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PARAMETRIC UPDATES IN PARAMETRIC TIMED AUTOMATA

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Abstract. We introduce a new class of Parametric Timed Automata (PTAs) where we allow clocks to be compared to parameters in guards, as in classic PTAs, but also to be updated to parameters. We focus here on the EF-emptiness problem: "is the set of parameter valuations for which some given location is reachable in the instantiated timed automaton empty?". This problem is well-known to be undecidable for PTAs, and so it is for our extension. Nonetheless, if we update all clocks each time we compare a clock with a parameter and each time we update a clock to a parameter, we obtain a syntactic subclass for which we can decide the EF-emptiness problem and even perform the exact synthesis of the set of rational valuations such that a given location is reachable. To the best of our knowledge, this is the first non-trivial subclass of PTAs, actually even extended with parametric updates, for which this is possible.

1. Introduction

Timed automata (TAs) are a powerful formalism to model and verify timed concurrent systems, both expressive enough to model many interesting systems and enjoying several decidability properties. In particular, the reachability of a discrete state is decidable and PSPACE-complete [AD94]. In TAs, clocks can be compared with constants in guards, and can be updated to 0 ("reset") along edges.

Timed automata may turn insufficient to verify systems where the timing constants themselves are subject to some uncertainty, or when they are simply not known at the early design stage. Parametric timed automata (PTAs) [AHV93] address this drawback by allowing parameters (unknown constants) in the timing constraints; this high expressive

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power comes at the cost of the undecidability of most interesting problems (see *e.g.* [And19]). In particular, the basic problem of EF-emptiness ("is the set of valuations for which a given location is reachable in the instantiated timed automaton empty?") is "robustly" undecidable: even for a single rational-valued [Mil00] or integer-valued parameter [AHV93, BBLS15], or when only strict constraints are used [Doy07]. A famous syntactic subclass of PTAs that enjoys limited decidability is L/U-PTAs [HRSV02], where the parameters set is partitioned into lower-bound and upper-bound parameters, *i. e.,* parameters that can only be compared to a clock as a lower-bound (resp. upper-bound). The EF-emptiness problem is decidable for L/U-PTAs [HRSV02, BL09] and for PTAs under several restrictions [BO14]; however, most other problems are undecidable (*e.g.* [BL09, Qua14, JLR15, ALR16, AL17, ALM20]) (see [And19] for a survey).

Recall that the EF-emptiness problem is decidable for L/U-PTAs [HRSV02, BL09] and for PTAs under several restrictions [BO14]; however, most other problems are undecidable (*e.g.* [BL09, Qua14, JLR15, ALR16, AL17]) (see [And19] for a survey).

1.1. **Contribution.** We investigate parametric updates, which can model an unknown timing configuration in a system where processes need to synchronize together on common events, as in *e.g.* programmable controller logic programs with concurrent tasks execution. We show that the EF-emptiness problem is decidable for PTAs augmented with parametric updates (*i. e.,* U2P-PTA), with the additional condition that whenever a clock is compared to a parameter in a guard or updated to a parameter, all clocks must be updated (possibly to parameters)—this gives R-U2P-PTA. This result holds when the parameters are *bounded rationals in guards*, and possibly *unbounded rationals in updates*. Non-trivial decidable subclasses of PTAs are a rarity (to the best of our knowledge, only L/U-PTAs [HRSV02] and integer-points (IP-)PTAs [ALR16]); this makes our positive result very welcome. In addition, not only the emptiness is decidable, but *exact synthesis* for bounded rational-valued parameters can be performed—which contrasts with L/U-PTAs and IP-PTAs for which synthesis was shown to be intractable [JLR15, ALR16].

About this manuscript. This is the extended version of [ALR19]. In addition to additional explanations and all proofs of our results, we added the whole new Section 7 adding stopwatches to our formalism.

1.2. **Related work.** Our construction is reminiscent of the parametric difference bound matrices (PDBMs) defined in [QSW17, section III.C] where the authors in this paper revisit the result of the binary reachability relation over both locations and clock valuations in TAs; however, parameters of [QSW17] are used to bound in time a run that reaches a given location, while we use parameters directly in guards and resets along the run, which make them active components of the run specifically for intersection with parametric guards, a key point not tackled in [QSW17]. Related DBMs with an additional parameter have been studied, such as shrunk DBMs [SBM14, BMRS19] and infinitesimally enlarged DBMs [San15].

Allowing parameters in clock updates is inspired by the updatable TA formalism defined in [BDFP04] where clocks can be updated not only to 0 ("reset") but also to rational constants ("update"). In [ALR18], we extended the result of [BDFP04] by allowing parametric updates (and no parameter elsewhere, *e.g.* in guards): the EF-emptiness is undecidable even in the restricted setting of bounded rational-valued parameters, but becomes decidable when parameters are restricted to (unbounded) integers.

Synthesis is obviously harder than EF-emptiness: only three results have been proposed to synthesize the exact set of valuations for subclasses of PTAs, but they are all concerned with *integer*-valued parameters [BL09, JLR15, ALR18]. More precisely, it is possible to synthesize unbounded integers for L- or U-PTAs (L/U-PTAs with only lower-bound, or only upper-bound, parameters) [BL09]; bounded integers for PTAs [JLR15] unbounded integers for timed automata with parametric updates [ALR18].

In contrast, we deal here with (bounded) rational-valued parameters—which makes this result the first of its kind. The idea of updating all clocks when compared to parameters comes from our class of *reset-PTAs* briefly mentioned in [ALR16], but not thoroughly studied.

Finally, updating clocks on each transition in which a parameter appears is reminiscent of the initialized rectangular hybrid automata formalism defined in [HKPV98], which remains one of the few decidable subclasses of hybrid automata. Indeed, timed automata can be defined as a subclass of initialized rectangular hybrid automata where clocks evolve at the same fixed rate, in which diagonal constraints are allowed but not systematically used in practice. However, besides the fact that in PTAs variables (clocks) evolve at the same rate, in initialized rectangular hybrid automata variables are reset whenever one of the derivatives of those variables changes, which is not at all the condition we use for global updates in our R-U2P-PTA.

1.3. **Outline.** Section 2 recalls preliminaries. Section 3 presents R-U2P-PTA along with our decidability result. Sections 4 and 5 introduce operations on our p–PDBMs and our extended region automaton. Section 6 proves the main decidability result. Section 7 extends our results to stopwatches. Section 8 gives a concrete application of our result. Section 9 concludes the paper.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the sets of non-negative integers, integers, non-negative rational numbers and non-negative real numbers respectively.

Throughout this paper, we assume a set $\mathbb{X} = \{x_1, \dots, x_H\}$ of *clocks*, *i. e.,* real-valued variables evolving at the same rate. A clock valuation is a function $w : \mathbb{X} \to \mathbb{R}_+$. We write $\vec{0}$ for the clock valuation that assigns 0 to all clocks. Given $d \in \mathbb{R}_+$, $w + d$ (resp. $w - d$) denotes the valuation such that $(w + d)(x) = w(x) + d$ (resp. $(w - d)(x) = w(x) - d$ if $w(x) - d > 0$, 0 otherwise), for all $x \in \mathbb{X}$. We assume a set $\mathbb{P} = \{p_1, \dots, p_M\}$ of *parameters*, *i. e.,* unknown constants. A parameter *valuation* v is a function $v : \mathbb{P} \to \mathbb{Q}_+$. We identify a valuation v with the point $(v(p_1), \dots, v(p_M))$ of $\mathbb{Q}_+^M$. Given $d \in \mathbb{N}$, $v + d$ (resp. $v - d$) denotes the valuation such that $(v + d)(p) = v(p) + d$ (resp. $(v - d)(p) = v(p) - d$ if $v(p) - d > 0$, 0 otherwise), for all $p \in \mathbb{P}$.

In the following, we assume $\triangleleft \in \{<, \leq\}$ and $\bowtie \in \{<, \leq, \geq, >\}$.

A *parametric guard* g is a constraint over $\mathbb{X} \cup \mathbb{P}$ defined as the conjunction of inequalities of the form $x \bowtie z$, where x is a clock and z is either a parameter or a constant in $\mathbb{Z}$. A *non-parametric guard* is a parametric guard without parameters (*i. e.,* over $\mathbb{X}$).

Given a parameter valuation v, $v(g)$ denotes the constraint over $\mathbb{X}$ obtained by replacing in g each parameter p with $v(p)$. We extend this notation to an *expression*: a sum or difference of parameters and constants. Likewise, given a clock valuation w, $w(v(g))$ denotes the expression obtained by replacing in $v(g)$ each clock x with $w(x)$. A clock valuation w

satisfies constraint $v(g)$ (denoted by $w \models v(g)$) if $w(v(g))$ evaluates to true. We say that v *satisfies* g, denoted by $v \models g$, if the set of clock valuations satisfying $v(g)$ is nonempty. We say that g is *satisfiable* if $\exists w, v$ s.t. $w \models v(g)$.

A *parametric update* is a partial function $u : \mathbb{X} \rightharpoonup \mathbb{N} \cup \mathbb{P}$ which assigns to some of the clocks an integer constant or a parameter. For v a parameter valuation, we define a partial function $v(u) : \mathbb{X} \rightharpoonup \mathbb{Q}_+$ as follows: for each clock $x \in \mathbb{X}$, $v(u)(x) = k \in \mathbb{N}$ if $u(x) = k$ and $v(u)(x) = v(p) \in \mathbb{Q}_+$ if $u(x) = p$ a parameter. A non-parametric update is $u_{np} : \mathbb{X} \rightharpoonup \mathbb{N}$. The term *reset* has been used for clock updates to values different from 0 in [BY03].For a clock valuation w and a parameter valuation v, we denote by $[w]_{v(u)}$ the clock valuation obtained after applying $v(u)$. We first define a new class of parametric timed automata and then define plain parametric timed automata and timed automata as special cases.

Definition 2.1. An update-to-parameter PTA (U2P-PTA) $\mathcal{A}$ is a tuple

$$\mathcal{A} = (\Sigma, L, \ell_0, \mathbb{X}, \mathbb{P}, \zeta),$$

where:

(1) Σ is a finite set of actions,
(2) L is a finite set of locations,
(3) $\ell_0 \in L$ is the initial location,
(4) $\mathbb{X}$ is a finite set of clocks,
(5) $\mathbb{P}$ is a finite set of parameters,
(6) ζ is a finite set of edges $e = \langle \ell, g, a, u, \ell' \rangle$ where $\ell, \ell' \in L$ are the source and target locations, g is a parametric guard, $a \in \Sigma$ and $u : \mathbb{X} \rightharpoonup \mathbb{N} \cup \mathbb{P}$ is a parametric update function.

An U2P-PTA is depicted in Figure 1. Note that all clocks are updated whenever there is a comparison with a parameter (as in newBlock) or a clock is updated to a parameter (as in $\mathrm{blockSolution}_{\mathrm{x}}$).

Given a parameter valuation v, we denote by $v(\mathcal{A})$ the structure where all occurrences of a parameter p_i have been replaced by $v(p_i)$. If $v(\mathcal{A})$ is such that all constants in guards and updates are integers, then $v(\mathcal{A})$ is a *updatable timed automaton* [BDFP04] but will be called *timed automaton* (TA) for the sake of simplicity in this paper. In the following, we may denote as a timed automaton any structure $v(\mathcal{A})$, by assuming a rescaling of the constants: by multiplying all constants in $v(\mathcal{A})$ by their least common denominator, we obtain an equivalent timed automaton (with integer constants).

A *bounded* U2P-PTA is a U2P-PTA with a bounded parameter domain that assigns to each parameter a minimum integer bound and a maximum integer bound. That is, each parameter p_i ranges in an interval $[a_i, b_i]$, with $a_i, b_i \in \mathbb{N}$. Hence, a bounded parameter domain is a hyperrectangle of dimension M.

A parametric timed automaton (PTA) [AHV93] is a U2P-PTA where, for any edge $e = \langle \ell, g, a, u, \ell' \rangle \in \zeta$, $u : \mathbb{X} \rightharpoonup \{0\}$.

Definition 2.2 (Concrete semantics of a TA)**.** Given a U2P-PTA $\mathcal{A} = (\Sigma, L, \ell_0, \mathbb{X}, \mathbb{P}, \zeta)$, and a parameter valuation v, the concrete semantics of $v(\mathcal{A})$ is given by the timed transition system $(S, s_0, \rightarrow)$, with

- $S = \{(\ell, w) \in L \times \mathbb{R}_+^H\}$, $s_0 = (\ell_0, \vec{0})$;
- $\rightarrow$ consists of the discrete and (continuous) delay transition relations:

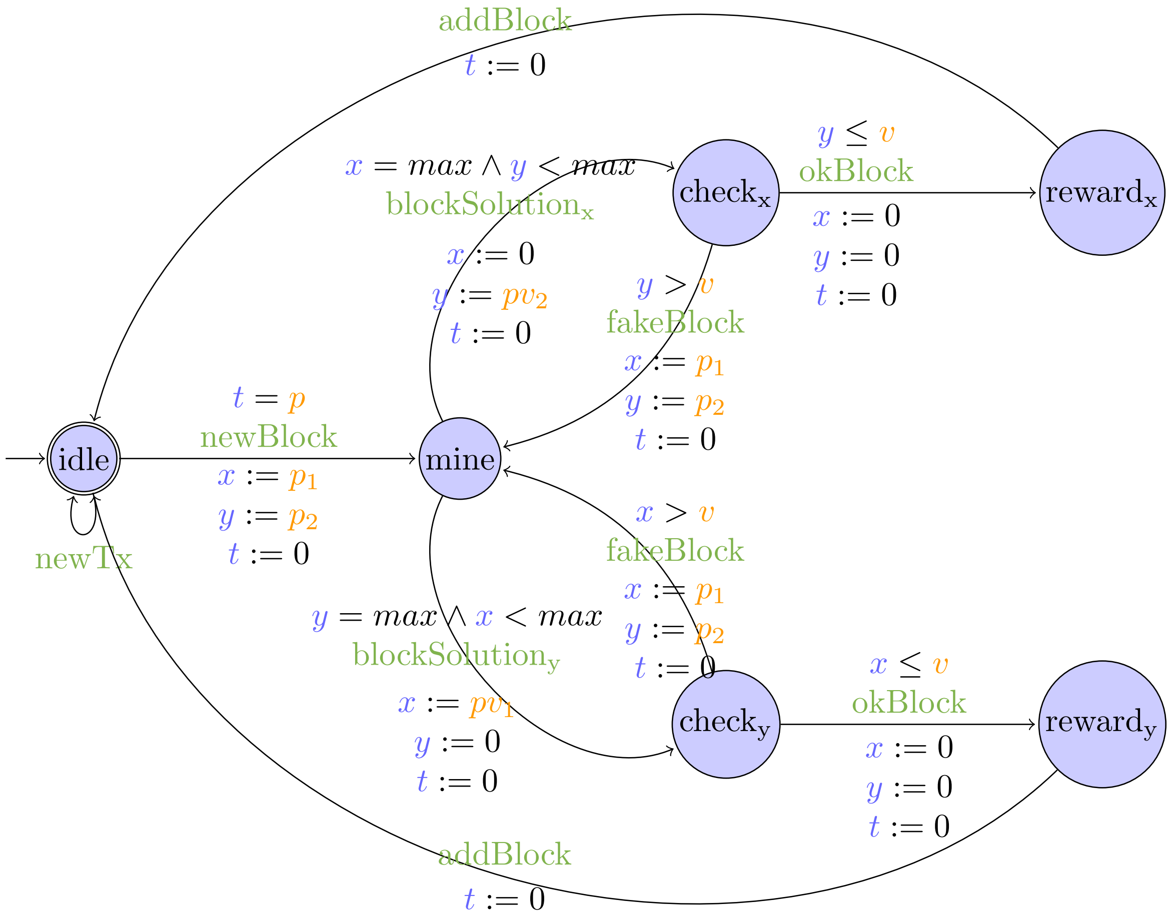


FIGURE 1. A blockchain proof-of-work modeled with a bounded R-U2P-PTA.

- discrete transitions: $(\ell, w) \overset{e}{\mapsto} (\ell', w')$, if $(\ell, w), (\ell', w') \in S$, there exists

$$e = \langle \ell, g, a, u, \ell' \rangle \in \zeta, \qquad w' = [w]_{v(u)}, \text{ and } \qquad w \models v(g)$$

- delay transitions: $(\ell, w) \overset{d}{\mapsto} (\ell, w + d)$, with $d \in \mathbb{R}_+$.

Moreover, we write $(\ell, w) \xrightarrow{e} (\ell', w')$ for a combination of a delay and discrete transitions where $((\ell, w), e, (\ell', w')) \in \rightarrow$ if $\exists d, w'' : (\ell, w) \overset{d}{\mapsto} (\ell, w'') \overset{e}{\mapsto} (\ell', w')$.

Given a TA $v(\mathcal{A})$ with concrete semantics $(S, s_0, \rightarrow)$, we refer to the states of S as the *concrete states* of $v(\mathcal{A})$. A (concrete) *run* of $v(\mathcal{A})$ is a possibly infinite alternating sequence of concrete states of $v(\mathcal{A})$ and edges starting from the initial concrete state s_0 of the form $s_0 \xrightarrow{e_0} s_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{m-1}} s_m \xrightarrow{e_m} \cdots$, such that for all $i = 0, 1, \ldots, e_i \in \zeta$, and $(s_i, e_i, s_{i+1}) \in \rightarrow$. Given a state $s = (\ell, w)$, we say that s is reachable (or that $v(\mathcal{A})$ reaches s) if s belongs to a run of $v(\mathcal{A})$. By extension, we say that ℓ is reachable in $v(\mathcal{A})$, if there exists a state (ℓ, w) that is reachable.

Throughout this paper, let K denotes the largest constant in a given U2P-PTA, *i. e.,* the maximum of the largest constant compared to a clock in a guard and the largest upper bound of a parameter (if the U2P-PTA is bounded).

Let us recall the notion of clock region [AD94]. Given a clock x and a clock valuation w, recall that $\lfloor w(x) \rfloor$ denotes the integer part of $w(x)$ while $\mathit{frac}(w(x))$ denotes its fractional part. We define the same notation for parameter valuations.

Definition 2.3 (clock region)**.** For two clock valuations w and w', $\sim$ is an equivalence relation defined by: $w \sim w'$ iff

(1) for all clocks x, either $\lfloor w(x) \rfloor = \lfloor w'(x) \rfloor$ or $w(x), w'(x) > K$;
(2) for all clocks x, y with $w(x), w(y) \leq K$, $\mathit{frac}(w(x)) \leq \mathit{frac}(w(y))$ iff $\mathit{frac}(w'(x)) \leq \mathit{frac}(w'(y))$;
(3) for all clocks x with $w(x) \leq K$, $\mathit{frac}(w(x)) = 0$ iff $\mathit{frac}(w'(x)) = 0$.

A *clock region* R_c is an equivalence class of $\sim$.

Two clock valuations in the same clock region (cf. Definition 2.3) reach the same regions by time elapsing, satisfy the same guards and can take the same transitions [AD94].

In this paper, we address the EF-emptiness problem: given a U2P-PTA $\mathcal{A}$ and a location ℓ, is set of valuations v such that there is a run in $v(\mathcal{A})$ reaching ℓ is empty? More formally, the problem can be written as:

EF-emptiness problem:
Input: a U2P-PTA $\mathcal{A}$ and a location ℓ
Problem: $\{v \mid \exists s_0 \xrightarrow{e_0} (\ell_1, w_1) \xrightarrow{e_1} \cdots \xrightarrow{e_{m-1}} (\ell, w) \text{ a run of } v(\mathcal{A})\} = \emptyset$?

3. A decidable subclass of U2P-PTAs

We now impose that, whenever a guard or an update along an edge contains parameters, then all clocks must be updated (to constants or parameters). Our main contribution is to prove that this restriction makes EF-emptiness decidable.

Definition 3.1. An *R-U2P-PTA* is a U2P-PTA where for any edge $\langle \ell, g, a, u, \ell' \rangle \in \zeta$, u is a total function whenever:

(1) g is a parametric guard, or
(2) $u(x) \in \mathbb{P}$ for some $x \in \mathbb{X}$.

Both conditions of Definition 3.1 are necessary. If we allow parametric guards to be passed without a full update of clocks, then we obtain a larger class of PTAs for which the EF-emptiness problem is undecidable as it is for regular PTAs [AHV93]. If we allow partial parametric updates of clocks, then we obtain a larger class of Reset-to-Parameter Timed Automaton defined in [ALR18] for which we proved the EF-emptiness problem is undecidable.

In the following we only consider either non-parametric, or (necessarily total) fully parametric update functions. A total update function which is not fully parametric (*i. e.,* an update of some clocks to parameters and all others to constants) can be encoded as a total fully parametric update immediately followed by a (partial) non-parametric update function.

The main idea for proving decidability is the following: given an R-U2P-PTA $\mathcal{A}$ we will construct a finite region automaton that bisimulates $\mathcal{A}$, as in TA [AD94]. Our regions will contain both clocks and parameters and will be a finite number, due to the finite number of parameter and their construction similar to clock regions [AD94]. Since parameters are allowed in guards, we need to construct parameter regions and more restricted clock regions.

We will define a form of Parametric Difference Bound Matrices (viz., p–PDBMs for precise PDBMs, inspired by [HRSV02]) in which, once valuated by a parameter valuation, two clock valuations have the same discrete behavior and satisfy the same non-parametric guards. A p–PDBM will define the *set of clocks and parameter valuations* that satisfies it, while once valuated by a parameter valuation, a valuated p–PDBM will define the *set of clock valuations* that satisfies it. A key point is that in our p–PDBMs the parametric constraints used in the matrix will be defined from a *finite* set of predefined expressions involving parameters and constants, and we will prove that this defines a finite number of p–PDBMs. Decidability will come from this fact: the region automaton will evolve in this finite and stable set of p–PDBMs under time elapsing and update operators.

We define this set of parametric constraints ($\mathcal{PLT}$ for parametric linear term) as follows: $\mathcal{PLT} = \{frac(p_i), 1 - frac(p_i), frac(p_i) - frac(p_j), frac(p_j) + 1 - frac(p_i), 1, 0, frac(p_i) - 1 - frac(p_j), -frac(p_i), frac(p_i) - 1\}$, for all $1 \leq i, j \leq M$. Given a parameter valuation v and $d \in \mathcal{PLT}$, we denote by $v(d)$ the term obtained by replacing in d each parameter p by $v(p)$. Let us now define an equivalence relation between parameter valuations v and v'.

Definition 3.2 (regions of parameters)**.** We write that $v \frown v'$ if

(1) for all parameters p, $\lfloor v(p) \rfloor = \lfloor v'(p) \rfloor$;
(2) for all $d_1, d_2, d_3 \in \mathcal{PLT}$, $v(d_1) \leq v(d_2) + v(d_3)$ iff $v'(d_1) \leq v'(d_2) + v'(d_3)$.

Parameter regions are defined as the equivalence classes of $\frown$, and we will use the notation R_p for parameter regions. The set of all *parameter regions* is denoted by $\mathcal{R}_p$. The definition is in a way similar to Definition 2.3 but also involves comparisons of sums of elements of $\mathcal{PLT}$. In fact, we will need this kind of comparisons to define our p–PDBMs. Nonetheless we do not need more complicated comparisons as in R-U2P-PTA whenever a parametric guard or updated is met the update is a total function: this preserves us from the parameter accumulation, *e.g.* obtaining expressions of the form $5frac(p_i) - 1 - 3frac(p_j)$ (that may occur in usual PTAs).

In the following, our p–PDBMs will be matrices of projections on parameters of parametric clock constraints, written as matrices of pairs of the form $D = (d, \triangleleft)$ where $d \in \mathcal{PLT}$. We therefore need to define comparisons on these pairs.

We define an associative and commutative operator $\oplus$ as $\triangleleft_1 \oplus \triangleleft_2 = <$ if $\triangleleft_1 \neq \triangleleft_2$, or $\triangleleft_1$ if $\triangleleft_1 = \triangleleft_2$. We define $D_1 + D_2 = (d_1 + d_2, \triangleleft_1 \oplus \triangleleft_2)$. Following the idea of parameter regions, we define the *validity* of a comparison between pairs of the form $(d_i, \triangleleft_i)$ within a given parameter region, *i. e.,* whether the comparison is true for all parameter valuations v in the parameter region R_p.

Definition 3.3 (validity of comparison)**.** Let R_p be a parameter region. Given any two linear terms d_1, d_2 over $\mathbb{P}$ (*i. e.,* of the form $\sum_i \alpha_i p_i + d$ with $\alpha_i, d \in \mathbb{Z}$), the comparison $(d_1, \triangleleft_1) \triangleleft (d_2, \triangleleft_2)$ is *valid for* R_p if:

(1) $\triangleleft = <$, and either
   (a) for all $v \in R_p$, $v(d_1) < v(d_2)$ evaluates to true regardless of $\triangleleft_1, \triangleleft_2$, or
   (b) for all $v \in R_p$, $v(d_1) \leq v(d_2)$ evaluates to true, $\triangleleft_1 = <$ and $\triangleleft_2 = \leq$;
(2) $\triangleleft = \leq$, and either
   (a) for all $v \in R_p$, $v(d_1) < v(d_2)$ evaluates to true regardless of $\triangleleft_1, \triangleleft_2$, or
   (b) for all $v \in R_p$, $v(d_1) \leq v(d_2)$ evaluates to true, and $\triangleleft_1 = \triangleleft_2$, or $\triangleleft_1 = <$.

Transitivity is immediate from the definition: if $D_1 \lhd_1 D_2$ and $D_2 \lhd_2 D_3$ are valid for R_p, $D_1(\lhd_1 \oplus \lhd_2)D_3$ is valid for R_p.

The following lemma derives from Definition 3.3:

Lemma 3.4 (validity of addition)**.** *Let $d_1, d_2, d_3, d_4 \in \mathcal{PLT}$. Let R_p be a parameter region. If $(d_1, \lhd_1) \leq (d_2, \lhd_2)$ and $(d_3, \lhd_3) \leq (d_4, \lhd_4)$ are valid for R_p then $(d_1, \lhd_1) + (d_3, \lhd_3) \leq (d_2, \lhd_2) + (d_4, \lhd_4)$ is valid for R_p.*

Proof. See Appendix A. □

We can now define our data structure, namely p–PDBMs (for *precise Parametric Difference Bound Matrices*), inspired by the PDBMs of [HRSV02]; PDBMs were themselves inspired by the DBMs of [Dil89]. However, our p–PDBM compare differences of *fractional parts* of clocks, instead of clocks as in classical DBMs; therefore, our p–PDBMs are closer to clock regions than to DBMs and *fully contained* into clock regions of [AD94]. A p–PDBM is a pair made of an integer vector (encoding the clocks integer part), and a matrix (encoding the parametric differences between any two clock fractional parts). Their interpretation also follows that of PDBMs and DBMs: for $i \neq 0$, the matrix cell $D_{i,0} = (d_{i,0}, \lhd_{i0})$ is interpreted as the constraint $\mathit{frac}(x_i) \lhd_{i0} d_{i,0}$, and $D_{0,i} = (d_{0,i}, \lhd_{0i})$ as the constraint $-\mathit{frac}(x_i) \lhd_{0i} d_{0,i}$. For $i \neq 0$ and $j \neq 0$, the matrix cell $D_{i,j} = (d_{i,j}, \lhd_{ij})$ is interpreted as $\mathit{frac}(x_i) - \mathit{frac}(x_j) \lhd_{ij} d_{i,j}$. Finally for all i, $D_{i,i} = (0, \leq)$.

Our p–PDBMs are partitioned into two types: open–p–PDBMs and point–p–PDBMs. A point–p–PDBM is a clock region defined by only parameters which contains only one clock valuation; that is, it corresponds to a set of inequalities of the form $x_i \leq p_j \wedge p_j \leq x_i$. In contrast, an open–p–PDBM is a clock region which can contain several clock valuations satisfying some possibly parametric constraints, or contain at least one clock valuation satisfying non-parametric constraints (as the corner-point of [AD94]). In particular, the initial clock region $\{0^H\}$ and any clock region $\{E_i^H\}$ where E_i is an integer for all clock x_i, is an open–p–PDBM.

Basically, only the first p–PDBM after a (necessarily total) parametric clock update will be a point–p–PDBM; any following p–PDBM will be an open–p–PDBM until the next (total) parametric update. The following two definitions impose several conditions to p–PDBMs that ensure we build satisfiable ones.

Definition 3.5 (open–p–PDBM)**.** Let R_p be a parameter region. An open–p–PDBM for R_p is a pair (E, D) with $E = (E_1, \dots, E_H)$ a vector of H integers (or ∞) which is the integer part of each clock, and D is an $(H+1)^2$ matrix where each element $D_{i,j}$ is a pair $(d_{i,j}, \lhd_{ij})$ for all $0 \leq i, j \leq H$, where $d_{i,j} \in \mathcal{PLT}$. Moreover, for all $0 \leq i \leq H$, $D_{i,i} = (0, \leq)$. In addition:

(1) For all i, $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p,
(2) For all $i \neq 0, j \neq 0$, either $(0, \leq) \leq D_{i,j} \leq (1, <)$ is valid for R_p and $(-1, <) \leq D_{j,i} \leq (0, \leq)$ is valid for R_p or $(0, \leq) \leq D_{j,i} \leq (1, <)$ is valid for R_p and $(-1, <) \leq D_{i,j} \leq (0, \leq)$ is valid for R_p.
(3) For all i, j, if $d_{i,j} = -d_{j,i}$ and is different from 1 then $\lhd_{ij} = \lhd_{ji} = \leq$, else $\lhd_{ij} = \lhd_{ji} = <$,
(4) For all i, j, k, $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form), and
(5) (a) There is at least one i s.t. $D_{i,0} = D_{0,i} = (0, \leq)$, or
    (b) there is at least one i s.t. $D_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \lhd_{0j})$, then we have $\lhd_{0j} = <$.

Condition 1 ensures fractional parts of clocks valuations have only non negative values. Condition 2 ensures the consistency of differences of clocks *i. e.,* $\mathit{frac}(x) - \mathit{frac}(y) \leq 0$ iff $0 \leq \mathit{frac}(y) - \mathit{frac}(x)$. Condition 3 ensures the only possible closed sets of clock valuations are parametric singleton of clock valuations. Condition 4 is the canonical form which ensures, as described in [HRSV02, BY03], that the open–p–PDBM has the tightest possible bounds *i. e.,* no constraint $\mathit{frac}(x) - \mathit{frac}(y) \triangleleft_{xy} d_{x,y}$ can be strengthened without losing solutions.

An open–p–PDBM satisfying condition 5a can be seen as a subregion of an open line segment or a corner point region of [AD94, fig. 9 example 4.4] (it can be seen as a *border* region) and one satisfying condition 5b can be seen as a subregion of an open region of [AD94, fig. 9 example 4.4] (it can be seen as a *center* region). Remark that sets of the form $\{\mathit{frac}(w(x)) \mid 0 \leq \mathit{frac}(w(x)) \leq 1\}$ are forbidden by Definition 3.5 (3), as in the regions of [AD94].

Let R_p be a parameter region. In the following, $p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ is the set of all possible open–p–PDBMs (E, D) for R_p. This definition is similar to that of [HRSV02, def. 3.1].

The second type is the point–p–PDBM. It represents the unique clock valuation (for a given parameter valuation) obtained after a total parametric update in an U2P-PTA.

Definition 3.6 (point–p–PDBM)**.** Let R_p be a parameter region. A point–p–PDBM for R_p is a pair (E, D) where D is an $(H+1)^2$ matrix where each element $D_{i,j}$ is a pair $(d_{i,j}, \leq)$ and for all $0 \leq i, j \leq H$, $d_{i,0} = \mathit{frac}(p_1) = -d_{0,i}$, and $d_{i,j} = \mathit{frac}(p_1) - \mathit{frac}(p_2) = -d_{j,i}$, for any $p_1, p_2 \in \mathbb{P}$. and for all $1 \leq i \leq H$, $E_i = \lfloor p_k \rfloor$ if $d_{i,0} = \mathit{frac}(p_k)$, for $1 \leq k \leq M$. In addition:

(1) For all i, $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p,
(2) For all i, j, k, $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form).

The fact that D is antisymmetric *i. e.,* for all i, j, $D_{i,j} = -D_{j,i}$, means that each clock is valuated to a parameter and each difference of clocks is valuated to a difference of parameters. Conditions 1 and 2 are the same as for open–p–PDBMs.

The set of all point–p–PDBM for R_p is denoted by $p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, and the set of all p–PDBMs for R_p is denoted by $p\text{–}\mathcal{PDBM}(R_p)$ (hence $p\text{–}\mathcal{PDBM}(R_p) = p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p) \cup p\text{–}\mathcal{PDBM}_{\odot}(R_p)$).

The use of *validity* ensures the consistency of the p–PDBM. We denote the set of all p–PDBMs that are *valid for* R_p by $p\text{–}\mathcal{PDBM}(R_p)$. Given a p–PDBM (E, D), it defines the subset of $\mathbb{R}^H \cup \mathbb{Q}^M$ satisfying the constraints $\bigwedge_{i,j \in [0,H]} \mathit{frac}(x_i) - \mathit{frac}(x_j) \triangleleft_{i,j} d_{i,j} \wedge \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i$. Given a p–PDBM (E, D) and a parameter valuation v, we denote by $(E, v(D))$ the *valuated* p*–PDBM*, *i. e.,* the set of clock valuations defined by the inequalities:

$$\bigwedge_{i,j \in [0,H]} \mathit{frac}(x_i) - \mathit{frac}(x_j) \triangleleft_{i,j} v(d_{i,j}) \wedge \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i.$$

For a clock valuation w, we write $w \in (E, v(D))$ if it satisfies all constraints of $(E, v(D))$[1].

The following two lemmas derive from the above definitions of point–p–PDBM and p–PDBMs:

Lemma 3.7 (positivity of reflexivity)**.** *Let R_p be a parameter region and (E, D) be a* p*–PDBM for R_p. For all clocks i, j, $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p.*

Proof. See Appendix B. □

[1] If v is a valuation assigning an *integer* to each parameter, then $(E, v(D))$ is DBM as defined in [BY03].

Lemma 3.8 (neutral element of the set of cells)**.** *Let R_p be a parameter region and (E,D) be a* p*–PDBM for R_p. For all clocks i,j, $D_{i,j} \leq D_{i,j} + D_{j,j}$ and $D_{i,j} \leq D_{i,i} + D_{i,j}$ are valid for R_p.*

Proof. See Appendix C. □

But let us first clarify our needs graphically. Intuitively, our p–PDBMs are partitioned into three types.

(1) The point–p–PDBM is a clock region defined by *only parameters* which contains only one clock valuation; it represents the unique clock valuation (for a given parameter valuation) obtained after a total parametric update in an U2P-PTA. Each clock is valuated to a parameter and each difference of clocks is valuated to a difference of parameters (it corresponds to constraints of the form $x = p$ and $x - y = p_i - p_j$).

Let v be a parameter valuation. We assume $\lfloor v(p_2) \rfloor = \lfloor v(p_1) \rfloor = k \in \mathbb{N}$ and $\mathit{frac}(v(p_1)) > \mathit{frac}(v(p_2))$. The p–PDBM obtained after an update $u(x) = v(p_2)$ and $u(y) = v(p_1)$ is represented using the following pair (where the indices $\mathbf{0}, \mathbf{x}, \mathbf{y}$ are shown for the sake of comprehension)

$$(E,D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{cc} & \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} & \begin{pmatrix} (0,\leq) & (-\mathit{frac}(p_2),\leq) & (-\mathit{frac}(p_1),\leq) \\ (\mathit{frac}(p_2),\leq) & (0,\leq) & (\mathit{frac}(p_2)-\mathit{frac}(p_1),\leq) \\ (\mathit{frac}(p_1),\leq) & (\mathit{frac}(p_1)-\mathit{frac}(p_2),\leq) & (0,\leq) \end{pmatrix} \end{array} \right)$$

Once valuated with v, it contains a unique clock valuation. We represent it as the black dot in Figure 2.

(2) In contrast, an open–p–PDBM satisfying condition (5a) is a clock region which can contain several clock valuations satisfying some possibly parametric constraints, or contain at least one clock valuation satisfying non-parametric constraints (as the corner-point region of [AD94]). In particular, the initial clock region $\{0^H\}$ and any clock region that is a single integer clock valuation is a p–PDBM. An open–p–PDBM satisfying condition 5a is characterized by at least one clock x s.t. $D_{x,0} = D_{0,x} = (0,\leq)$ and can be seen as a subregion of an open line segment or a corner point region of [AD94, fig. 9 example 4.4]. After an immediate update of x to k, the above p–PDBM (E,D) becomes

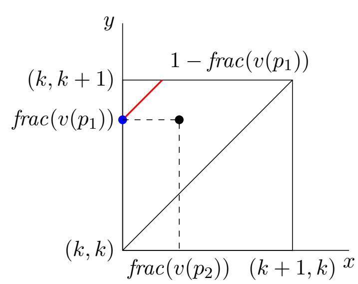


FIGURE 2. Graphical representations of p–PDBMs and [AD94] regions

$$(E,D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{cc} & \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} & \begin{pmatrix} (0,\leq) & (0,\leq) & (-\mathit{frac}(p_1),\leq) \\ (0,\leq) & (0,\leq) & (-\mathit{frac}(p_1),\leq) \\ (\mathit{frac}(p_1),\leq) & (\mathit{frac}(p_1),\leq) & (0,\leq) \end{pmatrix} \end{array} \right)$$

We represent it once valuated with v as the blue dot in Figure 2. The open line segment of [AD94, fig. 9 example 4.4] can be represented as

$$\left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{cc} & \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{array}{c} \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} & \begin{pmatrix} (0,\leq) & (0,\leq) & (0,<) \\ (0,\leq) & (0,\leq) & (0,<) \\ (1,<) & (1,<) & (0,\leq) \end{pmatrix} \end{array} \right)$$

and is depicted as the vertical left black line in Figure 2.

(3) An open–p–PDBM satisfying condition (5b) is a clock region which can contain several clock valuations satisfying some possibly parametric constraints (as the open region of [AD94]). An open–p–PDBM satisfying condition (5b) is characterized by at least one

clock y s.t. $D_{y,0} = (1, <)$ and for all x s.t. $D_{0,x} = (0, \lhd_{ox})$, then we have $\lhd_{ox} = <$ and can be seen as a subregion of an open region of [AD94, fig. 9 example 4.4]. After some time elapsing, and *before* any clock valuation reaches the next integer $k+1$—therefore the next open–p–PDBM satisfying condition 5a—, the above p–PDBM (E, D) becomes

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ (0, \leq) & (0, <) & (-\mathit{frac}(p_1), <) \\ (1 - \mathit{frac}(p_1), <) & (0, \leq) & (-\mathit{frac}(p_1), \leq) \\ (1, <) & (\mathit{frac}(p_1), \leq) & (0, \leq) \end{pmatrix} \Big)$$

We represent it once valuated with v as the red line in Figure 2. The open region of [AD94, fig. 9 example 4.4] can be represented as

$$\Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ (0, \leq) & (0, <) & (0, <) \\ (1, <) & (0, \leq) & (0, <) \\ (1, <) & (1, <) & (0, \leq) \end{pmatrix} \Big)$$

and is depicted as the top left black triangle in Figure 2.

Remark that sets of the form $\{\mathit{frac}(w(x)) \mid 0 \leq \mathit{frac}(w(x)) \leq 1\}$ are in contradiction with Definition 3.5 (3) and therefore cannot be part of a p–PDBM, as in the regions of [AD94]. Basically, only the first p–PDBM after a (necessarily total) parametric clock update will be a point–p–PDBM; any following p–PDBM will be a open–p–PDBM satisfying condition 5a or 5b until the next (total) parametric update.

The differentiation made in the previous paragraph between open–p–PDBMs satisfying condition 5a and 5b is intended to give an intuition to the reader about the inclusion of p–PDBMs into [AD94] clock regions. Technical details are given in the following Section 4. In the following subsections Sections 4.1 to 4.5, we are going to define operations on p–PDBMs (*i. e.,* update of clocks, time elapsing and guards satisfaction), and will show that the set of p–PDBMs is stable under these operations.

4. Operations on p–PDBMs

4.1. **Non-parametric update.** To apply a non-parametric update on a p–PDBM, following classical algorithms for DBMs [BY03], we define an update operator, given in Algorithm 1. Given a p–PDBM (E, D) and u_{np} a non-parametric update function that updates a clock x to $k \in \mathbb{N}$, $\mathit{update}((E, D), u_{np})$ defines a new p–PDBM by

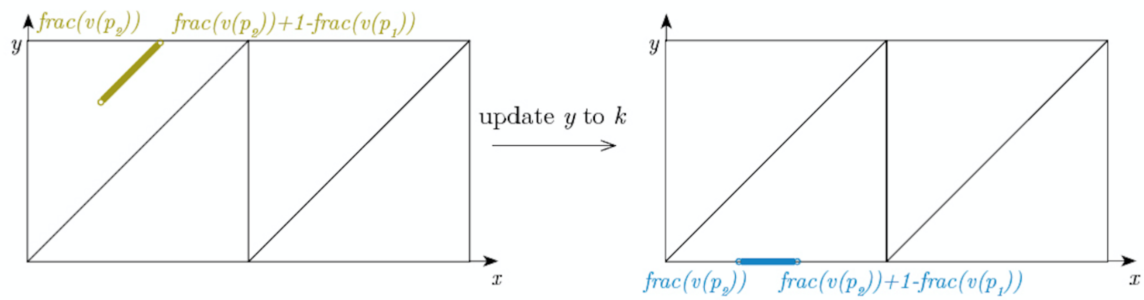
(1) updating E_x to k;
(2) setting the fractional part of x to 0: $D_{x,0} := D_{0,x} := (0, \leq)$;
(3) updating the new difference between fractional parts with all other clocks i, which is the range of values i can currently take: $D_{x,i} := D_{0,i}$ and $D_{i,x} := D_{i,0}$.

Example 4.1. Here is an open–p–PDBM satisfying condition 5b on the left of the figure below. Formally, it is written:

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ (0, \leq) & (-\mathit{frac}(p_2), <) & (-\mathit{frac}(p_1), <) \\ (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) & (0, \leq) & (-\mathit{frac}(p_1) + \mathit{frac}(p_2), \leq) \\ (1, <) & (\mathit{frac}(p_1) - \mathit{frac}(p_2), \leq) & (0, \leq) \end{pmatrix} \Big)$$

After an update of y to k prior to reaching $k+1$, here is the open–p–PDBM satisfying condition 5a obtained, on the right of the figure below. Formally, it is written:

$$(E,D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ \multicolumn{3}{c}{\begin{pmatrix} (0,\leq) & (-\mathit{frac}(p_2),<) & (0,\leq) \\ (\mathit{frac}(p_2)+1-\mathit{frac}(p_1),<) & (0,\leq) & (\mathit{frac}(p_2)+1-\mathit{frac}(p_1),<) \\ (0,\leq) & (-\mathit{frac}(p_2),<) & (0,\leq) \end{pmatrix}} \end{array} \right)$$



Algorithm 1: *update*(D, u_{np}): for all clock x where u_{np} is defined, update $\mathit{frac}(x) := 0$

---

foreach *x where $u_{np}(x)$ is defined* **do**
  $D_{x,0} := D_{0,x} = (0, \leq)$
  for *i from 1 to H* **do**
    $D_{x,i} = D_{0,i}$
    $D_{i,x} = D_{i,0}$
  end
end

Definition 4.2 (update of a p–PDBM)**.** Let u_{np} be a non-parametric update function. Given $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$, we define the *update* of (E, D), denoted by $(E', D') = \mathit{update}((E, D), u_{np})$ as: D' is the result of Algorithm 1 and for each clock x if $u_{np}(x)$ is defined $E'_x := u_{np}(x)$, $E'_x := E_x$ otherwise.

Lemma 4.3 (stability under update)**.** *Let R_p be a parameter region and*

$$(E, D) \in p\text{–}\mathcal{PDBM}(R_p).$$

Let u_{np} be a non-parametric update. Then $\mathit{update}((E, D), u_{np}) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.

Proof. Intuitively, we update in (E, D) the lower and upper bounds of some clocks to $(0, \leq)$ and the difference between two clocks $D_{i,j}$ to $D_{0,j}$ if x_i is updated: that is, the new difference between two clocks if one has been updated is just the lower/upper bound of the one that is not updated. This allows us to conserve the canonical form as we only "moved" some cells in D that already verified the canonical form. Therefore $\mathit{update}((E, D), u_{np})$ is a p–PDBM. See Appendix D for details. □

Applying a non-parametric *update* on any point–p–PDBM transforms it into an open–p–PDBM, and open–p–PDBMs are stable under *update*. It can seem a paradox that the (non-parametric) update of a point–p–PDBM becomes an open–p–PDBM; in fact, it remains geometrically speaking a point, *i. e.,* a singleton containing one clock valuation. Recall that our open–p–PDBMs include p–PDBMs geometrically corresponding to a point for each valuation. In contrast, point–p–PDBMs are also punctual (for each valuation), but are fully parametric.

The following lemma states that the update operator behaves as expected.

Lemma 4.4 (semantics of *update* on $p\text{–}\mathcal{PDBM}(R_p)$)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update. For all w, $[w]_{u_{np}} \in update((E, v(D)), u_{np})$ iff $w \in (E, v(D))$.*

Proof. The technical part is $(\Rightarrow)$. The idea is to prove that, given $w' \in update((E, v(D)), u_{np})$ there is a non-empty set of clock valuations w s.t. $w' = [w]_{u_{np}}$ that is precisely defined by the constraints in $(E, v(D))$. See Appendix E for details. □

4.2. **Parametric update.** Given $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$ we write $\overline{update}((E, D), u)$ to denote the update of (E, D) by u, when u is a total parametric update function, *i. e.,* updating the set of clocks exclusively to parameters. We therefore obtain a point–p–PDBM, containing the parametric constraints defining a unique clock valuation. The semantics is straightforward. Recall that a total update function which is not fully parametric (*i. e.,* an update of some clocks to parameters and some others to constants) can be encoded as a total parametric update immediately followed by a partial non-parametric update function.

4.3. **Time elapsing.** Given a parameter region R_p, recall that constraints satisfied by parameters are known, and we can order elements of $\mathcal{PLT}$. Thanks to this order, within a p–PDBM (E, D) the clocks with the (possibly parametric) largest fractional part *i. e.,* the clocks that have a larger fractional part than any other clock, can always be identified by their bounds in D. For a p–PDBM (E, D), we define the set of clocks with the largest fractional part (LFP) as $\mathsf{LFP}_{R_p}(D) = \{x \in [1, H] \mid (0, \leq) \leq D_{x,i}$ is valid for R_p, for all $0 \leq i \leq H\}$. Clocks belonging to LFP are the first to reach the upper bound 1 by letting time elapse.

Definition 4.5 (clocks with the largest fractional part in a p–PDBM)**.** Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. A *clock with the (possibly parametric) largest fractional part* is a clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p.

There is at least one clock with the (possibly parametric) largest fractional part:

Lemma 4.6 (existence of a clock with the largest fractional part)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. There is at least one clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p.*

Proof. See Appendix F. □

Note that several clocks may have the largest fractional parts (up to some syntactic replacements [2], in that case they satisfy the same constraints in (E, D)).

Suppose, given a parameter valuation $v \in R_p$, we have two different syntactic expressions that are equal once valuated (such as, given p, $v(p) = 1 - v(p)$ and by Definition 3.2 if it is

[2]Let $v \in R_p$ and suppose, we have two different syntactic expressions, such as $p, 1 - p$ that are equal once valuated *i. e.,* $v(p) = 1 - v(p)$. From Definition 3.2 remark that if it is for v, it is for any $v' \in R_p$. We choose one *e.g.* $1 - v(p)$ and replace the second, $v(p)$, everywhere it appears.

for v, it is for any $v' \in R_p$). Then we choose one and replace the second in every constraint where it appears (*e.g.* replace $1-v(p)$ by $v(p)$ everywhere). For a p–PDBM (E, D), we define the set of clocks with the largest fractional part (LFP) as $\mathsf{LFP}_{R_p}(D) = \{x \in \mathbb{X} \mid 0 \leq D_{x,i}$ is valid for R_p, for all $0 \leq i \leq H\}$.

As we are able, thanks to the parameter regions, to order our parameter valuations (*i. e.,* whether one is greater or less than another one), we can define LFP from the constraints defined in the point–p–PDBM. We will define and apply successively two time-elapsing algorithms: the first one starts from a point–p–PDBM or an open–p–PDBM respecting condition Definition 3.5 (5a). We will prove that we obtain an open–p–PDBM respecting condition Definition 3.5 (5b). The second one starts from an open–p–PDBM respecting condition Definition 3.5 (5b) and will define constraints defining the possible clocks valuations exactly when any clock of LFP has reached its upper bound 1. We will prove that we obtain an open–p–PDBM respecting condition Definition 3.5 (5a). As we will obtain at each iteration of the algorithm an open–p–PDBM respecting either condition Definition 3.5 (5a) or (5b), this will prove that we have a stable set of open–p–PDBMs. Now we explain our algorithms more precisely.

Clocks belonging to LFP are the first to reach the upper bound 1 by letting time elapse. Since LFP can contain multiple clocks and they have the same fractional part, we can consider any $x \in \mathsf{LFP}$.

Let $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$ and $x \in \mathsf{LFP}_{R_p}(D)$. To formalize time elapsing until the largest fractional part $\mathit{frac}(x)$ reaches 1, we define a time elapsing operator that will have two variants depending on the input: open–p–PDBM (Definition 3.5) satisfying condition (5a) and point–p–PDBM (Definition 3.6) or open–p–PDBM (Definition 3.5 satisfying condition (5b)).

Given an open–p–PDBM satisfying condition 5a or a point–p–PDBM (E, D) with $E_x = k$, $TE((E, D))$ described in Algorithm 2 and named $TE_<$, defines a new open–p–PDBM satisfying condition 5b by

(1) setting $D_{x,0} := (1, <)$ as x is the first one that will reach $k + 1$;
(2) updating the upper bound of all other clocks i, which has increased: $D_{i,0} := D_{i,x} + (1, <)$;
(3) updating all lower bounds as they have to leave the *border*: $D_{0,i} := D_{0,i} + (0, <)$ (x included).

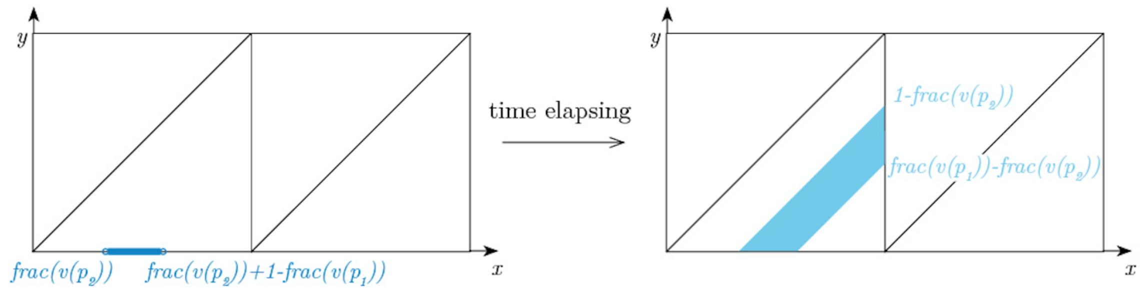
This gives the range of possible clock valuations *before* $\mathit{frac}(x)$ reaches 1. Intuitively it represents the transformation from an open line segment or the corner-point region of [AD94] into an open region of [AD94].

Example 4.7. Here is an open–p–PDBM satisfying condition 5a, on the left of the figure below. Formally, it is written:

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ (0, \leq) & (-\mathit{frac}(p_2), <) & (0, \leq) \\ (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) & (0, \leq) & (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) \\ (0, \leq) & (-\mathit{frac}(p_2), <) & (0, \leq) \end{pmatrix} \Big)$$

Time elapsing before $x \in \mathsf{LFP}$ reaches the next integer gives the following open–p–PDBM satisfying condition 5b, on the right of the figure below. Formally, it is written:

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{pmatrix} \mathbf{0} & \mathbf{x} & \mathbf{y} \\ (0, \leq) & (-\mathit{frac}(p_2), <) & (0, <) \\ (1, <) & (0, \leq) & (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) \\ (1 - \mathit{frac}(p_2), <) & (-\mathit{frac}(p_2), <) & (0, \leq) \end{pmatrix} \Big)$$



Algorithm 2: $TE_<((E,D))$: set upper bound of all $frac(x) \in \mathsf{LFP}_{R_p}(D)$ to 1

pick $x \in \mathsf{LFP}_{R_p}(D)$
for i *from* 1 *to* H **do**
  if $i \in \mathsf{LFP}_{R_p}(D)$ **then**
    $D_{i,0} := (1,<)$
  else
    $D_{i,0} := D_{i,x} + (1,<)$
  end
  $D_{0,i} := D_{0,i} + (0,<)$
end

The result of Algorithm 2 is denoted by $TE_<((E,D))$ and leaves E unchanged.

The time elapsing operator also operates the transformation from an open region of [AD94] to the upper open line segment or the corner-point region of [AD94], given in Algorithm 3 as $TE_=$. Given an open–p–PDBM (E,D) satisfying condition 5b where $E_x = k$, $TE((E,D))$ defines a new open–p–PDBM satisfying condition 5a by

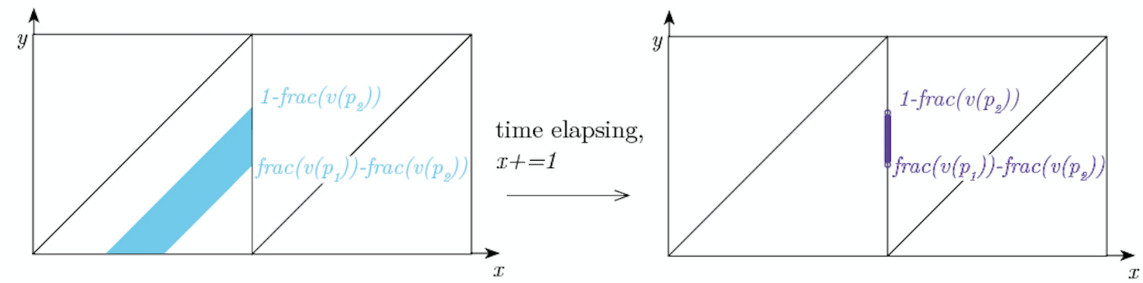
(1) setting $D_{x,0} := D_{0,x} := (0,\leq)$ (intuitively both became $(1,\leq)$) and $E_x = k+1$ (if $E_x \leq K+1$), as x is now in the upper *border*;
(2) updating the upper/lower bounds of all other clocks i: $D_{i,0} := D_{i,x} + (1,\leq)$ and $D_{0,i} := D_{x,i} + (-1,\leq)$;
(3) updating the new difference between fractional parts with all other clocks i, which is the range of values i can currently take (as in the update operator): $D_{x,i} := D_{0,i}$ and $D_{i,x} := D_{i,0}$.

Example 4.8. Here is an open–p–PDBM satisfying condition 5b, on the left of the figure below. Formally, it is written:

$$(E,D) = \left(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0,\leq) & (-frac(p_2),<) & (0,<) \\ (1,<) & (0,\leq) & (frac(p_2)+1-frac(p_1),<) \\ (1-frac(p_2),<) & (-frac(p_2),<) & (0,\leq) \end{pmatrix} \end{array} \right)$$

When $x \in \mathsf{LFP}$ reaches $k+1$, the open–p–PDBM satisfying condition 5a obtained is given on the right of the figure below. Formally, it is written:

$$(E,D) = \left(\begin{pmatrix} k+1 \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0,\leq) & (0,\leq) & (-frac(p_1)+frac(p_2),<) \\ (0,\leq) & (0,\leq) & (-frac(p_1)+frac(p_2),<) \\ (1-frac(p_2),<) & (1-frac(p_2),<) & (0,\leq) \end{pmatrix} \end{array} \right)$$



Algorithm 3: $TE_{=}((E, D))$: set upper and lower bound of all $frac(x) \in \mathsf{LFP}_{R_p}(D)$ to 1

pick $x \in \mathsf{LFP}_{R_p}(D)$
for i *from* 1 *to* H **do**
  if $i \in \mathsf{LFP}_{R_p}(D)$ **then**
    $D_{i,0} := (0, \leq)$
    $D_{0,i} := (0, \leq)$
    $E_i := E_i + 1$
  else
    $D_{i,0} := D_{i,x} + (1, \leq)$
    $D_{0,i} := D_{x,i} + (-1, \leq)$
  end
end
for i *from* 1 *to* H **do**
  $D_{i,x} := D_{i,0}$
  $D_{x,i} := D_{0,i}$
end

The result of Algorithm 3 is denoted by $TE_{=}((E, D))$.

Definition 4.9 (time elapsing in a p–PDBM)**.** Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p) \cup p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. We define $(E', D') = TE((E, D))$ as applying either $TE_{<}$ if (E, D) respects condition 5a or $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, or $TE_{=}$ if (E, D) respects condition 5b.

Lemma 4.10 (stability under time elapsing)**.** *Let R_p be a parameter region. Let $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Then $TE((E, D)) \in p\text{–}\mathcal{PDBM}(R_p)$.*

Proof. Although we perform some additions such as $D_{j,i} + (1, <)$, we do not create new expressions that are not in $\mathcal{PLT}$. In fact, this addition is performed on a negative term (*e.g.* $frac(p) - 1$), as x_i is a clock with the largest fractional part and adding 1 transforms it into another term of $\mathcal{PLT}$. The intuition is similar when performing additions such as $D_{i,j} + (-1, \leq)$: as x_i is a clock with the largest fractional part, $d_{i,j}$ is a positive term. The canonical form is also preserved by the last setting operations of the algorithm, as in the update operator. Therefore $TE((E, D))$ is a p–PDBM. See Appendix G for details. □

Note that, by Lemma 4.10 (E', D') is a p–PDBM. open–p–PDBMs are stable under $TE_{<}$ and $TE_{=}$, switching the condition they respect (5a, 5b). Applying $TE_{<}$ on a point–p–PDBM transforms it into an open–p–PDBM.

The following proposition proves that time elapsing behaves as we expect.

Proposition 4.11 (semantics of p–PDBM under TE)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. There exists $w' \in TE((E, v(D)))$ iff there exist $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. This proof is quite technical. Intuitively, we bound the difference of each upper bound $v(d_{i,0})$ and $w(x_i)$ and each lower bound $v(d_{0,i})$ and $w(x_i)$. This allows us to take a delay δ inside these bounds that allows us to reach the next p–PDBM. See Appendix H for details. ☐

Running example: Figure 3 represents graphically different p–PDBMs obtained after an update $u(x) = v(p_2)$ and $u(y) = v(p_1)$ (figure 1). Time elapsing before $y \in \mathsf{LFP}$ reaches the next integer gives the open–p–PDBM satisfying condition 5b (figure 2)

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0, \leq) & (-\mathit{frac}(p_2), <) & (-\mathit{frac}(p_1), <) \\ (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) & (0, \leq) & (-\mathit{frac}(p_1) + \mathit{frac}(p_2), \leq) \\ (1, <) & (\mathit{frac}(p_1) - \mathit{frac}(p_2), \leq) & (0, \leq) \end{pmatrix} \end{array} \Big)$$

After an update of y to k prior to reaching $k + 1$, the open–p–PDBM satisfying condition 5a obtained is (figure 3)

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0, \leq) & (-\mathit{frac}(p_2), <) & (0, \leq) \\ (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) & (0, \leq) & (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) \\ (0, \leq) & (-\mathit{frac}(p_2), <) & (0, \leq) \end{pmatrix} \end{array} \Big)$$

Time elapsing before $x \in \mathsf{LFP}$ reaches the next integer gives the open–p–PDBM satisfying condition 5b (figure 4)

$$(E, D) = \Big(\begin{pmatrix} k \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0, \leq) & (-\mathit{frac}(p_2), <) & (0, <) \\ (1, <) & (0, \leq) & (\mathit{frac}(p_2) + 1 - \mathit{frac}(p_1), <) \\ (1 - \mathit{frac}(p_2), <) & (-\mathit{frac}(p_2), <) & (0, \leq) \end{pmatrix} \end{array} \Big)$$

When $x \in \mathsf{LFP}$ reaches $k + 1$, the open–p–PDBM satisfying condition 5a obtained is (figure 5)

$$(E, D) = \Big(\begin{pmatrix} k+1 \\ k \end{pmatrix}, \begin{array}{c} \\ \mathbf{0} \\ \mathbf{x} \\ \mathbf{y} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & \mathbf{x} & \mathbf{y} \end{array} \\ \begin{pmatrix} (0, \leq) & (0, \leq) & (-\mathit{frac}(p_1) + \mathit{frac}(p_2), <) \\ (0, \leq) & (0, \leq) & (-\mathit{frac}(p_1) + \mathit{frac}(p_2), <) \\ (1 - \mathit{frac}(p_2), <) & (1 - \mathit{frac}(p_2), <) & (0, \leq) \end{pmatrix} \end{array} \Big)$$

4.4. **Non-parametric guard.**

From [AD94, Section 4.2] we have that either every clock valuation of a clock region satisfies a guard, or none of them does. Note that a p–PDBM for R_p is contained into a clock region of Definition 2.3 (see Appendix I for more details), therefore we have that if $w \in (E, v(D))$ satisfies a non-parametric guard g, then for all $w' \in (E, v(D))$ we also have w' satisfies g.

Let $v \in R_p$. We define $v \in \mathit{guard}_\forall(g, E, D)$ iff for all $w \in (E, v(D))$, $w \models g$. As any two $v, v' \in R_p$ satisfy the same constraints, the following lemma is straightforward

Lemma 4.12. *Let (E, D) be a p–PDBM for R_p and $v \in R_p$. Let g be a non-parametric guard. If $v \in \mathit{guard}_\forall(g, E, D)$, then for all $v' \in R_p$, $v' \in \mathit{guard}_\forall(g, E, D)$.*

Proof. See Appendix I. ☐

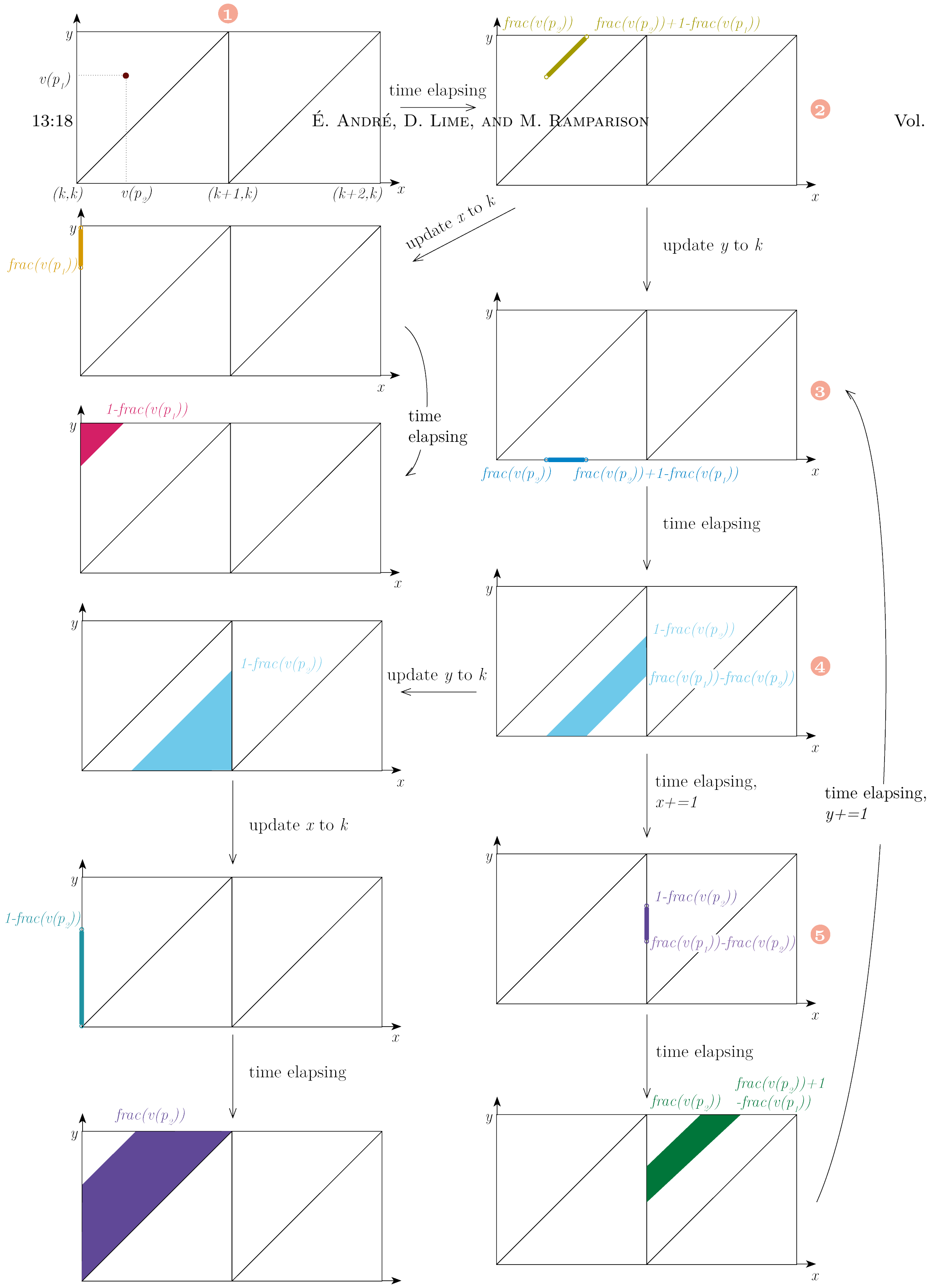


FIGURE 3. Representation of p–PDBMs in two dimensions with two clocks x, y, two parameters p_1, p_2 and v s.t. $\lfloor v(p_1) \rfloor = \lfloor v(p_2) \rfloor$ and $\mathit{frac}(v(p_1)) > \mathit{frac}(v(p_2))$.

4.5. **Parametric guard.** As for the previous result, using a projection on parameters *i. e.*, eliminating clocks, does not create new constraints on parameters that are not already in a parameter region R_p. Indeed, a parametric guard g only adds new constraints of the form $x \bowtie p$ which gives, when eliminating clocks in both a p–PDBM (E, D) and a parametric guard, again a comparison between elements of $\mathcal{PLT}$. Therefore, these new constraints already belong to $\mathcal{PLT}$ and we can decide whether the set of clock valuations satisfying these constraints is non-empty *i. e.,* given $v \in R_p$, $v(g)$ is satisfied by some clock valuation $w \in (E, v(D))$. This is a key point in the overall process of proving the decidability of our R-U2P-PTAs.

Note that there will also be additional constraints involving clocks (with other clocks, constants or parameters), but they will not be relevant as we immediately update all clocks, therefore replacing these constraints with new constraints encoding the clock updates.

Let $v \in R_p$. We define $v \in \textit{p-guard}_\exists(g, E, D)$ iff there is a $w \in (E, v(D))$ s.t. $w \models v(g)$.[3] Again, as any two $v, v' \in R_p$ satisfy the same constraints, the following lemma is straightforward

Lemma 4.13. *Let (E, D) be a p–PDBM for R_p and $v \in R_p$. Let g be a parametric guard. If $v \in \textit{p-guard}_\exists(g, E, D)$, then for all $v' \in R_p$, $v' \in \textit{p-guard}_\exists(g, E, D)$.*

Proof. See Appendix J. □

Now that we have defined useful operations on p–PDBMs, we are going, given a parameter region R_p, to construct a finite region automaton in which for any run, there is an equivalent concrete run in the R-U2P-PTA.

5. Parametric region automaton

Let $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$, we say $(E', D') \in \mathsf{Succ}((E, D)) \Leftrightarrow \exists\, i \geq 0$ s.t. $(E', v(D')) = TE^i((E, D))$. In other words, (E', D') is obtained after applying $TE((E, D))$ a finite number of times. $\mathsf{Succ}((E, D))$ is also called the *time successors* of (E, D).

In order to finitely simulate an R-U2P-PTA, we create a parametric region automaton.

Definition 5.1 (Parametric region automaton)**.** Let R_p be a parameter region. For an R-U2P-PTA $\mathcal{A} = (\Sigma, L, \ell_0, \mathbb{X}, \mathbb{P}, \zeta)$, given (E_0, D_0) the initial p–PDBM where all clocks are 0, the *parametric region automaton* $\mathcal{R}(\mathcal{A})$ over R_p is the tuple $(L', \Sigma, L'_0, \zeta')$ where:

(1) $L' = L \times p\text{–}\mathcal{PDBM}(R_p)$
(2) $L'_0 = (\ell_0, (E_0, D_0))$
(3) $\zeta' = \{\big((\ell, (E, D)), a, (\ell', (E', D'))\big) \in L' \times \Sigma \times L' \mid$ either $\exists e = \langle \ell, g, a, u_{np}, \ell' \rangle \in \zeta$, g is a non-parametric guard, $\exists (E'', D'') \in \mathsf{Succ}((E, D))$, $R_p \subseteq \textit{guard}_\forall(g, (E'', D''))$ and $(E', D') = \textit{update}(E'', D'', u_{np})$ is an open–p–PDBM, or $\exists e = \langle \ell, g, a, u, \ell' \rangle \in \zeta$, g is a parametric guard, $\exists (E'', D'') \in \mathsf{Succ}((E, D))$, $R_p \subseteq \textit{p-guard}_\exists(g, (E'', D''))$ and $(E', D') = \overline{\textit{update}}(E'', D'', u)$ is a point–p–PDBM.$\}$

Let R_p be a parameter region, $\mathcal{A}$ be an R-U2P-PTA and $\mathcal{R}(\mathcal{A}) = (L', \Sigma, L'_0, \zeta')$ its parametric region automaton over R_p. A run in $\mathcal{R}(\mathcal{A})$ is an untimed sequence
$\sigma : (\ell_0, (E_0, D_0))e_0(\ell_1, (E_1, D_1))e_1 \cdots (\ell_i, (E_i, D_i))e_i(\ell_{i+1}, (E_{i+1}, D_{i+1}))e_{i+1} \cdots$ such that for all i we have $\big((\ell_i, (E_i, D_i)), a_i, (\ell_{i+1}, (E_{i+1}, D_{i+1}))\big) \in \zeta'$, which we also write $(\ell_i, (E_i, D_i)) \xrightarrow{e_i} (\ell_{i+1}, (E_{i+1}, D_{i+1}))$. Note that we label our transitions with the edges of the R-U2P-PTA.

[3] Remark that here is why our construction works for EF-emptiness, but cannot be used for, *e.g.*, AF-emptiness ("is there a parameter valuation such that all runs reach a goal location ℓ"): unlike $\textit{guard}_\forall(g, E, D)$, not all clock valuations in a p–PDBM $(E, v(D))$ can satisfy a parametric guard if $v \in \textit{p-guard}_\exists(g, E, D)$.

$$\rightarrow (\ell_0, \vec{0}) \xrightarrow{e_0} (\ell_1, w_1) \xrightarrow{e_1} \cdots \xrightarrow{e_{i-1}} (\ell_i, w_i) \xrightarrow{e_i} (\ell_{i+1}, w_{i+1}) \xrightarrow{e_{i+1}} \cdots \xrightarrow{e_{j-1}} (\ell_j, w_j) \xrightarrow{e_j} (\ell_f, w_f)$$

(A) run of $\mathcal{A}$ with one parametric transition e_i

$$\rightarrow (\ell_0, (E_0, D_0)) \xrightarrow{e_0} (\ell_1, (E_1, D_1)) \xrightarrow{e_1} \cdots \xrightarrow{e_{i-1}} (\ell_i, (E_i, D_i)) \xrightarrow{e_i} (\ell_{i+1}, (E_{i+1}, D_{i+1})) \xrightarrow{e_{i+1}} \cdots \xrightarrow{e_{j-1}} (\ell_j, (E_j, D_j)) \xrightarrow{e_j} (\ell_f, (E_f, D_f))$$

(B) run of $\mathcal{R}(\mathcal{A})$ with one parametric transition e_i

Figure 4. A run in an R-U2P-PTA $\mathcal{A}$ (above) and its equivalent run in $\mathcal{R}(\mathcal{A})$ (below)

6. Decidability of EF-emptiness and synthesis

Using our construction of the parametric region automaton $\mathcal{R}(\mathcal{A})$ for a given R-U2P-PTA $\mathcal{A}$, we state the next proposition.

Proposition 6.1. *Let R_p be a parameter region. Let $\mathcal{A}$ be an R-U2P-PTA and $\mathcal{R}(\mathcal{A})$ its parametric region automaton over R_p. There is a run $\sigma : (\ell_0, (E_0, D_0)) \xrightarrow{e_0} (\ell_1, (E_1, D_1)) \xrightarrow{e_1} \cdots (\ell_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f))$ in $\mathcal{R}(\mathcal{A})$ iff for all $v \in R_p$ there is a run $\rho : (\ell_0, w_0) \xrightarrow{e_0} (\ell_1, w_1) \xrightarrow{e_1} \cdots (\ell_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (\ell_f, w_f)$ in $v(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.*

Proof. We prove this result by induction on the length of the run. It is quite direct as we construct runs without parametric guards. See Appendix K for details. □

Example 6.2. Consider Figure 4. Let $\mathcal{A}$ be an R-U2P-PTA, R_p a parameter region and $v \in R_p$. Suppose there is a run in $\mathcal{A}$, starting from the initial location $(\ell_0, \vec{0})$ reaching a goal location (ℓ_f, w_f). Along this run, all edges are non-parametric transitions but $e_i = \langle \ell_i, g, a_i, u, \ell_{i+1} \rangle$. That is, u is a total parametric update, and g is a possibly parametric guard.

The first part of this run, from $(\ell_0, \vec{0})$ to (ℓ_i, w_i) starts from $(l_0, (E_0, D_0))$ where (E_0, D_0) is the p–PDBM of the initial clock region $\{\vec{0}\}$, and ends in $(\ell_i, (E_i, D_i))$. The second part of this run, from (ℓ_{i+1}, w_{i+1}) to (ℓ_f, w_f) starts from $(l_{i+1}, (E_{i+1}, D_{i+1}))$ where (E_{i+1}, D_{i+1}) is a point–p–PDBM, and can reach $(\ell_f, (E_f, D_f))$ and further ends in $(\ell_s, (E_s, D_s))$.

These runs contain only non-parametric transitions, and as there is an edge in $\mathcal{A}$ from (ℓ_i, w_i) to (ℓ_{i+1}, w_{i+1}), we have to bisimulate this run in $\mathcal{R}(\mathcal{A})$ containing the parametric transition e_i, where $\overline{update}((E_i, D_i), u)$ gives (E_{i+1}, D_{i+1}).

From Proposition 6.1, we deduce that if there is a run reaching a goal location in an instantiated R-U2P-PTA, then for another parameter valuation in the same parameter region there is a run in the instantiated R-U2P-PTA with the same locations and transitions (but possibly different delays), reaching the same location.

Theorem 6.3. *Let $\mathcal{A}$ be an R-U2P-PTA. Let R_p be a parameter region and $v \in R_p$. If there is a run $\rho = (\ell_0, w_0) \xrightarrow{e_0} \cdots \xrightarrow{e_{i-1}} (\ell_i, w_i)$ in $v(\mathcal{A})$, then for all $v' \in R_p$ there is a run $\rho' = (\ell_0, w'_0) \xrightarrow{e_0} \cdots \xrightarrow{e_{i-1}} (\ell_i, w'_i)$ in $v'(\mathcal{A})$ with for all $0 \leq j \leq i$, there is $(E_j, D_j) \in \mathsf{p}$–$\mathcal{PDBM}(R_p)$ s.t. $w_j \in (E_j, v(D_j))$ and $w'_j \in (E_j, v'(D_j))$.*

Proof. Let $v \in R_p$ and ρ a run of $v(\mathcal{A})$ reaching (ℓ_i, w_i). From Proposition 6.1, there is a run σ in $\mathcal{R}(\mathcal{A})$ s.t. each clock valuation at a location j in ρ is in the p–PDBM (E_j, D_j) at the same location in σ. Still from Proposition 6.1, for all $v' \in R_p$ there is a run ρ' in $v'(\mathcal{A})$

reaching (ℓ_i, w'_i) s.t. each clock valuation at a location j in ρ' is in the p–PDBM (E_j, D_j) at the same location in σ (note that possibly $v = v'$). Therefore, we have for all $0 \leq j \leq i$, there is $(E_j, D_j) \in p\text{–}\mathcal{PDBM}(R_p)$ s.t. $w_j \in (E_j, v(D_j))$ and $w'_j \in (E_j, v'(D_j))$ and the expected result. □

Note that there is a finite number of p–PDBMs for each parameter region R_p. Let $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$ and consider $\mathcal{PLT}$: D is an $(H+1)^2$ matrix made of pairs $(d, \triangleleft)$ where $d \in \mathcal{PLT}$ and $\triangleleft \in \{\leq, <\}$. Therefore the number of possible D is bounded by

$$\left(2 \times \left(2 + 3 \times \binom{M}{2} + 4 \times M\right)\right)^{(H+1)^2}.$$

Moreover the number of E is unbounded, but only a finite subset of all values needs to be explored, *i. e.,* those smaller than $K+1$: indeed, following classical works on timed automata [AD94, BDFP04], (integer) values exceeding the largest constant used in the guards or the parameter bounds are equivalent.

To test EF-emptiness given an R-U2P-PTA $\mathcal{A}$ and a goal location ℓ, we first enumerate all parameter regions (which are a finite number), and apply for each R_p the following process: we pick $v \in R_p$ (*e.g.* using a linear programming algorithm [Kar84]). Then, we consider $v(\mathcal{A})$ which is an updatable timed automaton and test the reachability of ℓ in $v(\mathcal{A})$ [BDFP04]. Then EF-emptiness is false if and only if there is v and a run in $v(\mathcal{A})$ reaching ℓ.

Theorem 6.4. *The EF-emptiness problem is PSPACE-complete for bounded R-U2P-PTAs.*

Proof. Since a TA is a special case of R-U2P-PTA we have the PSPACE-hardness [AD94]. Now, let G be a set of goal locations of $\mathcal{A}$. We build a non-deterministic Turing machine that:

(1) takes $\mathcal{A}$, G and K as input;
(2) non-deterministically "guesses" a parameter region R_p;
(3) takes $v \in R_p$ and writes it to the tape;
(4) overwrites on the tape each parameter p by $v(p)$ giving the updatable TA $v(\mathcal{A})$;
(5) solves reachability in $v(\mathcal{A})$ for G;
(6) accepts iff the result of the previous step is "yes".

The machine accepts iff there is an integer valuation v bounded by K and a run in $v(\mathcal{A})$ reaching a location $\ell \in G$.

The size of the input is $|\mathcal{A}| + |G| + |K|$, using $|.|$ to denote the size in bits of the different objects. Moreover, the number of parameter regions is bounded (M is the number of parameters in $\mathcal{A}$) by $\left(M! \times 2^M \times \prod_{p \in \mathbb{P}}(2M+2)\right) \times \left(2 \times (2 + M(3\frac{M-1}{2} + 4))^3\right)$ since they are constructed as the clock regions of [AD94], the second part being the maximal number of constraints in a parameter region. Picking v at step (3) uses a PSPACE linear programming algorithm (*e.g.* [Kar84]). Storing the valuation at step (4) uses at most $M \times |K|$ additional bits, which is polynomial w.r.t. the size of the input. Step (5) also needs polynomial space from [BDFP04]. So globally this non-deterministic machine runs in polynomial space. Finally, by Savitch's theorem we have PSPACE = NPSPACE [Sav70], and the expected result. □

Given a goal location ℓ and a bounded R-U2P-PTA $\mathcal{A}$, we can exactly synthesize the parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching ℓ by enumerating each parameter region (of which there is a finite number) and test if ℓ is reachable for one of its parameter valuations. The result of the synthesis is the union of the parameter regions for which one valuation (and, from our results, all valuations in that region) indeed reaches the goal location in the instantiated TA.

Corollary 6.5. *Given a bounded R-U2P-PTA $\mathcal{A}$ and a goal location ℓ we can effectively compute the set of parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching ℓ.*

Proof. The procedure to obtain synthesis is as follows. We assume an R-U2P-PTA $\mathcal{A}$ and a goal location ℓ.

(1) enumerate all parameter regions (of which there is a finite number);
(2) for each R_p, pick a parameter valuation we pick $v \in R_p$ (*e.g.* using a linear programming algorithm [Kar84]);
(3) test the reachability of ℓ in the updatable timed automaton $v(\mathcal{A})$, which is decidable [BDFP04];
(4) if ℓ is reachable in $v(\mathcal{A})$, add R_p to the list of synthesized regions.

We finally return the union of all regions R_p that reach ℓ.

The correctness immediately comes from Theorems 6.3 and 6.4. □

Remark 6.6. By bounding parameter valuations in guards but not those used in updates, we still have a finite number of parameter regions. Indeed, an integer vector E with components E_x greater than $\lfloor K \rfloor + 1$ is equivalent to an integer vector E' with $E'_x = E_x$ if $E_x < \lfloor K \rfloor + 1$ and $E'_x = \lfloor K \rfloor + 1$ if $E_x \geq \lfloor K \rfloor + 1$. Moreover for all p, we have to replace each parameter valuation v used in an update by $v(p) = v'(p)$ if $v(p) \leq K$ and $v'(p) = K + 1$ if $v(p) > K$.

7. Parametric updates and stopwatches

In this section, we consider clocks in R-U2P-PTAs as *stopwatches* [CL00]: stopwatches can be stopped and started again on transitions. In the general case, stopwatches bring back undecidability in timed automata [BBR06]. Similarly to the "initialization" constraint of [HKPV98] we allow stopwatches to be stopped and started again only on specific transitions. We define $\mathcal{A}$ an R-U2P-PTA with stopwatches instead of clocks. We will prove by using our p–PDBMs structure that the EF-emptiness problem is decidable under the condition that stopwatches can be stopped at each full update function, and started again at the next full update function. Such a condition, as in Definition 3.1 is critical: allowing a partial (or empty) update of clocks ruins the efforts made to keep the set of p–PDBMs stable and allows accumulation of parameters, leading to the undecidability of the EF-emptiness problem.

Definition 7.1. A stopwatch reset update-to-parameter PTA (S-R-U2P-PTAs) $\mathcal{A}$ is a tuple $\mathcal{A} = (\Sigma, L, \ell_0, \mathbb{X}, \mathbb{P}, \zeta, \mathsf{stop})$, where:

(1) Σ is a finite set of actions,
(2) L is a finite set of locations,
(3) $\ell_0 \in L$ is the initial location,
(4) $\mathbb{X}$ is a finite set of stopwatches,
(5) $\mathbb{P}$ is a finite set of parameters,
(6) ζ is a finite set of edges $e = \langle \ell, g, a, u, \ell' \rangle$ where $\ell, \ell' \in L$ are the source and target locations, g is a parametric guard, $a \in \Sigma$ and $u : \mathbb{X} \rightharpoonup \mathbb{N} \cup \mathbb{P}$ is a parametric update function.

(7) $\mathsf{stop} : L \to 2^{\mathbb{X}}$ assigns to each location a set of stopwatches that are stopped at this location.

Moreover, u is a total function whenever:

(1) g is a parametric guard,
(2) $u(x) \in \mathbb{P}$ for some $x \in \mathbb{X}$, or
(3) $\mathsf{stop}(\ell) \neq \mathsf{stop}(\ell')$ for $e = \langle \ell, g, a, u, \ell' \rangle$.

The semantics is defined in a straightforward manner. The update and time elapsing operators are defined in a similar manner as for R-U2P-PTA.

Theorem 7.2. *The EF-emptiness problem is PSPACE-complete for bounded S-R-U2P-PTAs.*

Proof. To prove this result, we first remove stopped stopwatches from p–PDBMs, and show that we can still reason as in Theorem 6.4.

Note that after a full update, stopped stopwatches satisfy constraints defined by a p–$\mathcal{PDBM}_{\odot}(R_p)$. These constraints, defined in Definition 3.6, are

(1) For all i, $(-1, <) \leq D_{0,i} \leq (0, \leq)$ and $(0, \leq) \leq D_{i,0} \leq (1, <)$ are valid for R_p,
(2) For all i, j, k, $D_{i,j} \leq D_{i,k} + D_{k,j}$ is valid for R_p (canonical form).

Let x be a stopped stopwatch, we remove the column and the row corresponding to x in the point–p–PDBM, and we put these constraints aside.

(1) is still satisfied for stopwatches different from x as nothing changed. As both the column and the row are removed for x, constraints of the form $D_{x,j} \leq D_{x,k} + D_{k,j}$ and $D_{i,x} \leq D_{i,k} + D_{k,x}$ are removed as well. With the same argument, constraints of the form $D_{i,j} \leq D_{i,x} + D_{x,j}$ are also removed. Only constraints of the form $D_{i,j} \leq D_{i,k} + D_{k,j}$ where i, j, k are different from x remain, and therefore (2) is still satisfied. We apply the same reasoning for all stopped stopwatches.

After this modification, we still have a point–p–PDBM. Lemmas Lemmas 4.3, 4.4 and 4.10 and Proposition 4.11 are still applicable in this context.

Remains the comparison with non parametric guards Lemma 4.12 and parametric guards Lemma 4.13.

As stated, constraints on stopped stopwatches are put aside, call these constraints C_{stop}; as these constraints are the result of a full update it means there is one clock valuation that satisfies these constraints (it is not an interval, or a set of clock valuations). This argument is crucial in the following: we do not need to be able to bound difference of a stopped stopwatch and a running stopwatch (*i. e.*, $D_{x,y}$ in the p–PDBM for some stopped x and running y, or conversely) to compare separately the p–PDBM with a guard, parametric or not.

First, we compare the p–PDBM with a possibly parametric guard. Let R_p a parameter region and $v \in R_p$. Separately we compare the guard g with the constraints on stopped stopwatch C_{stop}:

- If g is a non parametric guard, let x be a stopped stopwatch appearing in g.
  - If x has been updated to a constant, we can test whether g is satisfied by testing the conjunction of the constraints $v(C_{\mathsf{stop}})$ projected on x and g projected on x.
  - If x has been updated to a parameter, we can test the same conjunction as the order between parameters is defined in R_p.
- If g is a parametric guard, let x be a stopped stopwatch appearing in g. In both cases (x updated to a parameter or a constant) we can test whether $v(g)$ is satisfied by testing

the conjunction of the constraints $v(C_{\mathsf{stop}})$ projected on x and $v(g)$ projected on x as the order between parameters and constants is defined in R_p.

Therefore, Lemma 4.12 and parametric guards Lemma 4.13 are still applicable in this context.

With our modified structure of p–PDBM with stopwatches, the core operators for clock updates, time elapsing and comparison with guards are still applicable. We obtain our result by a similar reasoning as Theorem 6.4. □

Similarly to Corollary 6.5, we obtain the following result:

Corollary 7.3. *Given a bounded S-R-U2P-PTA $\mathcal{A}$ and a goal location ℓ we can effectively compute the set of parameter valuations v s.t. there is a run in $v(\mathcal{A})$ reaching ℓ.*

8. Case study

We implemented EFsynth for R-U2P-PTAs in IMITATOR, a parametric model checker for (extensions of) PTAs [AFKS12].

Our class is the first for which synthesis is possible over bounded rational parameters. We believe our formalism is useful to model several categories of case studies, notably distributed systems with a periodic (global) behavior for which the period is unknown: this can be encoded using a parametric guard while resetting all clocks—possibly to other parameters.

Consider the R-U2P-PTA in Figure 1 with six locations, three clocks compared to parameters (x, y, t), one constant (max) and six parameters (p, p_1, p_2, v, pv_1, pv_2).

We consider the case of a network of peers exchanging transactions grouped by blocks, *e.g.* a blockchain, using the Proof-of-Work as a mean to validate new blocks to add. In this simplified example, we consider a set of two peers (represented by x, y) which have different computation power (represented by p_1, p_2). Peers write new transactions on the current block (newTx). If it is full ($t = p$), both peers try to add a new block (newBlock) to write the transaction on it. We update x to p_1, y to p_2, and t to 0 as the peers have a different computation power, and they start "mining" the block (find a solution to a computation problem). Either x or y will eventually offer a solution to the problem ($\mathrm{blockSolution_x}$ if $x = max$ or $\mathrm{blockSolution_y}$ if $y = max$). If y offers a solution, x will check whether the solution is correct: x is updated to pv_1 to represent its speed to verify an offer. x can refuse the offer if the verification is too long (fakeBlock if $x > v$) therefore the mining step restarts. x can approve the offer (okBlock if $x \leq v$), y is rewarded and the block is added to the blockchain (addBlock).

We are interested in a malicious peer x that wants to avoid y being rewarded for every new block. Therefore x asks: *"what are the possible computation power configurations and verification speed so that y can be rewarded"* ($EF(\mathrm{reward_y})$-synthesis), considered as a bug state in the automaton.

We run this R-U2P-PTA using IMITATOR [AFKS12][4]. We set $max = 30$ units of time and also the upper bound of p and $1 \geq v > 0$ unit of time. IMITATOR computes a disjunction of constraints so that $\mathrm{reward_y}$ is unreachable: we keep two relevant ones;

[4]Experiments were conducted with IMITATOR 2.10.4 "Butter Jellyfish" on a 2.4 GHz Intel Core i5 processor with 2 GiB memory. Computation time is less than 1 second. Sources, binaries, models and results are available at `imitator.fr/static/FORTE19/`

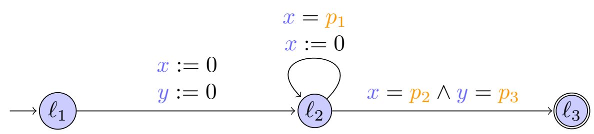


FIGURE 5. PTA of [AHV93, Fig. 1].

(1) $p_1 \geq p_2$: x has strictly more computation power than y in which case x always offers a block solution, or has the same computation power than y in which case the systems blocks. x should invest heavily into hardware to keep its computation power high;
(2) $pv_1 > v$: the malicious peer x is always faster to verify the solution offered by y and refuses it. The blockchain is probably compromised.

Using a parameter valuation respecting one of the previous constraints guarantees that y is never rewarded.

Remark 8.1. Even if we have to update all clocks whenever a parameter is met in a guard or in an update, the possibility to update clocks to unknown parameter offers an appreciable freedom in the range of system that can be modeled with R-U2P-PTA. Especially as our parameters can take unbounded rational values in updates and bounded rational values in guards.

However, such a restriction restricts the behavior that can be modeled. Consider the PTA of Figure 5 for which the set of possible parameter valuations s.t. ℓ_3 is reachable is $\{v \mid v(p_3) = nv(p_1) + v(p_2) \text{ for some } n \in \mathbb{N}\}$. This set cannot be computed from a R-U2P-PTA as in the loop transition over ℓ_2 both clocks x and y must be updated.

9. CONCLUSION AND PERSPECTIVES

Our class of R-U2P-PTAs is one of the few subclasses of PTAs (actually even extended with parametric updates) to enjoy decidability of EF-emptiness. In addition, R-U2P-PTAs are the first “subclass” of PTAs to allow exact synthesis of bounded *rational*-valued parameters.

In terms of future work, beyond reachability emptiness, we aim at studying unavoidability-emptiness and language preservation emptiness (“given a reference parameter valuation, does there exist another parameter valuation with the same untimed language” [ALM20]), as well as their synthesis.

We would also study the possibility to stop and start stopwatches without full updates; it would change the p–PDBM structure by creating new clock constraints, but seems promising and the reachability emptiness problem might be decidable as well.

Time bounded reachability is also interesting to study: given an R-U2P-PTA $\mathcal{A}$, a parameter max and a goal location ℓ, is there a parameter valuation v and a run in $v(\mathcal{A})$ reaching ℓ in less than $v(\mathsf{max})$ time?

Finally, we would like to investigate whether our parametric updates can be applied to decidable hybrid extensions of TAs [HKPV98, BDG+13]. For example, we shall find a subclass of hybrid automata that can be reduced to R-U2P-PTA as done with initialized rectangular hybrid automata and TAs in [HKPV98].

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Appendix A. Proof of Lemma 3.4

Lemma 3.4 (recalled). *Let $d_1, d_2, d_3, d_4 \in \mathcal{PLT}$. Let R_p be a parameter region. If $(d_1, \triangleleft_1) \leq (d_2, \triangleleft_2)$ and $(d_3, \triangleleft_3) \leq (d_4, \triangleleft_4)$ are valid for R_p then $(d_1, \triangleleft_1) + (d_3, \triangleleft_3) \leq (d_2, \triangleleft_2) + (d_4, \triangleleft_4)$ is valid for R_p.*

Proof. Four cases show up: for all $v \in R_p$,

- $v(d_1) < v(d_2)$ and $v(d_3) < v(d_4)$, then clearly $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 3.3 (2a).
- $v(d_1) < v(d_2)$ and $v(d_3) \leq v(d_4)$, then $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 3.3 (2a).
- $v(d_1) \leq v(d_2)$ and $v(d_3) < v(d_4)$, then $v(d_1) + v(d_3) < v(d_2) + v(d_4)$ and we have our result from Definition 3.3 (2a).
- $v(d_1) \leq v(d_2)$ and $v(d_3) \leq v(d_4)$, then $v(d_1) + v(d_3) \leq v(d_2) + v(d_4)$ and
  (1) if $\triangleleft_1 = \triangleleft_2$ and $\triangleleft_3 = \triangleleft_4$ then $\triangleleft_1 \oplus \triangleleft_3 = \triangleleft_2 \oplus \triangleleft_4$ and we have our result from Definition 3.3 (2b).
  (2) if $\triangleleft_1 = \triangleleft_2$ and $\triangleleft_3 = <$, $\triangleleft_4 = \leq$ then $\triangleleft_1 \oplus \triangleleft_3 = <$ and $\triangleleft_2 \oplus \triangleleft_4$ is either $<$ or $\leq$ and we have our result from Definition 3.3 (2b).
  (3) if $\triangleleft_1 = <$, $\triangleleft_2 = \leq$ and $\triangleleft_3 = \triangleleft_4$ then $\triangleleft_1 \oplus \triangleleft_3 = <$ and $\triangleleft_2 \oplus \triangleleft_4$ is either $<$ or $\leq$ and we have our result from Definition 3.3 (2b).
  (4) if $\triangleleft_1 = \triangleleft_3 =<$ and $\triangleleft_2 = \triangleleft_4 =\leq$ then $\triangleleft_1 \oplus \triangleleft_3 =<$ and $\triangleleft_2 \oplus \triangleleft_4 =\leq$ and we have our result from Definition 3.3 (2b).

From Definition 3.3 (2a, 2b) we have that $(d_1, \triangleleft_1) + (d_3, \triangleleft_3) \leq (d_2, \triangleleft_2) + (d_4, \triangleleft_4)$ is valid for R_p. ☐

Appendix B. Proof of Lemma 3.7

Lemma 3.7 (recalled). *Let R_p be a parameter region and (E, D) be a* p*–PDBM for R_p. For all clocks i, j, $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p.*

Proof. By condition (4) in Definition 3.5 and Definition 3.6 (2), we have that $D_{i,i} \leq D_{i,j}+D_{j,i}$ is valid for R_p; the result follows from the fact that $D_{i,i} = (0, \leq)$ (again from Definition 3.5 and Definition 3.6). ☐

Appendix C. Proof of Lemma 3.8

Lemma 3.8 (recalled). *Let R_p be a parameter region and (E, D) be a* p*–PDBM for R_p. For all clocks i, j, $D_{i,j} \leq D_{i,j} + D_{j,j}$ and $D_{i,j} \leq D_{i,i} + D_{i,j}$ are valid for R_p.*

Proof. Let R_p be a parameter region and (E, D) be a p–PDBM for R_p. Let $D_{i,j} = (d_{i,j}, \triangleleft_{ij})$ with $d_{i,j} \in \mathcal{PLT}$. By Definition 3.5 and Definition 3.6 for all clock i, $D_{i,i} = (0, \leq)$. We have $D_{j,i}+D_{i,i} = (d_{j,i}+0, \triangleleft_{ij} \oplus \leq) = D_{j,i}$. Moreover from Definition 3.3 (2b) $D_{i,j} \leq D_{i,j}$ is valid for R_p. Hence $D_{i,j} \leq D_{i,i}+D_{i,j}$ is valid for R_p. The same way we prove $D_{i,j} \leq D_{i,j}+D_{j,j}$ is valid for R_p. ☐

Appendix D. Proof of Lemma 4.3

Lemma 4.3 (recalled). *Let R_p be a parameter region and*

$$(E, D) \in p\text{–}\mathcal{PDBM}(R_p).$$

Let u_{np} be a non-parametric update. Then $update((E, D), u_{np}) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.

Proof. The case of a trivial non-parametric update *i. e.,* that updates no clock, is straightforward.

We split this proof in two parts: the first one treats the case of point–p–PDBMs and the second one of open–p–PDBMs.

First we show that applying an *update* on any point–p–PDBM transforms it into an open–p–PDBM.

Claim D.1 ($p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ becomes $p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ after *update*)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Let u_{np} be a non-parametric update. Then $update((E, D), u_{np}) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.*

Proof. Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Consider $(E', D') = update((E, D), u_{np})$. After applying Algorithm 1, for all clock x_i of (E, D) where u_{np} is defined, $E'_i = u_{np}(x_i)$; moreover for all clock j, $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$. First note that if x_i, x_j have been updated, $D'_{i,j} = D'_{j,i} = D'_{0,j} = D'_{j,0} = D'_{0,i} = D'_{i,0} = (0, \leq) = D_{0,0}$. For all clocks i, j, k, the following inequalities are valid for R_p:

(1) (a) if x_i is updated: $D'_{i,0} = (0, \leq) = D'_{0,i}$ and therefore trivially it holds that $-1 \leq D'_{0,i} \leq 0$ and $0 \leq D'_{i,0} \leq 1$ are valid for R_p;
  (b) if x_i is not updated: $D'_{i,0} = D_{i,0}$ and therefore $-1 \leq D'_{0,i} \leq 0$ and $0 \leq D'_{i,0} \leq 1$ are valid for R_p because these constraints were already satisfied in (E, D).

(2) For all x_i, x_j, if neither x_i nor x_j is updated, $D_{i,j}$ and $D_{j,i}$ are not modified so condition Definition 3.5 (2) still holds. If either x_i is updated, as $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$ condition Definition 3.5 (2) still holds as it holds for $D_{0,j}$ and $D_{j,0}$ and we apply the same reasoning if x_j is updated. If both x_i, x_j are updated, condition Definition 3.5 (2) trivially holds.

(3) For all x_i, if it is updated then $D'_{0,i} = D'_{i,0} = (0, \leq)$, hence $d_{0,i} = -d_{i,0} = 0$ and $\triangleleft_{0i} = \triangleleft_{i0} = \leq$; condition Definition 3.5 (3) holds. For all x_i, x_j, if neither x_i nor x_j is updated, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$ so condition Definition 3.5 (3) holds as it holds for $D_{i,j}$ and $D_{j,i}$. If either x_i is updated, as $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$, condition Definition 3.5 (3) holds as it holds for $D_{0,j}$ and $D_{j,0}$. We treat the case where x_j is updated similarly. If both x_i, x_j are updated, condition Definition 3.5 (3) trivially holds.

(4) Canonical form is preserved:
  (a) if x_i, x_j, x_k are not updated: since no clock is updated we have $D'_{i,j} = D_{i,j}$, $D'_{j,k} = D_{j,k}$ and $D'_{i,k} = D_{i,k}$ since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore it remains valid.
  (b) if x_k is updated and x_i, x_j are not updated: $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,0}$, $D'_{i,k} = D_{i,0}$ because x_k is updated. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,0} \leq D_{i,j} + D_{j,0}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(c) if x_j is updated and x_i, x_k are not updated: then $D'_{i,k} = D_{i,k}$ because neither x_i nor x_k are updated; since x_k is updated we have $D'_{j,k} = D_{0,k}$ and $D'_{i,j} = D_{i,0}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(d) if x_j, x_k are updated and x_i is not updated: then $D'_{i,k} = D_{i,0}$ because x_k is updated; since x_j is updated we have $D'_{i,j} = D_{i,0}$ and $D'_{j,k} = D_{0,0}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2) and Lemma 3.8, we know that $D_{i,0} \leq D_{i,0} + D_{0,0}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(e) if x_i is updated and x_j, x_k are not updated: then $D'_{i,k} = D_{0,k}$, $D'_{i,j} = D_{0,j}$ because x_i is updated; since x_j, x_k are not updated, we have $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p; therefore $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(f) if x_i, x_k are updated and x_j is not updated: we have $D'_{i,k} = (0, \leq) = D_{0,0}$, $D'_{i,j} = D_{0,j}$ and $D'_{j,k} = D_{j,0}$ because x_i, x_k are updated. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{0,0} \leq D_{0,j} + D_{j,0}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(g) if x_i, x_j are updated and x_k is not updated: we have $D'_{i,k} = D_{0,k}$, $D'_{i,j} = (0 <\leq) = D_{0,0}$ and $D'_{j,k} = D_{0,k}$ because x_i, x_j are updated. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2) and Lemma 3.8, we know that $D_{0,k} \leq D_{0,0} + D_{0,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(h) if x_i, x_j, x_k are updated: we have $D'_{i,k} = D_{0,0}$, $D'_{i,j} = D_{0,0}$ and $D'_{j,k} = D_{0,0}$ because x_i, x_j, x_k are updated. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2) and Lemma 3.8, we know that $D_{0,0} \leq D_{0,0} + D_{0,0}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

(5) there is at least one clock x s.t. $D'_{x,0} = D'_{0,x} = (0, \leq)$.

Therefore, $(E', D') \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. □

Now we show that applying an *update* on any open–p–PDBM transforms it into an open–p–PDBM respecting Definition 3.5 (2).

Claim D.2 (stability of $p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ under *update*)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. Let u_{np} be a non-parametric update. Then update$((E, D), u_{np}) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.*

Proof. Most cases are similar to the proof of Claim D.1.

The remaining cases to treat are the cases of Definition 3.5 (2). If i, j are different from 0, and

(1) if i, j are not updated then $D'_{i,j} = D_{i,j}$ and since it is the case in (E, D), condition Definition 3.5 (2) holds.

(2) if j is updated and i is not updated then $D'_{i,j} = D_{i,0}$ and $D'_{j,i} = D_{0,i}$ and as condition Definition 3.6 (1) holds for $D_{i,0}$ and $D_{0,i}$ in (E, D), condition Definition 3.5 (2) holds in (E', D').

(3) if i is updated and j is not updated then $D'_{i,j} = D_{0,j}$ and $D'_{j,i} = D_{j,0}$ and as condition Definition 3.6 (1) holds for $D_{j,0}$ and $D_{0,j}$ in (E, D), condition Definition 3.5 (2) holds in (E', D').

(4) if i, j are updated then trivially $D'_{i,j} = D'_{j,i} = (0, \leq)$ and condition Definition 3.5 (2) holds. $\square$

$\square$

Appendix E. Proof of Lemma 4.4

Lemma 4.4 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update. For all w, $[w]_{u_{np}} \in update((E, v(D)), u_{np})$ iff $w \in (E, v(D))$.*

Proof. We first treat the case of open–p–PDBMs, the case of point–p–PDBMs will be handled similarly at the end. We also prove this lemma for a singleton update (only one clock, say x_i) since updating several clocks can be done by applying several singleton updates in a 0 delay.

$\Longrightarrow$ **for open–p–PDBMs**

Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = update((E, v(D)), u_{np})$ and suppose $w' \in (E', D')$. We want to construct a valuation $w \in (E, v(D))$ s.t. $w' = u_{np}(w)$.

Let w be a clock valuation s.t. for all clock x_j where $i \neq j$, $w(x_j) = w'(x_j)$. That means that for all $j \neq i$,

$$frac(w(x_j)) \lhd_{j0} v(d_{j,0}), \quad -frac(w(x_j)) \lhd_{0j} v(d_{0,j}) \quad \text{and} \quad \lfloor w(x_j) \rfloor = E_j$$

hold from Definition 4.2 since it is the case in (E', D') and these values are left untouched by the update. Moreover for all $j \neq i$, $k \neq i$,

$$frac(w(x_j)) - frac(w(x_k)) \lhd_{jk} v(d_{j,k}) \quad \text{and} \quad frac(w(x_k)) - frac(w(x_j)) \lhd_{kj} v(d_{k,j})$$

again hold from Definition 4.2 since it is the case in (E', D') and these values are left untouched by the update.

We want a valuation for $w(x_i)$ s.t.

$$frac(w(x_i)) \lhd_{i0} v(d_{i,0}) \quad -frac(w(x_i)) \lhd_{0i} v(d_{0,i}) \quad \text{and} \quad \lfloor w(x_i) \rfloor = E_i$$

hold, and for all $j \neq i$, $k \neq i$,

$$frac(w(x_i)) - frac(w(x_j)) \lhd_{ij} v(d_{i,j}) \quad \text{and} \quad frac(w(x_k)) - frac(w(x_i)) \lhd_{ki} v(d_{k,i}) \tag{E.1}$$

hold. Let us prove that such a valuation w exists. We set $\lfloor w(x_i) \rfloor = E_i$.

The following lemma proves transitivity of constraints on clocks with respect to constraints in a p–PDBM.

Lemma E.1. *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. Let $w \in (E, v(D))$. For all clocks i, j, k, $frac(w(x_j)) - frac(w(x_k))(\lhd_{ji} \oplus \lhd_{ik})v(d_{j,i}) + v(d_{i,k})$.*

Proof. Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Let $v \in R_p$. Let $w \in (E, v(D))$.

Since $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$, for all i, j, k we have from Definition 3.5 (4),

$$D_{j,k} \leq D_{j,i} + D_{i,k}$$

is valid for R_p hence since $v \in R_p$, we have $v(D_{j,k}) \leq v(D_{j,i}) + v(D_{i,k})$. Precisely that is $(v(d_{j,k}), \triangleleft_{jk}) \leq (v(d_{j,i}), \triangleleft_{ji}) + (v(d_{i,k}), \triangleleft_{ik})$ *i. e.*,

$$(v(d_{j,k}), \triangleleft_{jk}) \leq (v(d_{j,i}) + v(d_{i,k}), \triangleleft_{ji} \oplus \triangleleft_{ik}).$$

For all clocks j, k satisfying constraints of (E, D),

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k}).$$

Then for all i, j, k, either:

- from Definition 3.3 (2a): $v(d_{j,k}) < v(d_{j,i}) + v(d_{i,k})$ and then, regardless of $\triangleleft_{jk}$ and $\triangleleft_{ji} \oplus \triangleleft_{ik}$ we have $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k))(\triangleleft_{ji} \oplus \triangleleft_{ik})v(d_{j,i}) + v(d_{i,k})$, or
- from Definition 3.3 (2b):
  - $v(d_{j,k}) \leq v(d_{j,i}) + v(d_{i,k})$ and $\triangleleft_{jk} = <$, $\triangleleft_{ji} \oplus \triangleleft_{ik} = \leq$ and then we have

    $$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k))(\triangleleft_{ji} \oplus \triangleleft_{ik})v(d_{j,i}) + v(d_{i,k}),$$

    or
  - $v(d_{j,k}) \leq v(d_{j,i}) + v(d_{i,k})$ and $\triangleleft_{jk} = \triangleleft_{ji} \oplus \triangleleft_{ik}$ and then we have

    $$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k))(\triangleleft_{ji} \oplus \triangleleft_{ik})v(d_{j,i}) + v(d_{i,k})$$

    which completes the proof.

This completes the proof of Lemma E.1. ☐

For all $j \neq i$ and $k \neq i$, since $v(D_{j,k}) \leq v(D_{j,i}) + v(D_{i,k})$ from Definition 3.5 (4), we have $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k)) \triangleleft_{jk} v(d_{j,k})$ and

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k))(\triangleleft_{ji} \oplus \triangleleft_{ik})v(d_{j,i}) + v(d_{i,k})$$

holds from Lemma E.1. Hence

$$\mathit{frac}(w(x_j)) - v(d_{j,i})(\triangleleft_{ji} \oplus \triangleleft_{ik})\mathit{frac}(w(x_k)) + v(d_{i,k}) \tag{E.2}$$

holds. Note that $\triangleleft_{ji} \oplus \triangleleft_{ik}$ is either $\leq$ or $<$. Note the following trick is inspired by [HRSV02, Proof of Lemma 3.5] and [HRSV02, Proof of Lemma 3.13]. Hence

$$I = \{t \in \mathbb{R}_+ \mid \mathit{frac}(w(x_j)) - v(d_{j,i}) \leq t \leq \mathit{frac}(w(x_k)) + v(d_{i,k}) \text{ for all clocks } j, k\}$$

is a non empty set. That means that choosing a $\mathit{frac}(w(x_i))$ with respect to constraints (E.1), recall that they are

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) \triangleleft_{ji} v(d_{j,i}) \quad \text{and} \quad \mathit{frac}(w(x_i)) - \mathit{frac}(w(x_k)) \triangleleft_{ik} v(d_{i,k})$$

is equivalent to choose a $\mathit{frac}(w(x_i))$ s.t.

$$\mathit{frac}(w(x_j)) - v(d_{j,i}) \triangleleft_{ji} \mathit{frac}(w(x_i)) \quad \text{and} \quad \mathit{frac}(w(x_i)) \triangleleft_{ik} \mathit{frac}(w(x_k)) + v(d_{i,k})$$

which is a nonempty set from formula (E.2). Finally we choose a $\mathit{frac}(w(x_i)) \in I$, then $w \in (E, v(D))$ and it completes the proof.

$\Longleftarrow$ for open–p–PDBMs

Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = \mathit{update}((E, v(D)), u_{np})$. Now suppose $w \in (E, v(D))$ and let $w' = [w]_{u_{np}}$.

- for x_i, since u_{np} is defined, $w'(x_i) = u_{np}(x_i) = E'_{x_i}$ (*i. e.*, $\mathit{frac}(w'(x_i)) = 0$) by applying *update* as defined in Definition 4.2. By applying *update* as defined in Definition 4.2, $D'_{i,0} = D'_{0,i} = (0, \leq)$, hence
$$-\mathit{frac}(w'(x_i)) \lhd_{0i} v(d'_{0,i}) \quad \text{and} \quad \mathit{frac}(w'(x_i)) \lhd_{i0} v(d'_{i,0})$$
hold from Definition 4.2 and Claim D.1. Moreover we know that for all $j \neq i$
$$-v(D'_{i,j}) = -v(D'_{0,j}) \quad \text{and} \quad v(D'_{j,i}) = v(D'_{j,0}) \tag{E.3}$$
holds from Definition 4.2, and we also know that
$$\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) = \mathit{frac}(w'(x_j)) \tag{E.4}$$
since $\mathit{frac}(w'(x_i)) = 0$. Hence, combining (E.3) and (E.4), clearly since
$$-\mathit{frac}(w'(x_j)) \lhd_{0j} v(d'_{0,j}) \quad \text{and} \quad \mathit{frac}(w'(x_j)) \lhd_{j0} v(d'_{j,0})$$
hold in (E', D'),
$$\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) \lhd_{ji} v(d'_{j,i}) \quad \text{and} \quad \mathit{frac}(w'(x_i)) - \mathit{frac}(w'(x_j)) \lhd_{ij} v(d'_{i,j})$$
hold.
- for any two clocks x_j, x_k where u_{np} is not defined, $w(x_j) = w'(x_j)$ and $w(x_k) = w'(x_k)$. Hence
$$-v(D'_{0,j}) \lhd_{0j} \mathit{frac}(w'(x_j)) \lhd_{j0} v(D'_{j,0})$$
and
$$-v(D'_{k,j}) \lhd_{kj} \mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_k)) \lhd_{jk} v(D'_{j,k})$$
hold from Definition 4.2 and Claim D.1 since bounds remain unchanged.

Then $w' \in \mathit{update}((E, v(D)), u_{np})$.

This concludes the case $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.

Let us now treat the case $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$.

$\Longrightarrow$ **for point–p–PDBMs**

Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Let $v \in R_p$. Let u_{np} be a non-parametric update which updates x_i to an integer n and lets the value of other clocks unchanged. Consider $(E', D') = \mathit{update}((E, v(D)), u_{np})$ and suppose $w' \in (E', D')$. We want to construct a valuation
$$w \in (E, v(D)) \quad \text{s.t.} \quad w' = u_{np}(w)$$
Let w be a clock valuation s.t. for all clock x_j where $j \neq i$, $w(x_j) = w'(x_j)$. That means for all $j \neq i$,
$$\mathit{frac}(w(x_j)) \lhd_{j0} v(d_{j,0}), \quad -\mathit{frac}(w(x_j)) \lhd_{0j} v(d_{0,j}) \quad \text{and} \quad \lfloor w(x_j) \rfloor = E_j$$
hold from Definition 4.2 since it is the case in (E', D') and bounds remain unchanged *i. e.*, $D_{0,j} = D'_{0,j}$ and $D_{j,0} = D'_{j,0}$. Moreover for all $k \neq i$ and $k \neq j$,
$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_k)) \lhd_{jk} v(d_{j,k}) \quad \text{and} \quad \mathit{frac}(w(x_k)) - \mathit{frac}(w(x_j)) \lhd_{kj} v(d_{k,j})$$
also hold from Definition 4.2 since it is the case in (E', D') and bounds remain unchanged *i. e.*, $D_{k,j} = D'_{k,j}$ and $D_{j,k} = D'_{j,k}$.

Recall that (E, D) contains only one clock valuation for each parameter valuation $v \in R_p$.

Let $frac(w(x_i)) = v(d_{i,0})$ (or equivalently $frac(w(x_i)) = -v(d_{0,i})$ since by Definition 3.6 we have $(d_{i,0}, \triangleleft_{i0}) = (-d_{0,i}, \triangleleft_{0i})$). Then, as it is the case in (E, D),

$$frac(w(x_i)) \triangleleft_{i0} v(d_{i,0}), \quad -frac(w(x_i)) \triangleleft_{0i} v(d_{0,i}) \quad \text{and} \quad \lfloor w(x_i) \rfloor = E_i$$

hold, and for all $j \neq i$, $k \neq i$,

$$frac(w(x_i)) - frac(w(x_j)) \triangleleft_{ij} v(d_{i,j}) \quad \text{and} \quad frac(w(x_k)) - frac(w(x_i)) \triangleleft_{ki} v(d_{k,i})$$

hold, which completes the proof, as $w \in (E, v(D))$ and $w' = u_{np}(w)$.

$\Longleftarrow$ for point–p–PDBMs

This case is straightforward and similar to the case ($\Leftarrow$) above of open–p–PDBMs. $\square$

Appendix F. Proof of Lemma 4.6

Lemma 4.6 (recalled). *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. There is at least one clock x s.t. for all $0 \leq i \leq H$, $(0, \leq) \leq D_{x,i}$ is valid for R_p.*

Proof. Reductio ad absurdum: Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ with at least 2 clocks i, j. Suppose for all clock x_i there is another clock x_j s.t. $D_{i,j} < 0$ is valid for R_p. Let $v \in R_p$. Then $v(D_{i,j}) < 0$.

- Suppose for x_j, x_i is the clock s.t. $D_{j,i} < 0$ is valid for R_p. Then $v(D_{j,i}) < 0$. We have $v(D_{i,j}) + v(D_{j,i}) < 0$ holds, therefore $0 \leq v(D_{i,j}) + v(D_{j,i})$ does not hold, and hence $0 \leq D_{i,j} + D_{j,i}$ is not valid for R_p. Then (E, D) does not respect Lemma 3.7 and violates condition (4) of Definition 3.5. So $(E, D) \notin p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.
- Suppose for x_j, a third clock x_k is the clock s.t. $D_{j,k} < 0$ is valid for R_p. Then $v(D_{j,k}) < 0$. Suppose we have only three clocks. Then for x_k, either x_i or x_j is the clock s.t. $D_{k,i} < 0$ is valid for R_p.
  - Assume this is x_i. Then $v(D_{k,i}) < 0$. We have $v(D_{k,i}) + v(D_{i,j}) < 0$ and $v(D_{k,j}) \leq v(D_{k,i}) + v(D_{i,j})$ by Definition 3.5 (4). Follows that $v(D_{k,j}) + v(D_{j,k}) < 0$ and $0 \leq D_{k,j} + D_{j,k}$ is not valid for R_p. Then (E, D) does not respect Lemma 3.7 and violates condition (4) of Definition 3.5. So $(E, D) \notin p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$.
  - Assume this is x_j. This case is similar (and simpler).

We apply the same reasoning for more than 3 clocks. Now suppose $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. We apply the same reasoning, replacing the argument of condition (4) of Definition 3.5 by the fact from Definition 3.6 that D is antisymmetric. $\square$

Appendix G. Proof of Lemma 4.10

Lemma 4.10 (recalled). *Let R_p be a parameter region. Let $(E, D) \in p\text{–}\mathcal{PDBM}(R_p)$. Then $TE((E, D)) \in p\text{–}\mathcal{PDBM}(R_p)$.*

Proof. We prove our lemma for the two types of open–p–PDBMs and for point–p–PDBMs.

G.0.1. $\rightarrow$ *Definition 3.5 type (5a) to (5b).*

Claim G.1 (modification of an open–p–PDBM respecting condition (5a) under $TE_<$)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respecting condition 5a, then $TE_<((E, D)) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respecting condition 5b.*

Proof. Suppose $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respects condition (5a) of Definition 3.5, *i. e.,* we have at least an x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$. Since, in R_p, we know which parameters have the largest fractional part, we can determine $\mathsf{LFP}_{R_p}(D)$ from Lemma 4.6. If more than one clock belong to $\mathsf{LFP}_{R_p}(D)$ then their valuations have the same fractional part. Indeed, from Definition 4.5 if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ then both $(0, \leq) \leq D_{i,j}$ and $(0, \leq) \leq D_{j,i}$ are valid for R_p, and from Definition 3.5 (2) we must have $D_{i,j} = D_{j,i} = (0, \leq)(\star)$.

Let $v \in R_p$. Assume $x_i \in \mathsf{LFP}_{R_p}(D)$ and $w \in (E, v(D))$, by letting time elapse, $\mathit{frac}(w(x_i))$ is the first that might reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $\mathit{frac}(w(x_j))$ cannot reach 1 before $\mathit{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_<((E, D))$, which will be an open–p–PDBM respecting condition 5b of Definition 3.5. While detailing the procedure of $TE_<$, we are going to prove that Definition 3.5 (1) and (2) hold for (E', D'). Further we will prove that (4) and (5b) also hold.

proof that Definition 3.5 (1) holds.

According to the definition of $TE_<$ (Algorithm 2) the first step is to set a new upper bound

$$D'_{i,0} = (1, <) \quad \text{for all } x_i \in \mathsf{LFP}_{R_p}(D)$$

and obviously $(0, \leq) \leq D'_{i,0} \leq (1, \leq)$ is valid for R_p. Then we set new upper bounds for all other clock $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$ by setting

$$D'_{j,0} = D_{j,i} + (1, <).$$

Indeed, $D_{j,i}$ is the constraint on the lower bound of $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i))$ and since the upper bound of x_i has increased, this gives the new upper bound of x_j. Note that since $x_i \in \mathsf{LFP}_{R_p}(D)$, from Definition 4.5 and Definition 3.5 (2) we have that $-1 \leq D_{j,i} \leq 0$ is valid for R_p for all clock x_j. Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p. Hence as $d_{j,i} + 1 \in \{1, 1 - p_1, p_2 + 1 - p_1, p_1 - p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{PLT}$, $\lhd_{ji'} = \lhd_{ji} \oplus < = <$ so $(0, \leq) \leq D'_{j,0} \leq (1, <)$ is valid for R_p.

Note that we cannot have $(d_{j,i}, \lhd_{ji}) = (-1, <)$ because even if $(d_{i,j}, \lhd_{ij}) = (1, <)$, since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ we do not have have $0 \leq D_{j,i} + D_{i,j}$ is valid for R_p from Definition 3.5 (4) and Lemma 3.7.

Secondary we set for all clock x regardless of whether they are in $\mathsf{LFP}_{R_p}(D)$

$$D'_{0,x} = D_{0,x} + (0, <).$$

Since some time elapsed, lower bounds of all clocks are increased. Moreover, as $(-1, <) \leq D_{0,x} \leq (0, \leq)$ is valid for R_p from Definition 3.5 (1), $(-1, \leq) \leq D'_{0,x} \leq (0, \leq)$ is also valid for R_p.

Therefore, Definition 3.5 (1) holds.

proof that Definition 3.5 (2) holds

Third we set for all clocks x, y regardless of whether they are in $\mathsf{LFP}_{R_p}(D)$

$$D'_{x,y} = D_{x,y}$$

so as Definition 3.5 (2) holds in (E, D), it still does. More intuitively since no fractional part has reached 1, constraints on differences of clocks and integer parts remain unchanged.

proof that Definition 3.5 (3) holds

For all x_i:

- if $x_i \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$, condition Definition 3.5 (3) holds;
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p, we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$ and condition Definition 3.5 (3) holds.

For all x_i, x_j:

- if $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition Definition 3.5 (3) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p, we have $d'_{i,0} \neq d'_{0,i}$ and $\triangleleft_{i0'} = \triangleleft_{0i'} = <$, condition Definition 3.5 (3) holds. The case $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_i \in \mathsf{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\triangleleft_{ij'} = \triangleleft_{ji'} = \leq$ and condition Definition 3.5 (3) holds.

proof that Definition 3.5 (4) holds

Now we prove that Definition 3.5 (4) holds, *i. e.*, for all clocks x_i, x_j, x_k, valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p. Indeed, when time elapses, all clocks have the same behavior, hence the difference between two clocks does not change without an update. Precisely, for all clocks x_i, x_j, x_k, are valid for R_p:

(1) if $x_i, x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: let $x \in \mathsf{LFP}_{R_p}(D)$ and
  - if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
  - if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p; then $D_{i,x} + (1, <) \leq D_{i,j} + D_{j,x} + (1, <)$ is valid for R_p from Lemma 3.4 and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
  - if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that

$$D_{i,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \tag{G.1}$$

  is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (1), we know that

$$D_{x,0} \leq (1, <); \tag{G.2}$$

  moreover we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <) \tag{G.3}$$

Since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{x,k} \leq D_{x,0} + D_{0,k}$ is valid for R_p; combining with (G.2) and (G.3) we obtain

$$D_{x,k} \leq (1,<) + D_{0,k} + (0,<). \tag{G.4}$$

Now, since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p and combining with (G.4) we obtain (G.1) and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1,<)$; from Definition 3.3 (2b) we have that
$$D_{i,x} + (1,<) \leq D_{i,x} + (1,<)$$
is valid for R_p. Hence from Lemma 3.8
$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$
is valid for R_p.
- if j,k are different from 0, $i = 0$, we have $D'_{0,k} = D_{0,k} + (0,<)$, $D'_{0,j} = D_{0,j} + (0,<)$ and $D'_{j,k} = D_{j,k}$; since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that
$$D_{0,k} + (0,<) = (d_{0,k},<) \quad \text{and} \quad D_{0,j} + (0,<) + D_{j,k} = (d_{0,j} + d_{j,k},<)$$
so we have from Definition 3.3 (2b)
$$D_{0,k} + (0,<) \leq D_{0,j} + (0,<) + D_{j,k}$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if j is different from 0, $i = k = 0$, we have $D'_{0,0} = (0,\leq)$, $D'_{0,j} = D_{0,j} + (0,<)$ and $D'_{j,0} = D_{j,x} + (1,<)$; since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,x} \leq D_{0,j} + D_{j,x}$ is valid for R_p; moreover, from Definition 3.3 (2b) and Lemma 3.4,
$$D_{0,x} + (0,<) \leq D_{0,j} + (0,<) + D_{j,x}$$
is valid for R_p. Recall that from Lemma 3.7 $(0,\leq) \leq D_{0,x} + D_{x,0}$ is valid for R_p and since $D_{x,0} \leq (1,<)$ from Definition 3.5 (1), we have
$$(0,\leq) \leq D_{0,x} + (1,<)$$
is valid for R_p. As we have $(1,<) + (0,<) = (1+0, < \oplus <) = (1,<)$, we obtain that
$$D_{0,x} + (1,<) \leq D_{0,j} + D_{j,x} + (1,<)$$
is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.
- if k is different from 0, $i = j = 0$, we have $D'_{0,k} = D_{0,k} + (0,<)$; From Definition 3.3 (2b) and Lemma 3.4 we have that
$$D_{0,k} + (0,<) \leq D_{0,k} + (0,<)$$
is valid for R_p. Hence from Lemma 3.8
$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$
is valid for R_p.
- if $i = j = k = 0$, from Definition 3.5 (4) and Lemma 3.8 we trivially have
$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$
is valid for R_p.

(2) if $x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $k \neq 0$ and

- if i, j are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,j}+D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.
- if $i \neq 0$, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,k} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that $D_{i,k} \leq D_{i,k} + (1, <) + D_{0,k} + (0, <)$ is valid for R_p, *i. e.*,
$$(0, \leq) \leq (1, <) + D_{0,k} + (0, <) \tag{G.5}$$
is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. We have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \tag{G.6}$$
Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $(0, \leq) \leq D_{0,k} + D_{k,0}$ is valid for R_p and from Definition 3.5 (1) that $D_{k,0} \leq (1, <)$ is valid for R_p; combining with (G.5) and (G.6) we obtain our result.
- if $i = 0$, $j \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. Moreover we have that
$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$
so we have from Definition 3.3 (2b)
$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if $i = j = 0$, from Definition 3.5 (4) and Lemma 3.8 we trivially have
$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$
is valid for R_p.

(3) if $x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0$ and
- if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j}+D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $i \neq 0$, $k = 0$, we have $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = (1, <)$; From Definition 3.3 (2b) we trivially have that $D_{i,j} + (1, <) \leq D_{i,j} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if $i = 0$, $k \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that
$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$
so we have from Definition 3.3 (2b) and Lemma 3.4
$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if $i = k = 0$, we have $D'_{0,0} = (0, \leq)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Lemma 3.7 we know that $(0, \leq) \leq D_{0,j} + D_{j,0}$ is valid for R_p, and since from Definition 3.5 (1) $D_{j,0} \leq (1, \leq)$ is valid for R_p, that means $(0, \leq) \leq D_{0,j} + (1, <)$ is valid for R_p. As we have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

we obtain that

$$(0, \leq) \leq D_{0,j} + (0,<) + (1,<)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.

(4) if $x_j, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and

- if i is different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.
- if $i = 0$, we have $D'_{0,k} = D_{0,k} + (0,<)$, $D'_{0,j} = D_{0,j} + (0,<)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. Moreover we have that

$$D_{0,k} + (0,<) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0,<) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$

so we have from Definition 3.3 (2b) and Lemma 3.4

$$D_{0,k} + (0,<) \leq D_{0,j} + (0,<) + D_{j,k}$$

is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.

(5) if $x_i \in \mathsf{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0$ and

- if j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.
- if $j \neq 0$, $k = 0$, we have $D'_{i,0} = (1,<)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,i} + (1,<)$; from Definition 3.5 (4) and Lemma 3.7 we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p. Since, from Definition 3.3 (2b) $(1,<) \leq (1,<)$ is valid for R_p, then from Lemma 3.4

$$(1,<) \leq D_{i,j} + D_{j,i} + (1,<)$$

is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if $j = 0$, $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1,<)$ and $D'_{0,k} = D_{0,k} + (0,<)$; we claim that

$$D_{i,k} \leq (1,<) + D_{0,k} + (0,<)$$

is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p; moreover, from Definition 3.5 (1), we know that $D_{i,0} \leq (1,<)$ is valid for R_p. We have

$$(1,<) + (0,<) = (1 + 0, < \oplus <) = (1,<)$$

so we obtain that

$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1,<) + D_{0,k} = (1,<) + D_{0,k} + (0,<)$$

is valid for R_p and therefore our result.
- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (1,<)$, $D'_{0,0} = (0, \leq)$; from Definition 3.3 (2b) we have that

$$(1,<) \leq (1,<)$$

is valid for R_p. Hence from Lemma 3.8

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p.

(6) if $x_i, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and

- if $j \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1,<)$ and $D'_{0,k} = D_{0,k} + (0,<)$; we claim that
$$D_{i,k} \leq (1,<) + D_{0,k} + (0,<)$$
is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p; moreover, from Definition 3.5 (1), we know that $D_{i,0} \leq (1,<)$ is valid for R_p. We have
$$(1,<) + (0,<) = (1+0, < \oplus <) = (1,<)$$
so we obtain that
$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1,<) + D_{0,k} = (1,<) + D_{0,k} + (0,<)$$
is valid for R_p and therefore our result.

(7) if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and
- if $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $k = 0$, we have $D'_{i,0} = (1,<)$, $D'_{i,j} = D_{i,j} = (0,\leq)$ since both $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ (cf. $(\star)$) and $D'_{j,0} = (1,<)$; then $(1,<) \leq (0,\leq) + (1,<)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.

(8) if $x_i, x_j, x_k \in \mathsf{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

proof that Definition 3.5 (5b) holds

Finally, for $x_i \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = (1,<)$ and for all clock j s.t. $D'_{0,j} = (0, \lhd_{0j'})$, then we have $\lhd_{0j'} = <$. Condition Definition 3.5 (5b) is satisfied.

We denote by (E,D') the obtained p–PDBM and $(E,D') \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. □

G.0.2. → *Definition 3.5 type (5b) to (5a).*

Lemma G.2. *Let $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$; let $x_i \in \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$. If $(d_{i,j}, \lhd_{ij}) = (0, \lhd)$, then $\lhd = <$*

Proof. Let $x_i \in \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$. Suppose $(d_{i,j}, \lhd_{ij}) = (0, \leq)$. From Definition 3.5 (2) we should have that $(d_{j,i}, \lhd_{ji}) = (0, \leq)$ so Lemma 3.7 is satisfied, and then $x_j \in \mathsf{LFP}_{R_p}(D)$. □

Claim G.3 (modification of an open–p–PDBM respecting condition 5b under $TE_=$)**.** *Let R_p be a parameter region and $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respecting condition 5b, then $TE_=(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respecting condition 5a.*

Proof. Suppose $(E,D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respects condition (5a) of Definition 3.5 *i. e.,* we have at least an x s.t. $D_{x,0} = (1,<)$ and for all other j s.t. $D_{0,j} = (0, \lhd_{0j})$, $\lhd_{0j} = <$. First we can determine $\mathsf{LFP}_{R_p}(D)$. Let $x \in \mathsf{LFP}_{R_p}(D)$. If more than one clock belong to $\mathsf{LFP}_{R_p}(D)$ then their valuations have the same fractional part. Indeed, from Definition 4.5 if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ then both $(0,\leq) \leq D_{i,j}$ and $(0,\leq) \leq D_{j,i}$ are valid for R_p, and from Definition 3.5 (2) we must have $D_{i,j} = D_{j,i} = (0,\leq)$.

Let $v \in R_p$. Let $x_i \in \mathsf{LFP}_{R_p}(D)$ and $w \in (E, v(D))$. By letting time elapse, $\mathit{frac}(w(x))$ is the first to actually reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $\mathit{frac}(w(x_j))$ cannot reach 1 before $\mathit{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_{=}((E,D))$ which is an open–p–PDBM respecting condition 5b. While detailing the procedure of $TE_{=}$, we are going to prove that Definition 3.5 (1) and (2) hold for (E', D'). Further we will prove that (4) and (5a) also hold.

proof that Definition 3.5 (1) holds

According to the definition of $TE_{=}$ Algorithm 3, the first step is to fix the value of $\mathit{frac}(x_i)$ to 0 by setting

$$D'_{i,0} = (0,\leq) \quad \text{and} \quad D'_{0,i} = (0,\leq) \quad \text{for all } x_i \in \mathsf{LFP}_{R_p}(D).$$

Indeed, when $\mathit{frac}(x_i)$ reaches 1, in the constraints expressed by $(E, v(D))$ we have to increase the integer part by 1 and set the new constraints on the fractional part to 0.

Secondary we set new upper and lower bound for all other clock $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$

$$D'_{0,j} = D_{i,j} + (-1,\leq) \quad \text{and} \quad D'_{j,0} = D_{j,i} + (1,\leq).$$

We have to force now upper and lower bounds for other clocks since we know the interval of time that elapsed when x_i reached 1.

Note that since $x_i \in \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$ from Definition 4.5 we have that $(0,\leq) \leq D_{i,j} \leq (1,<)$ is valid for R_p for all clock x_j. Nonetheless, since $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, we even have $D_{i,j} \neq (0,\leq)$: suppose $(d_{i,j}, \lhd_{ij}) = (0,\leq)$: from Definition 3.5 (2) we should have that $(d_{j,i}, \lhd_{ji}) = (0,\leq)$ so Lemma 3.7 is satisfied, and then $x_j \in \mathsf{LFP}_{R_p}(D)$. The same reasoning leads to $D_{j,i} \neq (0,\leq)$.

Obviously, we have $D_{i,j} \neq (0,<)$: suppose $D_{i,j} = (0,<)$, since $x_i \in \mathsf{LFP}_{R_p}(D)$ then from Definition 4.5 $(0,\leq) \leq D_{i,j}$ should be valid for R_p, which is not from Definition 3.3 (2b).

Precisely, $d_{i,j} \in \{1, 1-p_1, p_2+1-p_1, p_1-p_2, p_1\}$ for any two $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p. Hence as $-1 + d_{i,j} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$, we have that $D'_{0,j} \in \mathcal{PLT}$ and $(-1,<) \leq D'_{0,j} \leq (0,\leq)$ is valid for $R_p$from Lemma G.2.

Also note that since $x_i \in \mathsf{LFP}_{R_p}(D)$, from Definition 4.5 and Definition 3.5 (2) we have that $(-1,<) \leq D_{j,i} \leq (0,\leq)$ is valid for R_p for all clock x_j. Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p. Hence as $d_{j,i}+1 \in \{1, 1-p_1, p_2+1-p_1, p_1-p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{PLT}$ and $(0,\leq) \leq D'_{j,0} \leq (1,<)$ is valid for R_p.

Clearly Definition 3.5 (1) holds.

proof that Definition 3.5 (2) holds

Third we set for all two clocks i,j where $x_i \in \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$

$$D'_{i,j} = D'_{0,j} \quad \text{and} \quad D'_{j,i} = D'_{j,0},$$

for all two clocks $x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$

$$D'_{j,k} = D_{j,k}$$

and for all two clocks $x, y \in \mathsf{LFP}_{R_p}(D)$

$$D'_{x,y} = D'_{y,x} = (0, \leq).$$

Here as we have already proven above that $(-1, <) \leq D'_{0,j} \leq (0, \leq)$ and $(0, \leq) \leq D'_{0,j} \leq (1, <)$ are valid for R_p, Definition 3.5 (2) holds.

proof that Definition 3.5 (3) holds

For all x_i:

- if $x_i \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = (0, \leq)$, $D'_{0,i} = (0, \leq)$ hence $d'_{i,0} = -d'_{0,i}$ and $\triangleleft_{i0'} \triangleleft_{0i'} = \leq$, condition Definition 3.5 (3) holds;
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{0,i} = D_{x,i} + (-1, \leq)$ as condition Definition 3.5 (3) holds for $D_{i,x}$ and $D_{x,i}$ and $\triangleleft_{ij} \oplus \leq = \triangleleft_{ij}$, $\triangleleft_{ji} \oplus \leq = \triangleleft_{ji}$, condition Definition 3.5 (3) holds for $D'_{i,0}$ and $D'_{0,i}$.

For all x_i, x_j:

- if $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition Definition 3.5 (3) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j} + (1, \leq)$, $D'_{j,i} = D_{j,i} + (-1, \leq)$ condition Definition 3.5 (3) holds for $D_{i,j}$ and $D_{j,i}$ and $\triangleleft_{ij} \oplus \leq = \triangleleft_{ij}$, $\triangleleft_{ji} \oplus \leq = \triangleleft_{ji}$, condition Definition 3.5 (3) holds for $D'_{i,j}$ and $D'_{j,i}$. The case $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_i \in \mathsf{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\triangleleft_{ij'} \triangleleft_{ji'} = \leq$ and condition Definition 3.5 (3) holds.

proof that Definition 3.5 (4) holds

Now we prove that Definition 3.5 (4) holds, *i. e.*, for all clocks x_i, x_j, x_k, valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p. This is not trivial since, in this construction some clocks have been updated. Precisely, for all clocks x_i, x_j, x_k, are valid for R_p:

(1) if $x_i, x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: let $x \in \mathsf{LFP}_{R_p}(D)$ and
  - if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
  - if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p; then from Lemma 3.4 $D_{i,x} + (1, \leq) \leq D_{i,j} + D_{j,x} + (1, \leq)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
  - if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, \leq)$ and $D'_{0,k} = D_{x,k} + (-1, \leq)$; we claim that

    $$D_{i,k} \leq D_{i,x} + (1, \leq) + D_{x,k} + (-1, \leq) \tag{G.7}$$

    is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. We have

    $$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq) \tag{G.8}$$

    Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p; combining with (G.8) and since $D_{x,k} + (0, \leq) = D_{x,k}$, we obtain (G.7) and therefore our result.

- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1, \leq)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from Definition 3.3 (2b) that

$$D_{i,x} + (1, \leq) \leq D_{i,x} + (1, \leq)$$

is valid for R_p. Hence Lemma 3.8 gives that

$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$

is valid for R_p.
- if j, k are different from 0, $i = 0$, we have $D'_{0,k} = D_{x,k} + (-1, \leq)$, $D'_{0,j} = D_{x,j} + (-1, \leq)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{x,k} \leq D_{x,j} + D_{j,k}$ is valid for R_p. Moreover we have that

$$(-1, \leq) \leq (-1, \leq)$$

is valid for R_p so we have from Definition 3.3 (2b) and Lemma 3.4

$$D_{x,k} + (-1, \leq) \leq D_{x,j} + (-1, \leq) + D_{j,k}$$

is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if j is different from 0, $i = k = 0$, we have $D'_{0,j} = D_{x,j} + (-1, \leq)$ and $D'_{j,0} = D_{j,x} + (1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Lemma 3.7 we know that $(0, \leq) \leq D_{x,j} + D_{j,x}$ is valid for R_p; moreover, we have that

$$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq)$$

and $D_{j,x} + (0, \leq) = D_{j,x}$. Then we have from Lemma 3.4

$$(0, \leq) \leq D_{x,j} + (-1, \leq) + D_{j,x} + (1, \leq)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.
- if k is different from 0, $i = j = 0$, we have $D'_{0,k} = D_{x,k} + (-1, \leq)$, $D'_{i,j} = D'_{0,0} = (0, \leq)$; we have from Definition 3.3 (2b) that

$$D_{x,k} + (-1, \leq) \leq D_{x,k} + (-1, \leq)$$

is valid for R_p. Hence, as $D_{x,k} + (-1, \leq) + (0, \leq) = D_{x,k} + (-1, \leq)$ we have

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p.
- if $i = j = k = 0$, we trivially have from Definition 3.5 (4) and Lemma 3.8

$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$

is valid for R_p.

(2) if $x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $k \neq 0$ and
- if i, j are different from 0, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (1, \leq)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D'_{j,0} = D_{j,k} + (1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; moreover, since we have $(1, \leq) \leq (1, \leq)$ is valid for R_p then from Lemma 3.4

$$D_{i,k} + (1, \leq) \leq D_{i,j} + D_{j,k} + (1, \leq)$$

is valid for R_p, therefore we have $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

- if $i \neq 0$, $j = 0$, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (1, \leq)$, $D'_{i,0} = D_{i,k} + (1, \leq)$ and $D'_{0,k} = (0, \leq)$; clearly
$$(1, \leq) \leq (1, \leq) + (0, \leq)$$
and
$$D_{i,k} \leq D_{i,k}$$
are valid for R_p, then from Lemma 3.4 we obtain $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p.
- if $i = 0$, $j \neq 0$, we have $D'_{0,k} = (0, \leq)$, $D'_{0,j} = D_{k,j} + (-1, \leq)$ and $D'_{j,k} = D'_{j,0} = D_{j,k} + (1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Lemma 3.7 we know that $(0, \leq) \leq D_{k,j} + D_{j,k}$ is valid for R_p. Moreover we have that
$$(1, \leq) + (-1, \leq) = (1 + -1, \leq \oplus \leq) = (0, \leq)$$
so we have from Lemma 3.4
$$(0, \leq) \leq D_{k,j} + D_{j,k} + (0, \leq)$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if $i = j = 0$, we trivially have from Definition 3.5 (4) and Lemma 3.8
$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$
is valid for R_p.

(3) if $x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0$ and
- if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D'_{i,0} = D_{i,j} + (1, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{j,k} + (-1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; moreover, since we have
$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$
then as $D_{i,j} + D_{j,k} + (0, \leq) = D_{i,j} + D_{j,k}$, clearly $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $i \neq 0$, $k = 0$, we have $D'_{i,0} = D_{i,j} + (1, \leq)$, $D'_{i,j} = D'_{i,0} = D_{i,j} + (1, \leq)$ and $D'_{j,0} = (0, \leq)$; From Definition 3.3 (2b) we trivially have that $D_{i,j} + (1, \leq) \leq D_{i,j} + (1, \leq)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if $i = 0$, $k \neq 0$, we have $D'_{0,k} = D_{j,k} + (-1, \leq)$, $D'_{0,j} = (0, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{j,k} + (-1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. From Definition 3.3 (2b) we trivially have that $D_{j,k} + (-1, \leq) \leq D_{j,k} + (-1, \leq)$ is valid for R_p. As $(-1, \leq) + (0, \leq) = (-1, \leq)$, we have $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if $i = k = 0$, we have $D'_{0,j} = (0, \leq)$ and $D'_{j,0} = (0, \leq)$; As we have
$$(0, \leq) + (0, \leq) = (0, \leq)$$
we clearly have that $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.

(4) if $x_j, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and
- if i is different from 0, we have $D'_{i,k} = D'_{i,0} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = D'_{i,0} = D_{i,k} + (-1, \leq)$ and $D'_{j,k} = (0, \leq)$; we have that $(-1, \leq) + (0, \leq) = (-1, \leq)$ and
$$D_{i,k} + (-1, \leq) \leq D_{i,k} + (-1, \leq)$$
holds from Definition 3.3 (2b). Therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$.

- if $i = 0$, we have $D'_{0,k} = (0, \leq)$, $D'_{0,j} = (0, \leq)$ and $D'_{j,k} = (0, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ from Definition 3.5 (4), we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$. As we have
$$(0, \leq) + (0, \leq) = (0, \leq)$$
we clearly have that $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.

(5) if $x_i \in \mathsf{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0$ and
- if j, k are different from 0, we have $D'_{i,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Definition 3.5 (4) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; moreover, since we have
$$(-1, \leq) \leq (-1, \leq)$$
is valid for R_p then from Lemma 3.4 we have $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $j \neq 0$, $k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,0} = D_{j,i} + (1, \leq)$; from Lemma 3.7 we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p. Moreover, we have
$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$
then
$$(0, \leq) \leq D_{i,j} + D_{j,i} + (0, \leq)$$
is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if $j = 0$, $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, \leq)$ and $D'_{0,k} = D_{i,k} + (-1, \leq)$; we have that
$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$
and from Definition 3.3 (2b) that
$$D_{i,k} \leq D_{i,k} + (0, \leq)$$
is valid for R_p, which gives us our result.
- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from Definition 3.3 (2b) that
$$(0, \leq) \leq (0, \leq)$$
is valid for R_p. Hence
$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$
is valid for R_p.

(6) if $x_i, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and
- if $j \neq 0$, we have $D'_{i,k} = (0, \leq)$, $D'_{i,j} = D'_{0,j} = D_{i,j} + (-1, \leq)$ and $D'_{j,k} = D'_{j,0} = D_{j,i} + (1, \leq)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$, from Lemma 3.7 we know that $(0, \leq) \leq D_{i,j} + D_{j,i}$ is valid for R_p; we have
$$(1, \leq) + (-1, \leq) = (1 + (-1), \leq \oplus \leq) = (0, \leq)$$
and therefore from Lemma 3.4, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $j = 0$, we have $D'_{i,k} = (0, \leq)$, $D'_{i,0} = (0, \leq)$ and $D'_{0,k} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from Definition 3.3 (2b)
$$(0, \leq) \leq (0, \leq)$$
is valid for R_p. Therefore we obtain our result.

(7) if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and

- if $k \neq 0$, we have $D'_{i,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,k} = D'_{0,k} = D_{i,k} + (-1, \leq)$; we have that
$$D_{i,k} \leq D_{i,k}$$
is valid for R_p and from Lemma 3.4
$$(-1, \leq) \leq (-1, \leq)$$
is valid for R_p. Therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $k = 0$, we have $D'_{i,0} = (0, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,0} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from Definition 3.3 (2b)
$$(0, \leq) \leq (0, \leq)$$
is valid for R_p: therefore $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.

(8) if $x_i, x_j, x_k \in \mathsf{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = (0, \leq)$, $D'_{i,j} = (0, \leq)$ and $D'_{j,k} = (0, \leq)$; we have that $(0, \leq) + (0, \leq) = (0, \leq)$ and from Definition 3.3 (2b)
$$(0, \leq) \leq (0, \leq)$$
is valid for R_p: therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

proof that Definition 3.5 (5a) holds

Finally, there is at least one clock $x_i \in \mathsf{LFP}_{R_p}(D)$ s.t. $D_{0,i} = D_{i,0} = (0, \leq)$. Hence condition Definition 3.5 (5a) holds.

To conclude the proof, we set $E'_i = E_i + 1$ if $x_i \in \mathsf{LFP}_{R_p}(D)$ and $E'_j = E_j$ if $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$ We denote by (E, D') the obtained p–PDBM and $(E', D') \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$. $\square$

G.0.3. $\rightarrow$ *Definition 3.6 to Definition 3.5 type (5a).*

Claim G.4 ($p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ becomes $p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ after $TE_<$)**.** *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, then $TE_<\big((E, D)\big) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ respecting condition 5b.*

Proof. Suppose $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Since, in R_p, we know which parameters have the largest fractional part, we can determine $\mathsf{LFP}_{R_p}(D)$ from Lemma 4.6. If more than one clock belong to $\mathsf{LFP}_{R_p}(D)$ then their valuations have the same fractional part.

Indeed, from Definition 4.5 if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ then both $(0, \leq) \leq D_{i,j}$ and $(0, \leq) \leq D_{j,i}$ are valid for R_p, and from Definition 3.5 (2) we must have $D_{i,j} = D_{j,i} = (0, \leq)$.

Let $v \in R_p$. Let $x_i \in \mathsf{LFP}_{R_p}(D)$ and $w \in (E, v(D))$. By letting time elapse, $\mathit{frac}(w(x_i))$ is the first that might reach 1. Moreover, for all $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $\mathit{frac}(w(x_j))$ cannot reach 1 before $\mathit{frac}(w(x_i))$. We are going to construct a new $(E', D') = TE_<((E, D))$ which is an open–p–PDBM respecting condition 5b. While detailing the procedure of $TE_<$, we are going to prove that Definition 3.5 (1) and (2) hold for (E', D'). Further we will prove that (4) and (5b) also hold.

proof that Definition 3.5 (1) holds

According to the definition of $TE_<$ (Algorithm 2) the first step is to set a new upper bound
$$D'_{i,0} = (1, <) \quad \text{for all } x_i \in \mathsf{LFP}_{R_p}(D)$$

and obviously $(0, \leq) \leq D'_{i,0} \leq (1, <)$ is valid for R_p. Then we set new upper bounds for all other clock $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$ by setting

$$D'_{j,0} = D_{j,i} + (1, <).$$

Indeed, $D_{j,i}$ is the constraint on the lower bound of $w(x_j) - w(x_i)$ and since the upper bound of x_i has increased, this gives the new upper bound of x_j. Note that since $x_i \in \mathsf{LFP}_{R_p}(D)$, from Definition 4.5 we have for all clock x_j that $(-1, <) \leq D_{j,i} \leq (0, \leq)$ is valid for R_p. Precisely, $d_{j,i} \in \{0, -p_1, p_2 - p_1, p_1 - 1 - p_2, p_1 - 1\}$ for some $p_1, p_2 \in \mathbb{P}$ where $p_2 \leq p_1$ is valid for R_p. Hence as $d_{j,i}+1 \in \{1, 1-p_1, p_2+1-p_1, p_1-p_2, p_1\}$, we have that $d'_{j,0} \in \mathcal{PLT}$, $\lhd_{j0'} = \lhd_{j0'} \oplus < = <$ and $(0, \leq) \leq D'_{j,0} \leq (1, <)$ is valid for R_p. Note that we cannot have $(d_{j,i}, \lhd_{ji}) = (-1, <)$ because even if $(d_{i,j}, \lhd_{ij}) = (1, <)$, since $(E, D) \in p\text{–}\mathcal{PDBM}_{\blacksquare}(R_p)$ we do not have have $0 \leq D_{j,i} + D_{i,j}$ is valid for R_p from Definition 3.5 (4) and Lemma 3.7.

Secondary we set for all clock x regardless of whether they are in $\mathsf{LFP}_{R_p}(D)$

$$D'_{0,x} = D_{0,x} + (0, <).$$

Since some time elapsed, lower bounds of all clocks are increased. Moreover, from Definition 3.6 (1) as $(-1, <) \leq D_{0,x} \leq (0, \leq)$ is valid for R_p, $(-1, <) \leq D'_{0,x} \leq (0, \leq)$ is also valid for R_p.

proof that Definition 3.5 (2) holds

Third we set for all clocks x, y regardless of whether they are in $\mathsf{LFP}_{R_p}(D)$

$$D'_{x,y} = D_{x,y}$$

since no fractional part has reached 1, constraints on differences of clocks and integer parts remain unchanged. As it is the case in (E, D), Definition 3.5 (2) holds.

proof that Definition 3.5 (3) holds

For all x_i:

- if $x_i \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence $d'_{i,0} \neq d'_{0,i}$ and $\lhd_{i0'} \lhd_{0i'} = <$, condition Definition 3.5 (3) holds;
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p, we have $d'_{i,0} \neq d'_{0,i}$ and $\lhd_{i0'} \lhd_{0i'} = <$ and condition Definition 3.5 (3) holds.

For all x_i, x_j:

- if $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,i} = D_{j,i}$, condition Definition 3.5 (3) holds as it holds for $D_{i,j}$ and $D_{j,i}$.
- if $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{0,i} = D_{0,i} + (0, <)$ hence as $(0, \leq) \leq D'_{i,0}$ is valid for R_p and $D'_{0,i} \leq (0, \leq)$ is valid for R_p, we have $d'_{i,0} \neq d'_{0,i}$ and $\lhd_{i0'} \lhd_{0i'} = <$, condition Definition 3.5 (3) holds. The case $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $x_i \in \mathsf{LFP}_{R_p}(D)$ is treated similarly.
- if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,j} = D'_{j,i} = (0, \leq)$, hence $d'_{i,j} = -d'_{j,i} = 0$ and $\lhd_{ij'} \lhd_{ji'} = \leq$ and condition Definition 3.5 (3) holds.

proof that Definition 3.5 (4) holds

Now we prove that Definition 3.5 (4) holds, *i. e.,* for all clocks x_i, x_j, x_k valid conditions such as $D'_{i,j} \leq D'_{i,k} + D'_{k,j}$ remain valid in R_p. Indeed, when time elapses, all clocks have the same behavior, hence the difference between two clocks does not change without an update. Precisely, for all clocks x_i, x_j, x_k, are valid for R_p:

(1) if $x_i, x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: let $x \in \mathsf{LFP}_{R_p}(D)$ and

- if i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if i, j are different from 0, $k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,x} \leq D_{i,j} + D_{j,x}$ is valid for R_p; then from Lemma 3.4 $D_{i,x} + (1, <) \leq D_{i,j} + D_{j,x} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if i, k are different from 0, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,x} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that
$$D_{i,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \tag{G.9}$$
is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (1) we know that
$$D_{x,0} \leq (1, <) \tag{G.10}$$
is valid for R_p; moreover we have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \tag{G.11}$$
Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{x,k} \leq D_{x,0} + D_{0,k}$ is valid for R_p; combining with (G.10) and (G.11) we obtain $D_{x,k} \leq (1, <) + D_{0,k} + (0, <)$ is valid for R_p. As $D_{i,x} \leq D_{i,x}$ is valid for R_p, using Lemma 3.4 we obtain
$$D_{i,x} + D_{x,k} \leq D_{i,x} + (1, <) + D_{0,k} + (0, <) \tag{G.12}$$
is valid for R_p. Now, since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,x} + D_{x,k}$ is valid for R_p and combining with (G.12) we obtain (G.9) and therefore our result.
- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = D_{i,x} + (1, <)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; we have from Definition 3.3 (2b) that
$$D_{i,x} + (1, <) \leq D_{i,x} + (1, <)$$
is valid for R_p. Hence
$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$
is valid for R_p.
- if j, k are different from 0, $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that
$$D_{0,k} + (0, <) = (d_{0,k}, <) \quad \text{and} \quad D_{0,j} + (0, <) + D_{j,k} = (d_{0,j} + d_{j,k}, <)$$
so we have from Lemma 3.4
$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.

- if j is different from 0, $i = k = 0$, we have $D'_{i,k} = D'_{0,0} = (0, \leq)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = D_{j,x} + (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{0,x} \leq D_{0,j} + D_{j,x}$ is valid for R_p; moreover from Lemma 3.4,
$$D_{0,x} + (0, <) \leq D_{0,j} + (0, <) + D_{j,x}$$
is valid for R_p. As we have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$
we obtain from Lemma 3.4 that
$$D_{0,x} + (1, <) \leq D_{0,j} + D_{j,x} + (1, <)$$
is valid for R_p. Recall that from Lemma 3.7 $(0, \leq) \leq D_{0,x} + D_{x,0}$ is valid for R_p. Since from Definition 3.6 (1) $D_{x,0} \leq (1, <)$ is valid for R_p, we have $(0, \leq) \leq D_{0,x} + (1, <)$ is valid for R_p. Therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.
- if k is different from 0, $i = j = 0$, we have $D'_{i,k} = D'_{j,k} = D'_{0,k} = D_{0,k} + (0, <)$, $D'_{i,j} = D'_{0,0} = (0, \leq)$; we have from Definition 3.3 (2b) that
$$D_{0,k} + (0, <) \leq D_{0,k} + (0, <)$$
is valid for R_p. Hence from Lemma 3.8
$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$
is valid for R_p.
- if $i = j = k = 0$, we trivially have
$$D'_{0,0} \leq D'_{0,0} + D'_{0,0}$$
is valid for R_p.

(2) if $x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $k \neq 0$ and
- if i, j are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $i \neq 0$, $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = D_{i,k} + (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that $D_{i,k} \leq D_{i,k} + (1, <) + D_{0,k} + (0, <)$, *i. e.*,
$$0 \leq (1, <) + D_{0,k} + (0, <) \tag{G.13}$$
is valid for R_p, which is from Lemma 3.4 equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. We have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <). \tag{G.14}$$
Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $0 \leq D_{0,k} + D_{k,0}$ is valid for R_p and from Definition 3.6 (1) that $D_{k,0} \leq (1, <)$ is valid for R_p; combining with (G.14) we obtain (G.13) and therefore our result.
- if $i = 0$, $j \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have from Lemma 3.4
$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$
is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.

- if $i = j = 0$, from Definition 3.6 (2) we trivially have

$$D'_{0,k} \leq D'_{0,0} + D'_{0,k}$$

is valid for R_p.

(3) if $x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_i, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0$ and

- if i, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j}+D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $i \neq 0$, $k = 0$, we have $D'_{i,0} = D_{i,j} + (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = (1, <)$; from Definition 3.3 (2b) we trivially have that $D_{i,j} + (1, <) \leq D_{i,j} + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.
- if $i = 0$, $k \neq 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

holds from Definition 3.3 (2b). Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.
- if $i = k = 0$, we have $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,0} = (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Lemma 3.7 we know that $0 \leq D_{0,j} + D_{j,0}$ is valid for R_p, from Definition 3.6 (1) we know that $D_{j,0} \leq 1$ is valid for R_p which means $0 \leq D_{0,j} + (1, <)$ is valid for R_p. As we have

$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$

we obtain that

$$(0, \leq) \leq D_{0,j} + (0, <) + (1, <)$$

is valid for R_p and therefore $D'_{0,0} \leq D'_{0,j} + D'_{j,0}$ is valid for R_p.

(4) if $x_j, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_i \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $j \neq 0, k \neq 0$ and

- if i is different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $i = 0$, we have $D'_{0,k} = D_{0,k} + (0, <)$, $D'_{0,j} = D_{0,j} + (0, <)$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{0,k} \leq D_{0,j} + D_{j,k}$ is valid for R_p. Moreover we have that $(0, <) \leq (0, <)$ is valid for R_p so we have from Lemma 3.4

$$D_{0,k} + (0, <) \leq D_{0,j} + (0, <) + D_{j,k}$$

is valid for R_p. Hence $D'_{0,k} \leq D'_{0,j} + D'_{j,k}$ is valid for R_p.

(5) if $x_i \in \mathsf{LFP}_{R_p}(D)$ and $x_j, x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0$ and

- if j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j}+D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $j \neq 0$, $k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{i,j} = D_{i,j}$ and $D'_{j,0} = D_{j,i} + (1, <)$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Lemma 3.7 we know that $0 \leq D_{i,j} + D_{j,i}$. Then from Lemma 3.4

$$(1, <) \leq D_{i,j} + D_{j,i} + (1, <)$$

is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.

- if $j = 0$, $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that
$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$
is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p; moreover, from Definition 3.6 (1), we know that $D_{i,0} \leq (1, <)$ is valid for R_p. We have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$
We obtain that
$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$
is valid for R_p and therefore our result.
- if i is different from 0, $j = k = 0$, we have $D'_{i,0} = (1, <)$, $D'_{j,k} = D'_{0,0} = (0, \leq)$; from Definition 3.3 (2b) we have that
$$(1, <) \leq (1, <)$$
is valid for R_p. Hence
$$D'_{i,0} \leq D'_{i,0} + D'_{0,0}$$
is valid for R_p.

(6) if $x_i, x_k \in \mathsf{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, k \neq 0$ and
- if $j \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $j = 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,0} = (1, <)$ and $D'_{0,k} = D_{0,k} + (0, <)$; we claim that
$$D_{i,k} \leq (1, <) + D_{0,k} + (0, <)$$
is valid for R_p, which is equivalent to $D'_{i,k} \leq D'_{i,0} + D'_{0,k}$ is valid for R_p. Since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$ from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,0} + D_{0,k}$ is valid for R_p; moreover, from Definition 3.6 (1), we know that $D_{i,0} \leq (1, <)$ is valid for R_p. We have
$$(1, <) + (0, <) = (1 + 0, < \oplus <) = (1, <)$$
We obtain that
$$D_{i,k} \leq D_{i,0} + D_{0,k} \leq (1, <) + D_{0,k} = (1, <) + D_{0,k} + (0, <)$$
is valid for R_p and therefore our result.

(7) if $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ and $x_k \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: $i \neq 0, j \neq 0$ and
- if $k \neq 0$, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2), we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.
- if $k = 0$, since both $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$ we have $D'_{i,j} = D_{i,j} = (0, \leq)$, $D'_{i,0} = (1, <)$ and $D'_{j,0} = (1, <)$; trivially $(1, <) \leq (0, \leq) + (1, <)$ is valid for R_p and therefore, $D'_{i,0} \leq D'_{i,j} + D'_{j,0}$ is valid for R_p.

(8) if $x_i, x_j, x_k \in \mathsf{LFP}_{R_p}(D)$: i, j, k are different from 0, we have $D'_{i,k} = D_{i,k}$, $D'_{i,j} = D_{i,j}$ and $D'_{j,k} = D_{j,k}$; since $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$, from Definition 3.6 (2) we know that $D_{i,k} \leq D_{i,j} + D_{j,k}$ is valid for R_p; therefore, $D'_{i,k} \leq D'_{i,j} + D'_{j,k}$ is valid for R_p.

proof that Definition 3.5 (5b) holds

Finally, for $x_i \in \mathsf{LFP}_{R_p}(D)$, $D'_{i,0} = (1, <)$ and for all clock j s.t. $D'_{0,j} = (0, \triangleleft)$, then we have $\triangleleft = <$. Condition Definition 3.5 (5b) is satisfied.

We set $E' = E$ and denote by (E, D') the obtained p–PDBM, which is $(E, D') \in$ p–$\mathcal{PDBM}_{\blacksquare}(R_p)$. □

□

Appendix H. Proof of Proposition 4.11

Proposition 4.11 (recalled). *Let R_p be a parameter region and $(E, D) \in$ p–$\mathcal{PDBM}(R_p)$. Let $v \in R_p$. There exists $w' \in TE((E, v(D)))$ iff there exist $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. Note that this proof is inspired by [HRSV02, Proof of Lemma 3.13]. We treat first open–p–PDBMs and then point–p–PDBMs.

Claim H.1. *Let $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$. If (E, D) satisfies condition Definition 3.5 (5b) it has been obtained after applying Algorithm 2 on another* open–p–*PDBM satisfying condition Definition 3.5 (5a) or a* point–p–*PDBM.*

Let $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$. If (E, D) satisfies condition Definition 3.5 (5a) it has been obtained after applying Algorithm 3 on another open–p–*PDBM satisfying condition Definition 3.5 (5b) or after a non-parametric update applied on another* open–p–*PDBM or a* point–p–*PDBM.*

Proof. Let $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$ and suppose (E, D) satisfies condition Definition 3.5 (5b). Since for all y, if $d_{0,y} = 0$ we have $\triangleleft_{0y} = <$, from Claim D.1 and Claim D.2 it cannot be the result of a non-parametric update where there is at least a clock x update and $D_{x,0} = D_{0,x} = (0, \leq)$. From Claim G.3 it cannot be the result of Algorithm 3, as there must be at least a clock x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$. Then it is the result either from Claim G.1 of Algorithm 2 applied on an open–p–PDBM satisfying condition Definition 3.5 (5a), or from Claim G.4 of Algorithm 2 applied on a point–p–PDBM.

Let $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$ and suppose (E, D) satisfies condition Definition 3.5 (5a). Since there is at least a clock y s.t. $D_{y,0} = D_{0,y} = (0, \leq)$, from Claims G.1 and G.4 it cannot be the result of Algorithm 2, as for all x, if $d_{0,x} = 0$ we must have $\triangleleft_{ox} = <$. Then it is the result of either from Claim G.3 of Algorithm 3 applied on an open–p–PDBM satisfying condition Definition 3.5 (5b) or from Claims D.1 and D.2 of Algorithm 1 applied on an open–p–PDBM or a point–p–PDBM. □

Let R_p be a parameter region and $(E, D) \in p$–$\mathcal{PDBM}(R_p)$. We have to consider two different cases: $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$ and $(E, D) \in p$–$\mathcal{PDBM}_{\odot}(R_p)$.

Claim H.2. *Let R_p be a parameter region and $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$. There is $w' \in TE((E, v(D)))$ iff there is $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. Let R_p a parameter region and $(E, D) \in p$–$\mathcal{PDBM}_{\blacksquare}(R_p)$. Let $v \in R_p$.

$\Longrightarrow$ **open–p–PDBM respecting Definition 3.5 (5a)**

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition Definition 3.5 (5a), *i. e.,* suppose there is x_i s.t. $D'_{i,0} = -D'_{0,i} = (0, \leq)$. Let $w' \in (E', v(D'))$, for this x_i we have $w'(x_i) = 0$. We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$\mathit{frac}(w'(x_j)) - \delta - \mathit{frac}(w'(x_i)) + \delta \lhd_{ji} v(d_{j,i})$$

and

$$\mathit{frac}(w'(x_i)) - \delta - \mathit{frac}(w'(x_j)) + \delta \lhd_{ij} v(d_{i,j})$$

and

$$-\mathit{frac}(w'(x_j)) + \delta \lhd_{0j} v(d_{0,j}) \quad \text{and} \quad \mathit{frac}(w'(x_j)) - \delta \lhd_{j0} v(d_{j,0}).$$

In this proof we are going to define a δ which is different from 0, and give it an upper bound in order to show that constraints in (E, D) are satisfied while going backward of δ units of time from w'.

First we will prove that for all clock j, its constraints of lower bound $D_{0,j}$ and upper bound $D_{j,0}$ are satisfied. Second we will prove that for all i, bounds on their difference $D_{i,j}$ and $D_{j,i}$ are also satisfied.

We want to show that we have to go a little backward in time from w' to ensure the upper bounds $D_{j,0}$ of (E, D) hold. For this purpose, we are going to prove that for all x_j

$$D_{j,0} \leq D'_{j,0}$$

is valid for R_p. Intuitively this means upper bounds of clocks in (E', D') are greater than in (E, D), which is consistent as time is elapsing.

As (E', D') respects Definition 3.5 (5a) and precisely $(E', D') = TE_{=}((E, D))$, we know (E, D) is respecting condition Definition 3.5 (5b) from Claim G.1. As $\mathit{frac}(w'(x_i)) = 0$ it was in (E, D) a clock with the largest fractional part, *i. e.,* $x_i \in \mathsf{LFP}_{R_p}(D)$ and $D_{i,0} = (1, <)$.

By definition of $TE_<$ (cf. Algorithm 2), in (E, D) which is the open–p–PDBM obtained after the application of $TE_<$ on another p–PDBM (see Claim H.1), for each $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $D_{j,0} = D_{j,i} + (1, <)$ and for all $x_j \in \mathbb{X}$, we have $D_{j,0}$ is of the form $(d_{j,0}, <)$ for some $d_{j,0}$.

By definition of $TE_=$ applied to (E, D) (cf. Algorithm 3), in (E', D'), for each $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $D'_{j,0} = D_{j,i} + (1, \leq)$, *i. e.,* $d_{j,0} = d'_{j,0}$. Hence by Definition 3.3 (2b) and as $\lhd_{j0'}$ is either $\leq$ or $<$, we have

$$(d_{j,0}, <) = D_{j,0} \leq D'_{j,0} = (d_{j,0}, \lhd_{j0'})$$

is valid for R_p. Next we define the largest amount of time so that all upper bounds of (E, D) are satisfied.

We claim that for all x_j, $\mathit{frac}(w'(x_j)) - v(d_{j,0}) \leq 0$. Indeed, remark that by applying Algorithm 2 then 3, constraints on upper bounds of clocks in (E, D) and (E', D') differ only by their $\lhd$. As for $i \in \mathsf{LFP}_{R_p}(D)$ and $j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$ it we have $D_{j,0} = D_{j,i} + (1, <)$ in (E, D) and $D'_{j,0} = D_{j,i} + (1, \leq)$ in (E', D'), so $d_{j,0} = d'_{j,0}$. Since for any x, its fractional part is less or equal to its upper bound in D and therefore in D', any difference between a fractional part and its upper bound is either negative or null. For all x, since $\mathit{frac}(w'(x)) \lhd_{x0'} v(d'_{x,0})$ we have $\mathit{frac}(w'(x)) - v(d'_{x,0}) \lhd_{x0'} 0$. Since $v(d'_{x,0}) = v(d_{x,0})$, $\mathit{frac}(w'(x)) - v(d_{x,0}) \lhd_{x0'} 0$, therefore we have our result.

Now we claim that we have to go at least an $\epsilon > 0$ backward in time to ensure all bounds of (E, D) are met. Let $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$. As

$$frac(w'(x_j)) \triangleleft_{j0'} v(d_{j,0})$$

we have

– either $\triangleleft_{j0'} = <$ and we already have $frac(w'(x_j)) < v(d_{j,0})$,

– or $\triangleleft_{j0'} = \leq$ and for any $\epsilon > 0$ we have $frac(w'(x_j)) - \epsilon < v(d_{j,0})$.

It is also true for each $x_i \in \mathsf{LFP}_{R_p}(D)$: after applying $TE_<$ recall that we have $D_{i,0} = (1, <)$. We can take $\epsilon > 0$ and define $frac(w(x_i)) = 1 - \epsilon$, so we have $frac(w(x_i)) < v(d_{i,0})$.

Now that we know we have to go a little backward in time (at least an $\epsilon > 0$) so upper bounds of (E, D) are satisfied, we are going to give an upper bound to ϵ so that all lower bounds $D_{0,j}$ of (E, D) are also satisfied.

Let

$$t_1 = \min_{x \in \mathbb{X}} \{frac(w'(x)) + v(d_{0,x})\}$$

We want to prove that $t_1 > 0$.

Let us prove that for all x_j, $D'_{0,j} \leq D_{0,j}$ is valid for R_p. Recall that for $x_i \in \mathsf{LFP}_{R_p}(D)$, we have that $D_{i,0} = (1, <)$. Moreover, from Definition 3.5 (4) $D_{i,j} \leq D_{i,0} + D_{0,j}$ is valid for R_p, then we have

$$D_{i,j} \leq (1, <) + D_{0,j}$$

is valid for R_p. Recall that after applying Algorithm 3, $D'_{0,j} = D_{i,j} + (-1, \leq)$. By Definition 3.3 (2b) we have $(-1, \leq) \leq (-1, \leq)$. We invoke Lemma 3.4 which gives

$$D_{i,j} + (-1, \leq) \leq (1, <) + D_{0,j} + (-1, \leq) = D_{0,j} + (0, <) \text{ is valid for } R_p. \tag{H.1}$$

As, from Definition 3.3 (2b) we have $D_{0,j} + (0, <) \leq D_{0,j}$ is valid for R_p, we infer (H.1) and it gives

$$D'_{0,j} \leq D_{0,j} \text{ is valid for } R_p.$$

Since $w' \in (E', v(D'))$ we have $-frac(w'(x_j)) \triangleleft_{0j'} v(d'_{0,j})$,

$$0 \triangleleft_{0j'} frac(w'(x_j)) + v(d'_{0,j}).$$

Then we have that

$$0 \triangleleft_{0j'} frac(w'(x_j)) + v(d'_{0,j}) \leq frac(w'(x_j)) + v(d_{0,j})$$

where,

– either from Definition 3.3 (2a) $d'_{0,j} < d_{0,j}$;

– or from Definition 3.3 (2b), $d'_{0,j} \leq d_{0,j}$ and then $\triangleleft_{0j'} = \triangleleft_{0j} = <$. Indeed as $D'_{0,j} \leq D_{0,j}$ is valid for R_p, and since (E, D) is the open–p–PDBM obtained after the application of $TE_<$ (cf. Algorithm 2) on another p–PDBM (see Claim H.1), we have $\triangleleft_{0j} = <$.

To conclude we have that for all x_j either

$$0 \triangleleft_{0j'} frac(w'(x_j)) + v(d'_{0,j}) < frac(w'(x_j)) + v(d_{0,j})$$

or

$$0 < frac(w'(x_j)) + v(d'_{0,j}) \leq frac(w'(x_j)) + v(d_{0,j}).$$

As t_1 is by definition the minimum value of an expression $frac(w'(x_j)) + v(d_{0,j})$ for a given x_j, which as we just proved are all strictly positive, we have that for all x_j

$$0 < t_1 \leq frac(w'(x_j)) + v(d_{0,j}).$$

We proved that $t_1 > 0$, so we can set $\delta = \frac{t_1}{2}$ (therefore $\delta > 0$).

More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.

Now we are going to prove that for any clock x_j, its constraints on lower and upper bounds are satisfied, *i. e.,*

$$-v(d_{0,j}) \lhd_{0j} \mathit{frac}(w'(x_j)) - \delta \lhd_{j0} v(d_{j,0}).$$

First as $\delta < t_1$, we have

$$-\mathit{frac}(w'(x_j)) + \delta < -\mathit{frac}(w'(x_j)) + t_1 \leq -\mathit{frac}(w'(x_j)) + \mathit{frac}(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) < \mathit{frac}(w'(x_j)) - \delta$. Since (E, D) is the open–p–PDBM obtained after the application of $TE_<$ (cf. Algorithm 3) on another p–PDBM (see Claim H.1), we have $\lhd_{0j} = <$ so $-v(d_{0,j}) \lhd_{0j} \mathit{frac}(w'(x_j)) - \delta$. Secondary as $0 < \delta$, we have

$$\mathit{frac}(w'(x_j)) - \delta < \mathit{frac}(w)'(x_j) - 0 \leq \mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $\mathit{frac}(w'(x_j)) - \delta < v(d_{j,0})$. Since (E, D) is the open–p–PDBM obtained after the application of $TE_<$ (cf. Algorithm 3) on another p–PDBM (see Claim H.1), we have $\lhd_{j0} = <$ so $\mathit{frac}(w'(x_j)) - \delta \lhd_{j0} v(d_{j,0})$

Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w'.

Recall that in (E', D') we have for all clock x_j,

$$D'_{j,i} = D'_{j,0} = D_{j,i} + 1 \quad \text{and} \quad D'_{i,j} = D'_{0,j} = -1 + D_{i,j}.$$

In addition by definition of $TE_=$, for $x_i \in \mathsf{LFP}_{R_p}(D)$, $E_{x_i} = E'_{x_i} - 1$ and for $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$, $E_{x_j} = E'_{x_j}$.

We already treated the case whether i or j are 0, now suppose i, j are both different from 0.

- if $x_i, x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: let $x \in \mathsf{LFP}_{R_p}(D)$ and recall that after applying Algorithm 3, $D'_{i,j} = D_{i,j}$, $D'_{j,i} = D_{j,i}$; we have that $\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) \lhd_{ij'} d'_{j,i} = d_{j,i}$, and therefore $\mathit{frac}(w'(x_j)) - \delta - \mathit{frac}(w'(x_i)) + \delta \lhd_{ji} d_{j,i}$.
  We also have that $\mathit{frac}(w'(x_i)) - \mathit{frac}(w'(x_j)) \lhd_{ij'} d'_{i,j} = d_{i,j}$, therefore $\mathit{frac}(w'(x_i)) - \delta - \mathit{frac}(w'(x_j)) + \delta \lhd_{ij} d_{i,j}$;
- if $x_i \in \mathsf{LFP}_{R_p}(D)$ and $x_j \in \mathbb{X} \setminus \mathsf{LFP}_{R_p}(D)$: recall that after applying Algorithm 3, $D'_{j,0} = D_{j,i} + (1, \leq)$, and $D'_{0,j} = D_{i,j} + (-1, \leq)$. Observe that as we added $\leq$ which is the neutral element of the addition $\oplus$ between two operators $\lhd$, we have $\lhd_{j0'} = \lhd_{ji}$ and $\lhd_{0j'} = \lhd_{ij}$. Note that as $x_i \in \mathsf{LFP}_{R_p}(D)$, in (E', D') we have $D'_{0,i} = (0, \leq) = D'_{i,0}$ which means $\mathit{frac}(w'(x_i)) = 0$. Going backward in time of δ units of time from $w'(x_i)$ means that $\mathit{frac}(w(x_i)) = 1 - \delta$.
  We have that
  $$\mathit{frac}(w'(x_j)) \lhd_{j0'} v(d'_{j,0}) = v(d_{j,i}) + 1$$
  hence $\mathit{frac}(w'(x_j)) - 1 \lhd_{ji} v(d_{j,i})$ which is equivalent to
  $$\mathit{frac}(w'(x_j)) - \delta - 1 + \delta \lhd_{ji} v(d_{j,i}).$$
  The same way we have
  $$-\mathit{frac}(w'(x_j)) \lhd_{0j'} v(d'_{0,j}) = v(d_{i,j}) - 1$$
  hence $1 - \mathit{frac}(w'(x_j)) \lhd_{ij} v(d_{i,j})$ which is equivalent to
  $$1 - \delta - \mathit{frac}(w'(x_j)) + \delta \lhd_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j s.t. $D'_{0,j} \neq (0, \leq)$ and $D'_{j,0} \neq (0, \leq)$

$$w(x_j) = w'(x_j) - \delta$$

and for all x_i s.t. $D'_{0,i} = (0, \leq) = D'_{i,0}$

$$w(x_i) = (w'(x_i) - 1) + 1 - \delta$$

and clearly, $w \in (E, v(D))$.

$\Longrightarrow$ open–p–**PDBM respecting Definition 3.5 (5b)**

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition Definition 3.5 (5b), *i. e.,* suppose there is at least an x_i s.t. $D'_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \triangleleft_{0j})$, then we have $\triangleleft_{0j} = <$. Let $w' \in (E', v(D'))$.

We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$\mathit{frac}(w'(x_j)) - \delta - \mathit{frac}(w'(x_i)) + \delta \triangleleft_{ji} v(d_{j,i})$$

and

$$\mathit{frac}(w'(x_i)) - \delta - \mathit{frac}(w'(x_j)) + \delta \triangleleft_{ij} v(d_{i,j})$$

and

$$-\mathit{frac}(w'(x_j)) + \delta \triangleleft_{0j} v(d_{0,j}) \quad \text{and} \quad \mathit{frac}(w'(x_j)) - \delta \triangleleft_{j0} v(d_{j,0}).$$

As done previously we are going to define a δ which is different from 0 so we satisfy condition Definition 3.5 (5a), and show that constraints in (E, D) are satisfied while going backward of δ units of time from w'.

We define the largest and the least amount of time so that all upper bounds of (E, D) are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, \mathit{frac}(w'(x)) - v(d_{x,0})\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\mathit{frac}(w'(x)) + v(d_{0,x})\}.$$

We want to prove that $t_0 = t_1 > 0$. For this purpose, let us first show that for all i, j we have $\mathit{frac}(w'(x_j)) - v(d'_{j,0}) \leq \mathit{frac}(w'(x_i)) + v(d'_{0,i})$, which is $t_0 \leq t_1$.

First note that for all i, j

$$\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) \triangleleft_{ji'} v(d'_{j,i}).$$

By applying $TE_<$ (Algorithm 2) to (E, D), we have that $D'_{j,i} = D_{j,i}$, *i. e.,* $(d_{i,j}, \triangleleft_{ij}) = (d'_{i,j}, \triangleleft_{ij'})$, and from Definition 3.5 (4) we have that $D_{j,i} \leq D_{j,0} + D_{0,i}$ is valid for R_p.

Hence, we have from Definition 3.3 (2b) that either $v(d_{j,i}) < v(d_{j,0}) + v(d_{0,i})$ or $v(d_{j,i}) \leq v(d_{j,0}) + v(d_{0,i})$ and $\triangleleft_{ji} = \triangleleft_{j0} \oplus \triangleleft_{0i}$ or $\triangleleft_{ji} = <$ and $\triangleleft_{j0} \oplus \triangleleft_{0i} = \leq$.

We can then write that

$$\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i))(\triangleleft_{j0} \oplus \triangleleft_{0i}) v(d_{j,0}) + v(d_{0,i})$$

which is equivalent to

$$\mathit{frac}(w'(x_j)) - v(d_{j,0})(\triangleleft_{j0} \oplus \triangleleft_{0i}) \mathit{frac}(w'(x_i)) + v(d_{0,i})$$

so we obtain our result, as $(\triangleleft_{j0} \oplus \triangleleft_{0i})$ is either $\leq$ or $<$.

Now, recall that (E, D) respects condition Definition 3.5 (5a) so we have at least an x s.t. $D_{x,0} = D_{0,x} = (0, \leq)$.

For this clock x we have that $\mathit{frac}(w'(x)) = \mathit{frac}(w'(x)) - v(d_{x,0}) \leq t_0$ and that $t_1 \leq \mathit{frac}(w'(x)) + v(d_{0,x}) = \mathit{frac}(w'(x))$.

Hence $t_0 = t_1 = \mathit{frac}(w'(x))$.

As $\lhd_{x0} = \leq$, we have $(\lhd_{x0} \oplus \lhd_{0i}) = \lhd_{0i}$ and $(\lhd_{j0} \oplus \lhd_{0x}) = \lhd_{j0}$, which gives

$$\mathit{frac}(w'(x)) = \mathit{frac}(w'(x)) - v(d_{x,0}) \lhd_{0i} \mathit{frac}(w'(x_i)) + v(d_{0,i})$$

and

$$\mathit{frac}(w'(x_j)) - v(d_{j,0}) \lhd_{j0} \mathit{frac}(w'(x)) + v(d_{0,x}) = \mathit{frac}(w'(x)).$$

Moreover in (E', D') we have that $\mathit{frac}(w'(x)) \lhd_{0x'} v(d'_{0,x})$. Since (E', D') respects condition Definition 3.5 (5b), if $D'_{0,x} = (0, \lhd_{0x'})$ then $\lhd_{0x'} = <$. Hence $0 < \mathit{frac}(w'(x))$ and

$$0 < t_0 = t_1.$$

Let $\delta = t_0 = t_1$. More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.

First we have

$$-\mathit{frac}(w'(x_j)) + \delta \leq -\mathit{frac}(w'(x_j)) + t_1 \lhd_{j0} -\mathit{frac}(w'(x_j)) + \mathit{frac}(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) \lhd_{j0} \mathit{frac}(w'(x_j)) - \delta$.

Secondary we have

$$\mathit{frac}(w'(x_j)) - \delta \leq \mathit{frac}(w)'(x_j) - t_0 \lhd_{0j} \mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $\mathit{frac}(w'(x_j)) - \delta \lhd_{0j} v(d_{j,0})$.

Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w'

Recall that in (E', D') from the definition of Algorithm 2 we have for all clocks x_i, x_j,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) \lhd_{ji'} v(d'_{j,i}) = v(d_{j,i})$, and therefore as $\lhd_{ji'} = \lhd_{ji}$,

$$\mathit{frac}(w'(x_j)) - \delta - \mathit{frac}(w'(x_i)) + \delta \lhd_{ji} v(d_{j,i}).$$

We also have that $\mathit{frac}(w'(x_i)) - \mathit{frac}(w'(x_j)) \lhd_{ij'} v(d'_{i,j}) = v(d_{i,j})$, therefore as $\lhd_{ij'} = \lhd_{ij}$,

$$\mathit{frac}(w'(x_i)) - \delta - \mathit{frac}(w'(x_j)) + \delta \lhd_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j

$$w(x_j) = w'(x_j) - \delta$$

and clearly, $w \in (E, v(D))$.

Conversely, let $w \in (E, v(D))$,

$\Longleftarrow$ open–p–**PDBM respecting Definition 3.5 (5b)**

Suppose in (E, D) there is at least an x_i s.t. $D_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \lhd)$, we have $\lhd = <$. Let x_i be such a clock and $v \in R_p$.

Now consider $(E', D') = \mathit{TE}((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\mathit{frac}(w(x_j)) + \delta - \mathit{frac}(w(x_i)) - \delta \lhd_{ji'} v(d'_{j,i})$$

and

$$\mathit{frac}(w(x_i)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ij'} v(d'_{i,j})$$

and

$$-\mathit{frac}(w(x_j)) - \delta \lhd_{0j'} v(d'_{0,j}) \quad \text{and} \quad \mathit{frac}(w(x_j)) + \delta \lhd_{j0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w.

Recall that $x_i \in \mathsf{LFP}_{R_p}(D)$ and let $\delta = 1 - \mathit{frac}(w(x_i))$ which we will prove is the exact amount of time so that all upper bounds of (E', D') are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{-\mathit{frac}(w(x)) - \mathit{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\mathit{frac}(v(d'_{x,0})) - \mathit{frac}(w(x))\}.$$

Recall that since (E, D) respects condition Definition 3.5 (5b), for all j s.t. $D_{0,j} = (0, \lhd_{0j})$, we have $\lhd_{0j} = <$. Hence as $-\mathit{frac}(w(x_i)) < v(d_{0,j})$, $\mathit{frac}(w(x_i)) \neq 0$. Using the same reasoning as before, we are going to prove that $t_0 \leq \delta \leq t_1$.

First we will prove that $t_0 \leq \delta$. Consider $x_i \in \mathsf{LFP}_{R_p}(D)$. For all clock x_j, since $w \in (E, v(D))$ we have $\mathit{frac}(w(x_i)) - \mathit{frac}(w(x_j)) \lhd_{ij} \mathit{frac}(v(d_{i,j}))$.

From Algorithm 3 applied to (E, D) and since $x_i \in \mathsf{LFP}_{R_p}(D)$ we obtain in (E', D') that $D'_{0,j} = D_{i,j} + (-1, \leq)$. Clearly we have $\lhd_{0j'} = \lhd_{ij} \oplus \leq\, = \lhd_{ij}$. It gives that

$$\mathit{frac}(w(x_i)) - \mathit{frac}(w(x_j)) - 1 (\lhd_{ij} \oplus \leq) \mathit{frac}(v(d_{i,j})) - 1$$

which is equivalent to $\mathit{frac}(w(x_i)) - \mathit{frac}(w(x_j)) - 1 \lhd_{0j'} \mathit{frac}(v(d'_{0,j}))$ which is equivalent to

$$\mathit{frac}(w(x_i)) - 1 \lhd_{0j'} \mathit{frac}(v(d'_{0,j})) + \mathit{frac}(w(x_j)).$$

This gives us our first result.

Second we will prove that $\delta \leq t_1$. Consider $x_i \in \mathsf{LFP}_{R_p}(D)$. For all clock x_j, from Definition 3.5 (4) we have $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) \lhd_{ji} \mathit{frac}(v(d_{j,i}))$. We have

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) + 1 \lhd_{ji} \mathit{frac}(v(d_{j,i})) + 1.$$

From Algorithm 3 applied to (E, D) and since $x_i \in \mathsf{LFP}_{R_p}(D)$ we obtain in (E', D') that $D'_{j,0} = D_{j,i} + (1, \leq)$. Clearly we have $\lhd_{j0'} = \lhd_{ji} \oplus \leq = \lhd_{ji}$. Then we can write that $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) + 1 \lhd_{j0'} \mathit{frac}(v(d'_{j,0}))$ which is equivalent to

$$1 - \mathit{frac}(w(x_i)) \lhd_{j0'} \mathit{frac}(v(d'_{j,0})) - \mathit{frac}(w(x_j)).$$

This gives us our second result.

Now for all clock x_j, we obtain two results. First we have

$$-\mathit{frac}(w(x_j)) - \delta \lhd_{0j'} -\mathit{frac}(w(x_j)) - t_1 \leq -\mathit{frac}(w(x_j)) + \mathit{frac}(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \lhd_{0j'} \mathit{frac}(w(x_j)) + \delta$.

Secondary we have

$$\mathit{frac}(w(x_j)) + \delta \lhd_{j0'} \mathit{frac}(w(x_j)) + t_0 \leq \mathit{frac}(w(x_j)) - \mathit{frac}(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $\mathit{frac}(w(x_j)) + \delta \lhd_{j0'} v(d'_{j,0})$.

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0.

Note that if both $x_i, x_j \in \mathsf{LFP}_{R_p}(D)$, as $\mathit{frac}(w(x_i)) = \mathit{frac}(w(x_j))$, $D_{i,j} = D'_{i,j} = (0, \leq)$ and $D_{j,i} = D'_{j,i} = (0, \leq)$ from Definition 4.5. Hence $\mathit{frac}(w(x_i)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ij'} \mathit{frac}(v(d'_{i,j}))$ and $\mathit{frac}(w(x_j)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ji'} \mathit{frac}(v(d'_{j,i}))$.

The same way, if both $x_i, x_j \notin \mathsf{LFP}_{R_p}(D)$ we have $D_{i,j} = D'_{i,j}$ and $D_{j,i} = D'_{j,i}$ and again our result. If either x_i or x_j is in $\mathsf{LFP}_{R_p}(D)$, the case is similar to $D'_{0,j}$ or $D'_{i,0}$.

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$.

$\Longleftarrow$ open–p–**PDBM respecting Definition 3.5 (5a)**

Suppose in (E, D) there is at least an x_j s.t. $D_{j,0} = D_{0,j} = (0, \leq)$ Let $v \in R_p$, and $x_i \in \mathsf{LFP}_{R_p}(D)$.

Now consider $(E', D') = TE((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\mathit{frac}(w(x_j)) + \delta - \mathit{frac}(w(x_i)) - \delta \lhd_{ji'} v(d'_{j,i})$$

and

$$\mathit{frac}(w(x_i)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ij'} v(d'_{i,j})$$

and

$$-\mathit{frac}(w(x_j)) - \delta \lhd_{0j'} v(d'_{0,j}) \quad \text{and} \quad \mathit{frac}(w(x_j)) + \delta \lhd_{j0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w.

Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, -\mathit{frac}(w(x)) - \mathit{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\mathit{frac}(v(d'_{x,0})) - \mathit{frac}(w(x))\}.$$

We want to prove that $t_0 \leq t_1$. For this purpose, we are going to prove for all clocks i, j that $-\mathit{frac}(w(x_j)) - v(d'_{j,0}) \leq v(d'_{0,i}) - \mathit{frac}(w(x_i))$.

First note that

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) \lhd_{ji} v(d_{j,i})$$

By definition of $TE_<$ applied to (E, D), we have that $D'_{j,i} = D_{j,i}$, and from Definition 3.5 (4) we have that $D'_{j,i} \leq D'_{j,0} + D'_{0,i}$.

Hence, we have from Definition 3.3 (2b) that either $d'_{j,i} < d'_{j,0} + d'_{0,i}$ or $d'_{j,i} = d'_{j,0} + d'_{0,i}$ and $\lhd_{ji'} = \lhd_{j0'} \oplus \lhd_{0i'}$ or $\lhd_{ji'} = <$ and $\lhd_{j0'} \oplus \lhd_{0i'} = \leq$.

We can then write that

$$\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i))(\lhd_{j0'} \oplus \lhd_{0i'}) v(d'_{j,0}) + v(d'_{0,i})$$

which is equivalent to

$$-\mathit{frac}(w(x_i)) - v(d'_{0,i})(\lhd_{j0'} \oplus \lhd_{0i'}) v(d'_{j,0}) - \mathit{frac}(w(x_j))$$

Now we prove that $t_0 = 0$. Clearly from Definition 3.5 for any clock i we have that $-\mathit{frac}(w(x_i)) \lhd_{0i} v(d_{0,i})$ which is equivalent to $-\mathit{frac}(w(x_i)) - v(d_{0,i}) \lhd_{0i} 0$.

Hence if as (E, D) there is at least an x_j s.t. $D_{j,0} = D_{0,j} = (0, \leq)$, for this clock j we have $-\mathit{frac}(w(x_j)) - v(d_{0,j}) = 0$.

By definition of $TE_<$ applied to (E, D), we have that $D'_{0,i} = D_{0,i} + (0, <)$. In order to respect the constraint $-\mathit{frac}(w(x_i)) - \delta \lhd_{0i'} v(d'_{0,i})$ which is, as $\lhd_{0i'} =<$, $-\mathit{frac}(w(x_i)) - \delta < v(d'_{0,i})$ and especially for j where $v(d'_{0,j}) = 0$ we have to find a $\delta > 0$.

In order to find an upper bound for δ, we are going to prove that $t_1 > 0$. From Definition 3.5 (4) we have in (E, D) that for any clocks i, j $D_{j,0} \leq D_{j,i} + D_{i,0}$. Let $x_i \in \mathsf{LFP}_{R_p}(D)$. From Definition 3.5 (1), we have that $D_{i,0} \leq (1, <)$. This gives that $D_{j,i} + D_{i,0} \leq D_{j,i} + (1, <)$.

By definition of $TE_<$ applied to (E, D), we have that $D'_{j,0} = D_{j,i} + (1, <)$. Hence we have $D_{j,0} \leq D'_{j,0}$.

Now as $frac(w(x_i)) \lhd_{i0} v(d_{i,0})$ we can write $frac(w(x_i)) \lhd_{i0'} v(d'_{i,0})$ and then $0 \lhd_{i0'} v(d'_{i,0}) - frac(w(x_i))$ where $\lhd_{i0'} =<$, which prove our result.

We define $\delta = \frac{t_1}{2}$, therefore $t_0 < \delta < t_1$. Now for all clock x_j, we obtain two results. First we have

$$-frac(w(x_j)) - \delta < -frac(w(x_j)) - t_1 \lhd_{0j'} -frac(w(x_j)) + frac(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \lhd_{0j} frac(w(x_j)) + \delta$ as $\lhd_{0j'} = <$.

Secondary we have

$$frac(w(x_j)) + \delta < frac(w(x_j)) + t_0 \lhd_{j0'} frac(w(x_j)) - frac(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $frac(w(x_j)) + \delta \lhd_{j0} v(d'_{j,0})$ as $\lhd_{0j'} = <$.

Now we prove that constraints defined in (E', D') on differences of clocks are also satisfied by going forward of δ units of time from w

Recall that in (E', D') from the definition of Algorithm 2 we have for all clock x_j,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $frac(w(x_j)) - frac(w(x_i)) \lhd_{ji} v(d_{j,i}) = v(d'_{j,i})$, and therefore as $\lhd_{ji'} = \lhd_{ji}$,

$$frac(w(x_j)) + \delta - frac(w(x_i)) - \delta \lhd_{ji'} v(d'_{j,i}).$$

We also have that $frac(w(x_i)) - frac(w(x_j)) \lhd_{ij} v(d_{i,j}) = v(d'_{i,j})$, therefore as $\lhd_{ij'} = \lhd_{ij}$,

$$frac(w(x_i)) + \delta - frac(w(x_j)) - \delta \lhd_{ij'} v(d'_{i,j}).$$

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$. □

Claim H.3. *Let R_p be a parameter region and $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Let $v \in R_p$. There is $w' \in TE((E, v(D)))$ iff there is $w \in (E, v(D))$ and a delay δ s.t. $w' = w + \delta$.*

Proof. $(\Longleftarrow)$ **for point–p–PDBMs**

Let $v \in R_p$. Consider $(E', D') = TE((E, D))$ respecting condition Definition 3.5 (5b), *i. e.,* suppose there is at least an x_i s.t. $D'_{i,0} = (1, <)$ and for all j s.t. $D_{0,j} = (0, \lhd_{0j})$, then we have $\lhd_{0j} = <$. Let $w' \in (E', v(D'))$.

We need to find a value δ s.t. $w' - \delta \in (E, v(D))$ which is equivalent to prove for all x_i, x_j

$$frac(w'(x_j)) - \delta - frac(w'(x_i)) + \delta \lhd_{ji} v(d_{j,i})$$

and

$$frac(w'(x_i)) - \delta - frac(w'(x_j)) + \delta \lhd_{ij} v(d_{i,j})$$

and

$$-frac(w'(x_j)) + \delta \lhd_{0j} v(d_{0,j}) \quad \text{and} \quad frac(w'(x_j)) - \delta \lhd_{j0} v(d_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going backward of δ units of time from w'.

We define the largest and the least amount of time so that all upper bounds of (E, D) are satisfied. Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, frac(w'(x)) - v(d_{x,0})\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{frac(w'(x)) + v(d_{0,x})\}.$$

We want to prove that $t_0 = t_1 > 0$. For this purpose, let us first show that for all i, j we have $frac(w'(x_j)) - v(d'_{j,0}) \leq frac(w'(x_i)) + v(d'_{0,i})$, which is $t_0 \leq t_1$.

First note that for all i, j

$$frac(w'(x_j)) - frac(w'(x_i)) \lhd_{ji'} v(d'_{j,i}).$$

By applying $TE_<$ (Algorithm 2) to (E, D), we have that $D'_{j,i} = D_{j,i}$, *i. e.*, $(d_{i,j}, \lhd_{ij}) = (d'_{i,j}, \lhd_{ij'})$, and from Definition 3.6 (2) we have that $D_{j,i} \leq D_{j,0} + D_{0,i}$ is valid for R_p.

Hence, we have from Definition 3.3 (2b) that either $v(d_{j,i}) < v(d_{j,0}) + v(d_{0,i})$ or $v(d_{j,i}) \leq v(d_{j,0}) + v(d_{0,i})$ and $\lhd_{ji} = \lhd_{j0} \oplus \lhd_{0i}$ or $\lhd_{ji} = <$ and $\lhd_{j0} \oplus \lhd_{0i} = \leq$.

We can then write that

$$frac(w'(x_j)) - frac(w'(x_i))(\lhd_{j0} \oplus \lhd_{0i})v(d_{j,0}) + v(d_{0,i})$$

which is equivalent to

$$frac(w'(x_j)) - v(d_{j,0})(\lhd_{j0} \oplus \lhd_{0i})frac(w'(x_i)) + v(d_{0,i})$$

so we obtain our result, as $(\lhd_{j0} \oplus \lhd_{0i})$ is either $\leq$ or $<$.

Now, recall that in (E, D) for all x we have $d_{0,x} = -d_{x,0}$ and $\lhd_{0x} = \lhd_{x0}$.

For any clock x we have that $frac(w'(x)) - v(d_{x,0}) \leq t_0$ and that $t_1 \leq frac(w'(x)) + v(d_{0,x}) = frac(w'(x)) - v(d_{x,0})$.

Hence $t_0 = t_1$.

As for all x, $\lhd_{x0} = \leq$, we have for all i, j that $(\lhd_{x0} \oplus \lhd_{0i}) = \lhd_{0i}$ and $(\lhd_{j0} \oplus \lhd_{0x}) = \lhd_{j0}$, which gives

$$t_1 \lhd_{0i} frac(w'(x_i)) + v(d_{0,i})$$

and

$$frac(w'(x_j)) - v(d_{j,0}) \lhd_{j0} t_0.$$

Moreover in (E', D') we have that $frac(w'(x)) \lhd_{0x'} v(d'_{0,x})$. From Claim H.1, (E', D') is obtained after applying Algorithm 2 and therefore $\lhd_{0x'} = <$. Hence $0 < frac(w'(x))$ and

$$0 < t_0 = t_1.$$

Let $\delta = t_0 = t_1$. More intuitively δ is the value right in the middle of the least and the largest amount of time s.t. we can go backward in time from w' and respect all constraints defined in $(E, v(D))$.

First we have

$$-frac(w'(x_j)) + \delta \leq -frac(w'(x_j)) + t_1 \lhd_{j0} -frac(w'(x_j)) + frac(w'(x_j)) + v(d_{0,j}) = v(d_{0,j})$$

which is $-v(d_{0,j}) \lhd_{j0} frac(w'(x_j)) - \delta$.

Secondary we have

$$frac(w'(x_j)) - \delta \leq frac(w)'(x_j) - t_0 \lhd_{0j} frac(w'(x_j)) - frac(w'(x_j)) + v(d_{j,0}) = v(d_{j,0})$$

which is $frac(w'(x_j)) - \delta \lhd_{0j} v(d_{j,0})$.

Now we prove that constraints defined in (E, D) on differences of clocks are also satisfied by going back of δ units of time from w'

Recall that in (E', D') from the definition of Algorithm 2 we have for all clocks x_i, x_j,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\mathit{frac}(w'(x_j)) - \mathit{frac}(w'(x_i)) \lhd_{ji'} v(d'_{j,i}) = v(d_{j,i})$, and therefore as $\lhd_{ji'} = \lhd_{ji}$,

$$\mathit{frac}(w'(x_j)) - \delta - \mathit{frac}(w'(x_i)) + \delta \lhd_{ji} v(d_{j,i}).$$

We also have that $\mathit{frac}(w'(x_i)) - \mathit{frac}(w'(x_j)) \lhd_{ij'} v(d'_{i,j}) = v(d_{i,j})$, therefore as $\lhd_{ij'} = \lhd_{ij}$,

$$\mathit{frac}(w'(x_i)) - \delta - \mathit{frac}(w'(x_j)) + \delta \lhd_{ij} v(d_{i,j}).$$

To conclude, we define for all x_j

$$w(x_j) = w'(x_j) - \delta$$

and clearly, $w \in (E, v(D))$.

($\Longrightarrow$) **for point–p–PDBMs**

Assume in $(E, D) \in p\text{–}\mathcal{PDBM}_{\odot}(R_p)$. Let $v \in R_p$, and $x_i \in \mathsf{LFP}_{R_p}(D)$.

Now consider $(E', D') = \mathit{TE}((E, D))$. We need to find a value δ s.t. $w + \delta \in (E', v(D'))$. which is equivalent to prove for all x_i, x_j

$$\mathit{frac}(w(x_j)) + \delta - \mathit{frac}(w(x_i)) - \delta \lhd_{ji'} v(d'_{j,i})$$

and

$$\mathit{frac}(w(x_i)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ij'} v(d'_{i,j})$$

and

$$-\mathit{frac}(w(x_j)) - \delta \lhd_{0j'} v(d'_{0,j}) \quad \text{and} \quad \mathit{frac}(w(x_j)) + \delta \lhd_{j0'} v(d'_{j,0}).$$

As done previously we are going to define a δ which is different from 0, and show that constraints in (E, D) are satisfied while going forward of δ units of time from w.

Let

$$t_0 = \max_{x \in \mathbb{X}} \{0, -\mathit{frac}(w(x)) - \mathit{frac}(v(d'_{0,x}))\}$$

and

$$t_1 = \min_{x \in \mathbb{X}} \{\mathit{frac}(v(d'_{x,0})) - \mathit{frac}(w(x))\}.$$

We prove that $t_1 \leq t_0$.

for any clock i we have that $D_{i,0} = (\mathit{frac}(p), \leq)$ and $D_{i,0} = (-\mathit{frac}(p), \leq)$ *i. e.*, $d_{0,i} = -d_{i,0}$ for some p, hence $-\mathit{frac}(w(x_i)) - v(d_{0,i}) = -\mathit{frac}(w(x_i)) + v(d_{i,0})$.

By definition of $\mathit{TE}_<$ applied to (E, D), we have that $D'_{0,i} = D_{0,i} + (0, <)$. In order to respect the constraint $-\mathit{frac}(w(x_i)) - \delta \lhd_{0i'} v(d'_{0,i})$ which is, as $\lhd_{0i'} = <$, $-\mathit{frac}(w(x_i)) - \delta < v(d'_{0,i})$, we have to find a $\delta > 0$.

In order to find an upper bound for δ, we are going to prove that $t_1 > 0$. From Definition 3.6 (2) we have in (E, D) that for any clocks i, j $D_{j,0} \leq D_{j,i} + D_{i,0}$. Let $x_i \in \mathsf{LFP}_{R_p}(D)$. From Definition 3.6 (1), we have that $D_{i,0} \leq (1, <)$. This gives that $D_{j,i} + D_{i,0} \leq D_{j,i} + (1, <)$.

By definition of $\mathit{TE}_<$ applied to (E, D), we have that $D'_{j,0} = D_{j,i} + (1, <)$. Hence we have $D_{j,0} \leq D'_{j,0}$.

Now as $\mathit{frac}(w(x_i)) \lhd_{i0} v(d_{i,0})$ we can write $\mathit{frac}(w(x_i)) \lhd_{i0'} v(d'_{i,0})$ and then $0 \lhd_{i0'} v(d'_{i,0}) - \mathit{frac}(w(x_i))$ where $\lhd_{i0'} = <$, which prove our result.

We define $\delta = \frac{t_1}{2}$, therefore $t_0 < \delta < t_1$. Now for all clock x_j, we obtain two results. First we have

$$-\mathit{frac}(w(x_j)) - \delta < -\mathit{frac}(w(x_j)) - t_1 \lhd_{0j'} -\mathit{frac}(w(x_j)) + \mathit{frac}(w(x_j)) + v(d'_{0,j}) = v(d'_{0,j})$$

which is $-v(d'_{0,j}) \lhd_{0j} \mathit{frac}(w(x_j)) + \delta$ as $\lhd_{0j'} = <$.

Secondary we have

$$\mathit{frac}(w(x_j)) + \delta < \mathit{frac}(w(x_j)) + t_0 \lhd_{j0'} \mathit{frac}(w(x_j)) - \mathit{frac}(w(x_j)) + v(d'_{j,0}) = v(d'_{j,0})$$

which is $\mathit{frac}(w(x_j)) + \delta \lhd_{j0} v(d'_{j,0})$ as $\lhd_{0j'} = <$.

Now we prove that constraints defined in (E', D') on differences of clocks are also satisfied by going forward of δ units of time from w

Recall that in (E', D') from the definition of Algorithm 2 we have for all clock x_j,

$$D'_{j,i} = D_{j,i} \quad \text{and} \quad D'_{i,j} = D_{i,j}.$$

Since we already treated the case whether i or j are 0, now suppose i, j are both different from 0. We have that $\mathit{frac}(w(x_j)) - \mathit{frac}(w(x_i)) \lhd_{ji} v(d_{j,i}) = v(d'_{j,i})$, and therefore as $\lhd_{ji'} = \lhd_{ji}$,

$$\mathit{frac}(w(x_j)) + \delta - \mathit{frac}(w(x_i)) - \delta \lhd_{ji'} v(d'_{j,i}).$$

We also have that $\mathit{frac}(w(x_i)) - \mathit{frac}(w(x_j)) \lhd_{ij} v(d_{i,j}) = v(d'_{i,j})$, therefore as $\lhd_{ij'} = \lhd_{ij}$,

$$\mathit{frac}(w(x_i)) + \delta - \mathit{frac}(w(x_j)) - \delta \lhd_{ij'} v(d'_{i,j}).$$

Finally, we define $w' = w + \delta$ and $w' \in (E', v(D'))$. □

□

Appendix I. Proof of Lemma 4.12

Lemma 4.12 (recalled). *Let (E, D) be a* p*–PDBM for R_p and $v \in R_p$. Let g be a non-parametric guard. If $v \in \mathit{guard}_\forall(g, E, D)$, then for all $v' \in R_p$, $v' \in \mathit{guard}_\forall(g, E, D)$.*

Proof. Our idea is to define a clock region "larger" than our p–PDBM (following Definition 2.3) and show that, even for this (larger) clock region, either all clock valuations satisfy the guard—or none does.

Definition I.1. Let R_p be a parameter region, $v \in R_p$. Let (E, D) be a p–PDBM for R_p. We define the *clock region containing* $(E, v(D))$, denoted by $[(E, v(D))]_{R_c}$, as follows: for all $w \in [(E, v(D))]_{R_c}$, for all clocks x_i, x_j,

- if $E_{x_i} < K$, $\lfloor w(x_i) \rfloor = E_{x_i}$, else if $E_{x_i} = \infty$, $w(x_i) \geq K$;
- if $(0, \leq) < D_{i,j}$ is valid for R_p and $E_{x_i} < K$, $\mathit{frac}(w(x_j)) < \mathit{frac}(w(x_i))$;
- if $(0, \leq) = D_{i,j}$ is valid for R_p and $E_{x_i} < K$, $\mathit{frac}(w(x_j)) = \mathit{frac}(w(x_i))$;
- if $D_{i,0} = D_{0,i} = (0, \leq)$ and $E_{x_i} < K$, $\mathit{frac}(w(x_i)) = 0$;
- if $D_{i,0} \neq (0, \leq), D_{0,i} \neq (0, \leq)$ and $E_{x_i} < K$, $\mathit{frac}(w(x_i)) \neq 0$.

Claim I.2. *Let (E, D) be a* p*–PDBM for R_p and $v \in R_p$. We have $(E, v(D)) \subseteq [(E, v(D))]_{R_c}$.*

Proof. Clock regions of Definition 2.3 define constraints on clocks of the form $0 = \mathit{frac}(x)$, $0 < \mathit{frac}(x) < 1$, $0 = \mathit{frac}(x) - \mathit{frac}(y)$ and $0 < \mathit{frac}(x) - \mathit{frac}(y) < 1$ for some x, y, and $\lfloor x \rfloor = k$ for some integer k. Let (E, D) be a p–PDBM for R_p and $v \in R_p$. It defines constraints

$$\bigwedge_{i,j \in [0,H]^2} \mathit{frac}(x_i) - \mathit{frac}(x_j) \lhd_{i,j} v(d_{i,j}) \quad \wedge \quad \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i.$$

Clearly, if $w \in (E, v(D))$ satisfies $\lfloor x_i \rfloor = E_i$ then it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$.

Consider the constraints $\mathit{frac}(x_i) - \mathit{frac}(x_j) \lhd_{i,j} v(d_{i,j})$ and $\mathit{frac}(x_j) - \mathit{frac}(x_i) \lhd_{j,i} v(d_{j,i})$.

- Suppose i, j are both different from 0. From Definition 3.5 (3) and Definition 3.6, either $d_{i,j} = d_{j,i}$ and then $\lhd_{i,j} = \leq = \lhd_{j,i}$, then if $d_{i,j} = d_{j,i} = 0$ it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$, or $d_{i,j}$ and $d_{j,i}$ are different from 0, as they are elements of $\mathcal{PLT}$ which are strictly smaller than 1, it satisfies either $0 < \mathit{frac}(x_i) - \mathit{frac}(x_j) < 1$ or $0 = \mathit{frac}(x_i) - \mathit{frac}(x_j)$ in $[(E, v(D))]_{R_c}$. Finally if $d_{i,j} \neq d_{j,i}$, then $\lhd_{i,j} = < = \lhd_{j,i}$ and it satisfies $0 < \mathit{frac}(x_i) - \mathit{frac}(x_j) < 1$ in $[(E, v(D))]_{R_c}$.
- Suppose i is different from 0 and $j = 0$. From Definition 3.5 (3) and Definition 3.6, either $d_{i,0} = d_{0,i}$ and then $\lhd_{i,0} = \leq = \lhd_{0,i}$, then if $d_{i,0} = d_{0,i} = 0$ it satisfies the same constraint defined in $[(E, v(D))]_{R_c}$, or $d_{i,0}$ and $d_{0,i}$ are different from 0, as they are elements of $\mathcal{PLT}$ which are strictly smaller than 1, it satisfies either $0 < \mathit{frac}(x_i) < 1$ or $0 = \mathit{frac}(x_i)$ in $[(E, v(D))]_{R_c}$. Finally if $d_{i,0} \neq d_{0,i}$, then $\lhd_{i,0} = < = \lhd_{0,i}$ and it satisfies $0 < \mathit{frac}(x_i) < 1$ in $[(E, v(D))]_{R_c}$.
- The case j is different from 0 and $i = 0$ is similar.
- Suppose both i, j are 0, the constraint is not taken into account as we have no x_0 in $[(E, v(D))]_{R_c}$.

Finally, we have that if $w \in (E, v(D))$ then $w \in [(E, v(D))]_{R_c}$. □

Now we come back to the proof of the main lemma:

Let (E, D) be a p–PDBM for R_p and $v \in R_p$. It defines constraints

$$\bigwedge_{i,j \in [0,H]^2} \mathit{frac}(x_i) - \mathit{frac}(x_j) \lhd_{i,j} v(d_{i,j}) \quad \wedge \quad \bigwedge_{i \in [1,H]} \lfloor x_i \rfloor = E_i.$$

Moreover, let g be a non-parametric guard. It defines constraints for a finite number of integer constants k_i with $i \in I \subseteq [1, H]$

$$\bigwedge_{i \in I} \mathit{frac}(x_i) \leq 0 \quad \wedge \quad \bigwedge_{i \in I} -\mathit{frac}(x_i) \leq 0 \quad \wedge \quad \bigwedge_{i \in I} \lfloor x_i \rfloor \bowtie k_i.$$

The intersection between the two is given by the conjunction of those constraints. We project this intersection on parameter variables (by elimination of clock variables) and we prove that the intersection does not create new constraints on parameters different from those we already have in $(E, v(D))$ (and therefore in R_p). For some set of clocks $I \subseteq [1, H]$ and $i \in I$, suppose we have the constraints $\mathit{frac}(x_i) \leq 0$ and $-\mathit{frac}(x_i) \leq 0$ in g. When eliminating x_i in any constraint of the form $\mathit{frac}(x_i) - \mathit{frac}(x_j) \lhd_{i,j} v(d_{i,j})$, it is clear that we proceed on $\mathcal{PLT}$ to the operation $(0, \leq) + (d_{i,j}, \lhd_{i,j}) = (0 + d_{i,j}, \leq \oplus \lhd_{i,j}) = (d_{i,j}, \lhd_{i,j})$. The same way on any constraint of the form $\mathit{frac}(x_i) \lhd_{i,0} v(d_{i,0})$, eliminating x_i gives the constraint $(0, \leq) + (d_{i,0}, \lhd_{i,0}) = (d_{i,0}, \lhd_{i,0})$. Hence it does not create new inequalities not belonging to R_p.

Now suppose $v \in \mathit{guard}_\forall(g, E, D)$. We have that all $w \in (E, v(D))$ satisfy g.

As no new constraints not in $\mathcal{PLT}$ have been created, all $v' \in R_p$ respect the same constraints on their fractional part and integer part as v and therefore, $(E, v'(D))$ is contained in the same clock region as $(E, v(D))$ is, *i. e.*, $[(E, v(D))]_{R_c} = [(E, v'(D))]_{R_c}$. Finally, $v' \in \mathit{guard}_\forall(g, E, D)$. □

Appendix J. Proof of Lemma 4.13

Lemma 4.13 (recalled). *Let (E, D) be a* p*–PDBM for R_p and $v \in R_p$. Let g be a parametric guard. If $v \in p\text{-}guard_{\exists}(g, E, D)$, then for all $v' \in R_p$, $v' \in p\text{-}guard_{\exists}(g, E, D)$.*

Proof. Let (E, D) be a p–PDBM for R_p and $v \in R_p$. Let g be a parametric guard and suppose $v \in p\text{-}guard_{\exists}(g, E, D)$. After applying a projection on parameters, we obtain constraints on elements of $\mathcal{PLT}$. By hypothesis, all these constraints are satisfied by v. Suppose $v' \in R_p$. By definition of our parameter regions, and since v and v' both belong to R_p, v' satisfies the same constraints on elements of $\mathcal{PLT}$. Therefore, the same constraints is satisfied by v' and $v' \in p\text{-}guard_{\exists}(g, E, D)$. ☐

Appendix K. Proof of Proposition 6.1

Proposition 6.1 (recalled). *Let R_p be a parameter region. Let $\mathcal{A}$ be an R-U2P-PTA and $\mathcal{R}(\mathcal{A})$ its parametric region automaton over R_p. There is a run $\sigma : (\ell_0, (E_0, D_0)) \xrightarrow{e_0} (\ell_1, (E_1, D_1)) \xrightarrow{e_1} \cdots (\ell_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f))$ in $\mathcal{R}(\mathcal{A})$ iff for all $v \in R_p$ there is a run $\rho : (\ell_0, w_0) \xrightarrow{e_0} (\ell_1, w_1) \xrightarrow{e_1} \cdots (\ell_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (\ell_f, w_f)$ in $v(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.*

Proof.$\Leftarrow$ By induction on the length of the run.

Let $v \in R_p$. As the basis for the induction, in the initial location $(\ell_0, \{0\}^H)$ the only valuation is reachable by an empty run of $v(\mathcal{A})$. Moreover $\{0\}^H \in (E_0, v(D_0))$ the initial p–PDBM containing only 0. Therefore the initial location $(\ell_0, (E_0, v(D_0)))$ is reachable by an empty run of $\mathcal{R}(\mathcal{A})$.

For the induction step, suppose for all v, there is run in $v(\mathcal{A})$ of length $f - 1$ we have our result.

Let $v \in R_p$ and $\rho = (\ell_0, w_0) \xrightarrow{e_0} \cdots \xrightarrow{e_{f-2}} (\ell_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (\ell_f, w_f)$ be a run of $v(\mathcal{A})$ of length f. By induction hypothesis, there is a run $\sigma = (\ell_0, (E_0, D_0)) \xrightarrow{e_0} \cdots \xrightarrow{e_{f-2}} (\ell_{f-1}, (E_{f-1}, D_{f-1}))$ in $\mathcal{R}(\mathcal{A})$ and for all $0 \leq i \leq f - 1$, $w_i \in (E_i, v(D_i))$.

Consider e_{f-1}. By Definition 5.1 of the parametric region automaton, it is also in its set of edges ζ'. Three cases show up:

- If $e_{f-1} = \langle \ell_{f-1}, a, g, u_{np}, \ell_f \rangle$ contains no parametric guard nor parametric update. Using Definition 2.2 there is a delay δ (possibly 0) s.t. $(\ell_{f-1}, w_{f-1}) \overset{\delta}{\mapsto} (\ell_{f-1} w'_{f-1}) \overset{e_{f-1}}{\mapsto} (\ell_f, w_f)$ where $w'_{f-1} \models g$ and $w_f = [w'_{f-1}]_{u_{np}}$. As $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ there is $(E'_{f-1}, D'_{f-1}) \in \mathsf{Succ}((E_{f-1}, D_{f-1}))$ s.t. from Proposition 4.11 we have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$. As $w'_{f-1} \models g$ by construction of our p–PDBMs (see Section 4.4) any other clock valuation belonging to $(E'_{f-1}, v(D'_{f-1}))$ satisfies g. Therefore $v \in guard_{\forall}(g, E'_{f-1}, D'_{f-1})$ and from Lemma 4.12, $R_p \subseteq guard_{\forall}(g, E'_{f-1}, D'_{f-1})$. Now, as $w_f = [w'_{f-1}]_{u_{np}}$ consider the open–p–PDBM $(E_f, D_f) = update((E'_{f-1}, D'_{f-1}), u_{np})$; from Lemma 4.4 we have $w_f \in (E_f, v(D_f))$. Finally there is an edge

$$(\ell_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f)).$$

– If $e_{f-1} = \langle \ell_{f-1}, a, g, u, \ell_f \rangle$ contains a parametric guard and a parametric update. Using Definition 2.2 there is a delay δ (possibly 0) s.t. $(\ell_{f-1}, w_{f-1}) \overset{\delta}{\mapsto} (\ell_{f-1}, w'_{f-1}) \overset{e_{f-1}}{\mapsto} (\ell_f, w_f)$ where $w'_{f-1} \models v(g)$ and $w_f = [w'_{f-1}]_{v(u)}$. As $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ there is $(E'_{f-1}, D'_{f-1}) \in \mathsf{Succ}((E_{f-1}, D_{f-1}))$ s.t. from Proposition 4.11 we have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$. As $w'_{f-1} \models v(g)$, $v \in p\text{-}guard_\exists(g, E'_{f-1}, D'_{f-1})$ and from Lemma 4.13, $R_p \subseteq p\text{-}guard_\exists(g, E'_{f-1}, D'_{f-1})$. Now, as $w_f = [w'_{f-1}]_{v(u)}$ consider the point–p–PDBM $(E_f, D_f) = \overline{update}((E'_{f-1}, D'_{f-1}), u)$; $(E_f, v(D_f))$ contains only one clock valuation, precisely defined by the fully parametric update $v(u)$ so we have $w_f \in (E_f, v(D_f))$. Finally there is an edge $(\ell_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f))$.

– The case where e_{f-1} contains a non parametric guard and a parametric update is similar to the previous one.

Finally, there is a run $\sigma' = \sigma \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f))$ of length f in $\mathcal{R}(\mathcal{A})$ s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

$\Rightarrow$ By induction on the length of the run.

Let $v \in R_p$. As the basis for the induction, the initial location $(\ell_0, (E_0, v(D_0)))$ is reachable by an empty run of $\mathcal{R}(\mathcal{A})$. Moreover, as $\{0\}^H \in (E_0, v(D_0))$, the initial location $(\ell_0, \{0\}^H)$ is reachable by an empty run of $v(\mathcal{A})$.

For the induction step, suppose it is true for all run in $\mathcal{R}(\mathcal{A})$ of length $f - 1$.

Let $v \in R_p$ and $\sigma = (\ell_0, (E_0, D_0)) \xrightarrow{e_0} \cdots \xrightarrow{e_{f-2}} (\ell_{f-1}, (E_{f-1}, D_{f-1})) \xrightarrow{e_{f-1}} (\ell_f, (E_f, D_f))$ be a run of $\mathcal{R}(\mathcal{A})$ of length f. Consider e_{f-1}. By Definition 5.1 of the parametric region automaton, it is also in the set of edges ζ of $\mathcal{A}$. Two cases show up:

– If $e_{f-1} = \langle \ell_{f-1}, a, g, u_{np}, \ell_f \rangle$ contains no parametric guard nor parametric update. By induction hypothesis, there is a run $\rho = (\ell_0, w_0) \xrightarrow{e_0} \cdots \xrightarrow{e_{f-2}} (\ell_{f-1}, w_{f-1})$ of $v(\mathcal{A})$ of length $f - 1$ s.t. for all $0 \leq i \leq f - 1$, $w_i \in (E_i, v(D_i))$. Using Definition 5.1 there is $(E'_{f-1}, D'_{f-1}) \in \mathsf{Succ}((E_{f-1}, D_{f-1}))$, $R_p \subseteq guard_\forall(g, E'_{f-1}, D'_{f-1})$ and $(E_f, D_f) = update((E'_{f-1}, D'_{f-1}), u_{np})$. From Proposition 4.11 we have $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ and a delay δ s.t. $w'_{f-1} = w_{f-1} + \delta$. As $R_p \subseteq guard_\forall(g, E'_{f-1}, D'_{f-1})$ from Lemma 4.12 we have $v \in guard_\forall(g, E'_{f-1}, D'_{f-1})$ and $w'_{f-1} \models g$. Moreover, since $(E_f, D_f) = update((E'_{f-1}, D'_{f-1}), u_{np})$, we define $w_f = [w'_{f-1}]_{u_{np}}$ and therefore from Lemma 4.4, $w_f \in (E_f, v(D_f))$. Finally there is an edge $(\ell_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (\ell_f, w_f)$ and a run $\rho' = \rho \xrightarrow{e_{f-1}} (\ell_f, w_f)$ in $v(\mathcal{A})$ of length f s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

– If $e_{f-1} = \langle \ell_{f-1}, a, g, u, \ell_f \rangle$ contains a parametric guard and a parametric update. Using Definition 5.1 there is $(E'_{f-1}, D'_{f-1}) \in \mathsf{Succ}((E_{f-1}, D_{f-1}))$, $R_p \subseteq p\text{-}guard_\exists(g, E'_{f-1}, D'_{f-1})$ and $(E_f, D_f) = \overline{update}((E'_{f-1}, D'_{f-1}), u)$. From Lemma 4.13 we can take $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ s.t. $w'_{f-1} \models v(g)$. Let $w_f = [w'_{f-1}]_{v(u)}$. Clearly, $(E_f, D_f) = \overline{update}((E'_{f-1}, D'_{f-1}), u)$ is a point–p–PDBM; as $(E_f, v(D_f))$ contains only one clock valuation precisely defined by the fully parametric update $v(u)$, we have $w_f \in (E_f, v(D_f))$. From Proposition 4.11 as $w'_{f-1} \in (E'_{f-1}, v(D'_{f-1}))$ there is a delay δ and a $w_{f-1} \in (E_{f-1}, v(D_{f-1}))$ s.t. $w'_{f-1} = w_{f-1} + \delta$. Using the induction hypothesis, there is a run $\rho = (\ell_0, w_0) \xrightarrow{e_0} \cdots \xrightarrow{e_{f-2}} (\ell_{f-1}, w_{f-1})$ of $v(\mathcal{A})$ of length $f - 1$ s.t. for all $0 \leq i \leq f - 1$,

$w_i \in (E_i, v(D_i))$. Finally there is an edge $(\ell_{f-1}, w_{f-1}) \xrightarrow{e_{f-1}} (\ell_f, w_f)$ and a run $\rho' = \rho \xrightarrow{e_{f-1}} (\ell_f, w_f)$ in $v(\mathcal{A})$ of length f s.t. for all $0 \leq i \leq f$, $w_i \in (E_i, v(D_i))$.

- The case where e_{f-1} contains a non parametric guard and a parametric update is similar to the previous one. $\square$