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Fitting Splines to a Parametric Function

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to Nina
for being willing to listen

Preface

This work began as an attempt to see whether it was possible to fit a spline curve to another parametric function in such a way that the spline curve would respond smoothly and continuously to changes in the shape of the parametric function. In other words, would the resulting fitted curve be suitable for use in an animation? Since then, the scope of the project has changed somewhat to include an investigation of the intersections that occur between three different areas of study that normally would not touch each other: least squares orthogonal distance fitting (ODF), spline theory, and topology. The ODF method has become the standard technique used to develop mathematical models of the physical shapes of objects, due to the fact that it produces a fitted result that is invariant with respect to the size and orientation of the object. It is typically applied in cases where there are thousands of discrete measurements available on the physical shape of an object. In this case, there are two implicit assumptions that are being made: namely, that we are interested only in one solution (the one that has the minimum error) and that we are unable to substantially change the shape of the object being fit. We will relax these two assumptions by fitting splines to a family of parametric functions whose shape can be continuously modified. In this way, we can investigate the response of the spline curve to changes in the shape of the curve we are trying to fit. The quality of this response may be particularly important in cases where one wishes to produce a smooth animation of the motion of an object. If the response is discontinuous, the quality of the animation will suffer as a result. During these exercises, it became increasingly clear that there are often a number of solutions that can exist and that the interaction between them is important. Different solutions can spontaneously coalesce and disappear, and it is sometimes necessary to arbitrarily switch from one to another in order to minimize the error. Therefore, it is not sufficient to focus only on the minimum error solution since the definition of this solution will change as the object's shape changes.

The second area of study that will be touched on is the theory of splines. We have used six different splines to fit the shape of a simple family of epitrochoid curves: two types of Bézier curve, two uniform B-splines, and two Beta-splines. In theoretical studies of these splines, the emphasis is usually on how to develop mathematical shapes in such a way that they can be usefully implemented by a designer, such as a draftsman operating CAD software. In other words, the emphasis is on developing user-friendly ways of manipulating these shapes in a way that is mathematically complete and consistent. For example, in the development of the theory of Beta-splines, new degrees of freedom have been introduced and have been associated with such concepts as *tension* and *bias* in order to make them accessible to the designer. We will borrow these results *as is* but will make a slight digression to apply them to a least squares optimization problem. In the case of Beta-splines, this is challenging and, to the best of our knowledge, has not been done before, due to nonlinear couplings that exist between the different adjustable parameters in the spline model.

The final area of interest is topology. There are often multiple solutions to the ODF method, and these solutions can always be classified as being either local minima or saddle points of different degree. We classify them according to their *Morse index*, which counts the number of negative eigenvalues of the second-order response matrix. Since there are many solutions, two topological questions immediately arise: are there rules that can be applied concerning the relative number of local minima and saddle points, and are there different mechanisms available by which solutions can either merge and disappear or cross over each other and interchange roles? We will propose some simple rules which can be used to determine if a given set of solutions is internally consistent in the sense that it has the appropriate number of each type of solution. The rule that relates the number of occurrences of each type can be viewed as an instance of *Euler's characteristic* equation for polyhedra. We will also observe experimentally two distinct mechanisms by which solutions can either merge or cross each other. The merge of solutions is an instance of a *fold catastrophe*, while the crossover of solutions does not appear to have any analog in *catastrophe theory*. A diagnostic test will be developed to allow us to easily determine which type of event is occurring.

The organization of the work is as follows. Chapter 2 presents a general derivation of the ODF method, customized for fitting a continuous parametric function. This contains some results which may be new or at least expressed in unfamiliar form. Chapter 3 summarizes some previously derived properties of splines. We have included only those results that are absolutely essential for the description of a uniform B-spline. The results of the ODF curve fit using two types of Bézier curve are given in Chaps. 4 and 8; two types of uniform B-spline fits are described in Chaps. 6 and 7; and Chaps. 9 and 10 present the Beta-spline curve fits. The cubic Bézier curve fit in Chap. 4 presents some interesting topological problems, so Chap. 5 represents a digression to discuss the process of how solutions coalesce and/or cross each other. Other than that, the chapters are ordered roughly according to the degree of computational difficulty of the fit, with the cubic Bézier being the easiest and the Beta1-spline the most difficult.

The original purpose of the study was to see if the fitted spline shapes would respond smoothly to changes in the shape of the curve we are fitting. This is of some importance when discussing the animation of shapes. In general, the answer to this question is "no," but it is hoped that the results may be helpful in determining which particular type of spline would be most useful in any case. In some cases, it may be desirable to use the spline curve that presents the least computational challenges, such as a uniform B-spline, and in other cases, there may be symmetry considerations that would justify the use of a more complex spline such as the Beta2-spline. In either case, one must be aware of the topological changes that can occur as the shape of the object changes.

It is a pleasure to acknowledge the inspiration provided by the developers of Inkscape, which stimulated the initial work in this area.

The ODF calculations were performed using the Java code at the repository: https://github.com/alvinpenner/Spiro2SVG/

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September 2018

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