Beginners' Quest to Formalize Mathematics: A Feasibility Study in Isabelle

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Abstract. How difficult are interactive theorem provers to use? We respond by reviewing the formalization of Hilbert's tenth problem in Isabelle/HOL carried out by an undergraduate research group at Jacobs University Bremen. We argue that, as demonstrated by our example, proof assistants are feasible for beginners to formalize mathematics. With the aim to make the field more accessible, we also survey hurdles that arise when learning an interactive theorem prover. Broadly, we advocate for an increased adoption of interactive theorem provers in mathematical research and curricula.

Keywords: Interactive theorem proving \cdot Isabelle \cdot Formalized Mathematics \cdot Hilbert's tenth problem

1 Introduction

The challenge to formalize all of mathematics, as issued by the QED Manifesto [3], might have seemed unrealistic for the 1990s but recent advances in theorem proving clearly demonstrate the feasibility of using theorem provers in mathematical research. Examples for this are the formalization of the odd-order theorem in Coq [9] and Kepler's conjecture in HOL Light [10]. Even though these tools provide the possibility of establishing mathematical truth once and for all, mathematicians are reluctant to use interactive theorem provers to verify the correctness of their proofs [5,6]. "Interactive theorem provers are written by computer scientists for computer scientists," the complaint goes, quickly followed by a comment on their infeasibility for non-experts.

In October 2017, twelve undergraduate students who just started their university studies were asked to verify a mathematical proof using an interactive theorem prover. Upon initiative of Yuri Matiyasevich, whose contribution [16] was key to solving Hilbert's tenth problem but who had no experience with proof assistants, the undergraduates set out to formalize the problem and its solution. Given the interactive theorem prover Isabelle [18]³ as "relatively easy to learn," the *Hilbert meets Isabelle* project was born.

³ In rest of the paper we write "Isabelle" to also mean "Isabelle/HOL".

Sixteen months and many ups and downs later, the project stands close to completion. The students have made many mistakes and the large workgroup has shrunk, but, most importantly, they all have learned a lot. We herewith present a feasibility study of interactive theorem provers for non-experts and disprove the concern raised earlier. From young students to senior scientists in mathematics, computer science, and engineering, everyone can pick up a proof assistant to formalize their work — it will be well worth the effort!

This paper reports about the ongoing project, reviews the tools and resources that were used, and reflects on the learning process of the group. With an emphasis placed on formalizing mathematics, we wish to analyze the hurdles of becoming a proficient user of an interactive theorem prover, scrutinize our mistakes, and share the lessons we learned in the process. We also give a list of suggestions to developers and future beginners to aid the interactive theorem proving community to grow and welcome more mathematicians in the future.

Overview This paper is organized as follows: In section 2 we provide context to the formalization. In particular, we briefly outline Hilbert's tenth problem and explain the background and motivations of those involved. Then in section 3 we analyze the process of formalization, identify our key mistakes, the lessons learned from those mistakes, and things we will do differently now. The current status of the formalization is also given in this section. Finally, based on our experience of learning Isabelle, in section 4 we provide recommendations to the theorem proving community and beginners interested in formalizing mathematics.

2 The Quest to Formalize

On a visit to Jacobs University Bremen one and a half years ago, Yuri Matiyasevich recruited students for a newly conceived research idea: to conduct a formal verification of his solution to Hilbert's tenth problem. In order to promote this project, he gave a series of talks on the problem, its negative solution, and related questions [13,14]. These got a collection of students curious and before long, a research group was formed. The project was co-initiated by Dierk Schleicher who supported, mentored, and supervised the workgroup.

However, neither Yuri Matiyasevich nor Dierk Schleicher had any previous experience with interactive theorem provers. Coq [4] was known as a well established, yet difficult to learn proof assistant, but Yuri Matiyasevich ultimately suggested Isabelle. Supposedly with a less steep learning curve and better documentation, this choice manifested. Thus began the quest to formalize.

Hilbert's tenth problem and the MRDP theorem Hilbert's tenth problem comes from a list of 23 famous mathematical problems posed by the German mathematician David Hilbert in 1900 [12]. Hilbert's tenth problem asks about Diophantine equations, which are polynomial equations with integer coefficients: Does there exist an algorithm that determines if a given Diophantine equation has a solution in the integers? [16] The Matiyasevich–Robinson–Davis–Putnam theorem (also known as the MRDP theorem, DPRM theorem, or Matiyasevich's theorem) finished in 1970 by Yuri Matiyasevich [15], which states that every recursively enumerable subset of the natural numbers is the solution set to a Diophantine equation, implies a negative solution to Hilbert's tenth problem.

For the proof, one first needs to develop the theory of Diophantine equations. This entails showing that statements such as inequalities, disjunctions, or conjunctions of polynomial equations can be represented in terms of Diophantine polynomials. Then, as the first major step in the proof, one shows that exponentiation also has such a Diophantine representation. Next, after developing the notion of a recursively enumerable set using a Turing-complete model of computation (for instance using register machines), one shows that this computation model, which accepts exactly the elements of recursively enumerable sets, can be arithmetized, i.e. simulated using Diophantine equations and exponentiation. Since there exist recursively enumerable (semi-decidable), and hence Diophantine, sets that are not decidable, any proposed algorithm would have to solve the halting problem in order to decide an arbitrary Diophantine solution set.

Students' Background and Parallelization of Work After the team acquainted itself with the proof, the workgroup was split accordingly: Team I worked on showing that exponentiation is Diophantine, Team II on register machines and their arithmetization. Figure 1 gives an overview of the structure of the project. For the first part, Matiyasevich [16] provides detailed proofs; however, the arguments in the second part were at a higher level of abstraction. While Team I could work on formalizing the first part with minimal Isabelle knowledge and the already detailed paper proof, the second part of the formalization turned out more challenging. The arithmetization of register machines required not only an understanding of all details omitted in the paper, it also required a good understanding of existing theories of already formalized mathematics and practice with Isabelle's tools — what definitions lend themselves to automation? What is the appropriate level of abstraction? What makes for a definition that can be used well in proofs? etc.

Especially with the diverse background of many group members, the above questions were not answered, let alone asked, immediately. The students involved were mainly first year undergraduates studying mathematics and computer science, who had not taken a course on theorem proving. Not only did the students lack any foundational knowledge in logic and type theory, some did not even have prior programming experience. Combined, these factors resulted in an approach to learning that can best be described as haphazard. However, unbeknownst to the workgroup, these also became the preconditions for a larger feasibility experiment in theorem proving — how a group of inexperienced undergraduates can learn an interactive theorem prover to formalize a non-trivial mathematical result. In broader terms, the next few sections report on this feasibility study.



Fig. 1. Simplified overview of the project's structure.

3 Sledgehammer Abuse, Foolish Definitions, and Reinventions

In the beginning, many definitions, functions, lemmas, and proofs were written without an overall understanding of their individual functionality and utility. Especially for proofs, a lack of this structural understanding prevented the formalization from advancing. This section reviews the gamut of work done on the formalization between October 2017 and February 2019 and analyses the key mistakes that were made in the process, as well as the lessons learned from them.

Reinventions and Sledgehammer Abuse Due to the different nature of the two parts of the proof, the two teams progressed with different speed and success. Team I started by implementing custom 2×2 matrices which are used frequently in the first part. Although they initially searched for a matrix datatype within existing Isabelle theories, this search turned out unsuccessful as few relevant

results appeared. And the results that were found did not allow the team to infer how to use the respective implementations for their own proofs. As the formalization progressed, many other definitions and statements started relying on these custom types. Not only did this result in a dependency-blow-up of elementary properties that needed proving, they also prevented Isabelle from automating many parts of the proofs.

The reimplementation of this basic type was followed by stating and assuming intermediate lemmas without proof using Isabelle's convenient **sorry** keyword. This allowed for parallelizing the work on separate parts. The number-theoretic nature of proofs made it easier to use tools like **sledgehammer** that call external automatic theorem provers to search for proofs. The general approach to prove a given statement was the following: state an intermediate step, then check if **sledgehammer** can find a proof, otherwise introduce a simpler sub-step and repeat.

Since the paper proof was understood in full detail and the internal Isabelle libraries were sufficiently sophisticated, the **sledgehammer**-and-repeat approach worked surprisingly well. In fact, much of the entire first part was successfully formalized using this approach. This, however, had two main flaws. First, the proofs themselves were generated by automated theorem provers without human insight, hence cumbersome and almost impossible to understand. Second, since the approach worked relatively well it didn't incentivize the members to learn more about the Isabelle system and understand the functionality it provides. Remaining a mysterious tool that could automagically prove theorems, **sledgehammer**'s capabilities, limitations, and output were never actually understood.

Foolish Definitions In parallel to the above, Team II worked on arithmetizing register machines and the results of the second part of the proof, which culminates in the statement that all recursively enumerable sets are exponentially Diophantine. The groundwork underlying this part of the implementation included a definition of register machines, in particular Minsky machines. This modeling task initially posed a major hurdle towards the formalization. In retrospect, the ideal implementation makes all variables used in the proof become readily accessible. The first implementation written was, however, the direct opposite of that as it made extensive use of lists, fold operations, and comprehensions. This approach, while easy to implement, turned out to be too unwieldy for any proofs. In the end, the implementation from Xu et al. [20] was used as model for the formalization. While they describe a Turing machine, compatible ideas were extracted and used to implement register machines.

Once a workable model of register machines was implemented, the group could set on the goal to actually prove lemmas that were only stated before. For Team II, this is where the actual challenge of learning Isabelle started all over. Although the register machine model succeeded in being strongly modular, its properties were inherently more complicated than the number-theoretic statements from the first part. In particular, most lemmas about the workings

of a register machine typically required one to fix some initial state, some set of instructions as well as to assume validity of all state transitions, etc. Breaking proofs down into smaller and smaller pieces, as is commonly done also in mathematics, hence became much more difficult. In some sense, the large size of the implemented machinery posed a mental barrier to tackling the stated lemmas. Extrapolating the **sledgehammer**-and-repeat strategy of Team I, Team II initially hoped for automated theorem provers to prove very extensive lemmas without much human help. In retrospect this was a ridiculous expectation.

Expecting Intuition from Isabelle To add to this, it turned out that the small details of intermediate proof steps were often not understood as well as the group thought. This lead to "proof-hacking" scenarios even after lemmas had been successfully split into smaller statements. Most prominently, Matiyasevich [16, Section 4.4.2] gives a central property of register machines without elaborate proof because it follows from an analogous special case. Due to the similarity of the properties both in writing and in function, this generalization was intuitively clear. However, collectively the team did not know how to convey this intuition to Isabelle. It took several months until a complete "paper-proof" of all intermediate steps, done by a member of the group, could suddenly give the formalization of this property new momentum. With a new straight-forward approach, its proof was seamlessly completed. Even though many proofs are conceptual, often to ease reading and understanding, every correct proof can be made formal by definition, on paper and hence in an interactive theorem prover.

Finally, the exact implementation of finite power series used in the proof posed one more difficulty. Throughout the project, three definitions of such series coexisted. One can define a finite sum directly, or alternatively define recursive functions which, in each iteration, add a term to the series from the left or right. Their equivalence can be easily proven; yet, depending on the specific use case within a proof, the right definition becomes pivotal. Exactly the above generalization benefited from an explicit definition of the power series as a finite sum, and would have taken significantly more effort with any of the other definitions. Incidentally, this is similar to conventional mathematics on paper but contrasts conventional programming where there often is no difference between two equivalent definitions.

In similar fashion, the complexity of proofs may considerably change depending on the facts which are added to the set of automatically used simplification rules. As such, both the right setup of definitions before proving as well as the right setup of the prover determine the (human) provers' success.

Current Status With all of this at hand, the second part of the formalization was only recently advanced. Additionally, many conceptual issues were resolved earlier this year with the dedication and input from Mathias Fleury. As of writing, only few lemmas in the second part — and hence in the entire Isabelle formalization of the MRDP theorem — remain to be completed. In particular,

these include more minor lemmas on register machines, proving Lucas's theorem on digit-wise representations of binomial coefficients in a prime base, and proving that certain relations like binary digit-wise multiplication are Diophantine. Table 1 lists some statistics from the current state of the formalization⁴. Once completed, the formalization is expected to be sent as a submission to the Archive of Formal Proofs [1].

Table 1. Statistics about the current progress (as of commit bea7403d) of the formalization.

Lines of code	7759
— of which for Register Machines	2692
— of which for Diophantine theories	3856
— of which for Positional Notation	1150
— of which for miscellaneous files	61
Number of definitions	48
Number of functions	41
Number of lemmas and theorems	295

Lessons Learned Throughout the above story, we⁵ learned many lessons which we share below. From discussions with Isabelle users of different background and at different levels, a small survey showed that these also are issues for most learners. One could call the following "trivial" and we would probably agree. However, these lessons are so essential that we recommend any future beginner to be absolutely aware of them.

- 1. Merely understanding the idea of a proof and knowing how it is carried out conceptually does not suffice for its formalization. As tempting as is might seem to start proving in Isabelle, the formalization should only be started after the proof has carefully been written down on paper in full detail.
- 2. Working with concepts that frequently pop up in mathematics, it is likely that someone else has worked on them before. Instead of reinventing the wheel, one should search the existing and extensive Isabelle libraries.
- 3. The exact implementations of functions and predicates can both facilitate but also impair the progress of any proof. The chosen definitions directly reflect the approach taken to the problem, which also has a big impact on conventional proofs. However, they additionally require an adequate level of abstraction so that human and proof assistant can work with them effectively.

⁴ The actual source code has been made available at https://gitlab.com/hilbert-10/ dprm under the GPLv3 license.

⁵ In rest of the text the authors use "we" to interchangeably refer to themselves as authors and as representatives of the workgroup.

What to Change Next Time Conjointly, reflection of our method of working and learning reveals several defects, which are presented below. We suspect that these, in turn, are very likely to have systematically caused the above mistakes, or at least delayed their mending. In particular, we view the following points as definitive "not to-dos" for any formalization project using interactive theorem prover.

The most valuable source for beginners is undoubtedly Tobias Nipkow's Concrete Semantics [17]. We would have learned much quicker and with more structure, had we had strictly followed this book and its exercises. Learning Isabelle on the fly, in a *learning-by-doing* fashion, and looking up commands as needed was futile and it remains questionable if any such method of learning can be successful. Given many group member's previous programming experience, we clearly overestimated our ability to transfer this knowledge to an interactive theorem prover.

Expanding on this note, our experience suggests that relying on programming experience is helpful but should be done in tandem with an awareness of the key differences between programming and proving. Most notably, theory files are not compiled nor executed and proofs need to be written with a much more mathematical and structured mindset, as compared to programming. Interactive theorem provers are not just "yet another programming language" and our failure to realize this has only lengthened the learning process.

We became aware of the two mistakes described above after connecting with the very approachable Isabelle community. Only then we realized how naïve our initial approach to learning Isabelle was. Hence, when working on a formalization for the first time, it is very helpful to have an expert around who can be consulted when more conceptual questions arise. We agree that this may not be ideal, which is why further discussion on this issue follows in the next section.

4 Decoupling Learning from Experienced Individuals

We all have learned a lot from different members of the Isabelle community. In this ongoing process, we gradually realize that there's more to interactive theorem proving than just having one's lemmas accepted by the computer. A prime example for this is knowing what definitions are useful in which scenario. We've observed both ourselves and others, that experienced users seem fundamental to one's Isabelle education. Most of this education goes beyond mere factual information and includes understanding the Isabelle system on a deeper level, developing a systematic methodology of writing proofs, and developing a "feeling" for the proof assistant. From this, we conjecture the following.

Conjecture. Learning Isabelle currently depends on having an experienced user in reach who can regularly answer questions.

We speculate that this can be generalized to other interactive theorem provers, too. While we agree that learning from another user or developer in person is certainly efficient, this becomes unsustainable as the community grows. This naturally begs the question: How can a beginner's learning process become more guided by resources and documentation, therein more independent? We do not, and possibly can not, answer this question exhaustively. Nevertheless, we ask this question both to ourselves and the community and present our attempt at answering it.

Expand Documentation Documentation plays a key role in helping new users get accustomed to a new tool. And accessible, readable and easy to navigate documentation, hence, is key to promote self-learning. As beginners our first point of contact with Isabelle's documentation system was the "Documentation" tab in the prover IDE. However, we found it difficult to navigate as there was no clear indication of which document is suitable for beginners. In retrospect we realize that working though Nipkow's tutorial [17] would have been the most ideal. However, we still feel that the current documentation system could be expanded as follows.

We identify four key parts of a systematic documentation system⁶. Tutorials that walk new users through specific parts of the Isabelle system, how-to guides for learning and using tools, topic guides that give the theoretical basis for many of the features, and finally a repository of references that document all necessary details. While the current documentation system addresses three of those parts, it still lacks a crucial link that connects them: the topic guide. This lack immediately implies that any deeper understanding of the system can only come from being around regular users – hence tightly coupling a successful learning experience to advanced users.

Maintain a Knowledge-Base In tandem with documentation, it helps to accumulate a knowledge-base of beginner- and intermediate-level questions and answers. The Isabelle-users mailing list currently hosts the entire range of questions and is definitely appropriate for advanced questions. However, the thread archives are not suitable nor effectively searchable as a database for questions which many more are likely to have. Hence, we encourage users to ask these questions on Stack Overflow or a similar online forum. Stack Overflow, for example, aspires to become an ultimate and exhaustive knowledge base, and has achieved this for many larger communities. Conventional programming languages strongly benefit from that any basic question — as elementary it may be — has been asked and answered before. The interactive theorem proving community can do this as well.

We suggest that introductory-level resources like Concrete Semantics or the Isabelle Community Wiki main page actively encourage users to ask questions on Stack Overflow to build such a knowledge-base. This way, every question will

⁶ This identification is, by no means, original. Many large open-source software projects are aware of this structure and routinely advocate for documentation that conforms to it. See for instance the Django documentation [8] and the Write the Docs project [2].

only need to be asked and answered once, and everyone can benefit from users who share their expertise.

Develop the Isabelle Community Wiki Thirdly, we suggest to expand the Isabelle Community Wiki in a similar fashion: by an organic community effort. In the initial stages of our project, we used an internal "Isabelle Cheat Sheet" to facilitate mutual knowledge exchange. This Cheat Sheet was meant to be a platform where common problems and their solutions could be presented to everyone. In this respect, the intention was very similar to the "Isabelle Community Wiki". Although, interestingly enough, our own Cheat Sheet and the Wiki were completely disjoint until we merged them.

While the Cheat Sheet initially only included very basic syntactical facts it was quickly extended by features of Isabelle that are not described in existing beginner-level resources, e.g. Concrete Semantics. This includes the possibility of passing arguments to sledgehammer or how to look up facts in the existing theories. Other facts on the Cheat Sheet include keywords that can be used for custom case distinctions. Coming back to the previous point, we found out about the latter specifically after asking the Isabelle community on Stack Overflow. The fact that the question and answer have received several upvotes indicates that questions like this are indeed relevant to a broader audience.

Adopt into University Curricula In a larger scope, all of the aforementioned would strongly benefit from a growing user base. Having a larger community means that more people will ask questions and thereby create documentation, as well as eventually become experts themselves, working on exciting projects. As a matter of fact, knowing how to use an interactive theorem prover can be valued highly in many fields. Clearly, there is academia with mathematics and computer science which both have an interest and sometimes even a need to formally verify [10]. But uses in industry and engineering are equally compelling: formally verified robots, airplanes, rockets, and nuclear plants prove attractive to many companies and governments. Just one example of this prevailing relevance is given by the annual NASA Formal Methods Symposium.

In order to connect potential new users to the interactive theorem proving community as early as possible, we think that initiatives like *Proving for Fun* [11], i.e. "competitive proving" challenges, are a great idea to popularize the métier. Well-established competitive programming contests range from the International Informatics Olympiad to tech giant sponsored events and attract students as young as middle school from all over the world.

More radically, we suggest this subject be adapted into mathematics, computer science, and engineering curricula at universities. For wide acceptance, in particular Bachelor students need to be exposed to these tools before they specialize. Otherwise, knowledge will keep being passed on from PhD student to PhD student within existing research groups, but not become decoupled from exactly these. Of course, this integration can happen step by step. Initially, there

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may be an small elective course on interactive theorem proving, or some part of a current course on logic can be dedicated to introducing an interactive theorem prover. Once this exists, much more becomes possible. In mathematics classes, there can be extra-credit problems for also formally verifying one's proof, or eventually a full exercise with this purpose. Theorem proving helps to teach what theorems and facts are precisely used in every step of a proof [7].

Some might classify this as a significant change in paradigm for universitylevel education. We argue that our suggestion may well be compared to computer algebra systems, which just entered Bachelor curricula less than two decades ago. In this regard, interactive theorem provers are the logical next step. With Mathematica, SAGE, and similar systems successfully assisting computation and visualization, it's now time to introduce interactive theorem provers like Isabelle to assist modelling and proving. Initially well-suited as educational tools, they might eventually also make their way into day-to-day research work.

5 Conclusion

Our experience shows that non-experts can indeed learn interactive theorem proving to an extent that allows them to formalize significant mathematical results. Within one and a half years, we gained enough Isabelle proficiency to formalize a core part of the solution to Hilbert's tenth problem. We are happy to have used Isabelle for this purpose, which we found to have a modest learning curve and be worth the time investment. To future projects of similar kind, we recommend that beginners approach learning an interactive theorem prover in a more structured way than we did. To this end, we found Tobias Nipkow's "Concrete Semantics" [17] the most helpful first introduction to Isabelle. In general, we recommend to use a single beginner-friendly resource which should also be clearly advertised as such by more experienced members of the interactive theorem proving community. For carrying out a formalization, we realize that it is most crucial to start with a detailed "paper proof" in order to then verify every single step in a proof assistant.

Moreover, we find that interactive theorem provers are attractive to many more fields and industries than their current user-base. Notably, the group we encourage most to adopt proof assistants are mathematicians, not least by incorporating them into university curricula. Our feasibility study showed that interactive theorem proving is doable and practical — now it is the time to start formalizing mathematics on a larger scale.

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