# Complexity of conjunctive regular path query homomorphisms* 

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#### Abstract

A graph database is a digraph whose arcs are labeled with symbols from a fixed alphabet. A regular graph pattern (RGP) is a digraph whose edges are labeled with regular expressions over the alphabet. RGPs model navigational queries for graph databases called conjunctive regular path queries (CRPQs). A match of a CRPQ in the database is witnessed by a special navigational homomorphism of the corresponding RGP to the database. We study the complexity of deciding the existence of a homomorphism between two RGPs. Such homomorphisms model a strong type of containment between the two corresponding CRPQs. We show that this problem can be solved by an EXPTIME algorithm (while general query containmement in this context is EXPSPACE-complete). We also study the problem for restricted RGPs over a unary alphabet, that arise from some applications like XPath or SPARQL. For this case, homomorphism-based CRPQ containment is in NP. We prove that certain interesting cases are in fact polynomial-time solvable.


## 1 Introduction

Graphs are a fundamental way to store and organize data. Most prominently, graph database systems have been developed for three decades and are widely used [37; recently, such systems have seen an increased interest both in academic research and in the industry [3]. A graph database can be seen as a directed graph with arc-labels (possibly also vertex-labels). Various methods are used to retrieve data in such systems, see for example the recently developed graph query languages G-CORE [2] and CYPHER 19 for graph databases, with one of the earliest ones being G 11. Classically, matching queries in graph databases can be modeled as graph homomorphisms [26]. In this setting, a query is itself a graph, and a match is modeled by a homomorphism of the query graph to the database graph, that is, a vertex-mapping that preserves the graph adjacencies and labels. Graph databases can be very large, thus it is important to study the algorithmic complexity of such queries. In modern applications, classic homomorphisms are often not powerful enough to model realistic graph data queries, indeed their nature is inherently local. In recent years, navigational queries, or path queries, have been developed 3]. Such queries are more powerful than classical queries, since they allow for non-local pattern matching, by means of arbitrary paths or walks instead of arcs. Such queries can also be modeled as a more general kind of homomorphism, that we call navigational homomorphisms. The most studied type of navigational queries is the one of regular path queries, that is based on regular expressions [3, 18, 29]. The study of the algorithmic complexity of such homomorphisms has been recently initiated in 33. In this paper, we continue this study by focusing on a strong form of navigational query containment, modeled by navigational homomorphisms between two queries. We study the general complexity of deciding the existence of a regular path navigational homomorphism between two graph queries. We also study some more restricted cases arising from relevant applications.

[^0]Graph databases, classical queries and homomorphisms. Let $\Sigma$ be a fixed countable alphabet. A graph database $B$ over $\Sigma$ is an arc-labeled digraph. More formally, we have $B=\left(D_{B}, E_{B}\right)$ where $D_{B}$ is a finite digraph with vertex set $V\left(D_{B}\right)$ and arc set $A\left(D_{B}\right)$, and $E_{B}: A\left(D_{B}\right) \rightarrow \Sigma$ is an arc-label function.

One type of queries of graph databases can be expressed as homomorphisms, see for example [26]. In this setting, a query $Q=\left(D_{Q}, E_{Q}\right)$ for a graph database $B=\left(D_{B}, E_{B}\right)$ over $\Sigma$ is an arc-labeled digraph over $\Sigma$. (As noted in the survey [15], in some applications, graph databases and queries may have vertexlabels, or even more complicated structures; however, such cases can in fact always be reduced to the setting of arc-labeled digraphs.)

We say that $B$ matches $Q$ if there exists a homomorphism of $Q$ to $B$, that is, an arc- and labelpreserving vertex-mapping. More formally, such a homomorphism is a mapping $f$ of $V\left(D_{Q}\right)$ to $V\left(D_{B}\right)$ with the property that for every $\operatorname{arc}(x, y)$ in $D_{Q}$, there is an $\operatorname{arc}(f(x), f(y))$ in $D_{B}$ with $E_{Q}(x, y)=$ $E_{B}(f(x), f(y))$. If such a homomorphism exists, we note $Q \rightarrow B$. Every homomorphic image $f(Q)$ of $Q$ to $B$ is a match of the query $Q$ in $B$. We refer to the book [22] for more details on the theory of graph homomorphisms.

Note that queries and graph databases are modeled by the same kinds of objects, thus we can also consider homomorphisms between queries. This is a way to model query containment. We say that a query $Q_{2}$ is contained in query $Q_{1}$ if for every graph database $B$, if $Q_{2} \rightarrow B$ then $Q_{1} \rightarrow B$. Note that if $Q_{1} \rightarrow Q_{2}$, then for any graph database $B$, to every homomorphism of $Q_{2}$ to $B$ corresponds a homomorphism of $Q_{1}$ to $B$ (by transitivity of homomorphism). Thus $Q_{2}$ is contained in $Q_{1}$, and homomorphism can be seen as a strong form of containment.

The most basic algorithmic problems related to queries of graph databases are the evaluation problem, which consists in deciding whether a query $Q$ has a match in a database $B$, and the containment problem, that is, to decide whether a query $Q_{1}$ is contained in a query $Q_{2}$. These two problems can be modeled by the following decision problem.

> | Hom |
| :--- |
| Instance: Two arc-labeled digraphs $G$ and $H$. |
| Question: Does $G$ admit a homomorphism to $H$ ? |

HOM is generally NP-complete, even when $H$ is a small fixed graph (for example a symmetric triangle, in which case Hom is equivalent to the graph 3 -colourability problem). To understand better the complexity of Hom, the following version has been studied extensively, where $H$ is a fixed arc-labeled digraph, called the non-uniform homomorphism problem. The graph $H$ is called the template.

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Hom(H)
Instance: An arc-labeled digraph G.
Question: Does G admit a homomorphism to H?
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As shown in [14], the set of $\operatorname{Hom}(H)$ problems captures the whole class of constraint satisfaction problems (CSPs), whose complexity has recently been classified into polynomial cases and NP-complete cases in [9, 38] independently.

Navigational queries, navigational homomorphisms and RGPs. In a digraph $D$, a directed walk is a sequence of arcs of the digraph, such that the head of each arc is the same vertex as the tail of the next arc. A directed path is a directed walk where each vertex occurs in at most two arcs in the sequence.

Standard homomorphisms are not powerful enough to model many types of queries used in modern graph database systems. In particular, a homomorphism of a query $Q$ to a database $B$ can only match a subgraph of $B$ that is no larger than the query $Q$ itself. To the contrary, navigational queries are types of queries where we may allow arbitrarily large subgraphs of the database to match the query. In this setting, we still model the query $Q$ (for database $B$ ) as an arc-labeled digraph, but the arcs are labeled with sets of words over the alphabet $\Sigma$, rather than single letters. Now, a match of $Q$ in $B$ is a vertex-mapping $f$ from $V\left(D_{Q}\right)$ to $V\left(D_{B}\right)$ such that for an $\operatorname{arc}(x, y)$ of $Q$ labeled with a set $E(x, y)$ of words, there exists a directed walk $W_{x y}$ in $D_{B}$ from $f(x)$ to $f(y)$ such that the concatenation of labels of the arcs of $W_{x y}$ is a word of $E(x, y)$.

[^1]Perhaps the most popular navigational queries are regular path queries (RPQs), studied in many contexts [3, 4, 12, 17, 25, 30, 33. These navigational queries are based on regular expressions: the labels on query arcs are regular expressions over the alphabet $\Sigma$. The advantage of considering such queries is that regular languages are a relatively simple yet powerful way of defining sets of words, that is both wellunderstood and sufficiently expressive for many applications. The combination of regular path queries of this type over the same variables is called a conjunctive regular path query (CRPQ) 3] and the underlying graph structure representing it is a regular graph pattern (RGP) [33]. We refer to the recent survey [15] for more details on this model.

For a fixed countable alphabet $\Sigma$, we denote by $\operatorname{Reg} \operatorname{Exp}(\Sigma)$ the set of regular expressions over alphabet $\Sigma$, with the symbols + (union),$*$ (Kleene star), and $\cdot($ concatenation; sometimes this symbol is omitted). Moreover, for a regular expression $X$, as a notation we let $X^{+}:=X \cdot X^{*}$. For any regular expression $X$ in $\operatorname{Reg} \operatorname{Exp}(\Sigma)$, we denote by $L(X)$ the regular language defined by $X$.

A RGP $P$ over an alphabet $\Sigma$ is a pair $\left(D_{P}, E_{P}\right)$, where $D_{P}$ is a digraph with vertex set $V\left(D_{P}\right)$ and arc set $A\left(D_{P}\right)$ and $E_{P}: A\left(D_{P}\right) \rightarrow \operatorname{Reg} \operatorname{Exp}(\Sigma)$ is an arc-label function.

Given a directed walk $W=a_{1,2} \ldots a_{k-1, k}$ in a RGP $P$, the label $E_{P}(W)$ of $W$ is the regular expression over $\Sigma$ formed by the concatenation $E_{P}\left(a_{1,2}\right) \ldots E_{P}\left(a_{k-1, k}\right)$.

We now define homomorphisms of RGPs. Given two RGPs $P$ and $Q$ over alphabet $\Sigma$, a navigational homomorphism ( $n$-homomorphism for short) of $P$ to $Q$ is a mapping $f$ of $V\left(D_{P}\right)$ to $V\left(D_{Q}\right)$ such that for each arc $(x, y)$ in $D_{P}$, there is a directed walk $W$ in $Q$ from $f(x)$ to $f(y)$ such that the language $L\left(E_{Q}(W)\right)$ is contained in the language $L\left(E_{P}(x, y)\right)$. When such an n-homomorphism exists, we write $P \xrightarrow{n} Q$.

This type of homomorphism was studied in 4, 33, see also 32. This notion has also been called embedding [16, a term that originated in the context of interconnection networks [13].

Note that this definition also applies to graph databases since, mathematically speaking, a graph database is a RGP whose labels are all regular expressions consisting of a unique symbol of $\Sigma$. We may thus define the associated decision problem, that correponds to the task of CRPQ evaluation.

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RGPHom
Instance: Two RGPs P and Q.
Question: Does P admit an n-homomorphism to Q?
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Again, we may also study the non-uniform version of RGPHom, defined as follows for a fixed RGP $Q$. It was introduced in 33 for the restricted case where $Q$ is a graph database.

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RGPHom(Q)
Instance: A RGP P.
Question: Does P admit an n-homomorphism to Q?
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In [33, the authors study $\operatorname{RGPHom}(Q)$ for the specific case where $Q$ is a fixed graph database. They make a connexion to the classic homomorphism problem $\operatorname{Hom}(H)$ and show that $\operatorname{RGPHom}(Q)$ admits a complexity dichotomy: for a specific $Q, \operatorname{RGPHom}(Q)$ is either polynomial-time or NP-complete. Indeed, they showed it to be equivalent to the (classical) homomorphism problems a.k.a. the Constraint Satisfaction Problems [14, whose complexity delineation follows a dichotomy based on specific algebraic properties of the template $Q$, as shown independently by Bulatov 9 and Zhuk 38 .

In this paper, we initiate the study of $\operatorname{RGPHom}(Q)$ in full generality, that is when $Q$ is not just a graph database, but any RGP. As we will see, for this case we cannot expect a polynomial-time/NPcomplete dichotomy in the style of the result of [33], since RGPHom $(Q)$ is in fact PSPACE-hard already for very simple cases, as it can model the problem of deciding the inclusion between regular languages. Thus, it appears that RGPHom $(Q)$ where $Q$ is a general RGP merits more investigation.

Also note that the complexity of general CRPQ containment (that is, not necessarily witnessed by a n-homomorphism) has been studied and is known to be EXPSPACE-complete [10, 18] (some special cases enjoy a lower complexity, see [16, and some extensions are even more intractable [31). The homomorphism-based version of CRPQ containment does not capture CRPQ containment in its full generality (see Figure 1 for an example). However, as we will see, the advantage is that $\operatorname{RGPHom}(Q)$ is computationally easier.

Our results and structure of the paper. We show in Section 3 that RGPHom and related problems are PSPACE-hard (this holds even for $\operatorname{RGPHom}(Q)$ for a simple template $Q$ ), by a reduction from Regular Language Inclusion.

We then show in Section 4 that RGPHom is decidable by an EXPTIME algorithm. This shows that checking homomorphism-based CRPQ containment is computationally more efficient than checking general CRPQ containment, which is known to be EXPSPACE-complete 10, 18.

Finally, in Section 5, we address the simpler case of a unary alphabet $\Sigma=\{a\}$, and where all arc labels are either " $a$ " or " $a^{+}$". This includes not only all classic homomorphism problems and CSPs, but also some kinds of queries over hierarchical data such as SPARQL and XPath, studied for example in [12, 17, 25, 30, (it is known that most queries used in practice are very simple [7]). For this type of RGPs, RGPHOM is in NP. We give a polynomial/NP-complete complexity dichotomy for RGPHOM $(Q)$ in the case of undirected (or symmetric) RGPs $Q$ of this class, in the style of Hell and Nešetřil's dichotomy for $\operatorname{Hom} H$ [20]. Furthermore, we show that even for arbitrary (directed) RGPs $Q$, RGPHom $(Q)$ follows a dichotomy, by relating it to (classical) homomorphism problems. We then relate the case of path templates that have only " $a$ " labels to an interesting (polynomial-time solvable) parallel scheduling problem. Finally, we show that for all directed path RGP templates $Q$ with arc labels " $a$ " or " $a+$ ", $\operatorname{RGPHom}(Q)$ is polynomial-time solvable. This special case is motivated by previous studies modelling XPath and SPARQL queries.

We start with some preliminary considerations in Section 2 and we conclude in Section 6 .

## 2 Preliminaries

We now give some definitions and useful results from the literature.

### 2.1 Regular languages

Given a fixed alphabet $\Sigma$, a language is a set of words over $\Sigma$. A regular expression over $\Sigma$ is defined recursively as follows. A symbol of $\Sigma$ is a regular expression. Given two regular expressions $E_{1}$ and $E_{2}$, $\left(E_{1}\right)+\left(E_{2}\right),\left(E_{1}\right) \cdot\left(E_{2}\right)$ and $\left(E_{1}\right)^{*}$ are regular expressions. For a regular expression $E$, we let $L(E)$ be the language associated to $E$ in the classic way, where "+" denotes the union, "." denotes the concatenation, and $" * "$ is the Kleene star. A language regular if it is the language associated to some regular expression. It is well-known that a language is regulat if and only if it can be recognized by a nondeterministic finite automaton (NFA). We refer to [35] for further details on these matters.

We will use the following decision problems for regular languages.
Regular Language Inclusion
Instance: Two regular expressions $E_{1}$ and $E_{2}$ (over the same alphabet).
Question: Is $L\left(E_{1}\right) \subseteq L\left(E_{2}\right)$ ?

Regular Language Universality
Instance: A regular expression $E$ over alphabet $\Sigma$.
Question: Is $L(E)=\Sigma^{*}$ ?
Note that Regular Language Universality is the special case of Regular Language IncluSION where $E_{1}=\Sigma^{*}$ and $E_{2}=E$. The following are classic results.

Theorem 1 ([1, 36). Regular Language Universality and Regular Language Inclusion are PSPACE-complete.

### 2.2 Cores and n-cores

A digraph $D$ is called a core if it does not admit a homomorphism to a proper sub-digraph of itself; in other words, every endomorphism is an automorphism. Thus, we define similarly the notion of a navigational core, n -core for short: a RGP $P$ is an n -core if it does not admit a n-homomorphism to a proper sub-RGP of itself. Cores and n-cores are useful because of the following fact (this is classic in the case of graphs and their cores, see the book [22]).

Proposition 2. Let $P$ be an $R G P$ that is not an $n$-core and $C$, a sub- $R G P$ of $P$ such that $P \xrightarrow{n} C$. Then, for any $R G P Q$, we have $P \xrightarrow{n} Q$ if and only if $C \xrightarrow{n} Q$.

Proof. It is clear that $P$ and $C$ are n-homomorphically equivalent $(P \xrightarrow{n} C$ by assumption and $C \xrightarrow{n} P$ by the identity map, since $C$ is a sub-RGP of $P$ ). Thus, by transitivity of n-homomorphisms, any n-homomorphism of one of them to $Q$ directly translates into an n-homomorphism of the other to $Q$.

Thus, when studying the problem $\operatorname{RGPHom}(Q)$, we may always assume that $Q$ is an n-core, since $\operatorname{RGPHom}(Q)$ has the same complexity as $\operatorname{RGPHom}\left(C_{Q}\right)$, where $C_{Q}$ is a sub-RGP of $Q$ that is an ncore. Unfortunately, it is coNP-complete to decide whether a graph is a core 21 (thus deciding whether a RGP is an $n$-core is coNP-hard, even if it is a graph database).

Note that, with respect to classic digraph homomorphisms, any digraph $G$ has (up to isomorphism) a unique minimal subgraph to which it admits a homomorphism, called the core of $G$. This is not the case for RGPs and n-homomorphisms. For example, any two RGPs each consisting of a unique directed cycle with all arc labels equal to " $a^{+}$" have an n-homomorphism to each other. Thus, if we consider two such cycles of different lengths and glue them at one vertex, we obtain a RGP $P$ with two minimal sub-RGPs of $P$ (the two cycles) to which $P$ has an n-homomorphism, and these two are not isomorphic. A similar phenomenon has been observed in the case of a special subclass of directed tree RGPs, where being an n -core has been called non-redundancy 12 .

### 2.3 Graph homomorphisms dichotomy

The following classic dichotomy result for $\operatorname{Hom}(H)$ problems will be useful.
Theorem 3 (Hell and Nešetřil [20]). For any undirected graph $H, \operatorname{Hom}(H)$ is polynomial if $H$ is bipartite or contains a loop, and NP-complete otherwise.

## 3 PSPACE-hardness of RGPHom and related problems

In this section, we show that RGPHom and related problems are PSPACE-hard.

### 3.1 The general RGPHom problem

We first present a very simple reduction from Regular Language Inclusion to RGPHom.
Proposition 4. RGPHOM is PSPACE-hard.
Proof. We reduce from Regular Language Inclusion, which is PSPACE-complete (Theorem 1). Given an input $E_{1}, E_{2}$ of Regular Language Inclusion, we construct two RGPs $P_{1}$ and $P_{2}$ on two vertices each, where $P_{1}$ contains a single arc labelled $E_{1}$ and $P_{2}$, a single arc labelled $E_{2}$. Now, we have $L\left(E_{1}\right) \subseteq L\left(E_{2}\right)$ if and only if $P_{2} \xrightarrow{n} P_{1}$.

As witnessed by the simplicity of the reduction given in Proposition 4, the PSPACE-hardness of RGPHOM is inherently caused by the hardness of the underlying regular language problem.

### 3.2 The non-uniform case

We now show that the non-uniform version of the problem still remains PSPACE-hard, even for a very simple template.

Proposition 5. Let $\Sigma$ be a fixed alphabet of size at least 2, and let $D_{2}^{\Sigma}$ be the $R G P$ of order 2 over $\Sigma$ consisting of a single arc labelled $\Sigma^{*}$. Then, $\operatorname{RGPHom}\left(D_{2}^{\Sigma}\right)$ is PSPACE-complete.

Proof. To see that the problem is in PSPACE, note that the homomorphism part of the problem is simple. Let $P$ be the input RGP. An n-homomorphism of $P$ to $D_{2}^{\Sigma}$ exists if and only if the underlying digraph of $P$ maps to a single arc (this is the case if and only if it is a balanced digraph of height 1 ), and for every arc label $E$ of $P, \Sigma^{*} \subseteq L(E)$ (that is, $\Sigma^{*}=L(E)$ ). Since Regular Language Universality is in PSPACE ( 1 , see Theorem [1), $\operatorname{RGPHom}\left(D_{2}^{\Sigma}\right)$ is in PSPACE.

To show that RGPHom $\left(D_{2}^{\Sigma}\right)$ is PSPACE-hard, we reduce Regular Language Universality to it. Given an input $E$ (over alphabet $\Sigma$ ) of Regular Language Universality, we construct the RGP $P$ on two vertices consisting of a single arc labelled $E$. Now, we have $L(E)=\Sigma^{*}$ if and only if $P \xrightarrow{n} D_{2}^{\Sigma}$.

### 3.3 Testing for being an n-core

We now give a similar reduction that shows that testing if an RGP is an n-core is also PSPACE-hard. We define the following decision problem:

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N-Core
Instance: A RGP P
Question: Is P an n-core?
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## Proposition 6. N-Core is PSPACE-hard.

Proof. We reduce Regular Language Inclusion to N-Core. Given an input $E_{1}, E_{2}$ of Regular Language Inclusion (over alphabet $\Sigma$ ), we construct an RGP $P\left(E_{1}, E_{2}\right)$ over alphabet $\Sigma \cup\{X\}$, where $X \notin \Sigma$. We have $V(P)=\{x, y, z\}$ and $P$ contains an arc $(x, y)$ labelled $E_{1}$ and an arc $(x, z)$ labelled $X+E_{2}$. Now, we have $L\left(E_{1}\right) \subseteq L\left(E_{2}\right)$ if and only if $P$ is not a core. Since PSPACE=coPSPACE, we are done.

## 4 An EXPTIME algorithm for RGPHom

In this section, we show that RGPHom is decidable by an EXPTIME algorithm.
Note that in certain models where simple directed paths rather than directed walks are considered, like in [29, or when the target RGP is acyclic, there is a simple PSPACE algorithm to decide RGPHOM. Indeed, in those cases, the length of a walk is polynomial. Assume we want to check whether $P \xrightarrow{n} Q$. We can iterate over all possible mappings and all possible walks: for a mapping $f: V\left(D_{P}\right) \rightarrow V\left(D_{Q}\right)$ and, for each mapped pair $(\{x, y\},\{f(x), f(y)\})$ of vertices and each walk $W$ from $f(x)$ to $f(y)$, we check in polynomial space whether $L\left(E_{Q}(W)\right) \subseteq E_{P}(x, y)$.

However, in general, the walks may be arbitrarily long. As we will see, we can still bound their maximum length and give an EXPTIME algorithm for RGPHom.

For a regular language $L$ over alphabet $\Sigma$ and a positive integer $n$, we denote by $L_{\mid n}$ the $n$-truncation of $L$, that is, the set of words of $L$ whose length is at most $n$.

Lemma 7. Let $A, B_{1}, \ldots B_{k}$ be a collection of regular expressions over alphabet $\Sigma$, and let $n_{A}, n_{i}$ be the minimum number of states of an NFA recognizing $L(A)$ and $L\left(B_{i}\right)$, respectively. Then, we have that $L\left(B_{1}\right) \cdots L\left(B_{k}\right) \subseteq L(A)$ if, and only if, $L\left(B_{1}\right)_{\mid n_{A} n_{1}} \cdots L\left(B_{k}\right)_{\mid n_{A} n_{k}} \subseteq L(A)$.
Proof. It is clear that if $L\left(B_{1}\right) \cdots L\left(B_{k}\right) \subseteq L(A)$, then also $L\left(B_{1}\right)_{\mid n_{A} n_{1}} \cdots L\left(B_{k}\right)_{\mid n_{A} n_{k}} \subseteq L(A)$, since $L\left(B_{i}\right)_{\mid n_{A} n_{i}} \subseteq L\left(B_{i}\right)$ for every $i$ with $1 \leq i \leq k$.

For the converse, we assume that $L\left(B_{1}\right)_{\mid n_{A} n_{1}} \cdots L\left(B_{k}\right)_{\mid n_{A} n_{k}} \subseteq L(A)$. That is, any word $w_{1} \cdots w_{k}$ of $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$ with $\left|w_{i}\right| \leq n_{A} n_{i}$ for every $i$ with $1 \leq i \leq k$, belongs to $L(A)$. We need to prove that all words of $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$ (without length restriction) belong to $L(A)$.

We proceed by induction on the vectors of subword lengths of words in $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$. For such a word $w_{1} \cdots w_{k}$, this associated vector is $\left(\left|w_{1}\right|, \ldots,\left|w_{k}\right|\right)$, and these vectors are ordered lexicographically. The induction hypothesis is that all words of $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$ whose associated vector is at most $\left(l_{1}, \ldots, l_{k}\right)$ (where for any $i$ with $1 \leq i \leq k, l_{i}$ is a positive integer), belongs to $A$. By our assumption, the case where $l_{i} \geq n_{A} n_{i}$ is true.

Now, consider a word $w=w_{1} \cdots w_{k}$ of $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$, whose associated vector is $\left(\left|w_{1}\right|, \ldots,\left|w_{k}\right|\right)$, and where for some $j \in\{1, \ldots, k\},\left|w_{j}\right|=l_{j}+1$; whenever $i \neq j,\left|w_{i}\right| \leq l_{i}$. Let $\mathcal{A}$ and $\mathcal{A}_{j}$ be two NFAs recognizing $A$ and $B_{j}$ with smallest numbers $n_{A}$ and $n_{j}$ of states, respectively.

We consider the product automaton $\mathcal{A} \times \mathcal{A}_{j}$ of $\mathcal{A}$ and $\mathcal{A}_{j}$, with set of states $S \times S_{j}$ (where $S$ and $S_{j}$ are the sets of states of $\mathcal{A}$ and $\mathcal{A}_{j}$, respectively), and a transition $\left(\left(s_{1}, s_{2}\right), a,\left(s_{1}^{\prime}, s_{2}^{\prime}\right)\right)$ only if we have the transitions $\left(s_{1}, a, s_{1}^{\prime}\right)$ and $\left(s_{2}, a, s_{2}^{\prime}\right)$ in $\mathcal{A}$ and $\mathcal{A}_{j}$, respectively (all other transitions are "dummy transitions" to a "garbage state"). Consider the run of $\mathcal{A} \times \mathcal{A}_{j}$ for the word $w_{j}$. The crucial observation
is that, because $\left|w_{j}\right|=l_{j}+1>n_{A} n_{j}$, this run necessarily visits two states of $\mathcal{A} \times \mathcal{A}_{j}$ twice, that is, the run contains a directed cycle. Consider the shorter run obtained by pruning this cycle. The two runs start and end at the same two states of $\mathcal{A} \times \mathcal{A}_{j}$. The shorter run corresponds to a word $w_{j}^{\prime}$ of length at most $\left|w_{j}\right|-1 \leq l_{j}$. Since $w_{j} \in L\left(B_{j}\right)$, the end state of these runs is a pair containing an accepting state of $\mathcal{A}_{j}$ (thus $w_{j}^{\prime}$ belongs to $L\left(B_{j}\right)$ as well). Thus, the word $w^{\prime}$ obtained from $w$ by replacing $w_{j}$ with $w_{j}^{\prime}$ belongs to $L\left(B_{1}\right) \cdots L\left(B_{k}\right)$, and $w^{\prime}$ satisfies the induction hypothesis. Thus, $w^{\prime}$ belongs to $L(A)$. But now, considering the pruned cycle in $\mathcal{A} \times \mathcal{A}_{j}$, we can build a valid run for $w_{j}$ in $\mathcal{A} \times \mathcal{A}_{j}$ that leads to a valid run for $w$ in $\mathcal{A}$. This proves the inductive step and concludes the proof.

We will now apply Lemma 7 to the case of matching a regular expression to a walk in an RGP.
Proposition 8. Let $E$ be a regular expression over alphabet $\Sigma$, and $\mathcal{A}_{E}$ an $N F A$ with $n_{E}$ states recognizing $L(E)$. Let $Q=(D, E)$ be an $R G P$ over $\Sigma$. For any two vertices $u$ and $v$ in $Q$, we can compute a walk $W$ from $u$ to $v$ satisfying $L(E(W)) \subseteq L(E)$ (if one exists), in time $2^{O\left(n_{E}|Q| \log (|E|+|Q|)\right)}$. Moreover, if such a walk exists, then there exists one of length at most $2^{n_{E}}|Q|$.

Proof. Since $E$ and $Q$ are finite, we will assume that $|\Sigma| \leq|E|+|Q|$ (if not, we simply remove the unused symbols from $\Sigma$.)

By Lemma 7 there is a walk $W$ in $Q$ from $u$ to $v$ such that $L(E(W)) \subseteq L(E)$ if and only if there exists one in the RGP $Q^{\prime}$ obtained from $Q$ by replacing each arc-label $E(x, y)$ by a regular expression defining the $n_{E} n_{B}$-truncation $L(E(x, y))^{\prime}$ of $L\left(E(x, y)\right.$ ) (where $n_{B}$ is the smallest number of states of an NFA recognizing $L(E(x, y)))$. Thus, we first compute $Q^{\prime}$. Note that $L(E(x, y))^{\prime}$ contains at most $|\Sigma|^{n_{E} n_{B}}$ words.

Next, we will construct an auxiliary digraph $G(E, Q, u, v)$. This digraph has vertex set $2^{\mathcal{S}} \times V(Q)$, where $\mathcal{S}$ is the set of states of $\mathcal{A}_{E}$.

Given two states $s_{1}$ and $s_{2}$ of $\mathcal{A}_{E}$ and a word $w$ over $\Sigma$, we say that $w$ reaches $s_{2}$ from $s_{1}$ in $\mathcal{A}_{E}$ if there exists a sequence of transitions of $\mathcal{A}_{E}$ starting at $s_{1}$ and ending at $s_{2}$ using the sequence of letters of $w$.

Now, for two vertices $\left(S_{1}, u\right)$ and $\left(S_{2}, v\right)$ of $G(E, Q, u, v)$, we create the $\operatorname{arc}\left(\left(S_{1}, x\right),\left(S_{2}, y\right)\right)$ if and only if, for each state $s$ of $S_{1}$ and each word $w$ of $L(E(x, y))^{\prime}, w$ reaches a state of $S_{2}$ in $\mathcal{A}_{E}$.

Deciding whether $\left(\left(S_{1}, x\right),\left(S_{2}, y\right)\right)$ is an arc of $G(E, Q, u, v)$ takes time at most $\left|S_{1}\right| 2^{n_{E}}\left|L(E(x, y))^{\prime}\right|$, which is at most $|\Sigma|^{O\left(n_{E}|Q|\right)}$. Since there are $\left(2^{n_{E}}|Q|\right)^{2}$ pairs of vertices of $G(E, Q, u, v)$, overall the construction of $G(E, Q, u, v)$ can be done in time $|\Sigma|^{O\left(n_{E}|Q|\right)}$.

Now, we claim that there exists a walk $W$ from $u$ to $v$ with $L(E(W)) \subseteq L(E(x, y))$ if and only if there is a directed path in $G(E, Q, u, v)$ from a vertex $\left(\left\{s_{0}\right\}, x\right)$ to a vertex $\left(S_{f}, y\right)$, where $s_{0}$ is the initial state of $\mathcal{A}_{E}$, and $S_{f}$ is a subset of the accepting states of $\mathcal{A}_{E}$. Indeed, such a path corresponds precisely to a walk $W$ from $u$ to $v$ in $Q$, such that all the words of $L(W)$ are accepted by $\mathcal{A}_{E}$.

This check can be done in linear time in the size of $G(E, Q, u, v)$ using a standard BFS search, thus we obtain an additional time complexity of $\left(2^{n_{E}}|Q|\right)^{2}$, which is also at most $|\Sigma| O\left(n_{E}|Q|\right)$. Since $|\Sigma| \leq|E|+|Q|$ we obtain $2^{O\left(n_{E}|Q| \log (|E|+|Q|)\right)}$.

Finally, it is clear that the length of an obtained directed path of $G(E, Q, u, v)$ is at most the number of vertices of $G(E, Q, u, v)$, which is $2^{n_{E}}|Q|$, as claimed. This completes the proof.

We are now ready to prove the main theorem of this section.

## Theorem 9. RGPHom is in EXPTIME.

Proof. We proceed as follows. First, we go through all possible vertex-mappings of $V(P)$ to $V(Q)$ (there are $|V(Q)|^{|V(P)|}$ such possible mappings). Consider such a vertex-mapping, $f$.

For each arc $(x, y)$ in $P$ with label $E(x, y)$, we proceed as follows. Let $\mathcal{A}$ be an NFA recognizing $L(E(x, y))$ with smallest possible number $n_{A}$ of states. We apply Proposition 8 to $E(x, y), \mathcal{A}$ and $Q$, with $u=f(x)$ and $v=f(y)$ : thus we can decide in time $2^{O\left(n_{A}|Q| \log (|E(x, y)|+|Q|)\right)}$ whether the mapping $f$ satisfies the definition of an n-homomorphism for the $\operatorname{arc}(x, y)$. If yes, we proceed to the next arc; otherwise, we abort and try the next possible mapping. If we find a valid mapping, we return YES. Otherwie, we return NO.

Our algorithm has a time complexity of $|V(Q)|^{|V(P)|} \cdot|P| \cdot 2^{O(|P||Q| \log (|P|+|Q|))}$. Let $n=|P|+|Q|$ be the input size. We obtain an overall running time of $2^{O\left(n^{2} \log n\right)}$, which is an EXPTIME running time.

We obtain the following corollary.

Corollary 10. N-Core is in EXPTIME.
Proof. To decide whether a given RGP $P$ is an n-core, it suffices to check, for each sub-RGP $Q$ of $P$, whether we have $P \xrightarrow{n} Q$. Thus, an exponential number of applications of our EXPTIME algorithm for RGPHom is a valid EXPTIME algorithm for N-Core.

Note that for two RGPs $P$ and $Q$, if $P \xrightarrow{n} Q$ then the query $Q$ is contained in the query $P$, but there are examples where the converse does not hold (see Figure 11). Thus, the problem RGPHom for two RGPs does not fully capture RGP Query Containment. Nevertheless, we will show that the former can be solved in EXPTIME, which is better than the (tight) EXPSPACE complexity of RGP QuERY Containment shown in [10, 18.


P

$Q$

Figure 1: Two n-core RGPs $P$ and $Q$ over alphabet $\{a, b\}$ which have no n-homomorphism in either direction. From $P$ to $Q$ because one can not map suitably the arc labelled by $b$, in the other direction because neither $b$ nor $a^{+}$is included in $a$. However, any database that matches the RGP $P$ would contain a walk of arcs all labelled by $a$ (because of the arc with label $a^{+}$in $P$ ). The database would clearly also match $Q$. So $Q$ is contained in $P$.

## 5 Specific RGP classes over a unary alphabet: the $\left\{a, a^{+}\right\}$case

In this section, we consider that the alphabet $\Sigma$ is unary, say, $\Sigma=\{a\}$. It is known that some unary languages are undecidable (because the set of unary languages is uncountable, while the set of decidable languages is countable). Thus, even unary languages are highly nontrivial and it is of interest to study them. Note that for unary regular languages, Regular Language Inclusion and Regular Language Universality are no longer PSPACE-complete but they are coNP-complete (see [23] and [36, respectively).

The case where all arc-labels of the considered RGPs are equal to " $a$ " is equivalent to the problem of classic digraph homomorphisms, and thus it captures all CSPss, see 14. When each label is either " $a$ " or " $a^{+}$", we have two kinds of constraints: arcs labeled " $a$ " must map in a classic, local, way, while for arcs labeled " $a^{+}$" can be mapped to an arbitrary (nontrivial) path in the target RGP. Thus, this setting is useful for example to model descendence relations in hierarchichal data such as XML. THis setting is for example used in languages like SPARQL or XPath for XML documents, that are tree-structured. We refer for example to the papers [12, 17, 25, 30]. For digraphs, this type of homomorphisms is related to the setting where one considers the graph power of the template digraph, a problem studied in 8 .

We first show how to transform such problems into a classic homomorphism problem. For an RGP $Q$ with arc-labels either " $a$ " or " $a^{+}$", let $D(Q)$ be the two-arc-labeled digraph obtained from $Q$ by leaving all arcs labeled " $a$ " untouched, and adding an arc with label " $t$ " from a vertex $x$ to a vertex $y$ if and only if there is a directed path (regardless of any labels) from $x$ to $y$ in $Q$ (that is, the arcs labeled " $t$ " induce the transitive closure of the digraph). This construction is computable in polynomial time. See Figure 2 for an example (we adopt the style of 12 by marking the " $a^{+}$" arcs as doubled).

Proposition 11. RGPHOM restricted to RGPs with arc-labels either " $a$ " or " $a$ " is polynomially reducible to Hom for two-arc-labeled digraphs.

Proof. Let $D_{P}^{\prime}$ be the directed tree obtained from $D_{Q}$ by replacing ll arc-labels " $a^{+}$" by labels " $t$ ". For any instance $P, Q$ of RGPHom, we have that $P \xrightarrow{n} Q$ (as a navigational homomorphism) if and only if $D_{P}^{\prime} \rightarrow D(Q)$ (as a classic homomorphism of arc-labeled digraphs). Indeed, the arcs labeled " $t$ " in $D(Q)$ join precisely those pairs to which the two vertices of an arc labeled " $a^{+}$" can map in a navigational homomorphism of $P$ to $Q$. There is no difference in the mapping of arcs labeled " $a$ ". This completes the proof.


Figure 2: A path RGP $Q$ with arc-labels in $\left\{a, a^{+}\right\}$(doubled arcs are labeled " $a^{+}$", the others are labeled " $a$ ") and the corresponding two-arc-labeled digraph $D(Q)$.

We show next that the special case we consider here is much easier than the general PSPACE-hard case.

Proposition 12. For RGPs over an alphabet $\Sigma$ with $a \in \Sigma$ whose arc-labels are in $\left\{a, a^{+}\right\}$, RGPHom is in NP.

Proof. It suffices to apply Proposition 11, by noting that Hom is in NP.

### 5.1 Undirected RGPs with edge-labels in $\left\{a, a^{+}\right\}$

We now consider undirected RGPs, where arcs are pairs of vertices, called edges (equivalently, for each arc from $x$ to $y$, we have its symmetric arc from $y$ to $x$ with the same label) with arc-labels " $a$ " and " $a+$ ". In an n-homomorphism of a RGP $P$ to a RGP $Q$ that both satisfy this constraint, two vertices $x$ and $y$ joined by an edge labelled " $a$ " in $P$ must be mapped to two vertices labelled " $a$ " in $Q$ (as in a classic graph homomorphism). If $x$ and $y$ are joined by an edge labelled " $a{ }^{+}$" in $P$, they simply need to be mapped to two vertices of $Q$ that are connected by some path in $Q$. Thus, this can be seen as an extension of classic graph homomorphisms with additional (binary) connectivity constraints.

Proposition 13. Let $Q$ be an undirected connected $n$-core $R G P$ over alphabet $\{a\}$ with arc-labels in $\left\{a, a^{+}\right\}$, let $\left\{S_{1}, \ldots, S_{p}\right\}$ be the set of connected components of the sub-RGP $Q^{a}$ of $Q$ induced by the edges labeled " $a$ ", and denote by $X_{i}$ the set of vertices of $S_{i}$ incident with an edge labeled " $a$ ". Then, the following properties hold.
(a) If $Q$ contains no edge labeled " $a$ ", then $Q$ has at most one edge.
(b) If $Q$ contains a loop at vertex $v$, then $V(Q)=\{v\}$.
(c) If $Q$ is loop-free, then every edge labeled " $a$ "" is a bridge.
(d) Let $S_{i}$ and $S_{j}$ be two distinct components of $Q^{a}$. If $\left|X_{i}\right| \leq 1$, then $S_{i}$ has no n-homomorphism to $S_{j}$.
(e) For any component $S_{i}$ of $Q^{a}$ and any n-endomorphism $f$ of $S_{i}$ satisfying that $f(x)=x$ for every vertex $x$ of $X_{i}, f$ is an n-automorphism of $S_{i}$.

Proof. (a) If all edges of $Q$ are labelled " $a^{+} "$ (and $Q$ has at least one edge), since $Q$ is connected, $Q$ maps to any of its subgraphs containing exactly one edge, a contradiction.
(b) If there is a loop at vertex $v$ that is labelled " $a$ ", then $Q$ maps to its subgraph induced by $\{v\}$, and so, $V(Q)=\{v\}$. If the loop is labelled " $a$ " " but there are at least two vertices in $Q$, then $Q$ maps to its subgraph with that loop removed, contradicting the fact that $Q$ is an n-core.
(c) If two distinct vertices $x, y$ of $Q$ are joined by an edge labelled " $a^{+}$" that is not a bridge, then $Q$ maps to its subgraph obtained by removing that edge, a contradiction.
(d) If $S_{i} \xrightarrow{n} S_{j}$ through a n-homomorphism $f$, we can use $f$ to define an endomorphism $f^{\prime}$ of $Q$ such that, for any vertex $v, f^{\prime}(v)=f(v)$ if $v \in S_{i}$ and $f(v)=v$ otherwise. Since $f^{\prime}$ is not an automorphism, its existence contradicts the fact that $Q$ is an n-core.
(e) If this was not the case, again we could extend $f$ to obtain an endomorphism of $Q$ that is not an automorphism, a contradiction.

An example of an undirected n-core RGP is given in Figure 3, where the graph induced by the edges labeled " $a$ " has three components: a single vertex, a triangle and the Grötzsch graph. The two latter ones are not homomorphic to each other, in accordance with Proposition 13(d).


Figure 3: An undirected n-core RGP over alphabet $\{a\}$ with edge-labels in $\left\{a, a^{+}\right\}$. Doubled edges are labeled " $a^{+}$", the others are labeled " $a$ ".

Theorem 14. Let $Q$ be an undirected and connected $n$-core $R G P$ over alphabet $\{a\}$ with arc-labels in $\left\{a, a^{+}\right\}$. If $Q$ has at most one edge, RGPHom $(Q)$ is solvable in polynomial time. Otherwise, RG$\operatorname{PHom}(Q)$ is $N P$-complete.

Proof. Polynomial-time part. Suppose that $Q$ has at most one edge. Then, it either consists of a single vertex with at most one loop, or of two vertices joined by a unique edge. Let $P$ be an input RGP over alphabet $\{a\}$ and with arc-labels in $\left\{a, a^{+}\right\}$. If $Q$ has one vertex and no loop, $P \xrightarrow{n} Q$ if and only if $P$ has no edge. If $Q$ has one vertex and a loop labelled " $a$ ", always $P \xrightarrow{n} Q$. If $Q$ has one vertex and a loop labelled " $a^{+}$", $P \xrightarrow{n} Q$ if and only if $P$ has no edge labelled " $a$ ". If $Q$ has two vertices and an edge labelled " $a^{+}$", also if and only if $P$ has no edge labelled " $a$ ". Finally, if $Q$ has two vertices and an edge labelled " $a$ ", $P \xrightarrow{n} Q$ if and only if the subgraph of $P$ induced by the edges labelled " $a$ " is bipartite. All these conditions can be checked in polynomial time.
$N P$-complete part. Suppose now that $Q$ has at least two edges. $\operatorname{RGPHom}(Q)$ is in NP by Corollary 12 , We reduce from $\operatorname{Hom}(H)$, where $H$ is the undirected graph obtained from $Q^{a}$ (the sub-RGP of $Q$ induced by the edges labelled " $a$ ") by removing all edge-labels. By Proposition 13(a)-(b), $H$ has no loop and at least one edge. By Proposition 13(c)-(e), $H$ has a connected component with an odd cycle, thus by Theorem 3, $\operatorname{Hom}(H)$ is NP-complete. We trivially reduce $\operatorname{Hom}(H)$ to RGPHom $(Q)$ as follows: for an input undirected graph $G$ of $\operatorname{Hom}(H)$, construct the undirected RGP $P(G)$ from $G$ by labelling each edge with " $a$ ". Now, $G \rightarrow H$ if and only if $P(G) \xrightarrow{n} Q$.

### 5.2 Directed path RGPs with all arcs labeled " $a$ "

In this section, we consider $\operatorname{RGPHom}(Q)$ when $Q$ is a directed path whose arc labels are all " $a$ " (arguably the simplest RGP directed graph example) and where instances can have labels in $\left\{a, a^{+}\right\}$. This case turns out to have an interesting connection to the following parallel job scheduling problem, studied in the book [34, Chapter 4.4, p.666].

Parallel Job Scheduling With Relative Deadlines can be solved in polynomial time by a reduction to a shortest path problem in an edge-weighted digraph, see [34, Chapter 4.4, p.666]. We will now show that when $Q$ is a directed path whose arc labels are in $\left\{a, a^{+}\right\}, \operatorname{RGPHom}(Q)$ can be modeled as such a scheduling problem.

We sau that a digraph is balanced if it is acyclic and its vertices can be partitioned into levels $L_{1}, \ldots, L_{k}$ such that for each $\operatorname{arc}(x, y)$ with $x \in L_{i}$, we have $y \in L_{i+1}$. A digraph is balanced if and only if it has a homomorphism to a directed path (the length of this path must be at least the number of levels minus one).

Theorem 15. Let $Q$ be a RGP whose underlying digraph is a directed path, and whose arc labels are all " $a$ ". Then, RGPHom $(Q)$ for instances with arc-labels in $\left\{a, a^{+}\right\}$can be reduced in polynomial time to Parallel Job Scheduling With Relative Deadlines.

Proof. Let $q_{0}, \ldots, q_{n-1}$ be the vertices of $Q$, with the $\operatorname{arc}\left(q_{i}, q_{i+1}\right)$ whenever $0 \leq i \leq n-2$. Let $P$ be an instance of $\operatorname{RGPHom}(Q)$. It is clear that the underlying digraph of $P$ must be acyclic, otherwise $P$ is a NO instance. Now, consider the digraphs $P^{A}$ and $Q^{A}$ which are the subdigaphs of the underlying digraphs of $P$ and $Q$ respectively, that are induced by the arcs labeled " $a$ ". In order for $P$ to be a YES-instance, we must necessarily have that $P^{A}$ has a homomorphism to $Q^{A}$, that is, $P^{A}$ must be a balanced digraph, with at most as many levels as the length of $Q$.

We will now transform $P$ into an equivalent instance $P^{\prime}$. Observe that in an n-homomorphism of $P$ to $Q$, necessarily all vertices of $P$ that belong to a same connected component of $P^{A}$ and to the same level in $P^{A}$ will be mapped to the same vertex of $Q$. Thus, we may identify all such vertices of $P$ into a single vertex to obtain $P^{\prime}$. In $P^{\prime}$, all connected components of $P^{\prime A}$ are directed paths. Now, observe that if there is some arc labeled " $a^{+}$" within some connected component of $P^{\prime A}$, either it creates a directed cycle (then $P$ is a NO instance and we return NO), or it is not useful since this arc can be mapped trivially. Thus, we may assume that there is no such arc in $P^{\prime}$. See Figure 4 for an illustration of this process.


Figure 4: Proof of Theorem 15, illustration of the two-step simplification of an input RGP $P$ into $P^{\prime}$ and $P^{\prime \prime}$, with three components in $P^{A}$. Doubled arcs are labeled " $a^{+}$", the others are labeled " $a$ ".

We now perform a second simplification and transform $P^{\prime}$ into yet another equivalent RGP $P^{\prime \prime}$. Consider two components $C_{1}$ and $C_{2}$ of $P^{\prime A}$. If there are two " $a^{+}$"-labeled arcs from $C_{1}$ to $C_{2}$, then one of them can be removed to obtain an equivalent instance. Indeed, assume that the vertices of $C_{1}$ are $u_{1}, u_{2}, \ldots, u_{\left|C_{1}\right|}$ and those of $C_{2}$ are $v_{1}, v_{2}, \ldots, v_{\left|C_{2}\right|}$ (where $\left(u_{i}, u_{i+1}\right)$ and $\left(v_{i}, v_{i+1}\right)$ are " $a$ "-labeled arcs for $i$ less than $\left|C_{1}\right|$ and $\left|C_{2}\right|$, respectively). An " $a^{+"}$-labeled $\operatorname{arc}\left(u_{i}, v_{j}\right)$ implies that in any homomrophism of $P^{\prime}$ to $Q$, if the vertex $u_{0}$ is mapped to the vertex $q_{a}$, then $v_{0}$ must be mapped to a vertex $q_{b}$ with
$b \geq a+i-j+1$. Thus, among all such arcs $\left(u_{i}, v_{j}\right)$, in $P^{\prime}$ we remove all of them but the one that maximizes $i-j$ and obtain the equivalent instance $P^{\prime \prime}$. See again Figure 4 for an illustration.

We can now build the equivalent instance of Parallel Job Scheduling With Relative DeadLines. We let $J$ be the set of connected components of $P^{\prime \prime A}$. We let $t_{\max }=n-1$. For a connected component $C$ of $P^{\prime \prime A}, d(C)$ (recall that $C$ is a directed path) is the number of arcs of $C$. For two components $C_{1}$ and $C_{2}$ with vertices $u_{1}, u_{2}, \ldots, u_{\left|C_{1}\right|}$ and $v_{1}, v_{2}, \ldots, v_{\left|C_{2}\right|}$ numbered as before, if there is an " $a^{+}$"-labeled $\operatorname{arc}\left(u_{i}, v_{j}\right)$ from $C_{1}$ to $C_{2}$ in $P^{\prime \prime}$ (recall there is at most one such arc), we let $r\left(C_{1}, C_{2}\right)=j-i-1$ (the job $C_{1}$ must start at least $j-i-1$ time units before the job $C_{2}$ ). If there is no arc from $C_{1}$ to $C_{2}$, then we let $r\left(C_{1}, C_{2}\right)=-n$, which imposes no time contraint.

It is now clear that any solution $s$ to this instance of Parallel Job Scheduling With Relative Deadlines corresponds to an n-homomorphism $f_{s}$ of $P^{\prime \prime}$ to $Q$ (and thus of $P$ to $Q$ ), and vice-versa, where the start time of a job $C$ in $s$ is equal to the index of the image by $f_{s}$ of the source of $C$.

We obtain the following immediate corollary.
Corollary 16. If $Q$ is an RGP whose underlying digraph is a directed path with all arcs labeled " $a$ ", RGPHom $(Q)$ can be solved in polynomial time for instances whose arc labels are in $\left\{a, a^{+}\right\}$.

As we will see in the next section, Corollary 16 can be generalized to all RGPs whose underlying digraph is a directed path and whose arc labels are in $\left\{a, a^{+}\right\}$.

### 5.3 Directed path RGPs with arc-labels in $\left\{a, a^{+}\right\}$

Our next result is more general than Corollary 16, as we use a stronger method. It also extends a result from [30], where the statement is proved only for input RGPs whose underlying digraphs are directed trees. Here we prove it for all kinds of inputs.

But first, we need to define some notions that are useful tools for proving alorithmic results for homomorphism problems. For an arc-labeled digraph $D$ and a positive integer $k$, we define the product digraph $D^{k}$ as the digraph on vertex set $V(D)^{k}$, with an arc of label $\ell$ from $\left(x_{1}, \ldots, x_{k}\right)$ to $\left(y_{1}, \ldots, y_{k}\right)$ if all pairs $\left(x_{i}, y_{i}\right)$ with $1 \leq i \leq k$ are arcs of label $\ell$ in $D$.

A homomorphism of $D^{k}$ to $D$ is called a ( $k$-ary) polymorphism of $D$. For a set $S$, a function $f$ from $S^{3}$ to $S$ is a majority function if for all $x, y$ in $S, f(x, x, y)=f(x, y, x)=f(y, x, x)=x$. We have the following theorem, that can be found in [24, Theorem 5.2.4].

Theorem 17 (24]). Let $D$ be an arc-labeled digraph that has a ternary polymorphism that is a majority function. Then, $\operatorname{Hom}(D)$ is polynomial-time solvable.

Theorem 18. Let $Q$ be an $R G P$ whose underlying digraph is a directed path with arc labels either " $a$ " or " $a^{+}$". Then, RGPHom $(Q)$ can be solved in polynomial time.

Proof. By Proposition 11, it suffices to show that $\operatorname{Hom}(D(Q))$, as defined before Proposition 11, is polynomial-time solvable. To do this, we show that $D(Q)$ admits a majority polymorphism $f: V(D(Q))^{3} \rightarrow$ $V(D(Q))$, and we will obtain the claimed result by Theorem 17

As previously, let $q_{0}, \ldots, q_{n-1}$ be the vertices of $Q$, with the $\operatorname{arc}\left(q_{i}, q_{i+1}\right)$ whenever $0 \leq i \leq n-2$. This induces a natural ordering of the vertices with $q_{i} \leq q_{j}$ if and only if $i \leq j$. The majority polymorphism $s$ we consider is the median: for three (not necessarily distinct) integers $i, j, k$ with $0 \leq i \leq j \leq k \leq n-1$ and $\{a, b, c\}=\{i, j, k\}$, we let $f\left(q_{a}, q_{b}, q_{c}\right)=q_{j}$.

First, it is clear that $f$ is a majority function by our definition since the median of $q_{i}, q_{i}$ and $q_{j}$ is always $q_{i}$.

It remains to show that $f$ is a polymorphism, that is, a homomorphism of $D(Q)^{3}$ to $D(Q)$. Let $\left(q_{a}, q_{b}, q_{c}\right)$ and $\left(q_{d}, q_{e}, q_{f}\right)$ be two vertices of $D(Q)^{3}$ such that $\left(\left(q_{a}, q_{b}, q_{c}\right),\left(q_{d}, q_{e}, q_{f}\right)\right)$ is an arc in $D(Q)^{3}$ (that is, $\left(q_{a}, q_{d}\right),\left(q_{b}, q_{e}\right)$ and $\left(q_{c}, q_{f}\right)$ are arcs in $D(Q)$ with the same label). Assume without loss of generality that $a \leq b \leq c$. Since $D(Q)$ has two arc labels, there are two cases to consider.

If the arc is labeled " $a$ ", we necessarily have $d=a+1, e=b+1$ and $f=c+1$; since $a \leq b \leq c$, also $d \leq e \leq f$. Thus, we have $f\left(q_{a}, q_{b}, q_{c}\right)=q_{b}$ and $f\left(q_{d}, q_{e}, q_{f}\right)=q_{e}=q_{b+1}$. Thus there is an arc labeled " $a$ " from $f\left(q_{a}, q_{b}, q_{c}\right)$ to $f\left(q_{d}, q_{e}, q_{f}\right)$, as required.

If the arc is labeled " $t$ ", we have $d \geq a+1, e \geq b+1$ and $f \geq c+1$. Since $a \leq b \leq c$, we will have $f\left(q_{d}, q_{e}, q_{f}\right)=q_{m}$ with $m \geq b+1$. Since $f\left(q_{a}, q_{b}, q_{c}\right)=q_{b}$, there is an arc with label " $t$ " from $f\left(q_{a}, q_{b}, q_{c}\right)$ to $f\left(q_{d}, q_{e}, q_{f}\right)$, as required. This completes the proof.

We remark that Theorem 18 also applies to RGPs with vertex-labels (where a vertex with a given label can only be assigned to a vertex with the same label). Indeed, a vertex-label is modeled as a unary relation, and unary relations trivially satisfy the properties for having a majority polymorphism.

Moreover, using the same method, Theorem 18 extends to labels of the form " $a^{*}$ ", " $a^{k}$ " or " $a \leq k$ " where $k \in \mathbb{N}$.

## 6 Conclusion

We have seen that RGPHom, which is generally PSPACE-hard (but in NP when the target RGP is a graph database), is in EXPTIME. This favorably compares to the general complexity of RGP query containment, which is EXPSPACE-complete [10, 18, and motivates the use of RGP n-homomorphisms to approximate query containment. It remains to close the gap between the PSPACE lower bound and the EXPTIME upper bound. This also holds for N-Core.

We have also seen that when all labels are " $a$ " or " $a^{+}$" (a case that is also in NP, and that corresponds to XPath and SPARQL queries), we have a complete classification of the NP-complete and polynomial cases for undirected RGPs, and all RGPs whose underlying digraph is a directed path are polynomial. It was proved in [30 that when both the input and target is a directed tree with arc-labels in $\left\{a, a^{+}\right\}$, RGPHOM is polynomial-time solvable. Is it true that (for general instances) RGPHOM $(Q)$ is polynomial when $Q$ is a directed tree with arc-labels in $\left\{a, a^{+}\right\}$? ${ }^{2}$ When all arc-labels are " $a$ ", then the only n-core RGPs whose underlying digraphs are directed trees are, in fact, the directed paths (such a directed tree maps to its longest directed path). But there are many more n-cores when both labels " $a$ " and " $a$ " are used, see Figure 5 for a simple example.


Figure 5: An n-core directed tree RGP with arc-labels in $\left\{a, a^{+}\right\}$. Doubled edges are labeled " $a^{+}$", the others are labeled " $a$ ".

It would be interesting to study further cases corresponding to relevant applications, such as the ones of "simple regular expressions" studied in [16]. An interesting case with a unary alphabet is the case where all labels are in $\left\{a^{k} \mid k \in \mathbb{N}\right\}$. This models a kind of weighted homomorphism, where an arc with weight $k$ (that is, with label " $a^{k}$ ") can only map to a directed walk with sum of weights equal to $k$. If we consider weights of the form $\left\{a^{\leq k} \mid k \in \mathbb{N}\right\}$, this would be close to the setting of homomorphisms to powers of digraphs, also related to the notion of dilation of graph embeddings [8].

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[^1]:    ${ }^{1}$ In some applications, instead of the more general walks, trails or simple paths are considered, see for example 28 .

[^2]:    ${ }^{2}$ Note that this would not be true for all acyclic RGPs, indeed there exists an oriented tree $T$ with 27 vertices such that $\operatorname{Hom}(T)$ is NP-complete [6]. Thus, for the RGP $Q(T)$ with $T$ as its underlying digraph and all arc-labels equal to " $a$ ", $\operatorname{RGPHom}(Q(T))$ is NP-complete.

