Preservation of Piecewise Constancy under TV Regularization with Rectilinear Anisotropy^{*}

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Abstract

A recent result by Lasica, Moll and Mucha about the ℓ^1 -anisotropic Rudin-Osher-Fatemi model in \mathbb{R}^2 asserts that the solution is piecewise constant on a rectilinear grid, if the datum is. By means of a new proof we extend this result to \mathbb{R}^n . The core of our proof consists in showing that averaging operators associated to certain rectilinear grids map subgradients of the ℓ^1 -anisotropic total variation seminorm to subgradients.

1 Introduction

This article is concerned with a variant of the Rudin-Osher-Fatemi (ROF) image denoising model [13]. More specifically, we consider minimization of

$$\frac{1}{2} \|u - f\|_{L^2}^2 + \alpha J(u), \tag{1}$$

where $J(u) = \int_{\Omega} \|\nabla u(x)\|_{\ell^1} dx$ is the total variation with ℓ^1 -anisotropy. This model and variations thereof have been used in imaging applications for data exhibiting a rectilinear geometry [2, 7, 14, 15]. Numerical algorithms for minimizing (1) have been studied, for example, in [6, 10, 12].

The ℓ^1 -anisotropic total variation has a special property from a theoretical point of view as well. It has been shown in [3, Thm. 3.4, Rem. 3.5] that approximation of a general $u \in BV \cap L^p$ by functions u_m piecewise constant on rectilinear grids, in the sense that

$$||u - u_m||_{L^p} \to 0$$
 and $J(u_m) \to J(u)$,

is not possible for $J(u) = \int_{\Omega} \|\nabla u(x)\|_{\ell^q} dx$, unless q = 1.

Let $\Omega \subset \mathbb{R}^n$ be a finite union of hyperrectangles, each aligned with the coordinate axes. Our main result, Theorem 3.6, states that if the given function $f : \Omega \to \mathbb{R}$ is piecewise constant on a rectilinear grid, then the minimizer of (1) is too. This extends a recent result by Lasica, Moll and Mucha about two-dimensional domains [11, Thm. 5]. Their proof is based on constructing the solution by means of its level sets and relies on minimization of an anisotropic Cheeger-type functional over subsets of Ω .

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The proof we present below is centred around the averaging operator A_G associated to the grid G on which f is piecewise constant. In addition to being a contraction, it has the crucial property of mapping subgradients of J to subgradients, that is, $A_G(\partial J(0)) \subset \partial J(0)$, see Theorem 3.2. Combined with the dual formulation of (1) we obtain that the minimizer must be piecewise constant on the same grid as f. While it might be possible to extend the techniques of [11] to higher dimensions, we believe that modifying the so-called "squaring step" in the proof of [11, Lem. 2] could lead to difficulties.

Theorem 3.6 implies that, if f is piecewise constant on a rectilinear grid, then minimization of functional (1) becomes a finite-dimensional problem. More precisely, in this case the solution can be found by minimizing a discrete energy of the form

$$\sum_{i} w_i |u_i - f_i|^2 + \alpha \sum_{i,j} w_{ij} |u_i - u_j|,$$

where the weights $w_i, w_{ij} \geq 0$ depend only on the grid. For problems of this sort there are many efficient algorithms, such as graph cuts [4, 5, 8]. Extending the preservation of piecewise constancy to domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, means that the discrete reformulation can also be exploited for processing higher dimensional data such as volumetric images or videos.

This article is organized as follows. Section 2 contains the basic concepts that will be required throughout. In Section 2.1 we define several spaces of piecewise constant functions, while Section 2.2 is devoted to the ℓ^1 -anisotropic ROF model. Section 3 is the main part of this paper. It starts with introducing the averaging operator A_G and ends with Theorem 3.6. The article is concluded in Section 4.

2 Mathematical preliminaries

2.1 PCR functions

In this section we introduce several notions related to functions which are piecewise constant on rectilinear subsets of \mathbb{R}^n . Some of these are *n*-dimensional analogues of notions from [11, Sec. 2.3].

A bounded set $R \subset \mathbb{R}^n$ which can be written as a Cartesian product of n proper intervals is called an *n*-dimensional hyperrectangle. Recall that an interval is proper, if it is neither empty nor a singleton. Finite unions of *n*-dimensional hyperrectangles will be referred to as *rectilinear n*-polytopes.

A rectilinear grid, or simply grid, is a finite family of affine hyperplanes, each being perpendicular to one of the coordinate axes of \mathbb{R}^n . For a rectilinear *n*-polytope P we denote by G(P) the smallest grid with the property that the union of all its affine hyperplanes contains the entire boundary of P.

Throughout this article $\Omega \subset \mathbb{R}^n$ is an *open* rectilinear *n*-polytope. A finite family of rectilinear *n*-polytopes $\mathcal{Q} = \{P_1, \ldots, P_N\}$ is called a *partition* of Ω , if they have pairwise disjoint interiors and the union of their closures equals $\overline{\Omega}$. Every grid G defines a partition $\mathcal{Q}(G)$ of Ω into rectilinear *n*-polytopes in the following way: $P \subset \Omega$ belongs to $\mathcal{Q}(G)$, if and only if its boundary is contained in $\partial\Omega \cup \bigcup G$ while int P and $\bigcup G$ are disjoint. Note that if G contains $G(\Omega)$, then $\mathcal{Q}(G)$ consists only of hyperrectangles.

We adopt the notation $PCR(\Omega)$, or simply PCR, from [11] for the set of all integrable functions $f: \Omega \to \mathbb{R}$ which can be written as finite linear combinations of indicator functions of rectilinear *n*-polytopes. That is, $f \in PCR$ if there is an

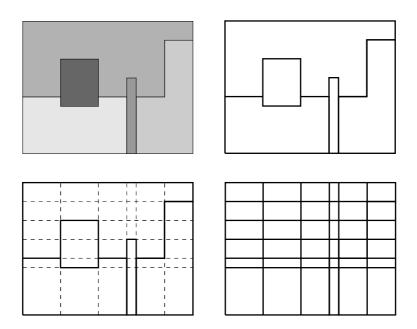


Figure 1: Upper left: A PCR function f on a planar domain Ω . Each level set of f is visualized using a different grey tone. Upper right: The boundaries of the level sets of f. Lower left: Extension of the boundaries of the level sets of f. Lower right: The minimal grid G_f (lines) and the partition $\mathcal{Q}(G_f)$ of Ω (rectangular cells).

 $N \in \mathbb{N}, c_i \in \mathbb{R}$ and rectilinear *n*-polytopes $P_i \subset \Omega, 1 \leq i \leq N$, such that

$$f = \sum_{i=1}^{N} c_i \mathbf{1}_{P_i} \tag{2}$$

almost everywhere. Here, $\mathbf{1}_A$ is the indicator function of the set A, defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We can assume, without loss of generality, that the values c_i are pairwise distinct and that the polytopes P_i are pairwise disjoint, which makes the representation (2) unique almost everywhere. To every $f \in PCR$ we associate its *minimal grid*, that is, the unique smallest grid covering the boundaries of all level sets of f, given by

$$G_f = \bigcup_{i=1}^N G(P_i).$$

Note that the partition $\mathcal{Q}(G_f)$ always consists of hyperrectangles only. See Figure 1 for an illustration of G_f and $\mathcal{Q}(G_f)$.

For a given grid G we denote by PCR_G the set of all functions in PCR which are equal almost everywhere to a finite linear combination of indicator functions of $P_i \in \mathcal{Q}(G)$.

The following notions are essential for the proof of Theorem 3.2. Fix an $i \in \{1, \ldots, n\}$ as well as coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. The set $\{x_i \in \mathbb{R} : (x_1, \ldots, x_n) \in \Omega\}$ is a union of finitely many disjoint intervals

$$I_i^k = (a_i^k, b_i^k), \quad k = 1, \dots, m.$$
 (3)

Note that the number of intervals m as well as the intervals I_i^k themselves depend on, and are uniquely determined by, the coordinates $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. For an illustration of the intervals I_i^k see Example 2.1 below. Next, let \mathcal{G} be the set of all rectilinear grids of \mathbb{R}^n . For every $G \in \mathcal{G}$ and $i \in \{1, \ldots, n\}$ we define

$$\Gamma_G^i = \left\{ g \in PCR_G : \sup_{s \in (a_i^k, b_i^k)} \left| \int_{a_i^k}^s g \, dx_i \right| \le 1, \int_{a_i^k}^{b_i^k} g \, dx_i = 0, 1 \le k \le m \right\}.$$

The restrictions on g are to be understood for almost every

 $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in\mathbb{R}^{n-1}$

such that there is an $x_i \in \mathbb{R}$ satisfying $(x_1, \ldots, x_n) \in \Omega$. The sum of the spaces Γ_G^i is denoted by

$$\Gamma_G = \left\{ \sum_{i=1}^n g_i : g_i \in \Gamma_G^i, 1 \le i \le n \right\},\$$

and we further set

$$\Gamma = \bigcup_{G \in \mathcal{G}} \Gamma_G.$$

Remark 2.1. The set Γ_G consists of divergences of certain piecewise affine vector fields. More precisely, Theorem 3.2 below implies that $\Gamma_G = \partial J(0) \cap PCR_G$, that is, Γ_G is the set of all subgradients of J which are piecewise constant on G.

Finally, those elements of Γ_G^i which have compact support in Ω are collected in the set $\Gamma_{G,c}^i$, and we define analogously

$$\Gamma_{G,c} = \left\{ \sum_{i=1}^{n} g_i : g_i \in \Gamma_{G,c}^i, 1 \le i \le n \right\},$$

$$\Gamma_c = \bigcup_{G \in \mathcal{G}} \Gamma_{G,c}.$$

Example 2.1. For the rectilinear 2-polytope Ω of Figure 2 we have the following intervals I_1^k and I_2^k

$$\{x_1 \in \mathbb{R} : (x_1, x_2) \in \Omega\} = \begin{cases} (0, 6), & \text{if } x_2 \in (0, 1) \cup (2, 3), \\ (0, 2) \cup (4, 6), & \text{if } x_2 \in [1, 2], \\ (3, 6), & \text{if } x_2 \in [3, 4), \end{cases}$$

and

$$\{x_2 \in \mathbb{R} : (x_1, x_2) \in \Omega\} = \begin{cases} (0, 3), & \text{if } x_1 \in (0, 2), \\ (0, 1) \cup (2, 3), & \text{if } x_1 \in [2, 3], \\ (0, 1) \cup (2, 4), & \text{if } x_1 \in (3, 4], \\ (0, 4), & \text{if } x_1 \in (4, 6). \end{cases}$$

2.2 The l^1 -anisotropic ROF model

The notion of anisotropic total variation was introduced in [1]. In this article we exclusively consider one particular variant.

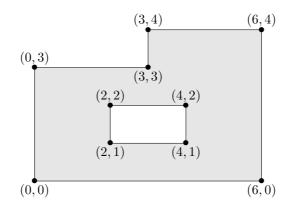


Figure 2: A rectilinear 2-polytope Ω .

For every $\alpha > 0$ we denote by \mathcal{B}_{α} the set of all smooth compactly supported vector fields on Ω whose components are bounded by α , that is,

$$\mathcal{B}_{\alpha} = \left\{ H \in C_{c}^{\infty}\left(\Omega, \mathbb{R}^{n}\right) : \max_{1 \leq i \leq n} |H_{i}(x)| \leq \alpha, \forall x \in \Omega \right\}.$$

The ℓ^1 -anisotropic total variation $J: L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is given by

$$J(u) = \sup_{H \in \mathcal{B}_1} \int_{\Omega} u \operatorname{div} H \, dx = \sup_{h \in \overline{\operatorname{div}} \mathcal{B}_1} \int_{\Omega} u h \, dx, \tag{4}$$

where the bar denotes closure in $L^2(\Omega)$. Thus, J is the support function of the closed and convex set $\overline{\operatorname{div} \mathcal{B}_1}$, which implies that $\overline{\operatorname{div} \mathcal{B}_1} = \partial J(0)$, or more generally

$$\overline{\operatorname{div}\mathcal{B}_{\alpha}} = \alpha \partial J(0) \tag{5}$$

for every $\alpha > 0$. If u is a Sobolev function, then $J(u) = \int_{\Omega} \|\nabla u(x)\|_{\ell^1} dx$. The next lemma states that the ℓ^1 -anisotropic ROF model is equivalent to constrained L^2 -minimization. However, the way it is formulated it actually applies to every support function J of a closed and convex subset of $L^2(\Omega)$.

Lemma 2.1. For every $\alpha > 0$ and $f \in L^2(\Omega)$ the minimization problem

$$\min_{u \in L^2(\Omega)} \frac{1}{2} \|u - f\|_{L^2}^2 + \alpha J(u)$$
(6)

is equivalent to

$$\min_{u \in f - \alpha \partial J(0)} \|u\|_{L^2}.$$

Proof. The dual problem associated to (6) is given by

$$\min_{w \in L^2(\Omega)} \frac{1}{2} \|w - f\|_{L^2}^2 + (\alpha J)^*(w), \tag{7}$$

where the asterisk stands for convex conjugation. The two solutions u_{α} and w_{α} of (6) and (7), respectively, satisfy the optimality conditions

$$u_{\alpha} = f - w_{\alpha},$$

$$w_{\alpha} \in \partial(\alpha J)(u_{\alpha}).$$
(8)

Concerning the derivation of (7) and (8) we refer to [9, Chap. III, Rem. 4.2]. Since αJ is the support function of the set $\alpha \partial J(0)$, recall (4), its conjugate is the characteristic function

$$(\alpha J)^*(w) = \begin{cases} 0, & w \in \alpha \partial J(0), \\ +\infty, & w \notin \alpha \partial J(0). \end{cases}$$

Therefore, problem (7) is equivalent to

$$\min_{w \in \alpha \partial J(0)} \|w - f\|_{L^2}$$

Finally, using the optimality condition (8) we get

$$||u_{\alpha}||_{L^{2}} = ||w_{\alpha} - f||_{L^{2}} = \min_{w \in \alpha \partial J(0)} ||w - f||_{L^{2}} = \min_{u \in f - \alpha \partial J(0)} ||u||_{L^{2}}.$$

3 The averaging operator A_G

Let Ω be a rectilinear *n*-polytope and *G* a grid. Define the averaging operator $A_G: L^1(\Omega) \to PCR_G(\Omega)$ by

$$A_G g = \sum_{i=1}^N \left(\frac{1}{|P_i|} \int_{P_i} g(s) ds \right) \mathbf{1}_{P_i},$$

where $P_i \in \mathcal{Q}(G)$ and $|P_i|$ is its *n*-dimensional volume.

Two properties of the operator A_G turn out to be important when establishing the main result of this paper, Theorem 3.6. The first one is

Lemma 3.1. For every $u \in L^1(\Omega)$ and convex $\varphi : \mathbb{R} \to \mathbb{R}$

$$\int_{\Omega} \varphi\left((A_G u)(x) \right) dx \leq \int_{\Omega} \varphi\left(u(x) \right) dx.$$

Proof. By applying Jensen's inequality we obtain

$$\int_{\Omega} \varphi\left((A_G u)(x)\right) dx = \sum_{i=1}^{N} \int_{P_i} \varphi\left((A_G u)(x)\right) dx = \sum_{i=1}^{N} \varphi\left(\frac{1}{|P_i|} \int_{P_i} u(x) dx\right) |P_i|$$
$$\leq \sum_{i=1}^{N} \int_{P_i} \varphi(u(x)) dx = \int_{\Omega} \varphi(u(x)) dx.$$

Remark 3.1. Lemma 3.1 implies in particular that A_G is a contraction,

$$||A_G||_{L^p \to L^p} \le 1, \quad 1 \le p < \infty.$$

The second property of A_G is that it maps subgradients of J to subgradients. Recall that $G(\Omega)$ is the smallest grid covering the entire boundary of Ω .

Theorem 3.2. Let G be a grid containing $G(\Omega)$. Then $A_G(\partial J(0)) \subset \partial J(0)$.

Proof. The proof is divided into three steps, each being proved in a separate lemma

$$A_G(\partial J(0)) \overset{\text{Lem. } 3.3}{\subset} \Gamma_G \subset \Gamma \overset{\text{Lem. } 3.4}{\subset} \overline{\Gamma_c} \overset{\text{Lem. } 3.5}{\subset} \partial J(0).$$

Note that the inclusion $\Gamma_G \subset \Gamma$ is trivial.

Lemma 3.3. Let G be a grid containing $G(\Omega)$. Then $A_G(\partial J(0)) \subset \Gamma_G$.

Proof. Throughout this proof we exploit the fact that $\partial J(0) = \overline{\operatorname{div} \mathcal{B}_1}$, recall equation (5).

First, note that PCR_G is a finite-dimensional subspace of L^2 and that the sets Γ_G^i , i = 1, ..., n, are bounded and closed subsets of PCR_G . It follows that Γ_G is a closed subset of PCR_G and in particular of L^2 . Therefore, it suffices to show $A_G(\operatorname{div} \mathcal{B}_1) \subset \Gamma_G$, as we then have $A_G(\operatorname{div} \mathcal{B}_1) \subset \overline{\Lambda_G} = \Gamma_G$, because A_G is continuous.

Take $H = (H_1, \ldots, H_n) \in \mathcal{B}_1$. We want to show that $A_G \partial H_i / \partial x_i \in \Gamma_G^i$, that is,

$$\sup_{s \in \left(a_{i}^{k}, b_{i}^{k}\right)} \left| \int_{a_{i}^{k}}^{s} A_{G} \frac{\partial H_{i}}{\partial x_{i}} dx_{i} \right| \leq 1, \quad \text{and} \quad \int_{a_{i}^{k}}^{b_{i}^{k}} A_{G} \frac{\partial H_{i}}{\partial x_{i}} dx_{i} = 0,$$

for i = 1, ..., n and each k, where (a_i^k, b_i^k) are the intervals defined in equation (3).

Consider the second integral first. From the definition of A_G it follows that (a_i^k, b_i^k) can be divided into a finite number of subintervals in such a way that the integrand is constant on each. In addition the assumption $G \supset G(\Omega)$ implies that the partition $\mathcal{Q}(G)$ consists of hyperrectangles only. Thus, after a potential relabelling of the $R_i \in \mathcal{Q}(G)$, we can write

$$\int_{a_i^k}^{b_i^k} A_G \frac{\partial H_i}{\partial x_i} dx_i = \sum_{j=1}^M \int_{s_{j-1}}^{s_j} \left(\frac{1}{|R_j|} \int_{R_j} \frac{\partial H_i}{\partial x_i} dx \right) dx_i$$
$$= \sum_{j=1}^M \frac{s_j - s_{j-1}}{|R_j|} \int_{R_j} \frac{\partial H_i}{\partial x_i} dx$$

for some $M \in \{1, \ldots, N\}$ and $a_i^k = s_0 < s_1 < \cdots < s_M = b_i^k$. Note that $|R_j|/(s_j - s_{j-1})$ is the (n-1)-dimensional volume of $\partial R_j \cap \partial R_{j+1}$, and that this volume is independent of $j \in \{1, \ldots, M\}$. In other words, the hyperrectangles R_j only differ in their extent in x_i -direction, compare Figure 1, bottom right. The reason is that $\mathcal{Q}(G)$ is not an arbitrary partition of Ω into hyperrectangles, but rather formed by a grid. Setting $C = (s_j - s_{j-1})/|R_j|$ we further obtain

$$= C \int_{\bigcup_j R_j} \frac{\partial H_i}{\partial x_i} \, dx$$

The remaining integral can be computed by turning it into an iterated one, integrating with respect to x_i first, and recalling that H is compactly supported

$$= C \underbrace{\int \cdots \int}_{n-1} \int_{a_i^k}^{b_i^k} \frac{\partial H_i}{\partial x_i} dx_i = C \int \cdots \int H_i \Big|_{x_i = a_i^k}^{x_i = b_i^k} = 0.$$

Here $F|_{x_i=a}^{x_i=b}$ stands for $F(x_i = b) - F(x_i = a)$, where $F(x_i = c)$ denotes the restriction of F to the affine hyperplane defined by $x_i = c$.

Now integrate up to an arbitrary $s \in (a_i^k, b_i^k]$. We can assume $s \in (s_{\ell-1}, s_{\ell}]$ for some $\ell \in \{1, \ldots, M\}$ and a brief computation similar to the one above shows that

$$\int_{a_{i}^{k}}^{s} A_{G} \frac{\partial H_{i}}{\partial x_{i}} dx_{i} = C \int \cdots \int H_{i} \Big|_{x_{i} = a_{i}^{k}}^{x_{i} = s_{\ell-1}} + \frac{s - s_{\ell-1}}{|R_{\ell}|} \int \cdots \int H_{i} \Big|_{x_{i} = s_{\ell-1}}^{x_{i} = s}$$
$$= C \int \cdots \int H_{i} (x_{i} = s_{\ell-1}) + \frac{s - s_{\ell-1}}{|R_{\ell}|} \int \cdots \int H_{i} \Big|_{x_{i} = s_{\ell-1}}^{x_{i} = s}.$$

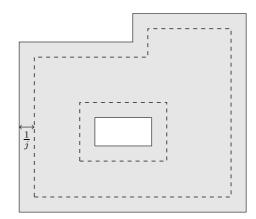


Figure 3: The construction of Ω_j by removing strips of width 1/j from Ω .

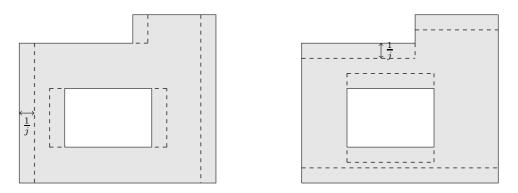


Figure 4: The construction of Ω_j^1 (left) and Ω_j^2 (right) by removing strips from Ω_j .

Recalling that we can write $C = (s_{\ell} - s_{\ell-1})/|R_{\ell}|$ we rearrange terms

$$= \frac{s_{\ell} - s}{|R_{\ell}|} \int \cdots \int H_i(x_i = s_{\ell-1}) + \frac{s - s_{\ell-1}}{|R_{\ell}|} \int \cdots \int H_i(x_i = s).$$

Finally, we estimate $H_i \leq 1$ and obtain

$$\leq \frac{s_{\ell} - s}{|R_{\ell}|} \frac{|R_{\ell}|}{s_{\ell} - s_{\ell-1}} + \frac{s - s_{\ell-1}}{|R_{\ell}|} \frac{|R_{\ell}|}{s_{\ell} - s_{\ell-1}} = 1.$$

Similarly, we get $\int_{a_i^k}^s A_G \frac{\partial H_i}{\partial x_i} dx_i \ge -1$. Thus we have $A_G \partial H_i / \partial x_i \in \Gamma_G^i$.

Lemma 3.4. $\Gamma \subset \overline{\Gamma_c}$.

Proof. Let $j \in \mathbb{N}$ and define $\Omega_j \subset \Omega$ by removing strips of width 1/j from the boundary of Ω . It is assumed that j is chosen large enough such that the strips are contained in Ω . See Figure 3 for an example of the construction of Ω_j in the plane. Next, for $i = 1, \ldots, n$ we define $\Omega_j^i \subset \Omega_j$, by removing strips of width 1/j from those parts of the boundary of Ω_j which are orthogonal to the x_i -axis. By choosing j large enough, the strips will be contained in Ω_j . For an illustration of the construction of Ω_j^i in the plane, see Figure 4.

Take $h \in \Gamma$. So, $h \in \Gamma_G$ for some $G \in \mathcal{G}$ and in particular $h = \sum_{i=1}^n h_i$ where

 $h_i \in \Gamma_G^i$. Let $g_j = \sum_{i=1}^n g_{j,i}$ where

$$g_{j,i}(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \Omega_j, \\ 2h_i(x), & \text{if } x \in \Omega_j \setminus \Omega_j^i, \\ h_i(x), & \text{otherwise.} \end{cases}$$

Note that there is a grid $G_j \supset G$ such that $g_{j,i} \in PCR_{G_j}$ for every $i = 1, \ldots, n$, and that for j large enough

$$\left| \int_{a_i^k}^s g_{j,i} \, dx_i \right| \le \left| \int_{a_i^k}^s h_i \, dx_i \right|$$

for every interval (a_i^k, b_i^k) , recall equation (3), and $s \in (a_i^k, b_i^k]$. It follows that $g_{j,i} \in \Gamma_{G_j,c}^i$ and therefore $g_j \in \Gamma_{G_j,c}$. Finally, it can be directly verified that

$$\lim_{j \to \infty} \|g_j - h\|_{L^2} = 0$$

As $h \in \Gamma$ was chosen arbitrarily we conclude that $\Gamma \subset \overline{\Gamma_c}$.

Lemma 3.5. $\overline{\Gamma_c} \subset \partial J(0)$.

Proof. As in Lemma 3.3 we use the fact that $\partial J(0) = \overline{\operatorname{div} \mathcal{B}_1}$.

Take $h \in \Gamma_c$. So there is a grid G such that $h = \sum_{i=1}^n h_i \in \Gamma_{G,c}$ where $h_i \in \Gamma_{G,c}^i$. From h we now construct a vector field $H = (H_1, \ldots, H_n)$. For every $i \in \{1, \ldots, n\}$ and $x \in \Omega$ there is a unique interval $I_i^k = (a_i^k, b_i^k)$ containing x_i , recall equation (3). Based on this observation we define the components of H by

$$H_i(x) = \int_{a_i^k}^{x_i} h_i(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_n) \, ds.$$

It follows that $||H_i||_{L^{\infty}} \leq 1$ and $\operatorname{supp}(H_i) \subset \Omega$. H is now modified into a vector field belonging to \mathcal{B}_1 . Let $\{\rho_j\}_{j\in\mathbb{N}}$ denote a sequence of mollifiers on \mathbb{R}^n supported on the closed Euclidean ball centred at 0 with radius 1/j. Recalling standard results regarding convolution and mollifiers, we derive $||H_i * \rho_j||_{L^{\infty}} \leq ||H_i||_{L^{\infty}} ||\rho_j||_{L^1} =$ $||H_i||_{L^{\infty}} \leq 1$ and moreover, for j large enough, $H_i * \rho_j \in C_c^{\infty}(\Omega)$. Hence, for $j \in \mathbb{N}$ large enough, the modification H_{ρ_j} of H given by

$$H_{\rho_j} = (H_1 * \rho_j, \dots, H_n * \rho_j)$$

is in \mathcal{B}_1 . It follows that $h \in \overline{\operatorname{div} \mathcal{B}_1}$, as

$$\begin{split} \left\| \operatorname{div} H_{\rho_j} - h \right\|_{L^2} &= \left\| \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} (H_i * \rho_j) - h_i \right) \right\|_{L^2} = \left\| \sum_{i=1}^n (h_i * \rho_j - h_i) \right\|_{L^2} \\ &\leq \sum_{i=1}^n \left\| h_i * \rho_j - h_i \right\|_{L^2} \xrightarrow{j \to \infty} 0. \end{split}$$

The element $h \in \Gamma_c$ was chosen arbitrarily and $\overline{\operatorname{div} \mathcal{B}_1}$ is closed, therefore $\overline{\Gamma_c} \subset \overline{\operatorname{div} \mathcal{B}_1}$.

3.1 Preservation of piecewise constancy

We are now ready to prove the following result.

Theorem 3.6. Given $f \in PCR$ with minimal grid G_f , the minimizer u_{α} of the corresponding anisotropic ROF functional

$$\min_{u \in L^2(\Omega)} \frac{1}{2} \|u - f\|_{L^2}^2 + \alpha J(u)$$

lies in PCR_{G_f} .

Proof. Recall that, according to Lemma 2.1, u_{α} is the unique element with minimal L^2 -norm in $f - \alpha \partial J(0)$. From Theorem 3.2 and the fact that $A_{G_f}f = f$ it follows that also $A_{G_f}u_{\alpha} \in f - \alpha \partial J(0)$. As $||A_{G_f}u_{\alpha}||_{L^2} \leq ||u_{\alpha}||_{L^2}$, because of Remark 3.1, we have $A_{G_f}u_{\alpha} = u_{\alpha}$. Therefore, $u_{\alpha} \in PCR_{G_f}$.

4 Conclusion

In [11, Thm. 5] the authors have shown that, for Ω being a rectilinear 2-polytope, $f \in PCR$ implies $u_{\alpha} \in PCR$. We have extended this preservation of piecewise constancy to rectilinear *n*-polytopes. Our proof can be summarized in the following way

$$\|A_{G_f}u_{\alpha}\|_{L^2} \stackrel{\text{Lem. 3.1}}{\leq} \|u_{\alpha}\|_{L^2} \stackrel{\text{Lem. 2.1}}{=} \min_{u \in f - \alpha \partial J(0)} \|u\|_{L^2} \stackrel{\text{Thm. 3.2}}{\leq} \|A_{G_f}u_{\alpha}\|_{L^2}.$$

The crucial step is Theorem 3.2, asserting that

$$A_G(\partial J(0)) \subset \partial J(0),$$

which exploits the fact that the anisotropy of J is compatible with the rectilinearity of the grid G.

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