Upper and Lower Bounds on Approximating Weighted Mixed Domination

Mingyu Xiao

Received: date / Accepted: date

Abstract A mixed dominating set of a graph G = (V, E) is a mixed set D of vertices and edges, such that for every edge or vertex, if it is not in D, then it is adjacent or incident to at least one vertex or edge in D. The mixed domination problem is to find a mixed dominating set with a minimum cardinality. It has applications in system control and some other scenarios and it is NP-hard to compute an optimal solution. This paper studies approximation algorithms and hardness of the weighted mixed dominating set problem. The weighted version is a generalization of the unweighted version, where all vertices are assigned the same nonnegative weight w_v and all edges are assigned the same nonnegative weight w_e , and the question is to find a mixed dominating set with a minimum total weight. Although the mixed dominating set problem has a simple 2-approximation algorithm, few approximation results for the weighted version are known. The main contributions of this paper include:

- 1. for $w_e \geq w_v$, a 2-approximation algorithm;
- 2. for $w_e \ge 2w_v$, inapproximability within ratio 1.3606 unless P = NP and within ratio 2 under UGC;
- 3. for $2w_v > w_e \ge w_v$, inapproximability within ratio 1.1803 unless P = NP and within ratio 1.5 under UGC;
- 4. for $w_e < w_v$, inapproximability within ratio $(1 \epsilon) \ln |V|$ unless P = NP for any $\epsilon > 0$.

Keywords Approximation algorithms · Inapproximability · Domination

Mingyu Xiao

School of Computer Science and Engineering,

University of Electronic Science and Technology of China,

E-mail: myxiao@gmail.com

Chengdu, China

1 Introduction

Domination is an important concept in graph theory. In a graph, a vertex *dominates* itself and all neighbors of it, and an edge *dominates* itself and all edges sharing an endpoint with it. The VERTEX DOMINATING SET problem [11] (resp., EDGE DOMINATING SET problem [22]) is to find a minimum set of vertices to dominate all vertices (resp., a minimum set of edges to dominate all edges) in a graph. These two domination problems have many applications in different fields. For example, in a network, structures like dominating sets play an important role in global flooding to alleviate the so-called broadcast storm problem. A message broadcast only in the dominating set is an efficient way to ensure that it is received by all transmitters in the network, both in terms of energy and interference [18]. More applications and introduction to domination problems can be found in the literature [10].

Domination problems are rich problems in the field of algorithms. Both VERTEX DOMINATING SET and EDGE DOMINATING SET are *NP*-hard [8,22]. There are several interesting algorithmic results about the polynomial solvability on special graph [23,15], approximation algorithms [13,7,6], parameterized algorithms [20,21] and so on.

In this paper, we consider a related domination problem, called the MIXED DOMINATION problem. Mixed domination is a mixture concept of vertex domination and edge domination, and MIXED DOMINATION requires to find a set of edges and vertices with the minimum cardinality to dominate other edges and vertices in a graph. MIXED DOMINATION was first proposed by Alavi et al. based on some specific application scenarios and it was named as the TO-TAL COVERING problem initially [2]. Although we prefer to call this problem a "domination problem" at present, it has some properties of "covering problems" and can also be treated as a kind of covering problems. For applications of MIXED DOMINATION, a direct application in system control was introduced by Zhao et al. [23]. They used it to minimize the number of phase measurement units (PMUs) needed to be placed and maintain the ability of monitoring the entire system. We can see that MIXED DOMINATION has drawn certain attention since its introduction [15, 16, 3, 23, 12].

MIXED DOMINATION is *NP*-hard even on bipartite and chordal graphs and planar bipartite graphs of maximum degree 4 [16]. Most of known algorithmic results of MIXED DOMINATION are about the polynomial-time solvable cases on special graphs. Zhao et al. [23] showed that this problem in trees can be solved in polynomial time. Lan et al. [15] provided a linear-time algorithm for MIXED DOMINATION in cacti, and introduced a labeling algorithm based on the primal-dual approach for MIXED DOMINATION in trees. Recently, MIXED DOMINATION was studied from the parameterized perspective [12]. Several parameterized complexity results under different parameters have been proved.

In terms of approximation algorithms, domination problems have also been extensively studied. It is easy to observe that a maximum matching in a graph is a 2-approximation solution to EDGE DOMINATING SET. But for VERTEX DOMINATING SET, the best known approximation ratio is $\log |V| + 1$ [13]. As a combination of EDGE DOMINATING SET and VERTEX DOMINATING SET, MIXED DOMINATION has a simple 2-approximation algorithm [9].

We will study approximation algorithms for weighted mixed domination problems. A mixed dominating set contains both edges and vertices. MIXED DOMINATION does not distinguish them in the solution set, and only considers the cardinality. However, edge and vertex are two different elements and they may have different contributions or prices in practice. In the application example in [23], we select vertices and edges to place phase measurement units (PMUs) on them to monitor their mixed neighbors' state variables in an electric power system. The price to place PMUs on edges and vertices may be different due to the different physical structures. It is reasonable to distinguish edge and vertex by setting different weights to them. So we introduce the following weighted version problem.

WEIGHTED MIXED DOMINATION (WMD)

Instance: A single undirected graph G = (V, E), and two nonnegative values w_v and w_e .

Question: To find a vertex subset $V_D \subseteq V$ and an edge subset $E_D \subseteq E$ such that

(i) any vertex in $V \setminus V_D$ is either an endpoint of an edge in E_D or adjacent to a vertex in V_D ;

(ii) any edge in $E \setminus E_D$ has at least one endpoint that is either an endpoint of an edge in E_D or a vertex in V_D ;

(iii) the value $w_v |V_D| + w_e |E_D|$ is minimized under the above constraints.

In WEIGHTED MIXED DOMINATION, all vertices (resp., edges) receive the same weight. Although the weight function may not be very general, the hardness of the problem increases dramatically, especially in approximation algorithms. It is easy to see that the 2-approximation algorithm for the unweighted version in [9] cannot be extended to the weighted version. In fact, for most domination problems, the weight version may become much harder. For example, it is trivial to obtain a 2-approximation algorithm for EDGE DOMINATING SET. But for the weighted version of EDGE DOMINATING SET, it took years to achieve the same approximation ratio [7]. In order to obtain more tractability results for WEIGHTED MIXED DOMINATION, we consider two cases: VERTEX-FAVORABLE MIXED DOMINATION (VFMD) and EDGE-FAVORABLE MIXED DOMINATION (EFMD). If we add one more requirement $w_v \leq w_e$ in Weighted Mixed Domination, then it becomes Vertex-FAVORABLE MIXED DOMINATION. EDGE-FAVORABLE MIXED DOMINATION is defined in a similar way by adding a requirement $w_e \leq w_v$. In fact, we will further distinguish two cases of VERTEX-FAVORABLE MIXED DOMINATION to study its complexity. We summarize our main algorithmic and complexity results for WEIGHTED MIXED DOMINATION in Table 1, where ε is any value > 0.

This paper is organized as follows. Sections 2 and 3 introduce some basic notations and properties. Section 4 deals with VERTEX-FAVORABLE MIXED

Problems		Approximation ratio		
		Upper bounds	Lower bounds	
VFMD	$2w_v \le w_e$		$10\sqrt{5} - 21 - \varepsilon$	if $P \neq NP$ (Theorem 3)
		9	$2-\varepsilon$	under UGC (Theorem 3)
	$w_v \le w_e < 2w_v$	(Theorems 2 and 5) $($	$5\sqrt{5} - 10 - \varepsilon$	if $P \neq NP$ (Theorem 6)
			$1.5 - \varepsilon$	under UGC (Theorem 6)
EFMD	$w_v > w_e$	-	$(1-\varepsilon)\ln n$	if $P \neq NP$ (Theorem 7)

Table 1: Upper and lower bounds on approximating WMD

DOMINATION. The results for the case that $2w_v \leq w_e$ are obtained by proving its equivalence to the VERTEX COVER problem. The case that $w_v \leq w_e < 2w_v$ is harder. Our 2-approximation algorithm is based on a linear programming for VERTEX COVER. The lower bounds are obtained by a nontrivial reduction from VERTEX COVER. Section 5 proves lower bounds for EDGE-FAVORABLE MIXED DOMINATION based on a reduction from the SET COVER problem. Finally, some concluding remarks are given in Section 6.

2 Preliminaries

In this paper, a graph G = (V, E) stands for an undirected simple graph with a vertex set V and an edge set E. We use n = |V| and m = |E| to denote the sizes of the vertex set and edge set, respectively. Let X be a subset of V. We use G - X to denote the graph obtained from G by removing vertices in Xtogether with all edges incident to vertices in X. Let G[X] denote the graph induced by X, i.e., $G[X] = G - (V \setminus X)$. For a subgraph or an edge set G', we use V(G') to denote the set of vertices in G'.

In a graph, a vertex *dominates* itself, all of its neighbors and all edges taking it as one endpoint; an edge *dominates* itself, the two endpoints of it and all other edges having a common endpoint. A mixed set of vertices and edges $D \subseteq V \cup E$ is called a *mixed dominating set*, if any vertex and edge are dominated by at least one element in D. For a mixed set D of vertices and edges, a vertex (resp., edge) in D is called a vertex element (resp., edge element) of D, and the set of vertex elements (resp., edge elements) may be denoted by V_D (resp., E_D). Thus $V_D = V(G) \cap D$. The set of vertices that appear in any form in D is denoted by V(D), i.e., $V(D) = \{v \in$ $V(G)|v \in D$ or v is adjacent to an edge in D}. It holds that $V_D \subseteq V(D)$. MIXED DOMINATION is to find a mixed dominating set of the minimum cardinality, and WEIGHTED MIXED DOMINATION is to find a mixed dominating set D such that $w_v |V_D| + w_e |E_D|$ is minimized. A weighted instance is a graph with each vertex assigned the same nonnegative weight w_v and each edge assigned the same nonnegative weight w_e . In a weighted instance, for a mixed set D of vertices and edges (it may only contain vertices or edges), we define $w(D) = w_v |D \cap V| + w_e |D \cap E|.$

A vertex set in a graph is called a *vertex cover* if any edge has at least one endpoint in this set and a vertex set is called an *independent set* if any pair of vertices in it are not adjacent in the graph. The VERTEX COVER problem is to find a vertex cover of the minimum cardinality. We may use S_{md} , S_{wmd} and S_{vc} to denote an optimal solution to MIXED DOMINATION, WEIGHTED MIXED DOMINATION and VERTEX COVER, respectively.

3 Properties

We introduce some basic properties of MIXED DOMINATION and WEIGHTED MIXED DOMINATION in this section.

Lemma 1 Any mixed dominating set of a graph contains all isolating vertices (i.e. the vertices of degree 0) as vertex elements.

This lemma follows from the definition of mixed dominating sets directly. Based on this lemma, we can simply include all isolating vertices in the graph to the solution set and assume the graph has no isolating vertices. We have said that MIXED DOMINATION is also related to covering problems. Next, we reveal some relations between MIXED DOMINATION and VERTEX COVER. By the definitions of vertex covers and mixed dominated sets, we get

Lemma 2 In a graph without isolating vertices, any vertex cover is a mixed dominating set.

Recall that for a mixed dominating set D, we use V(D) to denote the set of vertices appearing in D. On the other hand, we have that

Lemma 3 For any mixed dominating set D, the vertex set V(D) is a vertex cover.

Recall that S_{wmd} and S_{vc} denote an optimal solution to WEIGHTED MIXED DOMINATION and VERTEX COVER respectively. It is easy to get the following results from above lemmas.

Corollary 1 For any mixed dominating set D, it holds that

$$2|D| \ge |V_D| + 2|E_D| \ge |S_{vc}|.$$

Lemma 4 Let G be an instance of VERTEX-FAVORABLE MIXED DOMINA-TION having no isolating vertices. For any mixed dominating set D and vertex cover C in G, it holds that

$$w(S_{wmd}) \leq w(C)$$
 and $w(S_{vc}) \leq 2w(D)$.

Proof. The first inequality follows from Lemma 2 directly. By Corollary 1 and $w_v \leq w_e$, we have that $w(S_{vc}) = w_v |S_{vc}| \leq 2w_v |D| = 2w_v |V_D| + 2w_v |E_D| \leq 2w_v |V_D| + 2w_e |E_D| = 2w(D)$.

Corollary 2 Let G be an instance of VERTEX-FAVORABLE MIXED DOMINA-TION having no isolating vertices. It holds that

$$w(S_{wmd}) \le w(S_{vc}) \le 2w(S_{wmd}).$$

Lemma 4 and Corollary 2 imply the following result.

Theorem 1 For any $\alpha \geq 1$, given an α -approximation solution to VERTEX COVER, a 2α -approximation solution to VERTEX-FAVORABLE MIXED DOMINATION on the same graph can be constructed in linear time.

Proof. For a weighted instance G, let I be the set of degree-0 vertices in it. Let G' = G - I. Let C be an α -approximate solution to VERTEX COVER in G, which is also an α -approximate solution to VERTEX COVER in G'. Let S'_{vc} be a minimum vertex cover in G', and S'_{wmd} be an optimal solution to WEIGHTED MIXED DOMINATION in G'. We will show that $C \cup I$ is a 2α -approximation solution to VERTEX-FAVORABLE MIXED DOMINATION in G. By Lemmas 1 and 2, we know that $C \cup I$ is a mixed dominating set in G. By Corollary 2, we know that

$$w(C) \le \alpha w(S'_{vc}) \le 2\alpha w(S'_{wmd}).$$

In G, the set $S_{wmd} = S'_{wmd} \cup I$ is an optimal solution to WEIGHTED MIXED DOMINATION. We have

$$w(C) + w(I) \le 2\alpha w(S'_{wmd}) + w(I) \le 2\alpha (w(S'_{wmd}) + w(I)) = 2\alpha w(S_{wmd}),$$

which implies that $C \cup I$ is a 2α -approximation solution to VERTEX-FAVORABLE MIXED DOMINATION in G. Furthermore, the set I can be computed in linear time.

VERTEX COVER allows 2-approximation algorithms and then we have that

Corollary 3 VERTEX-FAVORABLE MIXED DOMINATION allows polynomialtime 4-approximation algorithms.

4 Vertex-Favorable Mixed Domination

We have obtained a simple 4-approximation algorithm for VERTEX-FAVORABLE MIXED DOMINATION. In this section, we improve the ratio to 2 and also show some lower bounds. We will distinguish two cases to study it: $2w_v \leq w_e$; $w_v \leq w_e < 2w_v$.

4.1 The case that $2w_v \leq w_e$

This is the easier case. In fact, we will reduce this case to VERTEX COVER and also reduce VERTEX COVER to it, keeping the approximation ratio. Thus, for this case we will get the same approximation upper and lower bounds as that of VERTEX COVER.

Lemma 5 Let G be a graph having no isolating vertices. Any minimum vertex cover S_{vc} in G is also an optimal solution to WEIGHTED MIXED DOMINATION with $w_e \geq 2w_v$ in G.

Proof. Let S_{wmd} be an optimal solution to WEIGHTED MIXED DOMINATION. The vertex set $V(S_{wmd})$ is still a mixed dominating set by Lemmas 3 and 2. It holds that $w(V(S_{wmd})) = w_v |V(S_{wmd})| \le w_v (|S_{wmd} \cap V| + 2|S_{wmd} \cap E|) \le w_v |S_{wmd} \cap V| + w_e |S_{wmd} \cap E| = w(S_{wmd})$. Then, $V(S_{wmd})$ is also an optimal solution to WEIGHTED MIXED DOMINATION. A minimum vertex cover S_{vc} is a mixed dominating set by Lemma 2. Note that $V(S_{wmd})$ is a vertex cover by Lemma 3 and then $w(S_{vc}) \le w(V(S_{wmd}))$. Thus, S_{vc} is an optimal solution to WEIGHTED MIXED DOMINATION.

Lemma 6 For a weighted instance G having no isolating vertices, if it holds that $w_e \geq 2w_v$, then any α -approximation solution to VERTEX COVER is also an α -approximation solution to WEIGHTED MIXED DOMINATION in G.

Proof. Let C be an α -approximation solution to VERTEX COVER. The set C is a vertex cover and then it is a mixed dominating set by Lemma 2. Next, we consider w(C). Let S_{wmd} and S_{vc} be an optimal solution to WEIGHTED MIXED DOMINATION and VERTEX COVER, respectively. Since $|C| \leq \alpha |S_{vc}|$, we have that $w(C) \leq \alpha w(S_{vc})$. By Lemma 5, we have that $w(S_{vc}) = w(S_{wmd})$. Thus, $w(C) \leq \alpha w(S_{wmd})$ and C is also an α -approximation solution to WEIGHTED MIXED MIXED DOMINATION.

The best known approximation ratio for VERTEX COVER is 2. Theorem 6 implies that

Theorem 2 WEIGHTED MIXED DOMINATION with $2w_v \leq w_e$ allows polynomialtime 2-approximation algorithms.

For lower bounds, we show a reduction from another direction.

Lemma 7 Let G be an instance having no isolating vertices, where $w_e \geq 2w_v$. For any α -approximation solution D to WEIGHTED MIXED DOMINATION in G, the vertex set V(D) is an α -approximation solution to VERTEX COVER in G.

Proof. Let S_{wmd} and S_{vc} be an optimal solution to WEIGHTED MIXED DOMI-NATION and VERTEX COVER, respectively. By Lemma 5, we have that $w(S_{wmd}) = w(S_{vc})$. Then $w(D) \leq \alpha w(S_{wmd}) = \alpha w(S_{vc}) = \alpha w_v |S_{vc}|$. Note that $w(D) = w_v |D_v| + w_e |D_e| \geq w_v |D_v| + 2w_v |D_e|$ and $|V(D)| \leq |D_v| + 2|D_e|$. Thus, $|V(D)| \leq \alpha |S_{vc}|$. Furthermore, V(D) is a vertex cover by Lemma 3. We know that V(D) is an α -approximation solution to VERTEX COVER. Dinur and Safra [4] proved that it is NP-hard to approximate VERTEX COVER within any factor smaller than $10\sqrt{5} - 21$. Khot and Regev [14] also prove that VERTEX COVER cannot be approximated to within $2 - \varepsilon$ for any $\varepsilon > 0$ under UGC. Those results and Lemma 7 imply

Theorem 3 For any $\varepsilon > 0$, WEIGHTED MIXED DOMINATION with $2w_v \le w_e$ is not $(10\sqrt{5} - 21 - \varepsilon)$ -approximable in polynomial time unless P = NP, and not $(2 - \varepsilon)$ -approximable in polynomial time under UGC.

4.2 The case that $w_v \leq w_e < 2w_v$

To simplify the arguments, in this section, we always assume the initial graph has no degree-0 vertices. Note that we can include all degree-0 vertices to the solution set directly according to Lemma 1, which will not affect our upper and lower bounds.

4.2.1 Upper bounds

We show that this case also allows polynomial-time 2-approximation algorithms. Our algorithm is based on a linear programming model for VERTEX COVER. Note that we are not going to build a linear programming for our problem WEIGHTED MIXED DOMINATION directly. Instead, we use a linear programming for VERTEX COVER.

Linear programming is a powerful tool to design approximation algorithms for VERTEX COVER and many other problems. Lemma 4 and Theorem 1 reveal some connections between WEIGHTED MIXED DOMINATION and VER-TEX COVER. Inspired by these, we investigate approximation algorithms for WEIGHTED MIXED DOMINATION starting from a linear programming model for VERTEX COVER. For a graph G = (V, E), we assign a variable $x_v \in \{0, 1\}$ for each vertex $v \in V$ to denote whether it is in the solution set. We can use the following integer programming model (IPVC) to solve VERTEX COVER:

$$\min \sum_{v \in V} x_v$$

s.t. $x_u + x_v \ge 1, \forall uv \in E$
 $x_v \in \{0, 1\}, \forall v \in V.$

If relax the binary variable x_v to $0 \le x_v \le 1$, we get a linear relaxation for VERTEX COVER, called LPVC. We will use $\mathcal{X}' = \{x'_v | v \in V\}$ to denote a feasible solution to LPVC and $w(\mathcal{X}')$ to denote the objective value under \mathcal{X}' on the graph G. LPVC can be solved in polynomial time. However, a feasible solution \mathcal{X}' to LPVC may not be corresponding to a feasible solution to VERTEX COVER since the values in \mathcal{X}' may not be integers. A feasible solution \mathcal{X}' to LPVC is *half integral* if $x'_v \in \{0, \frac{1}{2}, 1\}$ for all $x'_v \in \mathcal{X}'$. Nemhauser and Trotter [17] proved some important properties for LPVC.

Theorem 4 [17] Any basic feasible solution \mathcal{X}' to LPVC is half integral. A half-integral optimal solution to LPVC can be computed in polynomial time.

We use $\mathcal{X}^* = \{x_v^* | v \in V\}$ to denote a half-integral optimal solution to LPVC. We partition the vertex set V into three parts V_1 , $V_{\frac{1}{2}}$ and V_0 according to \mathcal{X}^* , which are the sets of vertices with the corresponding value x_v^* being 1, $\frac{1}{2}$ and 0, respectively. There are several properties for the half-integral optimal solution.

Lemma 8 [17] For a half-integral optimal solution, all neighbors of a vertex in V_0 are in V_1 , and there is a matching of size $|V_1|$ between V_0 and V_1 .

Lemma 8 implies that $(V_0, V_1, V_{\frac{1}{2}})$ is a *crown decomposition* (see [1] for the definition) and a half-integral optimal solution can be used to construct a 2-approximation solution and a 2k-vertex kernel for VERTEX COVER.

Lemma 9 For a half-integral optimal solution \mathcal{X} to LPVC, we use $G_{\frac{1}{2}}$ to denote the subgraph induced by The size of a minimum vertex cover in $G_{\frac{1}{2}}$ is at least $|V_{\frac{1}{2}}| - m$, where m is the size of a maximum matching in $G_{\frac{1}{2}}$.

Proof. Let M be a maximum matching in $G_{\frac{1}{2}}$, where |M| = m. We use V_M to denote the set of vertices appearing in M and $R = V_{\frac{1}{2}} \setminus V_M$, where $|R| = |V_{\frac{1}{2}}| - 2m$. Let C be a minimum vertex cover in $G_{\frac{1}{2}}$. We assume that $|C| < |V_{\frac{1}{2}}| - m$ and show a contradiction that \mathcal{X} is not optimal under the assumption.

We partition the vertex set $V_{\frac{1}{2}}$ into two parts C and $I = V_{\frac{1}{2}} \setminus C$. Note that C is a vertex cover and then I is an independent set. Let $R_I = R \setminus C$ and $R_C = R \cap C$. Since $|C| < |V_{\frac{1}{2}}| - m$ and C contains at least one vertex in each edge in M, we know that $R_I = C \setminus R$ is not an empty. A path P in $G_{\frac{1}{2}}$ that alternates between edges not in M and edges in M is called an M-alternating path. We use C_1 (resp., I_1) to denote the set of vertices in C (resp., in I) that are contained in some M-alternating paths beginning at a vertex in R_I . Let $C_2 = C \setminus C_1$ and $I_2 = I \setminus I_1$.

We show that

- (i) $|C_2 \cap V_M| \ge |I_2 \cap V_M|;$
- (ii) there is no edge between a vertex in I_1 and a vertex in C_2 .

For (i), if $|C_2 \cap V_M| < |I_2 \cap V_M|$, then there exists an edge $ab \in M$ such that $a \in C_1$ and $b \in I_2$. Note that $a \in C_1$ and then a is the end of an M-alternating path P beginning at a vertex in R_I . Since a is the endpoint of an edge ab in M, we know that the last edge in the path P is not in M. Thus, P plus edge ab is another M-alternating path beginning at a vertex in R_I and them b must be in I_1 instead of I_2 , a contradiction. So $|C_2 \cap V_M| \ge |I_2 \cap V_M|$.

For (ii), if there is an edge between $a \in I_1$ and $b \in C_2$, we will show a contradiction that M is not a maximum matching. First of all, we have that $a \notin R_I$ otherwise ab can be added into M to get a larger matching. So we know that a is the endpoint of an edge in M and this edge is between I_1 and C_1 . Furthermore, a is the end of an M-alternating path P beginning at a vertex in R_I since $a \in C_1$. So we can get an M-alternating path P' by adding edge ab at the end of P. Note that P' is an M-alternating path with the first edge

- 1. Compute a half-integral optimal solution \mathcal{X}^* for the input graph G and let $\{V_1, V_{\frac{1}{2}}, V_0\}$ be the vertex partition corresponding to \mathcal{X}^* .
- 2. Include all vertices in V_1 to the solution set as vertex elements and delete $V_0 \cup V_1$ from the graph (the remaining graph is the induced graph $G[V_{\frac{1}{2}}]$).
- 3. Find a maximum matching M in $G = G[V_{\frac{1}{2}}]$ and include all edges in M to the solution set as edge elements.
- 4. Add all remaining vertices in $V_{\frac{1}{2}} \setminus V(M)$ to the solution set as vertex elements.

Algorithm 2: The main steps of the 2-approximation algorithm

and the last edge not in M. Switching the edges in M and edges not in M on the path can yield a matching having one more edge than M, which is a contradiction to the maximum of M.

By $|C_2 \cap V_M| \ge |I_2 \cap V_M|$ and $|C| < |V_{\frac{1}{2}}| - m$, we can get that $|I_1| > |C_1|$. Note that any vertex in I_1 is only possible to adjacent to vertices in C_1 in $G_{\frac{1}{2}}$. In the whole graph G, the vertex set V_0 is an independent set of vertices with neighbors only in V_1 . So there is no edge between V_0 and I_1 . We know that $V_0 \cup I_1$ is an independent set of vertices with neighbors only in $V_1 \cup C_1$. Let $\mathcal{X}' = \{x'_v | v \in V\}$, where $x'_v = 0$ if $v \in V_0 \cup I_1$, $x'_v = 1$ if $v \in V_1 \cup C_1$ and $x'_v = \frac{1}{2}$ if $v \in V_{\frac{1}{2}} \setminus (I_1 \cup C_1)$. We can see that \mathcal{X}' is a feasible half integral solution to LPVC. Since $|I_1| > |C_1|$, we know that the objective value of \mathcal{X}' is an optimal half integral solution to LPVC. \Box

We are ready to describe our algorithm now. Our algorithm is based on a half-integral optimal solution \mathcal{X}^* to LPVC. We first include all vertices in V_1 to the solution set as vertex elements, which will dominate all vertices in $V_0 \cup V_1$ and all edges incident on vertices in V_1 . Next, we consider the subgraph $G[V_{\frac{1}{2}}]$ induced by $V_{\frac{1}{2}}$. We find a maximum matching M in $G[V_{\frac{1}{2}}]$ and include all edges in M to the solution set as edge elements. Last, for all remaining vertices in $V_{\frac{1}{2}}$ not appearing in M, include them to the solution set as vertex elements. The main steps of the whole algorithm are listed in Algorithm 2.

We prove the correctness of this algorithm. First, the algorithm can stop in polynomial time, because Step 1 uses polynomial time by Theorem 4 and all other steps can be executed in polynomial time. Second, we prove that the solution set returned by the algorithm is a mixed dominating set.

All vertices in $V_0 \cup V_1$ and all edges incident on vertices in $V_0 \cup V_1$ are dominated by vertices in V_1 because the graph has no degree-0 vertices and \mathcal{X}^* is a feasible solution to LPVC. All vertices and edges in $G[V_{\frac{1}{2}}]$ are dominated because all vertices in $V_{\frac{1}{2}}$ are included to the solution set either as vertex elements or as the endpoints of edge elements. We get the following lemma.

Lemma 10 Algorithm 2 runs in polynomial time and returns a mixed dominating set. Last, we consider the approximation ratio. Lemma 8 implies that the size of a minimum vertex cover in the induced subgraph $G[V_0 \cup V_1]$ is at least $|V_1|$. By Lemma 9, we know that the size of a minimum vertex cover in the induced subgraph $G[V_{\frac{1}{2}}]$ is at least $|V_{\frac{1}{2}}|-m$, where *m* is the size of a maximum matching in $G_{\frac{1}{2}}$. So the size of a minimum vertex cover of *G* is at least $|V_1| + |V_{\frac{1}{2}}| - m$, i.e.,

$$S_{vc}| \ge |V_1| + |V_{\frac{1}{2}}| - m. \tag{1}$$

Let D denote an optimal mixed dominating set in G. By Corollary 1, we have that $|V_D| + 2|E_D| \ge |S_{vc}|$. By this and $2w_v > w_e$, we have that

$$w(D) = |V_D|w_v + |E_D|w_e > \frac{w_e}{2}|V_D| + w_e|E_D| \ge \frac{w_e}{2}|S_{vc}|.$$
 (2)

Let D' denote a mixed dominating set returned by Algorithm 2. We have that $w(D') = |V_1|w_1 + mw_2 + (|V_1| - 2m)w_2$

$$w(D') = |V_1|w_v + mw_e + (|V_{\frac{1}{2}}| - 2m)w_v \\ \leq (|V_1| + |V_{\frac{1}{2}}| - m)w_e & \text{by } w_v \leq w_e \\ \leq |S_{vc}|w_e & \text{by } (1) \\ \leq 2w(D). & \text{by } (2) \end{cases}$$

Theorem 5 WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$ allows polynomial-time 2-approximation algorithms.

4.2.2 Lower bounds

In this section, we give lower bounds for WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$. These hardness results are also obtained by a reduction preserving approximation from VERTEX COVER. Lemma 1 shows that an α approximation algorithm for VERTEX COVER implies a 2α -approximation algorithm for VERTEX-FAVORABLE MIXED DOMINATION. For WEIGHTED MIXED DOMINATION with $w_e \geq 2w_v$, we have improved the expansion from 2α to α in Lemma 7. For WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$, it becomes harder. We will improve the expansion from 2α to $2\alpha - 1$.

Lemma 11 For any $\alpha \geq 1$, if there is a polynomial-time α -approximation algorithm for WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$, then there exists a polynomial-time $(2\alpha - 1)$ -approximation algorithm for VERTEX COVER.

Proof. For each instance G = (V, E) of VERTEX COVER, we construct |V| instances $G_i = (V_i, E_i)$ of WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$ such that a $(2\alpha - 1)$ -approximation solution to G can be found in polynomial time based on an α -approximation solution to each G_i .

For each positive integer $1 \le i \le |V|$, the graph $G_i = (V_i, E_i)$ is constructed in the same way. Informally, G_i contains a star T of 2n + 1 vertices and an auxiliary graph G'_i such that the center vertex c_0 of the star T is connected



Fig. 1: An illustration of the construction of G_3

to all vertices in G'_i , where G'_i contains a copy of G, an induced matching M_i with size $|M_i| = i$, and a complete bipartite graph between the vertices of G and the left part of the induced matching M_i . This is to say, $V_i = V \cup \{a_j\}_{j=1}^i \cup \{b_j\}_{j=1}^i \cup \{c_j\}_{j=0}^{2n}$ and $E_i = E \cup M_i \cup H_i \cup F_i$, where $M_i = \{a_j b_j\}_{j=1}^i$, $H_i = \{va_j | v \in V, j \in \{1, \ldots, i\}\}$, and $F_i = \{c_0 u | u \in V_i \setminus \{c_0\}\}$. We give an illustration of the construction of G_i for i = 3 in Figure 1. In the graphs G_i , the values of w_v and w_e can be any values satisfying $w_v \leq w_e < 2w_v$.

Let τ be the size of a minimum vertex cover of G. We first show that we can get a $(2\alpha - 1)$ -approximation solution to G in polynomial time based on an α -approximation solution to G_{τ} .

We define a function $w^*(G')$ on subgraphs G' of G as follows. For a subgraph G' of G,

$$w^*(G') = \min_{D \in \mathcal{D}} \{ w_v | V(G') \cap V_D | + \frac{1}{2} w_e | V(G') \cap V(E_D) | \}.$$

It is easy to see that

Lemma 12 Let S_{wmd} be an optimal solution to WEIGHTED MIXED DOMI-NATION on G. It holds that

$$w(S_{wmd}) \ge w^*(G),$$

and for any subgraph G' of G and any subgraph G_1 of G', it holds that

$$w^*(G') \ge w^*(G_1) + w^*(G' - V(G_1)).$$

Let D_{τ} be an optimal solution to G_{τ} and S_{vc} be a minimum vertex cover of G. By Lemma 12 and the definition of the function $w^*()$, we know that

$$w(D_{\tau}) \ge w^*(G_{\tau}) \ge w^*(T) + w^*(G'_{\tau}).$$

Note that T is a star and then $w^*(T) = w_v$. For G'_{τ} , we know that the size of a minimum vertex cover of it is at least 2τ because M_{τ} is an induced matching of size τ that needs at least τ vertices to cover all edges and the size of a minimum vertex cover of G is τ . By Lemma 3 and $w_e < 2w_v$, we know that $w^*(G'_{\tau}) \geq \tau w_e$. Thus, $w(D_{\tau}) \geq w_v + \tau w_e$.

On the other hand, $D'_{\tau} = \{c_0\} \cup M'$ is a mixed dominating set with $w(D'_{\tau}) = w_v + \tau w_e$, where M' is a perfect matching between S_{vc} and $\{a_j\}_{j=1}^{\tau}$ with size $|M'| = \tau$. So we have

$$w(D_{\tau}) = w_v + \tau w_e.$$

Let D^*_{τ} be an α -approximation solution to G_{τ} . We consider two cases. Case 1: the vertex c_0 is not a vertex element in D^*_{τ} . We will show that the whole vertex set V of G is of size at most $(2\alpha - 1)\tau$, which implies that the whole vertex set is a $(2\alpha - 1)$ -approximation solution to G. For all the degree-1 vertices $\{c_j\}_{j=1}^{2n}$ in G_{τ} , Since all the degree-1 vertices $\{c_j\}_{j=1}^{2n}$ in G_{τ} should be dominated and their only neighbor c_0 is not a vertex element in the mixed dominating set, we know that $\{c_j\}_{j=1}^{2n} \subseteq V(D^*_{\tau}) \cap V(T)$. For G'_{τ} , an induced subgraph of G_{τ} , the size of a minimum vertex cover of it is at least 2τ . Let $D''_{\tau} \subseteq D^*_{\tau}$ be the set of vertices and edges in G'_{τ} . By $w_e < 2w_v$, we know that $w(D''_{\tau}) \geq \tau w_e$. Thus,

$$w(D^*_{\tau}) \ge 2nw_v + \tau w_e > (n+\tau)w_e.$$

On the other hand, we have that

$$w(D_{\tau}^*) \le \alpha w(D_{\tau}) = \alpha(w_v + \tau w_e) \le \alpha(1 + \tau)w_e.$$

Therefore, $(n + \tau)w_e < \alpha(1 + \tau)w_e$. Thus, $n < \alpha + \alpha\tau - \tau \le (2\alpha - 1)\tau$.

Case 2: the vertex c_0 is a vertex element in D_{τ}^* . For this case, we show that $U_{\tau} = V(D_{\tau}^*) \cap V(G)$ is a vertex cover of G with size at most $(2\alpha - 1)\tau + (2\alpha - 1)$. Since $w(D_{\tau}^*) \leq \alpha(w_v + \tau w_e)$ and $w_v \leq w_e < 2w_v$, we know that $|V(D_{\tau}^*)|$ is at most $\alpha(2 + 2\tau)$. Since M_{τ} is an induced matching and T is a star, we know that $V(D_{\tau}^*)$ contains at least τ vertices in M_{τ} and at least one vertex in T. Therefore,

$$U_{\tau} \leq \alpha (2+2\tau) - \tau - 1 = (2\alpha - 1)\tau + 2\alpha - 1.$$

We know that U_{τ} is a $(2\alpha - 1 + \epsilon)$ -approximation algorithm for G, where $\epsilon = \frac{2\alpha - 1}{\tau}$. In fact, we can also get rid of ϵ in the above ratio by using one more trick. We let G' be $2\lceil \alpha \rceil$ copies of G, and construct G_i in the same way by taking G' as G. The size of the minimum vertex cover of G' is $2\lceil \alpha \rceil \tau$ now. For this case, we will get $|U_{\tau}| \leq (2\alpha - 1)2\lceil \alpha \rceil \tau + 2\alpha - 1$. Due to the similarity of each copy of G in G', we know that for each copy of G the number of vertices is an integer. So we know that $U_{\tau} \cap V(G)$ is a vertex cover of G with size at most $(2\alpha - 1)\tau$.

However, it is *NP*-hard to compute the size τ of the minimum vertex cover of *G*. we cannot construct G_{τ} in polynomial time directly. Our idea is to compute U_i for each G_i with $i \in \{1, \dots, |V(G)|\}$ and return the minimum one U_{i^*} . Therefore, U_{i^*} is a vertex cover of *G* with size $|U_{i^*}| \leq |U_{\tau}|$.

VERTEX COVER cannot be approximated within any factor smaller than $10\sqrt{5}-21$ in polynomial time unless P = NP [4] and cannot be approximated within any factor smaller than 2 in polynomial time under UGC [14]. These results and Lemma 11 imply that

Theorem 6 For any $\varepsilon > 0$, WEIGHTED MIXED DOMINATION with $w_v \leq w_e < 2w_v$ is not $(5\sqrt{5} - 10 - \varepsilon)$ -approximable in polynomial time unless P = NP, and not $(\frac{3}{2} - \varepsilon)$ -approximable in polynomial time under UGC.

5 Edge-Favorable Mixed Domination

We show that EDGE-FAVORABLE MIXED DOMINATION does not allow polynomialtime constant-ratio approximation algorithms if $P \neq NP$. The hardness result is obtained by a reduction from the SET COVER problem.

In an instance of SET COVER, we are given a set of elements $U = \{1, 2, ..., n\}$ and a collection S of m nonempty subsets of U whose union equals U, and the problem is to find a smallest number of subsets in S whose union equals U. For an instance I of SET COVER, we construct an instance $I' = (G, w_v, w_e)$ of EDGE-FAVORABLE MIXED DOMINATION. The graph $G = (V = V_S \cup V_U, E)$ is a bipartite graph containing $m + n(q^2 + 1)$ vertices, where $q = \lfloor m \ln n \rfloor$. The set V_S contains m vertices and each vertex in V_S is corresponding to a subset in S. The set V_U contains $n(q^2+1)$ vertices in total and $V_U = V_1 \cup V_2 \cdots \cup V_{q^2} \cup V_{q^2+1}$, where $|V_i| = n$ and each vertex in V_i is corresponding to an element in U for each $i \in \{1, 2, \ldots, q^2 + 1\}$. A vertex $v \in V_S$ is adjacent to a vertex $u \in V_U$ if and only if the subset corresponding to v contains the element corresponding to u. Thus, if a subset contains x elements, then the corresponding vertex in V_S has degree exactly $x(q^2 + 1)$. Let $w_v = 1$ and $w_e = \frac{1}{q}$. We first prove the following result.

Property 1: For any ratio $\delta \leq \ln n$, a δ -approximation solution D^* to I' will hold that

(i) $V_S \subseteq V(D^*)$, and

(ii) the set of subsets corresponding to $V_{D^*} \cap V_S$ is a set cover of U.

Assume to the contrary that there is a vertex $v \in V_S$ such that v is not in $V(D^*)$. Then all neighbors of v should be in $V(D^*)$. Since v has at least $q^2 + 1$ neighbors in V_U , which are not adjacent to each other, we know that D^* contains at least $q^2 + 1$ elements and $w(D^*) \ge w_e(q^2 + 1) > q$. Note that the vertex set V_S is a mixed dominating set and then $w(S_{wmd}) \le m$ for an optimal solution S_{wmd} to I'. Therefore, $\frac{w(D^*)}{w(S_{wmd})} > \frac{q}{m} \ge \ln n$, a contradiction. Also assume to the contrary that the set of subsets corresponding to $V_{D^*} \cap$

Also assume to the contrary that the set of subsets corresponding to $V_{D^*} \cap V_S$ is not a set cover of U. Thus there is a vertex $u \in V_U$ such that no neighbor of it is a vertex element in D^* , which implies that u and its q^2 twins (vertices in V_D corresponding to the same element in U) are in $V(D^*)$. Therefore, D^* contains at least $q^2 + 1$ elements and $w(D^*) \ge w_e(q^2 + 1) > q$. In the same way, we can show a contradiction. So Property 1 holds.

Recall that we use S_{sc} to denote a minimum set cover to I and S_{wmd} denote an optimal mixed dominating set to I'. We show that

$$w(S_{wmd}) = |S_{sc}| + \frac{m - |S_{sc}|}{q}.$$
(3)

The optimal solution S_{wmd} can be regarded as a 1-approximation solution to I'. By Property 1, we know that S_{wmd} contains at least m elements in total and at least $|S_{sc}|$ vertex elements. Therefore,

$$w(S_{wmd}) \ge w_v |S_{sc}| + w_e(m - |S_{sc}|) = |S_{sc}| + \frac{m - |S_{sc}|}{q}.$$

Next, we can construct a mixed dominating set D' such that $w(D') = |S_{sc}| + \frac{m-|S_{sc}|}{q}$. The mixed dominating set D' is constructed as follows: for each vertex in V_S corresponding to a set in S_{sc} , we include it to D' as a vertex element; for each other vertex in V_S , we include an arbitrary edge incident on it to D' as an edge element. The set D' constructed above is a mixed dominating set because S_{sc} is a set cover (and thus, all vertices in V_U are dominated by vertices in V_S) and all vertices in V_S have been included to D' (and thus, all edges will be dominated). It holds that $w(D') = w_v |S_{sc}| + w_e (m - |S_{sc}|) = |S_{sc}| + \frac{m-|S_{sc}|}{q}$. Then the optimal value for I' is exactly $|S_{sc}| + \frac{m-|S_{sc}|}{q}$, and (3) holds.

Equipped with Property 1 and (3), we are ready to prove the final result. Let D^* be an α -approximation solution to I' and V_{D^*} be the set of vertex elements in D^* , where $\alpha \leq \ln n$. We prove that the set C^* of subsets corresponding to $V_{D^*} \cap V_S$ is an α -approximation solution to I. By Property 1, we know that C^* is a set cover. Next, we analyze the size of C^* . Since D^* is an α -approximation solution to I', we know that $w(D^*) \leq \alpha(|S_{sc}| + \frac{m - |S_{sc}|}{q}) \leq \alpha|S_{sc}| + \frac{(m - |S_{sc}|) \ln n}{\lfloor m \ln n \rfloor} < \alpha|S_{sc}| + 1$. Thus, D^* contains at most $\alpha|S_{sc}|$ vertex elements and then $|V_{D^*} \cap V_S| \leq \alpha|S_{sc}|$. So the set C^* of subsets corresponding to $V_{D^*} \cap V_S$ is an α -approximation solution to I.

Lemma 13 For any $\alpha \leq \ln n$, if EDGE-FAVORABLE MIXED DOMINATION can be approximated in polynomial time within a factor of α , then SET COVER can be approximated in polynomial time within a factor of α .

It is known that for any $\epsilon > 0$, SET COVER cannot be approximated to $(1-\epsilon) \ln n$ in polynomial time unless P = NP [5]. By this result together with Lemma 13, we get a lower bound for EDGE-FAVORABLE MIXED DOMINATION.

Theorem 7 EDGE-FAVORABLE MIXED DOMINATION cannot be approximated to $(1 - \epsilon) \ln n$ in polynomial time unless P = NP, for any $\epsilon > 0$.

6 Concluding Remarks

Domination problems are important problems in graph theory and graph algorithms. In this paper, we give several approximation upper and lower bounds on WEIGHTED MIXED DOMINATION, where all vertices have the same weight and all edges have the same weight. For the general weighted version of MIXED DOMINATION such that each vertex and edge may receive a different weight, the hardness results in this paper show that it will be even harder and we may not be easy to get significant upper bounds. For further study, it will be interesting to reduce the gap between the upper and lower bounds in this paper.

Acknowledgements

This work was supported by the National Natural Science Foundation of China, under grants 61772115 and 61370071.

References

- F.N. Abu-Khzam, M.R. Fellows, M.A. Langston, W.H. Suters: Crown structures for vertex cover kernelization. *Theory Comput. Syst.*, 41 (3):411–430 (2007)
- Y. Alavi, M. Behzad, L. M. Lesniak-Foster, and E. A. Nordhaus. Total matchings and total coverings of graphs. *Journal of Graph Theory*, 1(2):135–140 (1977)
- Y. Alavi, J. Liu, J. Wang, and Z. Zhang. On total covers of graphs. Discrete Mathematics, 100(1-3):229–233 (1992)
- 4. I. Dinur and M. Safra. The importance of being biased. In *Proc. STOC'02*, pages 33–42, 2002
- 5. I. Dinur and D. Steurer. Analytical approach to parallel repetition. In *Proceedings of* the 46th annual ACM symposium on Theory of computing, pages 624–633, 2014
- B. Escoffier, J. Monnot, V. Th. Paschos and M. Xiao. New Results on Polynomial Inapproximability and Fixed Parameter Approximability of Edge Dominating Set. *Theory Comput. Syst.*, 56(2): 330–346 (2015)
- T. Fujito and H. Nagamochi. A 2-approximation algorithm for the minimum weight edge dominating set problem. Disc. Appl. Math., 118(3):199–207 (2002)
- M. R. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman and company, 1979
- P. Hatami. An approximation algorithm for the total covering problem. Discussiones Mathematicae Graph Theory, 27(3):553-558 (2007)
- T. W. Haynes, S. Hedetniemi and P. Slater. Fundamentals of Domination in Graphs. CRC Press, Boca Raton, 1998
- S. T. Hedetniemi and R. C. Laskar. Bibliography on domination in graphs and some basic definitions of domination parameters. *Discrete Mathematics*, 86(1):257–277 (1991)
- P. Jain, M. Jayakrishnan, F. Panolan and A. Sahu. Mixed Dominating Set: A Parameterized Perspective. In WG, LNCS 10520, pages 330–343, 2017
- D. S. Johnson. Approximation algorithms for combinatorial problems. J. Comput. System Sci., 9(3):256–278, (1973)
- S. Khot and O. Regev. Vertex cover might be hard to approximate to within 2 ε. J. Comput. System Sci., 74(3):335–349 (2008)
- J. K. Lan and G. J. Chang. On the mixed domination problem in graphs. Theoretical Computer Science, 476: 84–93 (2013)
- D. F. Manlove. On the algorithmic complexity of twelve covering and independence parameters of graphs. Discrete Applied Mathematics, 91(1-3):155–175 (1999)
- 17. G. L. Nemhauser and L. E. Trotter. Properties of vertex packing and independence system polyhedra. *Mathematical Programming*, 6(1):48–61 (1974)
- T. Nieberg and J. Hurink. A PTAS for the Minimum Dominating Set Problem in Unit Disk Graphs. In WAOA, LNCS 3879, pages 296–306, 2005
- R. Raz and S. Safra. A sub-constant error-probability low-degree test, and a subconstant error-probability PCP characterization of NP. In *Twenty-Ninth ACM Sympo*sium on Theory of Computing, pages 475–484, 1997
- M. Xiao, T. Kloks, and S. H. Poon. New parameterized algorithms for edge dominating set. *Theoretical Computer Science*, 511:147–158 (2013)

- 21. M. Xiao and H. Nagamochi. Parameterized edge dominating set in graphs with degree bounded by 3. Theoretical Computer Science, 508:2-15 (2013)
- Yanakakis and F. Gavril. Edge dominating sets in graphs. SIAM Journal on Applied Mathematics, 38(3):364–372 (1980)
 Y. Zhao, L. Kang, and M. Y. Sohn. The algorithmic complexity of mixed domination
- in graphs. Theoretical Computer Science, 412(22):2387–2392 (2011)