Ambiguity measures for preference-based decision viewpoints *

Camilo Franco[1], J. Tinguaro Rodríguez[2], Javier Montero[2], Daniel Gómez[3], and Ronald R. Yager[4]

[1] Department of Industrial Engineering, Andes University, Bogotá, 111711
c.franco31@uniandes.edu.co
[2] Department of Operations Research and Statistics, Complutense University, Madrid 28040
{jtrodrig,monty}@mat.ucm.es
[3] Department of Statistics, Complutense University, Madrid 28040
dagomez@estad.ucm.es
[4] Machine Intelligence Institute, Iona College, New Rochelle, NY 10801
yager@panix.com

Abstract. This paper examines the ambiguity of subjective judgments, which are represented by a system of pairwise preferences over a given set of alternatives. Such preferences are valued with respect to a set of reasons, in favor and against the alternatives, establishing a complete judgment, or *viewpoint*, on how to solve the decision problem. Hence, viewpoints entail particular decisions coming from the system of preferences, where the preference-based reasoning of a given viewpoint holds according to its *soundness* or *coherence*. Here we explore such a coherence under the frame of *ambiguity measures*, aiming at learning viewpoints with highest preference-score and minimum ambiguity. We extend existing measures of ambiguity into a multi-dimensional fuzzy setting, and suggest some future lines of research towards measuring the coherence or (ir)rationality of viewpoints, exploring the use of information measures in the context of preference learning.

Keywords: Ambiguity · Fuzziness · Rationality · Preference structures

1 Introduction

Any subjective decision process requires considering the formation of opinions, like preferences over a given set of possible choices. Examples of such a process can be political elections, an investment project competition, or much simpler choices like deciding which product to buy. Then, based on a set of attributes, e.g. the nutritional attributes of different alternatives, referring to calories, minerals, and proteins, preferences are formed by considering the reasons in favor and

* Supported by the Government of Spain (grant TIN2015-66471-P), the Government of Madrid (grant S2013/ICE-2845, CASICAM-CM), and Complutense University research group (910149)

against the alternatives, on how the amount of calories, minerals or proteins make us prefer and/or reject one alternative over the other. In general, the set of reasons act as *arguments* on which plan of action to take, like matching reasons one against the other, trying to balance the overall preference under a definite *viewpoint*. Hence, preferences have a key role in understanding the most suitable decision(s), and the less uncertain or *ambiguous* ones.

Traditionally, preference-based decision-making has been understood according to the theoretical (uni-polar) concept of *wants/desires*, as presented by classical decision theory (see e.g. [4]). Nonetheless, it has been suggested that decisions can be better understood according to the *pair* of explanatory concepts of *wants* and *needs* (see [9, 13]), acting as drivers for decision-making. Following this line of research, the components of wants and needs can be inferred from global preferences under more general, opposite-paired semantics [10, 12, 16]. Besides, such a semantics also allows understanding different indecision states that can explain the choice of, actually, not making a decision [10]. In this same line, it is proposed that opposite-paired preferences allow representing the emotional meaning associated to judgments, stressing that emotion goes hand in hand with rational decision-making [10].

In this way, this study focuses on how to measure the ambiguity of preference arguments, addressing the question on how *irrational* can a decision be. For doing so, the next section introduces the frame for subjective fuzzy preferences and the preference-aversion model. Then we present in Section 3 our proposal on decision viewpoints, obtaining a decision outcome and its overall ambiguity. In order to measure ambiguity, we explore in Section 4 the notion of ambiguity as it has been studied in decision theory literature, and extend classical ambiguity measures over multiple dimensions by fuzzy logic operators. Finally, some final comments are given for future research.

2 Decision modeling by fuzzy preferences

Given a set of alternatives $\mathbb{A}$, standard preference modeling understands the preference predicate $R(a, b)$ as *"a is not worse than b"* or *"a is at least as wanted as b"* [8]. The pairwise relation representing such a predicate, namely the weak or global preference relation $R \in \{0, 1\}$, can be decomposed into four distinct relations. These relations are *strict preference* P, the inverse strict preference P^{-1}, *indifference* I, and *incomparability* J, such that $P = R \cap \neg R^{-1}$, $I = R \cap R^{-1}$, and $J = \neg R \cap \neg R^{-1}$, where $\neg R = 1 - R$. Also consider that $R^d = \neg R^{-1} = (\neg R)^{-1}$. Under this classical/crisp setting, the relations I and J are assumed to be symmetrical, such that $I(a, b) = I(b, a)$ and $J(a, b) = J(b, a)$; I is assumed to be reflexive, such that $I(a, a) = 1$; J irreflexive, such that $J(a, a) = 0$; and P is assumed to be asymmetrical, such that $P(a, b)$ and $P(b, a)$ cannot hold simultaneously true.

2.1 The standard fuzzy model

Following the classical/crisp preference setting, the standard preference structure $\langle P, I, J\rangle$ consists in the mutually exclusive relations P, I, J: $P \cap I = \emptyset$, $P \cap J = \emptyset$, and $J \cap I = \emptyset$, partitioning the valuation space in the following way:

$$P \cup I = R, \tag{1}$$

$$P \cup I \cup P^{-1} = R \cup R^{-1}, \tag{2}$$

$$P \cup J = R^d, \tag{3}$$

$$P \cup P^{-1} \cup I \cup J = \mathbb{A}^2. \tag{4}$$

Allowing the different relations to simultaneously co-exist with different intensities, the standard structure can be extended through fuzzy logic, affirming the existence of continuous functions $p, p^{-1}, i, j : [0,1]^2 \to [0,1]$ that maintain the classical properties Eqs. (1)-(4) as much as possible [8]. In this way, the preference relation R can be represented as a fuzzy relation, such that

$$R(a,b) = \{\langle a, b, \mu_R(a,b)\rangle | a, b \in \mathbb{A}\},$$

where $\mu_R(a,b) : \mathbb{A}^2 \to [0,1]$ is the membership function of R, measuring the degree or intensity in which the pair $(a,b) \in \mathbb{A}^2$ verifies the preference predicate represented by R.

Recalling some traditional principles of social choice theory (see e.g. [1, 8]), the standard fuzzy model assumes *independence of irrelevant alternatives*, such that for every pair of alternatives $a, b \in \mathbb{A}$, the values of $P(a,b)$, $P^{-1}(a,b)$, $I(a,b)$ and $J(a,b)$ depend only on the pair (a,b), through the weak preference functions $x = \mu_R(a,b)$ and $y = \mu_R(b,a)$. So, it holds that $P(a,b) = p(x,y), P^{-1}(a,b) = p(y,x), I(a,b) = i(x,y)$ and $J(a,b) = j(x,y)$. Besides, other classical principles are assumed, like *monotonicity* or *positive association*, stating that functions $p(x, n(y)), p^{-1}(n(x), y), i(x,y)$ and $j(n(x), n(y))$ are non-decreasing over both arguments, where n is a strict negation, and *symmetry*, affirming the symmetry of the functions $i(x,y)$ and $j(x,y)$ (see again [8], but also [17]).

2.2 Preference-Aversion fuzzy model

The standard fuzzy model can be generalized under a paired setting, aiming at clarifying the semantics for the preference predicate, and completely specify its valuation space [9, 10]. Hence, two separate sources of information can be considered, representing the positive and the negative aspects of alternatives, with respect to the available criteria or attributes, serving as reasons or arguments for evaluating the verification of the preference predicate.

In this way, $R^+(a,b) = R(a,b) =$*a is at least as wanted as b* and its inverse $R^+(b,a)$, evaluate the source of positive information, while the negative counterpart allows evaluating the negative preference predicate $R^-(a,b) =$*a is at least as rejected as b* and its inverse $R^-(b,a)$. To simplify notation, from now on we say that $R^+(a,b) = Q(a,b)$, $R^+(b,a) = Q(b,a) = Q^{-1}(a,b)$, $R^-(a,b) = V(a,b)$ and $R^-(b,a) = V(b,a) = V^{-1}(a,b)$.

These positive and negative preference components can be aggregated into more complex structures, such as the Partial Comparability or the Preference-Aversion (P-A) structures. Following [10], the P-A structure in fact allows distinguishing between *wants* and *needs*, measuring positive aspects according to (Q, Q^{-1}), and measuring the negative ones through (V, V^{-1}).

Like a semi-dual or opposite of *preference*, the *aversion structure* can be seen as a separate, negative counterpart of the standard preference structure. In this sense, $\forall (a,b) \in \mathbb{A}^2$, the weak aversion predicates (V, V^{-1}) can be decomposed into the three relations Z, G, H, where $Z(a,b)$ holds if a is more rejected than b, $G(a,b)$ holds if a is as much as rejected as b, and $H(a,b)$ holds if a cannot be compared with b regarding their negative aspects (see e.g. [10]). Hence, the fuzzy P-A model can be defined by representing the predicates Q and V, as in

$$\mu_Q, \mu_V : \mathbb{A}^2 \to [0,1],$$

and defining fuzzy preference-aversion relations

$$p, i, j, z, g, h : [0,1]^2 \to [0,1]$$

such that [8, 9]

$$\begin{aligned}
P = p(\mu_Q, \mu_{Q^{-1}}) &= T(\mu_Q, n(\mu_{Q^{-1}})),\\
I = i(\mu_Q, \mu_{Q^{-1}}) &= T(\mu_Q, \mu_{Q^{-1}}),\\
J = j(\mu_Q, \mu_{Q^{-1}}) &= T(n(\mu_Q), n(\mu_{Q^{-1}})),\\
Z = z(\mu_V, \mu_{V^{-1}}) &= T(\mu_V, n(\mu_{V^{-1}})),\\
G = g(\mu_V, \mu_{V^{-1}}) &= T(\mu_V, \mu_{V^{-1}}),\\
H = h(\mu_V, \mu_{V^{-1}}) &= T(n(\mu_V), n(\mu_{V^{-1}})),
\end{aligned}$$

where T is a (conjunctive operator) t-norm, used for aggregating pairs of values of the same positive or negative nature, and i, j, g and h are symmetrical functions. Then, by means of a (disjunctive operator) t-conorm S, and following Eqs. (1)-(4), the classical properties can be formulated in fuzzy terms as

$$S(p, i) = \mu_Q, \tag{5}$$

$$S(p, i, p^{-1}) = S(\mu_Q, \mu_{Q^{-1}}), \tag{6}$$

$$S(p, j) = n(\mu_Q), \tag{7}$$

$$S(p, p^{-1}, i, j) = 1, \tag{8}$$

while the *aversion* ones are stated as

$$S(z, g) = \mu_V, \tag{9}$$

$$S(z, g, z^{-1}) = S(\mu_V, \mu_{V^{-1}}), \tag{10}$$

$$S(z, h) = n(\mu_V), \tag{11}$$

$$S(z, z^{-1}, g, h) = 1. \tag{12}$$

Some solutions for functions (p, i, j) and (z, g, h), fulfilling (5)-(8) and (9)-(12), are given by the De-Morgan triple (T^L, S^L, n) [8],

$$p(\mu_Q, \mu_{Q^{-1}}) = T^M(\mu_Q, n(\mu_{Q^{-1}})),$$

$$i(\mu_Q, \mu_{Q^{-1}}) = T^L(\mu_Q, \mu_{Q^{-1}}),$$

$$j(\mu_Q, \mu_{Q^{-1}}) = T^L(n(\mu_Q), n(\mu_{Q^{-1}})),$$

and

$$z(\mu_V, \mu_{V^{-1}}) = T^M(\mu_V, n(\mu_{V^{-1}})),$$

$$g(\mu_V, \mu_{V^{-1}}) = T^L(\mu_V, \mu_{V^{-1}}),$$

$$h(\mu_V, \mu_{V^{-1}}) = T^L(n(\mu_V), n(\mu_{V^{-1}})),$$

where n now denotes a strong negation, and $\forall x, y \in [0, 1], T^L(x, y) = \max(x+y-1, 0), T^M = \min(x, y)$, and $S = S^L = \min(x+y, 1)$. Besides, the functions i and j are mutually exclusive, as well as g and h. Other solutions regarding t-(co)norms allow modeling p and z as strongly asymmetrical relations (by means of T^L), where the preference (5), (7)-(8), and the aversion conditions (9), (11)-(12), are satisfied, but not (6) nor (10). One last solution is given by the multiplicative De Morgan triple (T^p, S^p, N), such that $\forall x, y \in [0, 1], T^p(x, y) = x \cdot y$ and $S = S^p = x+y-x \cdot y$, where all basic relations in (p, i, j) and (z, g, h) can simultaneously co-exist, but without fulfilling (5)-(7) nor (9)-(11). This last solution only satisfies completeness (8) and (12) [18].

2.3 Modeling decisions by opposite concepts

The decision problem can now be understood in terms of *opposites* [12, 16], where the positive and negative dimensions of the preference-aversion structure have to be aggregated into a unified outcome. One possibility for doing so is by inferring the behavioral components of preference, which guide the decision process. In this sense, wants and needs can be estimated from the preference statements, distinguishable under the P-A model (as shown in [10]). In this way, $W =$ *wants* and $D =$ *needs* are respectively defined with respect to the opposite poles of preference for $Q =$*wanting* and for V=*rejecting*, such that

$$W = (Q \cap \neg Q^{-1}) \tag{13}$$

and

$$D = Q \cap \neg V, \tag{14}$$

where W and D are *paired concepts* in the sense of [16].

In this sense, the want-component W distinguishes a priority on only wanted alternatives, based on the positive reasons, while the need component D distinguishes a priority on wanted and non-rejected alternatives, based on both positive and negative reasons. As it has been shown in [10], the need component can be estimated from the specific preference semantics of the P-A structure, offering a decision-making criterion for a given system of preferences.

An operational example for estimating needs from preferences is given $\forall a, b \in \mathbb{A}$, by

$$\mu^D(a,b) = \phi_O[T(\mu_Q, n(\mu_{Q^{-1}})), n(T(\mu_V, n(\mu_{V^{-1}})))], \tag{15}$$

which can be simplified, in case n is an involutive strict negation, i.e. a strong negation, as

$$\mu^D(a,b) = \phi_O[T(\mu_Q, n(\mu_{Q^{-1}})), T(n(\mu_V), \mu_{V^{-1}})], \tag{16}$$

where $\phi_O : [0,1]^2 \to [0,1]$ is an overlapping (conjunctive, not necessarily associative) operator [2], modelling the intersection of positive (Q), and negative (V) preferences.

Let us note that the non-associativity of the aggregation operator ϕ_O is used here to stress that positive and negative reasons cannot be grouped arbitrarily without affecting the final outcome [10]. On the other hand, the want component W is given by $\mu^W = T(\mu_Q, n(\mu_{Q^{-1}})$.

Hence, a decision can obey to the P-A overall preferential situations, making a choice because of wanting or needing, but also postponing it, due to the verification of some incomparable component, perhaps identifying *ambivalence* or lacking reasons for comparing alternatives on solid grounds [9, 10]. Such reasoning for a preferential-argument should be explicit, by the following proposal on decision viewpoints.

3 Decision viewpoints

We can understand the decision making process as an analytical search for the alternatives that allow satisfying, or even maximizing a given set of attributes or criteria. In general, decision making looks for a set of relevant reasons, or the one compelling reason, for choosing the most suitable alternatives. From a purely subjective perspective, by comparing pairs of alternatives regarding a set of attributes or criteria, the individual finds reasons, or arguments, assigning a positive or negative character to the specific property being measured by those criteria, having a paired-opposite meaning for each criterion/attribute. In this sense, a preference argument or *viewpoint*, refers to a whole system of preferences (entailing a global order or decision), which has an associated pair of values: one measuring the total value of preference (like an overall expected outcome), and another measuring the coherence of the preference argument.

It is worth noticing that this proposal can be related to a more standard multi-criteria decision approach on viewpoints [1], where a group of related criteria make up the more general concept of a *viewpoint*, allowing to independently compare the expected decision outcomes of different viewpoints. In this sense, we propose here a different approach, referring to *reasons* instead of groups of criteria, and considering besides the outcome of a viewpoint, an information measure on the (ir)rationality of that viewpoint.

In this way, a viewpoint V_h is defined on pairwise preference relations $\mu \in U$, where U is the set of all fuzzy preferences relations, and $\mu = (\mu_1, \mu_2, ..., \mu_m)$, being m the number of reasons, such that

$$V_h : U \to \Omega_h. \tag{17}$$

In consequence, every viewpoint V_h obtains an outcome that is here measured on a bi-variate space Ω_h, like e.g. $\Omega_h = [0,1]^2$, where one variate ω_1^h takes on the total suitability-outcome of the system of preferences μ, and the other ω_2^h, measures the inconsistency or ambiguity of the resulting preference order or chain of preferences (what we call the preference argument or viewpoint).

The suitability-outcome ω_1^h expresses a global (positive and negative) total preference value for viewpoint V_h (estimating the overall intensity of preference-aversion). For instance, by denoting by m^+ and m^- the number of positive and negative reasons, respectively, the global suitability-outcome could be directly computed by

$$\omega_1^h = \sum_{i=1}^{m^+} \sum_{j=1}^{k} w_i^+ \mu_{ij}^{(+)} - \sum_{i=1}^{m^-} \sum_{j=1}^{k} w_i^- \mu_{ij}^{(-)}, \tag{18}$$

where w_i^+, w_i^- are weights assigned to each reason; k is the number of comparisons among all the alternatives (excluding the diagonal/identity) on the preference and aversion dimensions of the P-A model, such that $k = |\mathbb{A}|(|\mathbb{A}| - 1)$; and for any pair $a, b \in \mathbb{A}$, indexed by $j = 1, ..., k$, under a specific reason $i = 1, ..., m$, it holds that $\mu_{ij}^{(+)} = \mu_{Q_i}(a,b)$, and $\mu_{ij}^{(-)} = \mu_{V_i}(a,b)$.

Notice that outcome ω_1^h can also be understood according to the want and need components of the P-A structure (as in Eq. (16)), defined for the need-viewpoint $V_{h=D}$. Then, by taking e.g. a *grouping* multidimensional operator over M dimensions [5], $\phi_G : [0,1]^M \to [0,1]$, we can compute the overall outcome $\omega_1^{h=D}$, by

$$\omega_1^D = \phi_G\left(\mu_{ij}^D\right). \tag{19}$$

In summary, the constitution of a decision viewpoint V_h requires the estimation of a pair of outcomes $\{\omega_1, \omega_2\}_h$. The first one, ω_1, the suitability score, expresses the overall intensity of preference which has to be computed from the system of pairwise relations U. The second one, ω_2, measures the soundness of the viewpoint. In order to address this second value, we explore next *ambiguity measures* (which have to be extended from their original formulations to operate on the P-A structure), referring to the incoherence or irrationality of a given viewpoint. In this sense, following Eq. (18), the decision problem can be solved by finding the viewpoint (or need-viewpoint according to Eq. (19)) with maximum outcome ω_1^h, while minimizing its *ambiguity* ω_2^h.

4 Ambiguity measures for decision modeling

4.1 Fuzziness and ambiguity measures

Thinking about the notion of *consistency* for pairwise preference relations from a fuzzy perspective, it can be explored as how close a pairwise relation is from its crisp/extreme values. In this sense, it relates to *fuzziness*, and it serves as a first approximation to measuring how far away can a preference relation, and later a whole viewpoint be, from a completely *rational* strict judgment.

Examining the concept of fuzziness as the lack of distinction between a set Q and its negation $\neg Q$ [19], the intersection of both sets (both its affirmation and its negation), suggests the idea of blurry frontiers or fuzziness. Such fuzziness F can be defined as the intersection between the sets Q and $n(Q)$, such that

$$F(Q) = T(Q, n(Q)).$$

The relation between ambiguity and fuzziness becomes more clear when modeling *opposition* among concepts [12], identifying different types of opposition which may result not necessarily in a strict negation. Let's recall that an opposition operator [12] is defined for all relations $\mu \in [0,1]$, by the function $O : [0,1] \to [0,1]$, such that O is involutive (i.e., $O^2 = \mathbf{I}$, where $\mathbf{I}$ is the Identity), and $\forall x, y, x', y' \in [0,1]$, if $\mu(x,y) \leq \mu(x',y')$, then $O(\mu)(x',y') \leq O(\mu)(x,y)$ holds true. In this way, assuming a particular negation operator n^*, it can be said that the opposition will be of an antonym type, such that $O \leq n^*$, or on the contrary, O will be a sub-antonym operator such that $O > n^*$. This operator allows us to model a semantic relation of opposition regarding any chosen negation n^*, obtaining one of two main families of opposites, that of being antonym or sub-antonym.

Therefore, the antonym or the sub-antonym of a concept can be modeled by $O(Q)$, like e.g. $Q = \textit{high}$, $O(Q) = \textit{low}$ or $V = \textit{small}$, $O(V) = \textit{big}$, by means of the opposite operator O, and overlapping operator ϕ_O, such that

$$F(Q) = \phi_O(Q, O(Q)).$$

In this sense, fuzziness $F(Q)$ refers to an *imprecise* frontier for understanding the difference between both paired concepts Q and $O(Q)$, and the *middle state(s)* in between, capturing the *ambiguity* of the concept Q.

Ambiguity due to ignorance has been studied since the works of Knight [15] and Keynes [14], where the interest was explicitly focused on what they called, *measurable* and *unmeasurable* probabilistic-uncertainty. It was argued that ambiguity, understood as lack of knowledge [6, 7], played an important role in rational learning for decision processes. Thereafter, Fishburn [7] suggested a function of ambiguity, which was later extended to the fuzzy setting [19].

In its original version, an ambiguity measure (α^*) is defined for a reference set X, and $\forall B, C \in X$, by

$$\alpha^* : X \to [0,1]$$

if, and only if, it satisfies the following three axioms [7]:

1. A1. $\alpha^*(\emptyset) = 0$
2. A2. $\alpha^*(B) = \alpha^*(\neg B)$
3. A3. $\alpha^*(B \cup C) + \alpha^*(B \cap C) \leq \alpha^*(B) + \alpha^*(C)$

As mentioned above, these axioms have been extended into a fuzzy setting [19], using the min and max operators for representing conjunction and disjunction, respectively.

4.2 Fuzzy-ambiguity measures

Here we propose an extension of Fishburn's ambiguity axioms into a fuzzy setting by means of overlapping, grouping and opposition operators. In this way, ambiguity measures can be extended for measuring the overall ambiguity of fuzzy (P-A) relations.

Given a reference set of all fuzzy sets X (or all fuzzy preferences, where $X = U$), grouping and overlap operators ϕ_G, ϕ_O, and an opposition operator O, the function

$$\alpha : X \to [0, 1]$$

is a *fuzzy-ambiguity measure* if, and only if, it satisfies the following three axioms ($\forall \mu_B, \mu_C \in X$):

1. F1. $\alpha(\emptyset) = 0$
2. F2. $\alpha(\mu_B) = \alpha(O(\mu_B))$
3. F3. $\alpha(\phi_G(\mu_B, \mu_C)) + \alpha(\phi_O(\mu_B, \mu_C)) \leq \alpha(\mu_B) + \alpha(\mu_C)$

Notice that such a fuzzy formulation of ambiguity depends on the specific operator used to model opposition, and at the same time, on the overlap and grouping operators ϕ_O, ϕ_G, which allow the triangle inequality (F3) to hold.

In this way, for any pair of alternatives and their corresponding preference relations $\mu \in U$, some examples for fuzzy-ambiguity measures (modeling opposition by a strong negation) are Shannon's entropy (also a measure of fuzziness [19])

$$\alpha^{ent}(\mu) = -\left(\sum_{i=1}^{m} \mu_i \ln(\mu_i) + \sum_{i=1}^{m} n(\mu_i) \ln(n(\mu_i))\right),$$

or the sum of specificity measures [19],

$$\alpha^{sp}(\mu) = \frac{m-1}{m}\left(Sp(\mu) + Sp(n(\mu))\right)$$

where *specificity* measures the degree in which any set (of preferences) consists of one and only one element [19].

4.3 Ambiguity measures for decision viewpoints

In this section we will examine ambiguity measures for estimating the soundness of decision viewpoints, denoted earlier by ω_2^h. Given a viewpoint V_h, its ambiguity is computed by extending the examples presented above for ambiguity measures for all the preference relations under V_h. That is, by extending the m-dimensional space for considering every pairwise comparison (between all pairs of alternatives) on $\mathbb{A}^2$.

Hence, the ambiguity of a given decision viewpoint can be computed by the entropy-ambiguity

$$\alpha_h^{ent}(V_h) = \frac{-1}{km}\left(\sum_{i=1}^{m}\sum_{j=1}^{k}\mu_{ij}\ln(\mu_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{k}n(\mu_{ij}))\ln(n(\mu_{ij})))\right),$$

where the km constant stands for the number of k comparisons and m reasons.

Example Consider a democratic voting example, where citizens vote for a public authority among three candidates, such that $\mathbb{A} = \{a, b, c\}$. Each candidate has their own proposals, and the subjective evaluations of those proposals constitute the reasons for valuing the preference-aversion intensities μ. That is, from each proposal there is an associated positive meaning, given by μ_Q, and a negative one, given by μ_V, acting in favor or against each candidate. The corresponding preference and aversion relations for two types of voters, an indecisive and a crisp voter, are given in Table 1.

Table 1. Preference and aversion relations for each pair of candidates (μ_Q, μ_V)

Indecisive	a	b	c
a		(0.3,0.8)	(0.5,0.3)
b	(0.9,0.2)		(0.3,0.6)
c	(0.9,0.4)	(0.4,0.5)	

Strict	a	b	c
a		(1,0)	(1,0)
b	(0,1)		(0,1)
c	(0,1)	(1,0)	

Taking equal weights for the positive and negative reasons in Eq. (18), the score for the indecisive viewpoint is $\omega_1^{indecisive} = 0.5$, and if we take the need-outcome of Eq. (19), with a strong negation $n(x) = 1 - x$, and the overlap and grouping operators $\phi_O = \min$ and $\phi_G = \max$, respectively, we obtain a score of $\omega_1^{indecisive-need} = 0.7$. On the other hand, computing $\omega_2^{indecisive}$ according to the entropy-ambiguity α^{ent}, we obtain that $\alpha^{ent} = 0.57$. Then it can be verified that there is presence of ambiguity in the preferences of this indecisive voter.

Under the same setting, if we take a second viewpoint $h = strict$, of a strictly crisp voter (with only strict transitive preferences), we obtain that $\omega_1^{strict} = 1 = \omega_1^{strict-need}$, and $\omega_2^{strict} = \alpha^{ent} = 0$. Then, for this type of strict voter with no ambiguous judgments, his preferences reflect that a clear decision follows from his complete argumentation.

As a result, the indecisive viewpoint reflects the greater ambiguity of the voter's preferences, having higher fuzziness than a strict voter, whose preferences are all strict judgments. As it could be expected, as preferences come closer to the higher fuzzy intensities of 0.5, their ambiguity becomes higher, and the decision problem appears to be more complex, or less clear.

Examining with greater detail the outcomes of the different viewpoints, it would be relevant to study the reasons that receive greater ambiguity, and understand the decision problem according to those dimensions that become more difficult to judge. Such dimensions may be of a complex nature, referring to a set of *simpler* reasons that could be evaluated separately, aiming at explaining the problem for the indecisive type of individuals.

5 Final comments

We have explored the proposal on decision viewpoints, which guide and explain the decision making process according to its overall preference score, together with information on its levels of coherence or (ir)rationality. For this purpose we have examined ambiguity measures, presenting some examples for measuring the ambiguity of decision viewpoints. Focusing on the proposal for decision viewpoints, it is left for future research to examine viewpoints on specific strict relations (taking $\mu = p$), or only the indifference ($\mu = I$) or incomparability ($\mu = j$) relations, or even considering separate viewpoints for the preference and aversion structures. Then we could have strict, indifferent or incomparable viewpoints, as well as preference or aversion viewpoints.

Further studies over information measures and viewpoints could go beyond ambiguity, and consider also the coherence or rationality of arguments. Thinking about the ambiguity of a single viewpoint, we can also think of measuring the degree of concordance or coherence between groups of viewpoints. Then the question would be how *coherent* the different viewpoints are. This coherence could be understood as the consensus or dissention between two converging or diverging opinions.

Another approach for thinking on the soundness of decisions is to explore the *rationality* of judgments. In this sense, it would be desirable to think of fuzzy rationality measures [3] and their extension for the P-A framework, where the presence of cycles in preference chains entail lower degrees of rationality. Besides, considering absolutely opposite opinions, or complementary opinions μ and $\neg\mu$, it is observed that complementary opinions should obtain the same value of rationality (irrespective if we look at μ or $\neg\mu$). In fact, this observation can be understood as a foundational basis for the notion of *rationality*. As this is a *symmetry* condition between μ and $\neg\mu$, rationality can be characterized in future work under both a symmetric and an asymmetric type of behaviors, modeling opposite opinions, or *arguments*, by μ and $O(\mu)$.

References

1. D. Bouyssou, T. Marchant, M. Pirlot, A. Tsoukiàs, Ph. Vincke. Evaluation and Decision Models with Multiple Criteria. Springer, Heidelberg, 2006.

2. H. Bustince, M. Pagola, R. Mesiar, E. Hullermeier, F. Herrera. Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. IEEE Transactions on Fuzzy Systems 20, 405-415, 2012.
3. V. Cutello, J. Montero. Fuzzy rationality measures. Fuzzy Sets and Systems 62, 39-54, 1994.
4. G. Debreu, Theory of Value. An Axiomatic Approach of Economic Equilibrium. Yale University Press, New York, 1959.
5. L. De Miguel, D. Gómez, J.T. Rodríguez, J. Montero, H. Bustince, G. Dimuro, J. Sanz. General overlap functions. Fuzzy Sets and Systems. DOI: 10.1016/j.fss.2018.08.003.
6. D. Ellsberg. Risk ambiguity and the savage axioms. Quarterly Journal of Economics 75, 643-669, 1961.
7. P. Fishburn. The axioms and algebra of ambiguity. Theory and Decision 34, 119-137, 1993.
8. J. Fodor, M. Roubens. Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht, 1994.
9. C. Franco, J. Montero, J.T. Rodríguez. A fuzzy and bipolar approach to preference modelling with application to need and desire. Fuzzy Sets and Systems 214, 2013, 20-34.
10. C. Franco, J.T. Rodríguez, J. Montero. Building the meaning of preference from logical paired structures. Knowledge-Based Systems 83, 2015, 32-41.
11. C. Franco, J.T. Rodríguez, J. Montero. Learning preferences from paired opposite-based semantics. International Journal of Approximate Reasoning 86, 2017, 80-91.
12. C. Franco, J.T. Rodríguez, J. Montero, D. Gómez. Modeling opposition with restricted paired structures. Journal of Multi-Valued Logic & Soft Computing 30, 2018, 239-262.
13. N. Georgescu-Roegen. The pure theory of consumers behavior. Quarterly Journal of Economics 50, 545593, 1936.
14. J.M. Keynes. A Treatise on Probability. MacMillan, London, 1963.
15. F. Knight. Risk, Uncertainty, and Profit. University of Chicago Press, Chicago, 1971.
16. J. Montero, H. Bustince, C. Franco, J.T. Rodríguez, D. Gómez, M. Pagola, J. Fernandez, E. Barrenechea. Paired structures in knowledge representation, Knowledge Based Systems 100, 2016, 50-58.
17. J. Montero, J. Tejada, C. Cutello. A general model for deriving preference structures from data. European Journal of Operational Research 98, 98-110, 1997.
18. B. Van der Walle, B. de Baets, E. Kerre. Characterizable fuzzy preference structures. Annals of Operational Research 80 105136, 1998.
19. R.R. Yager. On a measure of ambiguity. International Journal of Intelligent Systems 10, 1001-1019, 1995.