m}\right) \sim \mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right)
$$

Initially $\operatorname{inv}(\mathcal{A})$ is trivially true, and it also trivially establishes the final post-condition. We need to prove that both choices in the algorithm preserve the invariant.

$$
\begin{equation*}
\operatorname{inv}(\mathcal{A}) \wedge k>1 \wedge \operatorname{perm}\left(\mathcal{A},\left(B_{1}, \ldots, B_{k}\right) \cdot \mathcal{C}\right) \Rightarrow \operatorname{inv}\left(\left[\operatorname{FB}\left(B_{1}\| \| \ldots \mid \| B_{k}\right)\right] \cdot \mathcal{C}\right) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{inv}(\mathcal{A}) \wedge \operatorname{perm}(\mathcal{A},(A, B) \cdot \mathcal{C}) \Rightarrow \operatorname{inv}([\mathrm{FB}(\mathrm{FB}(A) ; ; \mathrm{FB}(B))] \cdot \mathcal{C}) \tag{2}
\end{equation*}
$$

The properties (1) and (2) are obtained by applying the Hoare rule for the nondeterministic choice, and then the rule for nondeterministic assignment (Lemma 1).

We prove (1). Assume

$$
\mathcal{A}=\left(C_{1}, \ldots, C_{m}\right) \text { and } \operatorname{inv}(\mathcal{A})=\operatorname{io}-\operatorname{distinct}(\mathcal{A}) \wedge \mathrm{FB}\left(C_{1}\| \| \ldots \| C_{m}\right) \sim \mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) .
$$

Let $D_{1}=\mathrm{FB}\left(B_{1}\| \| \ldots B_{k}\right)$, and $\mathcal{C}=\left(D_{2}, \ldots, D_{u}\right)$. It follows that io $-\operatorname{distinct}\left(D_{1}, \ldots, D_{u}\right)$. We prove now that $\mathrm{FB}\left(D_{1}| ||\ldots|| | D_{u}\right) \sim \mathrm{FB}\left(A_{1}|\| \ldots| \mid A_{n}\right)$.

$$
\operatorname{FB}\left(D_{1}\| \| \ldots \| D_{u}\right)
$$

$=\quad\{$ Theorem 2 and $\| \|$ is associative $\}$
$\operatorname{FB}\left(\mathrm{FB}\left(D_{1}\right) ; ; \operatorname{FB}\left(D_{2}\| \| \ldots \| D_{u}\right)\right)$
$=\quad\left\{\right.$ Definition of $\left.D_{1}\right\}$
$\operatorname{FB}\left(\operatorname{FB}\left(\operatorname{FB}\left(B_{1}| ||\ldots| \mid B_{k}\right)\right) ; \operatorname{FB}\left(D_{2}|\| \ldots|| | D_{u}\right)\right)$
$=\{$ Theorem 2$\}$
$\operatorname{FB}\left(\operatorname{FB}\left(B_{1}| ||\ldots| \mid B_{k}\right) ; ; \operatorname{FB}\left(D_{2}| ||\ldots| \mid D_{u}\right)\right)$
$=\{$ Theorem 2 and $\| \|$ is associative $\}$
$\operatorname{FB}\left(B_{1}| ||\ldots|| | B_{k}| |\left|D_{2}\right|| | \ldots| | \mid D_{u}\right)$
$\sim\left\{\right.$ Lemma 8 and $\left.\operatorname{perm}\left(\left(B_{1}, \ldots, B_{k}, D_{2}, \ldots, D_{u}\right),\left(C_{1}, \ldots, C_{m}\right)\right)\right\}$
$\operatorname{FB}\left(C_{1}\| \| \| C_{m}\right)$
$\sim$ \{Assumptions $\}$
$\operatorname{FB}\left(A_{1}| ||\ldots|| | A_{n}\right)$
Property (2) can be reduced to property (1) by applying Theorem 2.

## 7 Proving Equivalence of Two Translation Strategies

To demonstrate the usefulness of our framework, we return to our original motivation, namely, the open problem of how to prove equivalence of the translation strategies introduced in [14]. Two of the translation strategies of [14], called feedback-parallel and incremental translation, can be seen as a determinizations (or refinements) of the abstract algorithm of Section 6, and therefore can be shown to be equivalent and correct with respect to the abstract semantics. (The third strategy proposed in [14], called feedbackless, is significantly different and is presented in the next section.)

The feedback-parallel strategy is the implementation of the abstract algorithm where we choose $k=|\mathcal{A}|$. Intuitively, all diagram components are put in parallel and the common inputs and outputs are connected via feedback operators. On the running example from Figure 1c, this strategy will generate the following component:

$$
\begin{aligned}
& \left((u, s),\left(v, s^{\prime}\right), \text { feedback }^{3}([z, x, y, u, s \leadsto z, u, x, s, y]\right. \\
& \left.\left.\quad ; \mathrm{D}(\text { Add }) \| \mathrm{D}(\text { Delay }) \| \mathrm{D}(\text { Split }) ;\left[x, y, s^{\prime}, z, v \leadsto z, x, y, v, s^{\prime}\right]\right)\right)
\end{aligned}
$$

The switches are ordering the variables such that the feedback variables are first and in the same order in both input and output lists.

The incremental strategy is the implementation of the abstract algorithm where we use only the second choice of the algorithm and the first two components of the list $\mathcal{A}$. This strategy is dependent on the initial order of $\mathcal{A}$, and we order $\mathcal{A}$ topologically (based on the input - output connections) at the beginning, in order to reduce the number of switches needed.

Again on the running example, assume that this strategy composes first Add with Delay, and the result is composed with Split. The following component is then obtained:

$$
\left((u, s),\left(v, s^{\prime}\right), \text { feedback(D(Add) \| Id ; D(Delay) ; D(Split) \| Id }\right)
$$

The Add and Split components are put in parallel with Id for the unconnected input and output state respectively. Next all components are connected in series with one feedback operator for the variable $z$.

The next theorem shows that the two strategies are equivalent, and that they are independent of the initial order of $\mathcal{A}$.

Theorem 4. If $A$ and $B$ are the result of the feedback-parallel and incremental strategies on $\mathcal{A}$, respectively, then $A$ and $B$ are input - output equivalent $(A \sim B)$. Moreover both strategies are independent of the initial order of $\mathcal{A}$.

Proof. Both strategies are refinements of the nondeterministic algorithm. Therefore, using Lemma 3, they satisfy the same correctness conditions (Theorem 3), i.e.

$$
A \sim \mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \text { and } B \sim \mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right)
$$

where $\mathcal{A}=\left(A_{1}, \ldots, A_{n}\right)$. From this, since $\sim$ is transitive and symmetric, we obtain $A \sim B$.
For the second part, we use a similar reasoning. Let $\mathcal{A}=\left[A_{1}, \ldots, A_{n}\right]$, and $\mathcal{B}=\left[B_{1}, \ldots, B_{n}\right]$ a permutation of $\mathcal{A}$. If $A$ and $B$ are the outputs of feedback-parallel on $\mathcal{A}$ and $\mathcal{B}$, respectively, then we prove $A \sim B$. Using Theorem 3 again we have:

$$
A \sim \mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \text { and } B \sim \mathrm{FB}\left(B_{1} \mid\|\ldots\| B_{n}\right)
$$

Moreover, because $\mathcal{B}$ is a permutation of $\mathcal{A}$, using Lemma 8 we have

$$
\operatorname{FB}\left(A_{1} \mid\|\ldots\| A_{n}\right) \sim \operatorname{FB}\left(B_{1}\| \| \ldots| | B_{n}\right)
$$

Therefore $A \sim B$. The same holds for the incremental strategy.
Since both strategies are refinements of the nondeterministic algorithm, they both satisfy the same correctness conditions of Theorem 3.

## 8 Proving Equivalence of A Third Translation Strategy

The abstract algorithm for translating HBDs, as well as the two translation strategies presented in Section 7 , use the feedback operator when translating diagrams. As discussed in [14], expressions that contain the feedback operator are more complex to process and simplify. For this reason, we wish to avoid using the feedback operator as much as possible. Fortunately, in practice, diagrams such as those obtained from Simulink are deterministic and algebraic loop free. As it turns out, such diagrams can be translated into algebraic expressions that do not use the feedback operator at all [14]. This can be done using the third translation strategy proposed in [14], called feedbackless.

While the two translation strategies presented in Section 7 can be modeled as refinements of the abstract algorithm, the feedbackless strategy is significantly more complex, and cannot be captured as such a refinement. We therefore treat it separately in this section. In particular, we formalize the feedbackless strategy and we show that it is equivalent to the abstract algorithm, namely, that for the same input, the results of the two algorithms are io-equivalent.

### 8.1 Deterministic and Algebraic-Loop-Free Diagrams

Before we introduce the feedbackless strategy, we need some additional definitions.
Definition 7. A diagram $S$ is deterministic if

$$
[x \leadsto x, x] ;(S \| S)=S ;[y \sim y, y] .
$$

An io-diagram $A$ is deterministic if $\mathrm{D}(A)$ is deterministic.
The definition of deterministic diagram corresponds to the following intuition. If we execute two copies of $S$ in parallel using the same input value $x$, we should obtain the same result as executing one $S$ for the same input value $x$.

The deterministic property is closed under the serial, parallel, and switch operations of the HBD Algebra.
Lemma 9. If $S, T \in \operatorname{Dgr}$ are deterministic and $x, y$ are list of variables such that $x$ are distinct and $\operatorname{set}(y) \subseteq \operatorname{set}(x)$, then

1. $[x \sim y]$ is deterministic
2. $S ; T$ is deterministic

## 3. $S \| T$ is deterministic

It is not obvious whether we can deduce from the axioms that the deterministic property is closed under the feedback operation. However, since we do not use the feedback operation in this algorithm, we don't need this property.
Definition 8. The output input dependency relation of an io-diagram $A$ is defined by

$$
\text { oi_rel }(A)=\operatorname{set}(\mathrm{O}(A)) \times \operatorname{set}(\mathrm{I}(A))
$$

and the output input dependency relation of a list $\mathcal{A}=\left[A_{1}, \ldots, A_{n}\right]$ of io-diagrams is defined by

$$
\text { oi_rel }(\mathcal{A})=\text { oi_rel }\left(A_{1}\right) \cup \ldots \cup \text { oi_rel }\left(A_{n}\right)
$$

A list $\mathcal{A}$ of io-diagrams is algebraic loop free, denoted loop_free $(\mathcal{A})$, if

$$
\left(\forall x:(x, x) \notin(\text { oi_rel }(\mathcal{A}))^{+}\right)
$$

where $(\text { oi_rel }(\mathcal{A}))^{+}$is the reflexive and transitive closure of relation (oi_rel $(\mathcal{A})$ ).

If we apply this directly to the list of io-diagrams from our example $\mathcal{A}=$ [Add, Delay, Split] we obtain

$$
\text { oi_rel }(\mathcal{A})=\left\{(x, u),(x, z),(y, x),(y, s),\left(s^{\prime}, x\right),\left(s^{\prime}, s\right),(z, y),(v, y)\right\}
$$

and we have that $(z, z) \in\left(\text { oi_rel }^{(\mathcal{A})}\right)^{+}$because $(z, y),(y, x),(x, z) \in$ oi_rel $(\mathcal{A})$, therefore $\mathcal{A}$ is not algebraic loop free. However, the diagram from the example is accepted by Simulink, and it is considered algebraic loop free. In our treatment oi_rel $(\mathcal{A})$ contains pairs that do not represent genuine output input dependencies. For example output $y$ of Delay depends only on the input $s$, and it does not depend on $x$. Similarly, output $s^{\prime}$ of Delay depends only on $x$.

Before applying the feedbackless algorithm, we change the initial list of blocks into a new list such that the output input dependencies are recorded more accurately, and all elements in the new list have one single output. We split a basic block $A$ into a list of blocks $A_{1}, \ldots, A_{n}$ with single outputs such that $A \sim A_{1}\| \| \ldots \| A_{n}$. Basically every block with $n$ outputs is split into $n$ single output blocks.

We could do the splitting systematically by composing a block $A$ with all projections of the output. For example if $A=\left(x,\left(u_{1}, \ldots, u_{n}\right), S\right)$, then we can split $A$ into $A_{i}=\left(x, u_{i}, S ;\left[u_{1}, \ldots, u_{n} \sim u_{i}\right]\right)$. Such splitting is always possible as shown in the following lemma:
Lemma 10. If $A$ is deterministic, then $A_{1}, \ldots, A_{n}$ is a splitting of $A$, i.e.

$$
A \sim A_{1}\| \| \ldots \| A_{n}
$$

However, this will still introduce unwanted output input dependencies. We solve this problem by defining the splitting for every basic block, such that it accurately records the output input dependency. For example, we split the delay block into Delay ${ }_{1}$ and Delay ${ }_{2}$ :

$$
\begin{aligned}
& \text { Delay }_{1}=(s, y,[s \leadsto s])=(s, y, \text { Id }) \\
& \text { Delay }_{2}=\left(x, s^{\prime},[x \leadsto x]\right)=\left(x, s^{\prime}, \text { Id }\right)
\end{aligned}
$$

The Split block is split into Split ${ }_{1}$ and Split $_{2}$ :

$$
\begin{aligned}
& \text { Split }_{1}=(y, z,[y \sim y])=(y, z, \text { Id }) \\
& \text { Split }_{2}=(y, v,[y \sim y])=(y, v, \text { Id })
\end{aligned}
$$

The blocks Delay ${ }_{1}$, Delay $_{2}$, Split $_{1}$, and Split $_{2}$ are all the same, except the naming of the inputs and outputs. The Add block has one single output that depends on both inputs, so it remains unchanged.

After splitting, the list of single output blocks for our example becomes

$$
\mathcal{B}=\left(\text { Add }, \text { Delay }_{1}, \text { Delay }_{2}, \text { Split }_{1}, \text { Split }_{2}\right)
$$

and we have

$$
\text { oi_rel }(\mathcal{B})=\left\{(x, u),(x, z),(y, s),\left(s^{\prime}, x\right),(z, y),(v, y)\right\}
$$

Now $\mathcal{B}$ is algebraic loop free.
Definition 9. A block diagram is algebraic loop free if, after splitting, the list of blocks is algebraic loop free.

We assume that every splitting of a block $A$ into $B_{1}, \ldots, B_{k}$ is done such that $A \sim B_{1}\| \| \ldots \mid \| B_{k}$.
Lemma 11. If a list of blocks $\mathcal{A}=\left(A_{1}, \ldots, A_{n}\right)$ is split into $\mathcal{B}=\left(B_{1}, \ldots, B_{m}\right)$, then we have

$$
A_{1}\| \| \ldots\left\|A_{n} \sim B_{1}\right\|\|\ldots\| B_{m}
$$

For the feedbackless algorithm, we assume that $\mathcal{A}$ is algebraic loop free, all io-diagrams in $\mathcal{A}$ are single output and deterministic, and all outputs are distinct. We denote this by ok_fbless $(\mathcal{A})$.

Definition 10. For $\mathcal{A}$, such that ok_fbless $(\mathcal{A})$, a variable $u$ is internal in $\mathcal{A}$ if there exist $A$ and $B$ in $\mathcal{A}$ such that $\mathrm{O}(A)=u$ and $u \in \operatorname{set}(\mathrm{I}(B))$. We denote the set of internal variables of $\mathcal{A}$ by internal $(\mathcal{A})$.
Definition 11. If $A$ and $B$ are single output io-diagrams, then their internal serial composition is defined by

$$
A \triangleright B=\text { if } \operatorname{set}(\mathrm{O}(A)) \subseteq \operatorname{set}(\mathrm{I}(B)) \text { then } A ; ; B \text { else } B
$$

and

$$
A \triangleright\left(B_{1}, \ldots, B_{n}\right)=\left(A \triangleright B_{1}, \ldots, A \triangleright B_{n}\right)
$$

We use this composition when all io-diagrams have a single output, and for an io-diagram $A$, we connect $A$ in series with all io-diagrams from $B_{1}, \ldots, B_{n}$ that have $\mathrm{O}(A)$ as an input.

The internal serial composition satisfies some properties that are used in proving the correctness of the algorithm.

Lemma 12. If ok_fbless $(A, B, C)$ then $((A \triangleright B) \triangleright(A \triangleright C)) \sim((B \triangleright A) \triangleright(B \triangleright C))$
Lemma 13. If ok_fbless $(\mathcal{A})$ and $A \in \operatorname{set}(\mathcal{A})$ such that $\mathrm{O}(A) \in \operatorname{internal}(\mathcal{A})$ then

1. ok_fbless $(A \triangleright(\mathcal{A} \ominus A))$ and
2. internal $(A \triangleright(\mathcal{A} \ominus A))=\operatorname{internal}(\mathcal{A})-\{\mathrm{O}(A)\}$.

### 8.2 Functional Definition of the Feedbackless Strategy

Definition 12. For a list $x$ of distinct internal variables of $\mathcal{A}$, we define by induction on $x$ the function fbless $(x, \mathcal{A})$ by

$$
\begin{array}{ll}
\operatorname{fbless}(\epsilon, \mathcal{A}) & =\mathcal{A} \\
\operatorname{fbless}(u \cdot x, \mathcal{A}) & =\operatorname{fbless}(x, A \triangleright(\mathcal{A} \ominus A))
\end{array}
$$

where $A$ is the unique io-diagrams from $\mathcal{A}$ with $\mathrm{O}(A)=u$.
Lemma 13 shows that the function fbless is well defined.
The function fbless is the functional equivalent of the feedbackless iterative algorithm that we introduce in Subsection 8.3.

Theorem 5. If $\mathcal{A}=\left(A_{1}, \ldots, A_{n}\right)$ is a list of io-diagrams satisfying ok_fbless $(\mathcal{A}), x$ is a distinct list of all internal variables of $\mathcal{A}(\operatorname{set}(x)=$ internal $\mathcal{A})$, and $\left(B_{1}, \ldots, B_{k}\right)=\operatorname{fbless}(x, \mathcal{A})$ then

$$
\operatorname{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \sim\left(B_{1} \mid\|\ldots\| B_{n}\right)
$$

This theorem together with Lemma 11 show that the result of the fbless function is io-equivalent to the results of the nondeterministic algorithm. This theorem also shows that the result of fbless is independent of the choice of the order of the internal variables in $x$.

The proof of Theorem 5 is available in the RCRS formalization [17], and it is based on Lemmas 12 and 13 and other results.

### 8.3 The Feedbackless Translation Algorithm

The recursive function fbless calculates the feedbackless translation, but it assumes that the set of internal variables is given at the beginning in a specific order. We want an equivalent iterative version of this function, which at every step picks an arbitrary io-diagram $A$ with internal output, and performs one step:

$$
\mathcal{A}:=A \triangleright(\mathcal{A} \ominus A)
$$

The feedbackless algorithm is given in Alg. 2.

$$
\begin{aligned}
& \text { input: } \left.\mathcal{A}=\left(A_{1} \ldots, A_{n}\right) \quad \text { (list of io-diagrams satisfying ok_fbless }(\mathcal{A})\right) \\
& \text { while internal }(\mathcal{A}) \neq \emptyset: \\
& \qquad\left[\mathcal{A}:=\mathcal{A}^{\prime} \mid \exists A \in \operatorname{set}(\mathcal{A}): \mathrm{O}(A) \in \operatorname{internal}(\mathcal{A}) \wedge \mathcal{A}^{\prime}=A \triangleright(\mathcal{A} \ominus A)\right] \\
& A:=B_{1} \mid\|\ldots\| B_{k} \quad\left(\text { where } \mathcal{A}=\left(B_{1}, \ldots, B_{k}\right)\right)
\end{aligned}
$$

Alg. 2: Feedbackless algorithm for translating HBDs.

The feedbackless algorithm is also nondeterministic, because it allows choosing at every step one of the available io-diagrams with internal output. As we will see in Subsection 8.4, this nondeterminism allows for different implementations regarding the complexity of the generated expressions.

Theorem 6. If $\mathcal{A}=\left(A_{1} \ldots, A_{n}\right)$ is a list of io-diagrams satisfying ok_fbless $(\mathcal{A})$, then the feedbackless algorithm terminates for input $\mathcal{A}$, and if $A$ is the output of the algorithm on $\mathcal{A}$, then

$$
\operatorname{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \sim A
$$

Proof. Let Feedbackless be the predicate transformer of the feedbackless algorithm. We prove that choosing nondeterministically an order $x$ of the internal variables of $\mathcal{A}$, and calculating fbless $(x, \mathcal{A})$ is refined by Feedbackless. Formally we have:

```
{ok_fbless(\mathcal{A})};
```



```
    Feedbackless
```

$\sqsubseteq$

To prove this refinement we need to use the assertion $\{0$ __fbless $(\mathcal{A})\}$. Intuitively, this assertion restricts the refinement only for inputs $\mathcal{A}$ satisfying the property ok_fbless $(\mathcal{A})$.

Because of this refinement, the feedbackless algorithm terminates.
Using this refinement, Lemma 3 (connecting refinement to Hoare correctness triples), and Theorem 5, we obtain that the output of Feedbackless satisfies the desired property:

$$
\mathrm{FB}\left(A_{1}\| \| \| A_{n}\right) \sim A
$$

when the input satisfies ok_fbless $(\mathcal{A})$. Stated as a Hoare correctness triple, this property is:

$$
\left(\operatorname{ok} \_ \text {fbless }(\mathcal{A}) \wedge \mathcal{A}=\left(A_{1}, \ldots, A_{n}\right)\right)\{\mid \text { Feedbackless } \mid\}\left(\operatorname{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \sim A\right)
$$

The details are available in the RCRS formalization [17].
Theorem 7. For a deterministic and algebraic loop free block diagram, the feedbackless algorithm and the nondeterministic algorithm are equivalent.

Proof. Assume $\mathcal{A}=\left(A_{1} \ldots, A_{n}\right)$ is the initial set of blocks satisfying io $-\operatorname{distinct}(\mathcal{A})$, and $A$ is one possible output of the nondeterministic algorithm. We have $\mathrm{FB}\left(A_{1}\| \| \ldots \| A_{n}\right) \sim A$.

Assume that $\mathcal{B}=\left(B_{1} \ldots, B_{m}\right)$ is a splitting of $\mathcal{A}$ satisfying ok_fbless $(\mathcal{B})$ and $B$ is the output of the feedbackless algorithm for $\mathcal{B}$. We have $\operatorname{FB}\left(B_{1}| ||\ldots|| | B_{m}\right) \sim B$.

Because $\mathcal{B}$ is a splitting of $\mathcal{A}$ we also have $A_{1}| | \ldots| |\left|A_{n} \sim B_{1}\right| \| \ldots| | B_{m}$.
Finally, using Lemma 8, we obtain $A \sim B$.
If we apply the feedbackless algorithm to the example from Fig. 1a we obtain:


Figure 14: Example for efficient implementation of feedbackless.

```
    (Add, Delay \({ }_{1}\), Delay \(_{2}\), Split \(_{1}\), Split \(_{2}\) )
\(\mapsto \quad\{\) Variable \(x\) is internal and \(\mathrm{O}(\) Add \()=x\}\)
    ( Delay \(_{1},\left((z, u), s^{\prime}, \mathrm{D}(\right.\) Add \() ; \mathrm{D}\left(\right.\) Delay \(\left.\left._{2}\right)\right)\), Split \(_{1}\), Split \(\left._{2}\right)\)
\(\mapsto \quad\left\{\right.\) Variable \(y\) is internal and \(\mathrm{O}\left(\right.\) Delay \(\left.\left._{1}\right)=y\right\}\)
\(\left(\left((z, u), s^{\prime}, \mathrm{D}(\right.\right.\) Add \() ; \mathrm{D}\left(\right.\) Delay \(\left.\left._{2}\right)\right)\),
    \(\left(s, z, \mathrm{D}\left(\right.\right.\) Delay \(\left._{1}\right) ; \mathrm{D}\left(\right.\) Split \(\left.\left._{1}\right)\right),\left(s, v, \mathrm{D}\left(\right.\right.\) Delay \(\left._{1}\right) ; \mathrm{D}\left(\right.\) Split \(\left.\left.\left._{2}\right)\right)\right)\)
\(\mapsto \quad\) \{Variable \(z\) is internal \(\}\)
    \(\left(\left((s, u), s^{\prime},\left(\left(\mathrm{D}_{\left(\text {Delay }_{1}\right)}\right) \mathrm{D}\left(\right.\right.\right.\right.\) Split \(\left.\left.\left._{1}\right)\right) \| \mathrm{Id}\right) ; \mathrm{D}(\) Add \() ; \mathrm{D}\left(\right.\) Delay \(\left.\left._{2}\right)\right)\),
        \(\left(s, v, \mathrm{D}\left(\right.\right.\) Delay \(\left.\left.\left._{1}\right) ; \mathrm{D}\left(\mathrm{Split}_{2}\right)\right)\right)\)
\(\mapsto \quad\) \{There are no internal variables anymore \(\}\)
    \(\left((s, u),\left(s^{\prime}, v\right),[s, u \leadsto s, u, s] ;\left(\left(\left(\mathrm{D}\left(\right.\right.\right.\right.\right.\) Delay \(\left._{1}\right) ; \mathrm{D}\left(\right.\) Split \(\left.\left.\left._{1}\right)\right) \| \mathrm{Id}\right) ; \mathrm{D}(\) Add \() ; \mathrm{D}\left(\right.\) Delay \(\left.\left._{2}\right)\right)\)
\(\|\left(\mathrm{D}\left(\right.\right.\) Delay \(\left._{1}\right) ; \mathrm{D}\left(\right.\) Split \(\left.\left.\left._{2}\right)\right)\right)\)
\(=\{\) Simplifications \(\}\)
    \(\left((s, u),\left(s^{\prime}, v\right),[s, u \sim s, u, s] ;(((\operatorname{ld} ;\right.\) Id \() \| \operatorname{Id}) ; \mathrm{D}(\) Add \() ;\) Id \(\left.) \|(I d ; I d)\right)\)
\(=\{\) Simplifications \(\}\)
\(\left((s, u),\left(s^{\prime}, v\right),[s, u \sim s, u, s] ;(\mathrm{D}(\mathrm{Add}) \| \mathrm{Id})\right)\)
```


### 8.4 On the Nondeterminism of the Feedbackless Translation

We have seen already that different choices in the nondeterministic abstract algorithm result in different algebraic expressions, e.g., with different numbers of composition operators. We show in this section that the same is true for the feedbackless translation algorithm. In particular, consider a framework like the Refinement Calculus of Reactive Systems [14], where the intermediate results of the algorithm are symbolically simplified at every translation step. Different choices of the order of internal variables could result in different complexities of the simplification work. We illustrate this with the example from Figure 14.

After splitting the list of blocks for this example is

$$
\mathcal{A}=((u, a, A),(a, b, B),(b, c, \text { Id }),(b, d, \mathrm{Id}),(c, v, C),(d, w, D))
$$

and the set of internal variables is

$$
\text { internal }(\mathcal{A})=\{a, b, c, d\}
$$

If we choose the order $(c, d, b, a)$, then after first two steps (including intermediate simplifications) we obtain the list:

$$
((u, a, A),(a, b, B),(b, v, C),(b, w, D))
$$

After another step for internal variable $b$ we obtain:

$$
((u, a, A),(a, v, \operatorname{simplify}(B ; C)),(a, w, \operatorname{simplify}(B ; D)))
$$

where the function simplify models the symbolic simplification. Finally, after applying the step for the internal variable $a$ we obtain:

$$
\begin{equation*}
((u, v, \operatorname{simplify}(A ; \operatorname{simplify}(B ; C))),(u, w, \operatorname{simplify}(A ; \operatorname{simplify}(B ; D)))) \tag{3}
\end{equation*}
$$

In this order, we end up simplifying $A$ serially composed with $B$ twice. This is especially inefficient if $A$ and $B$ are complex. If we choose the order $(c, d, a, b)$, then in the first three steps we obtain:

$$
((u, b, \operatorname{simplify}(A ; B)),(b, v, C),(b, w, D))
$$

At this point the term $A ; B$ is simplified, and the simplified version is composed with $C$ and $D$ to obtain:

$$
\begin{equation*}
((u, v, \text { simplify }(\operatorname{simplify}(A ; B) ; C)),(u, w, \text { simplify }(\operatorname{simplify}(A ; B) ; D))) \tag{4}
\end{equation*}
$$

If we compare relations (3) and (4) we see the same number of occurrences of simplify, but in relation (4) there are two occurrences of the common subterm $\operatorname{simplify}(A ; B)$, and this is simplified only once.

As this example shows, different choices of the nondeterministic feedbackless translation strategy result in expressions of different quality, in particular with respect to simplification. It is beyond the scope of this paper to examine efficient deterministic implementations of the feedbackless translation. Our goal here is to prove the correctness of this translation, by proving its equivalence to the abstract algorithm. It follows that every refinement/determinization of the feedbackless strategy will also be equivalent to the abstract algorithm, and therefore a correct implementation of the semantics. Once we know that all possible refinements give equivalent results, we can concentrate in finding the most efficient strategy. In general, we remark that this way of using the mechanisms of nondeterminism and refinement are standard in the area of correct by construction program development, and are often combined to separate the concerns of correctness and efficiency, as is done here.

## 9 Implementation in Isabelle

Our implementation in Isabelle uses locales [6] for the axioms of the algebra. We use locale interpretations to show that these axioms are consistent. In Isabelle locales are a powerful mechanism for developing consistent abstract theories (based on axioms). To represent the algorithm we use monotonic predicate transformers and we use Hoare total correctness rules to prove its correctness.

The formalization contains the locale for the axioms, a theory for constructive functions, and one for proving that such functions are a model for the axioms. An important part of the formalization is the theory introducing the diagrams with named inputs and outputs, and their operations and properties. The formalization also includes a theory for monotonic predicate transformers, refinement calculus, Hoare total correctness rules for programs, and a theory for the nondeterministic algorithm and its correctness.

In total the formalization contains 14797 lines of Isabelle code of which 13587 lines of code for the actual problem, i.e., excluding the code for monotonic predicate transformers, refinement calculus, and Hoare rules.

## 10 Conclusions and Future Work

We introduced an abstract algebra for hierarchical block diagrams, and an abstract algorithm for translating HBDs to terms of this algebra. We proved that this algorithm is correct in the sense that no matter how its nondeterministic choices are resolved, the results are semantically equivalent. As an application, we closed a question left open in [14] by proving that the Simulink translation strategies presented there yield equivalent results. Our HBD algebra is reminiscent of the algebra of flownomials [40] but our axiomatization is more general, in the sense that our axioms are weaker. This implies that all models of flownomials are also models of our algebra. Here, we presented constructive functions as one possible model of our algebra. Our work applies to hierarchical block diagrams in general, and the de facto predominant tool for embedded system
design, Simulink. Proving the HBD translator correct is a challenging problem, and as far as we know our work is the only one to have achieved such a result.

We believe that our results are reusable in other contexts as well, in at least two ways. First, every other translation that can be shown to be a refinement/special case of our abstract translation algorithm, is automatically correct. For example, [35, 45] impose an order on blocks such that they use mostly serial composition and could be considered an instance of our abstract algorithm. Second, our algorithms translate diagrams into an abstract algebra. By choosing different models of this algebra we obtain translations into these alternative models.

As mentioned earlier, RCRS has been formalized in Isabelle [30]. The formalization is part of the RCRS toolset which is publicly available in a figshare repository [17]. The theories relevant to this paper are under RCRS/Isabelle/TranslateHBD. The RCRS toolset can be downloaded also from the RCRS web page: http://rcrs.cs.aalto.fi/. The RCRS formalization represents a significant amount of work. The entire formalization is close to 30000 lines of Isabelle code. The material for this paper consists of 14797 lines of Isabelle code, 864 lemmas and 25 theorems, and required an effort of 8 person-months excluding paper writing.

As future work we plan to investigate further HBD translation strategies, in addition to those studied above. As mentioned earlier, this work is part of the broader RCRS project, which includes a Translator of Simulink diagrams to RCRS theories implemented on top of Isabelle [14, 32, 15]. Currently the Translator can only handle diagrams without algebraic loops, i.e., without instantaneous circular dependencies. Extending the Translator and the corresponding determinacy proofs to diagrams with algebraic loops is left for future work. This is a non-trivial problem, because of subtleties in the definition of instantaneous feedback semantics, especially in the presence of non-deterministic and non-input-receptive systems [34]. For deterministic and input-receptive systems, however, the model of constructive functions that we use in this paper should be sufficient. Another future research goal is to unify the proof of the third translation strategy with that of the other two which are currently modeled as refinements of the abstract translation algorithm.

This work covers hierarchical block diagrams in general and Simulink in particular. Any type of diagram can be handled, however, we do assume a single-rate (i.e., synchronous) semantics. Handling multi-rate or event-triggered diagrams is left for future work. Handling hierarchical state machine models such as Stateflow is also left for future work.

Our work in this paper and in the RCRS project in general implicitly provides, via the translation, a formal semantics for the subset of Simulink described above. As already mentioned in $\S 2$, ultimately the semantics of Simulink is "what the simulator does". Since the code of the simulator is proprietary, the only way to validate a formal semantics such as ours is by simulation. Some preliminary work towards this goal is reported in [14], which also presents preliminary case studies, including a real-world automotive control benchmark provided by Toyota [24]. A more thorough validation of the semantics and experimentation with further case studies are future research topics.

As mentioned in $\S 2$, there are many existing translations from Simulink to other formalisms. It is beyond the scope of this paper to define and prove correctness of those translations, but this could be another future work direction. In order to do this, one would first need to formalize those translations. This in turn requires detailed knowledge of the algorithms or even access to their implementation, which is not always available. Our work and source code are publicly available and we hope can serve as a good starting point for others who may wish to provide formal correctness proofs of diagram translations.

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[^0]:    ${ }^{1}$ http://www.mathworks.com/products/simulink/

[^1]:    ${ }^{2}$ We omit several of the proofs of the results presented in this paper. These proofs and additional material used in the formalization and verification of our algorithms can be found in our Isabelle formalization [17].

