# Preservation of Piecewise Constancy under TV Regularization with Rectilinear Anisotropy* 

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#### Abstract

A recent result by Lasica, Moll and Mucha about the $\ell^{1}$-anisotropic Rudin-Osher-Fatemi model in $\mathbb{R}^{2}$ asserts that the solution is piecewise constant on a rectilinear grid, if the datum is. By means of a new proof we extend this result to $\mathbb{R}^{n}$. The core of our proof consists in showing that averaging operators associated to certain rectilinear grids map subgradients of the $\ell^{1}$-anisotropic total variation seminorm to subgradients.


## 1 Introduction

This article is concerned with a variant of the Rudin-Osher-Fatemi (ROF) image denoising model [13]. More specifically, we consider minimization of

$$
\begin{equation*}
\frac{1}{2}\|u-f\|_{L^{2}}^{2}+\alpha J(u) \tag{1}
\end{equation*}
$$

where $J(u)=\int_{\Omega}\|\nabla u(x)\|_{\ell^{1}} d x$ is the total variation with $\ell^{1}$-anisotropy. This model and variations thereof have been used in imaging applications for data exhibiting a rectilinear geometry [2, 7, 14, 15]. Numerical algorithms for minimizing (11) have been studied, for example, in [6, 10, 12].

The $\ell^{1}$-anisotropic total variation has a special property from a theoretical point of view as well. It has been shown in 3, Thm. 3.4, Rem. 3.5] that approximation of a general $u \in B V \cap L^{p}$ by functions $u_{m}$ piecewise constant on rectilinear grids, in the sense that

$$
\left\|u-u_{m}\right\|_{L^{p}} \rightarrow 0 \quad \text { and } \quad J\left(u_{m}\right) \rightarrow J(u)
$$

is not possible for $J(u)=\int_{\Omega}\|\nabla u(x)\|_{\ell^{q}} d x$, unless $q=1$.
Let $\Omega \subset \mathbb{R}^{n}$ be a finite union of hyperrectangles, each aligned with the coordinate axes. Our main result, Theorem [3.6 states that if the given function $f: \Omega \rightarrow \mathbb{R}$ is piecewise constant on a rectilinear grid, then the minimizer of (11) is too. This extends a recent result by Lasica, Moll and Mucha about two-dimensional domains [11. Thm. 5]. Their proof is based on constructing the solution by means of its level sets and relies on minimization of an anisotropic Cheeger-type functional over subsets of $\Omega$.

[^0]The proof we present below is centred around the averaging operator $A_{G}$ associated to the grid $G$ on which $f$ is piecewise constant. In addition to being a contraction, it has the crucial property of mapping subgradients of $J$ to subgradients, that is, $A_{G}(\partial J(0)) \subset \partial J(0)$, see Theorem 3.2. Combined with the dual formulation of (11) we obtain that the minimizer must be piecewise constant on the same grid as $f$. While it might be possible to extend the techniques of [11 to higher dimensions, we believe that modifying the so-called "squaring step" in the proof of [11, Lem. 2] could lead to difficulties.

Theorem 3.6 implies that, if $f$ is piecewise constant on a rectilinear grid, then minimization of functional (11) becomes a finite-dimensional problem. More precisely, in this case the solution can be found by minimizing a discrete energy of the form

$$
\sum_{i} w_{i}\left|u_{i}-f_{i}\right|^{2}+\alpha \sum_{i, j} w_{i j}\left|u_{i}-u_{j}\right|
$$

where the weights $w_{i}, w_{i j} \geq 0$ depend only on the grid. For problems of this sort there are many efficient algorithms, such as graph cuts [4, 5, 8]. Extending the preservation of piecewise constancy to domains $\Omega \subset \mathbb{R}^{n}, n \geq 3$, means that the discrete reformulation can also be exploited for processing higher dimensional data such as volumetric images or videos.

This article is organized as follows. Section 2 contains the basic concepts that will be required throughout. In Section 2.1 we define several spaces of piecewise constant functions, while Section 2.2 is devoted to the $\ell^{1}$-anisotropic ROF model. Section 3 is the main part of this paper. It starts with introducing the averaging operator $A_{G}$ and ends with Theorem [3.6. The article is concluded in Section 4 .

## 2 Mathematical preliminaries

### 2.1 PCR functions

In this section we introduce several notions related to functions which are piecewise constant on rectilinear subsets of $\mathbb{R}^{n}$. Some of these are $n$-dimensional analogues of notions from [11, Sec. 2.3].

A bounded set $R \subset \mathbb{R}^{n}$ which can be written as a Cartesian product of $n$ proper intervals is called an $n$-dimensional hyperrectangle. Recall that an interval is proper, if it is neither empty nor a singleton. Finite unions of $n$-dimensional hyperrectangles will be referred to as rectilinear n-polytopes.

A rectilinear grid, or simply grid, is a finite family of affine hyperplanes, each being perpendicular to one of the coordinate axes of $\mathbb{R}^{n}$. For a rectilinear $n$-polytope $P$ we denote by $G(P)$ the smallest grid with the property that the union of all its affine hyperplanes contains the entire boundary of $P$.

Throughout this article $\Omega \subset \mathbb{R}^{n}$ is an open rectilinear $n$-polytope. A finite family of rectilinear $n$-polytopes $\mathcal{Q}=\left\{P_{1}, \ldots, P_{N}\right\}$ is called a partition of $\Omega$, if they have pairwise disjoint interiors and the union of their closures equals $\bar{\Omega}$. Every grid $G$ defines a partition $\mathcal{Q}(G)$ of $\Omega$ into rectilinear $n$-polytopes in the following way: $P \subset \Omega$ belongs to $\mathcal{Q}(G)$, if and only if its boundary is contained in $\partial \Omega \cup \bigcup G$ while int $P$ and $\bigcup G$ are disjoint. Note that if $G$ contains $G(\Omega)$, then $\mathcal{Q}(G)$ consists only of hyperrectangles.

We adopt the notation $P C R(\Omega)$, or simply $P C R$, from 11 for the set of all integrable functions $f: \Omega \rightarrow \mathbb{R}$ which can be written as finite linear combinations of indicator functions of rectilinear $n$-polytopes. That is, $f \in P C R$ if there is an


Figure 1: Upper left: A PCR function $f$ on a planar domain $\Omega$. Each level set of $f$ is visualized using a different grey tone. Upper right: The boundaries of the level sets of $f$. Lower left: Extension of the boundaries of the level sets of $f$. Lower right: The minimal grid $G_{f}$ (lines) and the partition $\mathcal{Q}\left(G_{f}\right)$ of $\Omega$ (rectangular cells).
$N \in \mathbb{N}, c_{i} \in \mathbb{R}$ and rectilinear $n$-polytopes $P_{i} \subset \Omega, 1 \leq i \leq N$, such that

$$
\begin{equation*}
f=\sum_{i=1}^{N} c_{i} \mathbf{1}_{P_{i}} \tag{2}
\end{equation*}
$$

almost everywhere. Here, $\mathbf{1}_{A}$ is the indicator function of the set $A$, defined by

$$
\mathbf{1}_{A}(x)= \begin{cases}1, & x \in A \\ 0, & x \notin A\end{cases}
$$

We can assume, without loss of generality, that the values $c_{i}$ are pairwise distinct and that the polytopes $P_{i}$ are pairwise disjoint, which makes the representation (2) unique almost everywhere. To every $f \in P C R$ we associate its minimal grid, that is, the unique smallest grid covering the boundaries of all level sets of $f$, given by

$$
G_{f}=\bigcup_{i=1}^{N} G\left(P_{i}\right)
$$

Note that the partition $\mathcal{Q}\left(G_{f}\right)$ always consists of hyperrectangles only. See Figure 1 for an illustration of $G_{f}$ and $\mathcal{Q}\left(G_{f}\right)$.

For a given grid $G$ we denote by $P C R_{G}$ the set of all functions in $P C R$ which are equal almost everywhere to a finite linear combination of indicator functions of $P_{i} \in \mathcal{Q}(G)$.

The following notions are essential for the proof of Theorem 3.2. Fix an $i \in$ $\{1, \ldots, n\}$ as well as coordinates $x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}$. The set $\left\{x_{i} \in \mathbb{R}\right.$ : $\left.\left(x_{1}, \ldots, x_{n}\right) \in \Omega\right\}$ is a union of finitely many disjoint intervals

$$
\begin{equation*}
I_{i}^{k}=\left(a_{i}^{k}, b_{i}^{k}\right), \quad k=1, \ldots, m \tag{3}
\end{equation*}
$$

Note that the number of intervals $m$ as well as the intervals $I_{i}^{k}$ themselves depend on, and are uniquely determined by, the coordinates $x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}$. For an illustration of the intervals $I_{i}^{k}$ see Example 2.1 below. Next, let $\mathcal{G}$ be the set of all rectilinear grids of $\mathbb{R}^{n}$. For every $G \in \mathcal{G}$ and $i \in\{1, \ldots, n\}$ we define

$$
\Gamma_{G}^{i}=\left\{g \in P C R_{G}: \sup _{s \in\left(a_{i}^{k}, b_{i}^{k}\right)}\left|\int_{a_{i}^{k}}^{s} g d x_{i}\right| \leq 1, \int_{a_{i}^{k}}^{b_{i}^{k}} g d x_{i}=0,1 \leq k \leq m\right\}
$$

The restrictions on $g$ are to be understood for almost every

$$
\left(x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}\right) \in \mathbb{R}^{n-1}
$$

such that there is an $x_{i} \in \mathbb{R}$ satisfying $\left(x_{1}, \ldots, x_{n}\right) \in \Omega$. The sum of the spaces $\Gamma_{G}^{i}$ is denoted by

$$
\Gamma_{G}=\left\{\sum_{i=1}^{n} g_{i}: g_{i} \in \Gamma_{G}^{i}, 1 \leq i \leq n\right\}
$$

and we further set

$$
\Gamma=\bigcup_{G \in \mathcal{G}} \Gamma_{G}
$$

Remark 2.1. The set $\Gamma_{G}$ consists of divergences of certain piecewise affine vector fields. More precisely, Theorem 3.2 below implies that $\Gamma_{G}=\partial J(0) \cap P C R_{G}$, that is, $\Gamma_{G}$ is the set of all subgradients of $J$ which are piecewise constant on $G$.

Finally, those elements of $\Gamma_{G}^{i}$ which have compact support in $\Omega$ are collected in the set $\Gamma_{G, c}^{i}$, and we define analogously

$$
\begin{aligned}
\Gamma_{G, c} & =\left\{\sum_{i=1}^{n} g_{i}: g_{i} \in \Gamma_{G, c}^{i}, 1 \leq i \leq n\right\} \\
\Gamma_{c} & =\bigcup_{G \in \mathcal{G}} \Gamma_{G, c} .
\end{aligned}
$$

Example 2.1. For the rectilinear 2-polytope $\Omega$ of Figure $\mathbf{Q}^{2}$ we have the following intervals $I_{1}^{k}$ and $I_{2}^{k}$

$$
\left\{x_{1} \in \mathbb{R}:\left(x_{1}, x_{2}\right) \in \Omega\right\}= \begin{cases}(0,6), & \text { if } x_{2} \in(0,1) \cup(2,3), \\ (0,2) \cup(4,6), & \text { if } x_{2} \in[1,2], \\ (3,6), & \text { if } x_{2} \in[3,4),\end{cases}
$$

and

$$
\left\{x_{2} \in \mathbb{R}:\left(x_{1}, x_{2}\right) \in \Omega\right\}= \begin{cases}(0,3), & \text { if } x_{1} \in(0,2) \\ (0,1) \cup(2,3), & \text { if } x_{1} \in[2,3] \\ (0,1) \cup(2,4), & \text { if } x_{1} \in(3,4] \\ (0,4), & \text { if } x_{1} \in(4,6)\end{cases}
$$

### 2.2 The $\ell^{1}$-anisotropic ROF model

The notion of anisotropic total variation was introduced in 1]. In this article we exclusively consider one particular variant.


Figure 2: A rectilinear 2-polytope $\Omega$.

For every $\alpha>0$ we denote by $\mathcal{B}_{\alpha}$ the set of all smooth compactly supported vector fields on $\Omega$ whose components are bounded by $\alpha$, that is,

$$
\mathcal{B}_{\alpha}=\left\{H \in C_{c}^{\infty}\left(\Omega, \mathbb{R}^{n}\right): \max _{1 \leq i \leq n}\left|H_{i}(x)\right| \leq \alpha, \forall x \in \Omega\right\}
$$

The $\ell^{1}$-anisotropic total variation $J: L^{2}(\Omega) \rightarrow \mathbb{R} \cup\{+\infty\}$ is given by

$$
\begin{equation*}
J(u)=\sup _{H \in \mathcal{B}_{1}} \int_{\Omega} u \operatorname{div} H d x=\sup _{h \in \operatorname{div} \mathcal{B}_{1}} \int_{\Omega} u h d x \tag{4}
\end{equation*}
$$

where the bar denotes closure in $L^{2}(\Omega)$. Thus, $J$ is the support function of the closed and convex set $\overline{\operatorname{div} \mathcal{B}_{1}}$, which implies that $\overline{\operatorname{div} \mathcal{B}_{1}}=\partial J(0)$, or more generally

$$
\begin{equation*}
\overline{\operatorname{div} \mathcal{B}_{\alpha}}=\alpha \partial J(0) \tag{5}
\end{equation*}
$$

for every $\alpha>0$. If $u$ is a Sobolev function, then $J(u)=\int_{\Omega}\|\nabla u(x)\|_{\ell^{1}} d x$.
The next lemma states that the $\ell^{1}$-anisotropic ROF model is equivalent to constrained $L^{2}$-minimization. However, the way it is formulated it actually applies to every support function $J$ of a closed and convex subset of $L^{2}(\Omega)$.

Lemma 2.1. For every $\alpha>0$ and $f \in L^{2}(\Omega)$ the minimization problem

$$
\begin{equation*}
\min _{u \in L^{2}(\Omega)} \frac{1}{2}\|u-f\|_{L^{2}}^{2}+\alpha J(u) \tag{6}
\end{equation*}
$$

is equivalent to

$$
\min _{u \in f-\alpha \partial J(0)}\|u\|_{L^{2} .} .
$$

Proof. The dual problem associated to (6) is given by

$$
\begin{equation*}
\min _{w \in L^{2}(\Omega)} \frac{1}{2}\|w-f\|_{L^{2}}^{2}+(\alpha J)^{*}(w) \tag{7}
\end{equation*}
$$

where the asterisk stands for convex conjugation. The two solutions $u_{\alpha}$ and $w_{\alpha}$ of (6) and (7), respectively, satisfy the optimality conditions

$$
\begin{align*}
& u_{\alpha}=f-w_{\alpha} \\
& w_{\alpha} \in \partial(\alpha J)\left(u_{\alpha}\right) . \tag{8}
\end{align*}
$$

Concerning the derivation of (7) and (8) we refer to [9, Chap. III, Rem. 4.2]. Since $\alpha J$ is the support function of the set $\alpha \partial J(0)$, recall (4), its conjugate is the characteristic function

$$
(\alpha J)^{*}(w)= \begin{cases}0, & w \in \alpha \partial J(0) \\ +\infty, & w \notin \alpha \partial J(0)\end{cases}
$$

Therefore, problem (7) is equivalent to

$$
\min _{w \in \alpha \partial J(0)}\|w-f\|_{L^{2}} .
$$

Finally, using the optimality condition (8) we get

$$
\left\|u_{\alpha}\right\|_{L^{2}}=\left\|w_{\alpha}-f\right\|_{L^{2}}=\min _{w \in \alpha \partial J(0)}\|w-f\|_{L^{2}}=\min _{u \in f-\alpha \partial J(0)}\|u\|_{L^{2}}
$$

## 3 The averaging operator $A_{G}$

Let $\Omega$ be a rectilinear $n$-polytope and $G$ a grid. Define the averaging operator $A_{G}: L^{1}(\Omega) \rightarrow P C R_{G}(\Omega)$ by

$$
A_{G} g=\sum_{i=1}^{N}\left(\frac{1}{\left|P_{i}\right|} \int_{P_{i}} g(s) d s\right) \mathbf{1}_{P_{i}}
$$

where $P_{i} \in \mathcal{Q}(G)$ and $\left|P_{i}\right|$ is its $n$-dimensional volume.
Two properties of the operator $A_{G}$ turn out to be important when establishing the main result of this paper, Theorem 3.6. The first one is

Lemma 3.1. For every $u \in L^{1}(\Omega)$ and convex $\varphi: \mathbb{R} \rightarrow \mathbb{R}$

$$
\int_{\Omega} \varphi\left(\left(A_{G} u\right)(x)\right) d x \leq \int_{\Omega} \varphi(u(x)) d x .
$$

Proof. By applying Jensen's inequality we obtain

$$
\begin{aligned}
\int_{\Omega} \varphi\left(\left(A_{G} u\right)(x)\right) d x & =\sum_{i=1}^{N} \int_{P_{i}} \varphi\left(\left(A_{G} u\right)(x)\right) d x=\sum_{i=1}^{N} \varphi\left(\frac{1}{\left|P_{i}\right|} \int_{P_{i}} u(x) d x\right)\left|P_{i}\right| \\
& \leq \sum_{i=1}^{N} \int_{P_{i}} \varphi(u(x)) d x=\int_{\Omega} \varphi(u(x)) d x
\end{aligned}
$$

Remark 3.1. Lemma 3.1 implies in particular that $A_{G}$ is a contraction,

$$
\left\|A_{G}\right\|_{L^{p} \rightarrow L^{p}} \leq 1, \quad 1 \leq p<\infty
$$

The second property of $A_{G}$ is that it maps subgradients of $J$ to subgradients. Recall that $G(\Omega)$ is the smallest grid covering the entire boundary of $\Omega$.

Theorem 3.2. Let $G$ be a grid containing $G(\Omega)$. Then $A_{G}(\partial J(0)) \subset \partial J(0)$.
Proof. The proof is divided into three steps, each being proved in a separate lemma

$$
A_{G}(\partial J(0)) \stackrel{\text { Lem. }}{\subset}{ }^{3.3} \Gamma_{G} \subset \Gamma \stackrel{\text { Lem. }}{\subset}{ }^{3.4} \overline{\Gamma_{c}} \stackrel{\text { Lem. }}{\subset}
$$

Note that the inclusion $\Gamma_{G} \subset \Gamma$ is trivial.

Lemma 3.3. Let $G$ be a grid containing $G(\Omega)$. Then $A_{G}(\partial J(0)) \subset \Gamma_{G}$.
Proof. Throughout this proof we exploit the fact that $\partial J(0)=\overline{\operatorname{div} \mathcal{B}_{1}}$, recall equation (5).

First, note that $P C R_{G}$ is a finite-dimensional subspace of $L^{2}$ and that the sets $\Gamma_{G}^{i}, i=1, \ldots, n$, are bounded and closed subsets of $P C R_{G}$. It follows that $\Gamma_{G}$ is a closed subset of $P C R_{G}$ and in particular of $L^{2}$. Therefore, it suffices to show $A_{G}\left(\operatorname{div} \mathcal{B}_{1}\right) \subset \Gamma_{G}$, as we then have $A_{G}\left(\overline{\operatorname{div} \mathcal{B}_{1}}\right) \subset \overline{A_{G}\left(\operatorname{div} \mathcal{B}_{1}\right)} \subset \overline{\Gamma_{G}}=\Gamma_{G}$, because $A_{G}$ is continuous.

Take $H=\left(H_{1}, \ldots, H_{n}\right) \in \mathcal{B}_{1}$. We want to show that $A_{G} \partial H_{i} / \partial x_{i} \in \Gamma_{G}^{i}$, that is,

$$
\sup _{s \in\left(a_{i}^{k}, b_{i}^{k}\right)}\left|\int_{a_{i}^{k}}^{s} A_{G} \frac{\partial H_{i}}{\partial x_{i}} d x_{i}\right| \leq 1, \quad \text { and } \quad \int_{a_{i}^{k}}^{b_{i}^{k}} A_{G} \frac{\partial H_{i}}{\partial x_{i}} d x_{i}=0
$$

for $i=1, \ldots, n$ and each $k$, where $\left(a_{i}^{k}, b_{i}^{k}\right)$ are the intervals defined in equation (31).
Consider the second integral first. From the definition of $A_{G}$ it follows that $\left(a_{i}^{k}, b_{i}^{k}\right)$ can be divided into a finite number of subintervals in such a way that the integrand is constant on each. In addition the assumption $G \supset G(\Omega)$ implies that the partition $\mathcal{Q}(G)$ consists of hyperrectangles only. Thus, after a potential relabelling of the $R_{j} \in \mathcal{Q}(G)$, we can write

$$
\begin{aligned}
\int_{a_{i}^{k}}^{b_{i}^{k}} A_{G} \frac{\partial H_{i}}{\partial x_{i}} d x_{i} & =\sum_{j=1}^{M} \int_{s_{j-1}}^{s_{j}}\left(\frac{1}{\left|R_{j}\right|} \int_{R_{j}} \frac{\partial H_{i}}{\partial x_{i}} d x\right) d x_{i} \\
& =\sum_{j=1}^{M} \frac{s_{j}-s_{j-1}}{\left|R_{j}\right|} \int_{R_{j}} \frac{\partial H_{i}}{\partial x_{i}} d x
\end{aligned}
$$

for some $M \in\{1, \ldots, N\}$ and $a_{i}^{k}=s_{0}<s_{1}<\cdots<s_{M}=b_{i}^{k}$. Note that $\left|R_{j}\right| /\left(s_{j}-\right.$ $\left.s_{j-1}\right)$ is the $(n-1)$-dimensional volume of $\partial R_{j} \cap \partial R_{j+1}$, and that this volume is independent of $j \in\{1, \ldots M\}$. In other words, the hyperrectangles $R_{j}$ only differ in their extent in $x_{i}$-direction, compare Figure 1, bottom right. The reason is that $\mathcal{Q}(G)$ is not an arbitrary partition of $\Omega$ into hyperrectangles, but rather formed by a grid. Setting $C=\left(s_{j}-s_{j-1}\right) /\left|R_{j}\right|$ we further obtain

$$
=C \int_{\bigcup_{j} R_{j}} \frac{\partial H_{i}}{\partial x_{i}} d x
$$

The remaining integral can be computed by turning it into an iterated one, integrating with respect to $x_{i}$ first, and recalling that $H$ is compactly supported

$$
=C \underbrace{\int \cdots \int}_{n-1} \int_{a_{i}^{k}}^{b_{i}^{k}} \frac{\partial H_{i}}{\partial x_{i}} d x_{i}=\left.C \int \cdots \int H_{i}\right|_{x_{i}=a_{i}^{k}} ^{x_{i}=b_{i}^{k}}=0 .
$$

Here $\left.F\right|_{x_{i}=a} ^{x_{i}=b}$ stands for $F\left(x_{i}=b\right)-F\left(x_{i}=a\right)$, where $F\left(x_{i}=c\right)$ denotes the restriction of $F$ to the affine hyperplane defined by $x_{i}=c$.

Now integrate up to an arbitrary $s \in\left(a_{i}^{k}, b_{i}^{k}\right]$. We can assume $s \in\left(s_{\ell-1}, s_{\ell}\right]$ for some $\ell \in\{1, \ldots, M\}$ and a brief computation similar to the one above shows that

$$
\begin{aligned}
\int_{a_{i}^{k}}^{s} A_{G} \frac{\partial H_{i}}{\partial x_{i}} d x_{i} & =\left.C \int \cdots \int H_{i}\right|_{x_{i}=a_{i}^{k}} ^{x_{i}=s_{\ell-1}}+\left.\frac{s-s_{\ell-1}}{\left|R_{\ell}\right|} \int \cdots \int H_{i}\right|_{x_{i}=s_{\ell-1}} ^{x_{i}=s} \\
& =C \int \cdots \int H_{i}\left(x_{i}=s_{\ell-1}\right)+\left.\frac{s-s_{\ell-1}}{\left|R_{\ell}\right|} \int \cdots \int H_{i}\right|_{x_{i}=s_{\ell-1}} ^{x_{i}=s}
\end{aligned}
$$



Figure 3: The construction of $\Omega_{j}$ by removing strips of width $1 / j$ from $\Omega$.


Figure 4: The construction of $\Omega_{j}^{1}$ (left) and $\Omega_{j}^{2}$ (right) by removing strips from $\Omega_{j}$.

Recalling that we can write $C=\left(s_{\ell}-s_{\ell-1}\right) /\left|R_{\ell}\right|$ we rearrange terms

$$
=\frac{s_{\ell}-s}{\left|R_{\ell}\right|} \int \cdots \int H_{i}\left(x_{i}=s_{\ell-1}\right)+\frac{s-s_{\ell-1}}{\left|R_{\ell}\right|} \int \cdots \int H_{i}\left(x_{i}=s\right)
$$

Finally, we estimate $H_{i} \leq 1$ and obtain

$$
\leq \frac{s_{\ell}-s}{\left|R_{\ell}\right|} \frac{\left|R_{\ell}\right|}{s_{\ell}-s_{\ell-1}}+\frac{s-s_{\ell-1}}{\left|R_{\ell}\right|} \frac{\left|R_{\ell}\right|}{s_{\ell}-s_{\ell-1}}=1 .
$$

Similarly, we get $\int_{a_{i}^{k}}^{s} A_{G} \frac{\partial H_{i}}{\partial x_{i}} d x_{i} \geq-1$. Thus we have $A_{G} \partial H_{i} / \partial x_{i} \in \Gamma_{G}^{i}$.

Lemma 3.4. $\Gamma \subset \overline{\Gamma_{c}}$.
Proof. Let $j \in \mathbb{N}$ and define $\Omega_{j} \subset \Omega$ by removing strips of width $1 / j$ from the boundary of $\Omega$. It is assumed that $j$ is chosen large enough such that the strips are contained in $\Omega$. See Figure 3 for an example of the construction of $\Omega_{j}$ in the plane. Next, for $i=1, \ldots, n$ we define $\Omega_{j}^{i} \subset \Omega_{j}$, by removing strips of width $1 / j$ from those parts of the boundary of $\Omega_{j}$ which are orthogonal to the $x_{i}$-axis. By choosing $j$ large enough, the strips will be contained in $\Omega_{j}$. For an illustration of the construction of $\Omega_{j}^{i}$ in the plane, see Figure (4)

Take $h \in \Gamma$. So, $h \in \Gamma_{G}$ for some $G \in \mathcal{G}$ and in particular $h=\sum_{i=1}^{n} h_{i}$ where
$h_{i} \in \Gamma_{G}^{i}$. Let $g_{j}=\sum_{i=1}^{n} g_{j, i}$ where

$$
g_{j, i}(x)= \begin{cases}0, & \text { if } x \in \Omega \backslash \Omega_{j} \\ 2 h_{i}(x), & \text { if } x \in \Omega_{j} \backslash \Omega_{j}^{i} \\ h_{i}(x), & \text { otherwise }\end{cases}
$$

Note that there is a grid $G_{j} \supset G$ such that $g_{j, i} \in P C R_{G_{j}}$ for every $i=1, \ldots, n$, and that for $j$ large enough

$$
\left|\int_{a_{i}^{k}}^{s} g_{j, i} d x_{i}\right| \leq\left|\int_{a_{i}^{k}}^{s} h_{i} d x_{i}\right|
$$

for every interval $\left(a_{i}^{k}, b_{i}^{k}\right)$, recall equation (3), and $s \in\left(a_{i}^{k}, b_{i}^{k}\right]$. It follows that $g_{j, i} \in \Gamma_{G_{j}, c}^{i}$ and therefore $g_{j} \in \Gamma_{G_{j}, c}$. Finally, it can be directly verified that

$$
\lim _{j \rightarrow \infty}\left\|g_{j}-h\right\|_{L^{2}}=0
$$

As $h \in \Gamma$ was chosen arbitrarily we conclude that $\Gamma \subset \overline{\Gamma_{c}}$.
Lemma 3.5. $\overline{\Gamma_{c}} \subset \partial J(0)$.
Proof. As in Lemma 3.3 we use the fact that $\partial J(0)=\overline{\operatorname{div} \mathcal{B}_{1}}$.
Take $h \in \Gamma_{c}$. So there is a grid $G$ such that $h=\sum_{i=1}^{n} h_{i} \in \Gamma_{G, c}$ where $h_{i} \in \Gamma_{G, c}^{i}$. From $h$ we now construct a vector field $H=\left(H_{1}, \ldots, H_{n}\right)$. For every $i \in\{1, \ldots, n\}$ and $x \in \Omega$ there is a unique interval $I_{i}^{k}=\left(a_{i}^{k}, b_{i}^{k}\right)$ containing $x_{i}$, recall equation (3). Based on this observation we define the components of $H$ by

$$
H_{i}(x)=\int_{a_{i}^{k}}^{x_{i}} h_{i}\left(x_{1}, \ldots, x_{i-1}, s, x_{i+1} \ldots, x_{n}\right) d s
$$

It follows that $\left\|H_{i}\right\|_{L^{\infty}} \leq 1$ and $\operatorname{supp}\left(H_{i}\right) \subset \Omega$. $H$ is now modified into a vector field belonging to $\mathcal{B}_{1}$. Let $\left\{\rho_{j}\right\}_{j \in \mathbb{N}}$ denote a sequence of mollifiers on $\mathbb{R}^{n}$ supported on the closed Euclidean ball centred at 0 with radius $1 / j$. Recalling standard results regarding convolution and mollifiers, we derive $\left\|H_{i} * \rho_{j}\right\|_{L^{\infty}} \leq\left\|H_{i}\right\|_{L^{\infty}}\left\|\rho_{j}\right\|_{L^{1}}=$ $\left\|H_{i}\right\|_{L^{\infty}} \leq 1$ and moreover, for $j$ large enough, $H_{i} * \rho_{j} \in C_{c}^{\infty}(\Omega)$. Hence, for $j \in \mathbb{N}$ large enough, the modification $H_{\rho_{j}}$ of $H$ given by

$$
H_{\rho_{j}}=\left(H_{1} * \rho_{j}, \ldots, H_{n} * \rho_{j}\right)
$$

is in $\mathcal{B}_{1}$. It follows that $h \in \overline{\operatorname{div} \mathcal{B}_{1}}$, as

$$
\begin{aligned}
\left\|\operatorname{div} H_{\rho_{j}}-h\right\|_{L^{2}} & =\left\|\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_{i}}\left(H_{i} * \rho_{j}\right)-h_{i}\right)\right\|_{L^{2}}=\left\|\sum_{i=1}^{n}\left(h_{i} * \rho_{j}-h_{i}\right)\right\|_{L^{2}} \\
& \leq \sum_{i=1}^{n}\left\|h_{i} * \rho_{j}-h_{i}\right\|_{L^{2}} \xrightarrow{j \rightarrow \infty} 0
\end{aligned}
$$

The element $h \in \Gamma_{c}$ was chosen arbitrarily and $\overline{\operatorname{div} \mathcal{B}_{1}}$ is closed, therefore $\overline{\Gamma_{c}} \subset$ $\overline{\operatorname{div} \mathcal{B}_{1}}$.

### 3.1 Preservation of piecewise constancy

We are now ready to prove the following result.

Theorem 3.6. Given $f \in P C R$ with minimal grid $G_{f}$, the minimizer $u_{\alpha}$ of the corresponding anisotropic ROF functional

$$
\min _{u \in L^{2}(\Omega)} \frac{1}{2}\|u-f\|_{L^{2}}^{2}+\alpha J(u)
$$

lies in $P C R_{G_{f}}$.
Proof. Recall that, according to Lemma 2.1 $u_{\alpha}$ is the unique element with minimal $L^{2}$-norm in $f-\alpha \partial J(0)$. From Theorem 3.2 and the fact that $A_{G_{f}} f=f$ it follows that also $A_{G_{f}} u_{\alpha} \in f-\alpha \partial J(0)$. As $\left\|A_{G_{f}} u_{\alpha}\right\|_{L^{2}} \leq\left\|u_{\alpha}\right\|_{L^{2}}$, because of Remark 3.1, we have $A_{G_{f}} u_{\alpha}=u_{\alpha}$. Therefore, $u_{\alpha} \in P C R_{G_{f}}$.

## 4 Conclusion

In [11, Thm. 5] the authors have shown that, for $\Omega$ being a rectilinear 2-polytope, $f \in P C R$ implies $u_{\alpha} \in P C R$. We have extended this preservation of piecewise constancy to rectilinear $n$-polytopes. Our proof can be summarized in the following way

$$
\left\|A_{G_{f}} u_{\alpha}\right\|_{L^{2}} \stackrel{\text { Lem. } 3.1}{\leq}\left\|u_{\alpha}\right\|_{L^{2}} \stackrel{\text { Lem. [2.1 }}{=} \min _{u \in f-\alpha \partial J(0)}\|u\|_{L^{2}} \stackrel{\text { Thm. }}{\leq}\left\|A_{G_{f}} u_{\alpha}\right\|_{L^{2}}
$$

The crucial step is Theorem 3.2, asserting that

$$
A_{G}(\partial J(0)) \subset \partial J(0)
$$

which exploits the fact that the anisotropy of $J$ is compatible with the rectilinearity of the grid $G$.

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