

The statistical Minkowski distances: Closed-form formula for Gaussian Mixture Models

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Abstract

The traditional Minkowski distances are induced by the corresponding Minkowski norms in real-valued vector spaces. In this work, we propose novel statistical symmetric distances based on the Minkowski's inequality for probability densities belonging to Lebesgue spaces. These statistical Minkowski distances admit closed-form formula for Gaussian mixture models when parameterized by integer exponents. This result extends to arbitrary mixtures of exponential families with natural parameter spaces being cones: This includes the binomial, the multinomial, the zero-centered Laplacian, the Gaussian and the Wishart mixtures, among others. We also derive a Minkowski's diversity index of a normalized weighted set of probability distributions from Minkowski's inequality.

Keywords: Minkowski ℓ_p metrics, L_p spaces, Minkowski's inequality, statistical mixtures, exponential families, multinomial theorem, statistical divergence, information radius, projective distance, scale-invariant distance, homogeneous distance.

1 Introduction and motivation

1.1 Statistical distances between mixtures

Gaussian Mixture Models (GMMs) are flexible statistical models often used in machine learning, signal processing and computer vision [41, 19] since they can arbitrarily closely approximate any smooth density. To measure the dissimilarity between probability distributions, one often relies on the principled information-theoretic Kullback-Leibler (KL) divergence [8], commonly called the relative entropy. However the lack of closed-form formula for the KL divergence between GMMs¹ has motivated various KL lower and upper bounds [16, 15, 37, 38] for GMMs or approximation techniques [10], and further spurred the *design* of novel distances that admit closed-form formula between GMMs [28]. To give a few examples, let us cite the statistical squared Euclidean distance [19, 21], the Jensen-Rényi divergence [41] (for the quadratic Rényi entropy), the

¹When the GMMs share the same components, it is known that the KL divergence between them amount to an equivalent Bregman divergence [35] that is however computationally intractable because its corresponding Bregman generator is the differential negentropy that does not admit a closed-form expression in that case.

Cauchy-Schwarz (CS) divergence [18, 20], and a statistical distance based on discrete optimal transport [22, 38].

A *distance* $D : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is a non-negative real-valued function D on the *product space* $\mathcal{X} \times \mathcal{X}$ such that $D(p, q) = D((p, q)) = 0$ iff. $p = q$. A distance $D(p : q)$ between p and q may not be symmetric: This fact is emphasized by the ':' delimiter notation: $D(p : q) \neq D(q : p)$. For example, the KL divergence is an oriented distance: $\text{KL}(p : q) \neq \text{KL}(q : p)$. Two usual symmetrizations of the KL divergence are the Jeffreys' divergence and the Jensen-Shannon divergence [27]. Informally speaking, a *divergence*² is a *smooth distance*³ that allows one to define an information-geometric structure [2]. In other words, a divergence is a smooth premetric distance [9].

Recently, the Cauchy-Schwarz divergence [18] has been generalized to Hölder divergences [39]. These Cauchy and Hölder distances $D(p : q)$ are said to be *projective* because $D(\lambda p : \lambda' q) = D(p : q)$ for any $\lambda, \lambda' > 0$. An important family of projective divergences for robust statistical inference are the γ -divergences [13, 33]. Interestingly, those projective distances do not require to handle normalized probability densities but only need to consider *positive densities* instead (handy in applications). The Hölder projective divergences do not admit closed-form formula for GMMs, except for the very special case of the CS divergence. The underlying reason is that the conjugate exponents $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ of Hölder divergences would need to be both integers. This constraint yields $\alpha = \beta = 1$, giving the special case of the CS divergence (the other integer exponent case is in the limit when $\alpha = 0$ and $\beta = \infty$).

1.2 Minkowski distances and Lebesgue spaces

The renown Minkowski distances are norm-induced metrics [9] measuring distances between d -dimensional vectors $x, y \in \mathbb{R}^d$ defined for $\alpha \geq 1$ by:

$$M_\alpha(x, y) := \|x - y\|_\alpha = \left(\sum_{i=1}^d |x_i - y_i|^\alpha \right)^{\frac{1}{\alpha}}, \quad (1)$$

where the Minkowski norms are given by $\|x\|_\alpha = \left(\sum_{i=1}^d |x_i|^\alpha \right)^{\frac{1}{\alpha}}$. The Minkowski norms can be extended to countably infinite-dimensional ℓ_α spaces of sequences (see [1], p. 68).

Let $(\mathcal{X}, \mathcal{F})$ be a measurable space where \mathcal{F} denotes the σ -algebra of \mathcal{X} , and let μ be a probability measure (with $\mu(\mathcal{X}) = 1$) with full support $\text{supp}(\mu) = \mathcal{X}$ (where $\text{supp}(\mu) := \text{cl}(\{F \in \mathcal{F} : \mu(F) > 0\})$ and cl denotes the set closure). Let \mathbb{F} be the set of all real-valued measurable functions defined on \mathcal{X} . We define the *Lebesgue space* [1] $L_\alpha(\mu)$ for $\alpha \geq 1$ as follows:

$$L_\alpha(\mu) := \left\{ f \in \mathbb{F} : \int_{\mathcal{X}} |f(x)|^\alpha d\mu(x) < \infty \right\}. \quad (2)$$

The Minkowski distance [25] of Eq. 1 can be generalized to probability densities belonging to Lebesgue $L_\alpha(\mu)$ spaces, to get the *statistical Minkowski distance* for $\alpha \geq 1$:

$$M_\alpha(p, q) := \left(\int_{\mathcal{X}} |p(x) - q(x)|^\alpha d\mu(x) \right)^{\frac{1}{\alpha}}. \quad (3)$$

²Also called a contrast function in [11].

³A Riemannian distance is not smooth but a squared Riemannian distance is smooth.

When $\alpha = 1$, we recover twice the *Total Variation* (TV) metric:

$$\text{TV}(p, q) := \frac{1}{2} \int |p(x) - q(x)| d\mu(x) = \frac{1}{2} \|p - q\|_{L_1(\mu)} = \frac{1}{2} M_1(p, q). \quad (4)$$

Notice that the statistical Minkowski distance does not admit closed-form formula in general because of the absolute value. The total variation is related to the probability of error in Bayesian statistical hypothesis testing [29].

In this work, we design novel distances based on the Minkowski's inequality (triangle inequality for $L_\alpha(\mu)$), which proves that $\|p\|_{L_\alpha(\mu)}$ is a norm (i.e., the L_α -norm), so that the statistical Minkowski's distance between functions of a Lebesgue space can be written as $M_\alpha(p, q) = \|p - q\|_{L_\alpha(\mu)}$. The space $L_\alpha(\mu)$ is a Banach space (ie., complete normed linear space).

1.3 Paper outline

The paper is organized as follows: Section 2 defines the new Minkowski distances by measuring in various ways the tightness of the Minkowski's inequality applied to probability densities. Section 3 proves that all these statistical Minkowski distances admit closed-form formula for mixture of exponential families with conic natural parameter spaces for integer exponents. In particular, this includes the case of Gaussian mixture models. Section 4 lists a few examples of common exponential families with conic natural parameter spaces. In Section 5, we define Minkowski's diversity indices for a normalized weighted set of probability distributions. Finally, section 6 concludes this work and hints at perspectives.

2 Distances from the Minkowski's inequality

Let us state Minkowski's inequality:

Theorem 1 (Minkowski's inequality). *For $p(x), q(x) \in L_\alpha(\mu)$ with $\alpha \in [1, \infty)$, we have the following Minkowski's inequality:*

$$\left(\int |p(x) + q(x)|^\alpha d\mu(x) \right)^{\frac{1}{\alpha}} \leq \left(\int |p(x)|^\alpha d\mu(x) \right)^{\frac{1}{\alpha}} + \left(\int |q(x)|^\alpha d\mu(x) \right)^{\frac{1}{\alpha}}, \quad (5)$$

with equality holding only when $q(x) = 0$ (almost everywhere, a.e.), or when $p(x) = \lambda q(x)$ a.e. for $\lambda > 0$ for $\alpha > 1$.

The usual proof of Minkowski's inequality relies on Hölder's inequality [40, 39]. Following [39], we define distances by measuring in several ways the tightness of the Minkowski's inequality. When clear from context, we shall write $\|\cdot\|_\alpha$ for short instead of $\|\cdot\|_{L_\alpha(\mu)}$. Thus Minkowski's inequality writes as:

$$\|p + q\|_\alpha \leq \|p\|_\alpha + \|q\|_\alpha. \quad (6)$$

Minkowski's inequality proves that the L_α -spaces are normed vector spaces.

Notice that when $p(x)$ and $q(x)$ are probability densities (i.e., $\int p(x)d\mu(x) = \int q(x)d\mu(x) = 1$), Minkowski's inequality becomes an equality iff. $p(x) = q(x)$ almost everywhere, for $\alpha > 1$. Thus we can define the following novel Minkowski's distances between probability densities satisfying the identity of indiscernibles:

Definition 2 (Minkowski difference distance). For probability densities $p, q \in L_\alpha(\mu)$, we define the Minkowski difference $D_\alpha(\cdot, \cdot)$ distance for $\alpha \in (1, \infty)$ as:

$$D_\alpha(p, q) := \|p\|_\alpha + \|q\|_\alpha - \|p + q\|_\alpha \geq 0. \quad (7)$$

Definition 3 (Minkowski log-ratio distance). For probability densities $p, q \in L_\alpha(\mu)$, we define the Minkowski log-ratio distance $L_\alpha(\cdot, \cdot)$ for $\alpha \in (1, \infty)$ as:

$$L_\alpha(p, q) := -\log \frac{\|p + q\|_\alpha}{\|p\|_\alpha + \|q\|_\alpha} = \log \frac{\|p\|_\alpha + \|q\|_\alpha}{\|p + q\|_\alpha} \geq 0. \quad (8)$$

By construction, all these Minkowski distances are symmetric distances: Namely, $M_\alpha(p, q) = M_\alpha(q, p)$, $D_\alpha(p, q) = D_\alpha(q, p)$ and $L_\alpha(p, q) = L_\alpha(q, p)$.

Notice that $L_\alpha(p, q)$ is *scale-invariant*⁴: $L_\alpha(\lambda p, \lambda q) = L_\alpha(p, q)$ for any $\lambda > 0$. Scale-invariance is a useful property in many signal processing applications. For example, the scale-invariant Itakura-Saito divergence (a Bregman divergence) has been successfully used in Nonnegative Matrix Factorization [12] (NMF). Distance $D_\alpha(p, q)$ is *homogeneous* since $D_\alpha(\lambda p, \lambda q) = |\lambda| D_\alpha(p, q)$ for any $\lambda \in \mathbb{R}$ (and so is distance $M_\alpha(p, q)$).

3 Closed-form formula for statistical mixtures of exponential families

In this section, we shall prove that D_α and L_α between statistical mixtures are in closed-form for all integer exponents (and M_α for all even exponents) for mixtures of exponential families with conic natural parameter spaces.

Let us first define the positively *weighed geometric integral* I of a set $\{p_1, \dots, p_k\}$ of k probability densities of $L_\alpha(\mu)$ as:

$$I(p_1, \dots, p_k; \alpha_1, \dots, \alpha_k) := \int_{\mathcal{X}} p_1(x)^{\alpha_1} \dots p_k(x)^{\alpha_k} d\mu(x), \quad \alpha \in \mathbb{R}_+^k. \quad (9)$$

An *exponential family* [7, 31] $\mathcal{E}_{t, \mu}$ is a set $\{p_\theta(x)\}_\theta$ of probability densities wrt. μ which densities can be expressed proportionally canonically as:

$$p_\theta(x) \propto \exp(t(x)^\top \theta), \quad (10)$$

where $t(x)$ is a D -dimensional vector of sufficient statistics [7]. The term $t(x)^\top \theta$ can be written equivalently as $\langle t(x), \theta \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on \mathbb{R}^D . Thus the normalized probability densities of $\mathcal{E}_{t, \mu}$ can be written as:

$$p_\theta(x) = \exp\left(t(x)^\top \theta - F(\theta)\right), \quad (11)$$

where

$$F(\theta) := \log \int_{\mathcal{X}} \exp(t(x)^\top \theta) d\mu(x), \quad (12)$$

⁴Like any distance based on the log ratio of triangle inequality gap induced by a homogeneous norm.

is called the *log-partition function* (also called cumulant function [7] or log-normalizer [31]). The natural parameter space is:

$$\Theta := \left\{ \theta \in \mathbb{R}^D : \int_{\mathcal{X}} \exp(t(x)^\top \theta) d\mu(x) < \infty \right\}. \quad (13)$$

Many common distributions (Gaussians, Poisson, Beta, etc.) belong to exponential families in disguise [7, 31].

Lemma 4. *For probability densities $p_{\theta_1}, \dots, p_{\theta_k}$ belonging to the same exponential family $\mathcal{E}_{t,\mu}$, we have:*

$$I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) = \exp \left(F \left(\sum_{i=1}^k \alpha_i \theta_i \right) - \sum_{i=1}^k \alpha_i F(\theta_i) \right) < \infty, \quad (14)$$

provided that $\sum_{i=1}^k \alpha_i \theta_i \in \Theta$.

Proof.

$$\begin{aligned} I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) &= \int \prod_{i=1}^k \left(\exp \left((t(x)^\top \theta_i - F(\theta_i)) \right) \right)^{\alpha_i} d\mu(x), \\ &= \int \exp \left(t(x)^\top \left(\sum_i \alpha_i \theta_i \right) - \sum_i \alpha_i F(\theta_i) + \underbrace{F \left(\sum_i \alpha_i \theta_i \right) - F \left(\sum_i \alpha_i \theta_i \right)}_{=0} \right) d\mu(x), \\ &= \exp \left(F \left(\sum_i \alpha_i \theta_i \right) - \sum_i \alpha_i F(\theta_i) \right) \underbrace{\int_{\mathcal{X}} \exp \left(t(x)^\top \left(\sum_i \alpha_i \theta_i \right) - F \left(\sum_i \alpha_i \theta_i \right) \right) d\mu(x)}_{=1}, \\ &= \exp \left(F \left(\sum_i \alpha_i \theta_i \right) - \sum_i \alpha_i F(\theta_i) \right), \end{aligned}$$

since $\int_{\mathcal{X}} \exp \left(t(x)^\top \left(\sum_i \alpha_i \theta_i \right) - F \left(\sum_i \alpha_i \theta_i \right) \right) d\mu(x) = \int_{\mathcal{X}} p_{\sum_i \alpha_i \theta_i}(x) d\mu(x) = 1$, provided that $\bar{\theta} := \sum_i \alpha_i \theta_i \in \Theta$ (and $p_{\bar{\theta}} \in \mathcal{E}_{t,\mu}$). \square

In particular, the condition $\sum_i \alpha_i \theta_i \in \Theta$ always holds when the natural parameter space Θ is a *cone*. In the remainder, we shall call those exponential families with natural parameter space being a cone, *Conic Exponential Families* (CEFs) for short. Note that when $\sum_i \alpha_i \theta_i \notin \Theta$, the integral $I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k)$ diverges (that is, $I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) = \infty$).

Observe that for a CEF density $p_{\theta}(x)$, we have $p_{\theta}(x)^\alpha$ in $L_\alpha(\mu)$ for *any* $\alpha \in [1, \infty)$.

Corollary 5. *We have $I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) = \exp \left(F \left(\sum_i \alpha_i \theta_i \right) - \sum_i \alpha_i F(\theta_i) \right) < \infty$ for probability densities belonging to the same exponential family with natural parameter space Θ being a cone.*

We also note in passing that $I(p_1, \dots, p_k; \alpha_1, \dots, \alpha_k) < \infty$ for $\alpha \in \mathbb{R}^k$ for probability densities belonging to the same exponential family with natural parameter space being an *affine space* (e.g., Poisson or isotropic Gaussian families [32]).

Let us define:

$$J_F(\theta_1, \dots, \theta_k; \alpha_1, \dots, \alpha_k) := \sum_i \alpha_i F(\theta_i) - F\left(\sum_i \alpha_i \theta_i\right). \quad (15)$$

This quantity is called the *Jensen diversity* [30] when $\alpha \in \Delta_k$ (the $(k-1)$ -dimensional standard simplex), or Bregman information⁵ in [5]. Although the Jensen diversity is non-negative when $\alpha \in \Delta_k$, this Jensen diversity of Eq. 15 maybe negative when $\alpha \in \mathbb{R}_+^k$. When $\alpha \in \mathbb{R}_+^k$, we thus call the Jensen diversity the *generalized Jensen diversity*. It follows that we have:

$$I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) = \exp(-J_F(\theta_1, \dots, \theta_k; \alpha_1, \dots, \alpha_k)) \quad (16)$$

The CEFs include the Gaussian family, the Wishart family, the Binomial/multinomial family, etc. [7, 31, 28].

Let us consider a finite positive mixture $\tilde{m}(x) = \sum_{i=1}^k w_i p_i(x)$ of k probability densities, where the weight vector $w \in \mathbb{R}_+^k$ are not necessarily normalized to one.

Lemma 6. *For a finite positive mixture $\tilde{m}(x)$ with components belonging to the same CEF, $\|\tilde{m}\|_{L_\alpha(\mu)}$ is finite and in closed-form, for any integer $\alpha \geq 2$.*

Proof. Consider the multinomial expansion $\tilde{m}(x)^\alpha$ obtained by applying the multinomial theorem [6]:

$$\tilde{m}(x)^\alpha = \sum_{\substack{\sum_{i=1}^k \alpha_i = \alpha \\ \alpha_i \in \mathbb{N}}} \binom{\alpha}{\alpha_1, \dots, \alpha_k} \prod_{j=1}^k (w_j p_j(x))^{\alpha_j}, \quad (17)$$

where

$$\binom{\alpha}{\alpha_1, \dots, \alpha_k} := \frac{\alpha!}{\alpha_1! \times \dots \times \alpha_k!}, \quad (18)$$

is the *multinomial coefficient* [4]. It follows that:

$$\int \tilde{m}(x)^\alpha d\mu(x) = \sum_{\substack{\sum_i \alpha_i = \alpha \\ \alpha_i \in \mathbb{N}}} \binom{\alpha}{\alpha_1, \dots, \alpha_k} \left(\prod_{j=1}^k w_j^{\alpha_j} \right) I(p_1, \dots, p_k; \alpha_1, \dots, \alpha_k). \quad (19)$$

Thus the term $\int \tilde{m}(x)^\alpha d\mu(x)$ amounts to a positively weighted sum of integrals of monomials that are positively weighted geometric means of mixture components. When $p_i = p_{\theta_i}$, since $I(p_{\theta_1}, \dots, p_{\theta_k}; \alpha_1, \dots, \alpha_k) < \infty$ using Eq. 5, we conclude that $\tilde{m} \in L_\alpha(\mu)$ for $\alpha \in \mathbb{N}$, and we get the formula:

$$\|\tilde{m}\|_{L_\alpha(\mu)} = \left(\sum_{\substack{\sum_i \alpha_i = \alpha \\ \alpha_i \in \mathbb{N}}} \binom{\alpha}{\alpha_1, \dots, \alpha_k} \left(\prod_{j=1}^k w_j^{\alpha_j} \right) \exp(-J_F(\theta_1, \dots, \theta_k; \alpha_1, \dots, \alpha_k)) \right)^{\frac{1}{\alpha}}, \quad (20)$$

for $\alpha \in \mathbb{N}$. □

⁵Because $\sum_i \alpha_i B_F(\theta_i : \bar{\theta}) = J_F(\theta_1, \dots, \theta_k; \alpha_1, \dots, \alpha_k)$ for the barycenter $\bar{\theta} = \sum_i \alpha_i \theta_i$, where $B_F(\theta : \theta') = F(\theta) - F(\theta') - (\theta - \theta')^\top \nabla F(\theta')$ is a Bregman divergence.

A naive multinomial expansion of $\tilde{m}(x)^\alpha$ yields k^α terms that can then be simplified. Using the multinomial theorem, there are $\binom{k+\alpha-1}{\alpha}$ integral terms in the formula of $\int (\sum_{i=1}^k w_i p_i(x))^\alpha d\mu(x)$. This number corresponds to the number of sequences of k disjoint subsets whose union is $\{1, \dots, \alpha\}$ (also called the number of ordered partitions but beware that some sets may be empty).

The multinomial expansion can be calculated efficiently using a generalization of Pascal's triangle, called *Pascal's simplex* [26], thus avoiding to compute from scratch all the multinomial coefficients.

We have the following generalized Pascal's recurrence formula for calculating the multinomial coefficients:

$$\binom{\alpha}{\alpha_1, \dots, \alpha_k} = \sum_{i=1}^k \binom{\alpha-1}{\alpha_1, \dots, \alpha_i-1, \dots, \alpha_k}, \quad (21)$$

with the terminal cases $\binom{\alpha}{\alpha_1, \dots, \alpha_k} = 0$ if there exists an $\alpha_i < 0$. Also by convention, we set conveniently $\binom{\alpha}{\alpha_1, \dots, \alpha_k} = 0$ if there exists $\alpha_i > \alpha$.

An efficient way to implement the multinomial expansion using nested iterative loops follows from this identity:

$$\left(\sum_{i=1}^k x_i \right)^\alpha = \sum_{\alpha_1=0}^{\alpha} \sum_{\alpha_2=0}^{\alpha_1} \dots \sum_{\alpha_{k-1}=0}^{\alpha_{k-2}} \binom{\alpha}{\alpha_1} \binom{\alpha_1}{\alpha_2} \dots \binom{\alpha_{k-1}}{\alpha_{k-2}} x_1^{\alpha-\alpha_1} x_2^{\alpha_1-\alpha_2} \dots x_{k-1}^{\alpha_{k-2}-\alpha_{k-1}} x_k^{\alpha_{k-1}}. \quad (22)$$

We are now ready to show when the statistical Minkowski's distances M_α, D_α and L_α are in closed-form for mixtures of CEFs using Lemma 6.

Theorem 7 (Closed-form formula for Minkowski's distances). *For mixtures $m = \sum_{i=1}^k w_i p_{\theta_i}$ and $m' = \sum_{j=1}^{k'} w'_j p_{\theta'_j}$ of CEFs $\mathcal{E}_{\mu,t}$, D_α and L_α admits closed-form formula for integers $\alpha \geq 2$, and M_α is in closed-form when $\alpha \geq 2$ is an even positive integer.*

Proof. For D_α and L_α , it is enough to show that $\|m\|_{L_\alpha(\mu)}, \|m'\|_{L_\alpha(\mu)}$ and $\|m+m'\|_{L_\alpha(\mu)}$ are all in closed-form. This follows from Lemma 6 by setting \tilde{m} to be m, m' and $m+m'$, respectively. The overall number of generalized Jensen diversity terms in the formula of D_α or L_α is $O\left(\binom{k+k'+\alpha-1}{\alpha}\right)$.

Now, consider distance M_α . To get rid of the absolute value in M_α for even integers α , we rewrite M_α as follows:

$$\begin{aligned} M_\alpha(m, m') &= \|m - m'\|_{L_\alpha(\mu)} = \left(\int |m(x) - m'(x)|^\alpha d\mu(x) \right)^{\frac{1}{\alpha}}, \\ &= \left(\int \left((m(x) - m'(x))^2 \right)^{\frac{\alpha}{2}} d\mu(x) \right)^{\frac{1}{\alpha}}. \end{aligned}$$

Let $\tilde{m}(x) = (m(x) - m'(x))^2$. We have:

$$\tilde{m}(x) = (m(x) - m'(x))^2, \quad (23)$$

$$= m(x)^2 + m'(x)^2 - 2m(x)m'(x), \quad (24)$$

$$= \left(\sum_{i=1}^k w_i p_{\theta_i}(x) \right)^2 + \left(\sum_{j=1}^{k'} w'_j p_{\theta'_j}(x) \right)^2 - 2 \sum_{i=1}^k \sum_{j=1}^{k'} w_i w'_j p_{\theta_i}(x) p_{\theta'_j}(x). \quad (25)$$

We have the density products $p_{\theta, \theta'} := p_{\theta} p_{\theta'} = I(p_{\theta}, p_{\theta'}; 1, 1) \in L_{\frac{\alpha}{2}}(\mu)$ (using Lemma 6) for any $\theta, \theta' \in \Theta$ and $\alpha \geq 2$. When $\alpha = 2$, $\frac{\alpha}{2} = 1$, and we easily reach a closed-form formula for $M_2(m, m')$. Otherwise, let us expand all the terms in Eq. 25, and rewrite $\tilde{m}(x) = \sum_{l=1}^K w_l'' p_{\theta_l, \theta'_l}$. Now, a key difference is that $w_l'' \in \mathbb{R}$, and not necessarily positive. Nevertheless, since $\frac{\alpha}{2} \in \mathbb{N}$, we can still use the multinomial theorem to expand $\tilde{m}(x)^{\frac{\alpha}{2}}$, distribute the integral over all terms, and compute elementary integrals $I(p_{\theta_1, \theta'_1}, \dots, p_{\theta_K, \theta'_K}; \alpha'_1, \dots, \alpha'_K)$ with $\sum_{l=1}^K \alpha'_l = \frac{\alpha}{2}$ in closed-form. Thus M_{α} is available in closed-form for mixtures of CEFs for all even positive integers $\alpha \geq 2$. The number of terms in the M_{α} formula is $O\left(\binom{\max(k^2, k'^2) + \alpha - 1}{\alpha}\right)$. \square

Note that there exists a generalization⁶ of the binomial theorem to *real* exponents $\alpha \in \mathbb{R}$ called *Newton's generalized binomial theorem* using an infinite series of general binomial coefficients:

$$(x_1 + x_2)^{\alpha} = \sum_{i=0}^{\infty} \binom{\alpha}{i} x_1^{\alpha-i} x_2^i, \quad (26)$$

with the generalized binomial coefficient defined by:

$$\binom{\alpha}{i} := \frac{\alpha(\alpha-1)\dots(\alpha-i+1)}{i!} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-i+1)\Gamma(i+1)},$$

where $\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt$ is the Gamma function extending the factorial: $\Gamma(n) = (n-1)!$. Equation 26 is only valid whenever the infinite series converge. That is, for $|x_1| \geq |x_2|$. When extending to mixture densities (i.e., $(w_1 p_1(x) + w_2 p_2(x))^{\alpha}$) and taking the integral, we therefore need to split the integral into two integrals depending on whether $w_1 p_1(x) \geq w_2 p_2(x)$, or not. Furthermore, we need to compute these integrals on truncated support domains: This becomes very tricky as the dimension of the support increase [14].

4 Some examples of conic exponential families

Let us report a few conic exponential families with their respective canonical decompositions. The measure μ is usually either the Lebesgue measure on the Euclidean space (i.e., $d\mu(x) = dx$), or the counting measure.

- **Bernoulli/multinomial families.** The Bernoulli density is $p(x; \lambda) = \lambda^x (1-\lambda)^{1-x}$ with $\lambda \in (0, 1) = \Delta_1$, for $\mathcal{X} = \{0, 1\}$. The natural parameter is $\theta = \log \frac{\lambda}{1-\lambda}$ and the conic natural parameter space is $\Theta = \mathbb{R}$. The log-partition function is $F(\theta) = \log(1 + e^{\theta})$. The sufficient statistics is $t(x) = x$.

The multinomial density generalizes the Bernoulli and the binomial densities. Here, we consider the categorical distribution also called “multinoulli” distribution. The multinoulli density is given by:

$$p(x; \lambda_1, \dots, \lambda_d) = \prod_{i=1}^d \lambda_i^{x_i},$$

⁶There also exists a generalization of the multinomial theorem to real exponents, however, this is much less known in the literature (see http://fractional-calculus.com/multinomial_theorem.pdf).

where $\lambda \in \Delta_d$, the $(d-1)$ -dimensional standard simplex. We have $\mathcal{X} = \{0, 1\}^d$. The sufficient statistic vector is $t(x) = (x_1, \dots, x_{d-1})$. The natural parameter is a $(d-1)$ -dimensional vector with natural coordinates $\theta = \left(\log \frac{\lambda_1}{\lambda_d}, \dots, \log \frac{\lambda_{d-1}}{\lambda_d}\right)$. The conic natural parameter space is $\Theta = \mathbb{R}^{d-1}$ (ie., a non-pointed cone). The log-partition function is $F(\theta) = \log(1 + \sum_{i=1}^{d-1} e^{\theta_i})$.

- **Zero-centered Laplacian family.** The density is $p(x; \sigma) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}$ and the sufficient statistic is $t(x) = |x|$. The natural parameter is $\theta = -\frac{1}{\sigma}$ with the conic parameter space $\Theta = (-\infty, 0) = \mathbb{R}_{--}$. The log-normalizer is $F(\theta) = \log\left(\frac{2}{-\theta}\right)$. See [3] for an application of Laplacian mixtures.

- **Multivariate Gaussian family.** The probability density of a d -variate Gaussian distribution is:

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left(-\frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{2}\right), \quad x \in \mathbb{R}^d$$

where $|\Sigma|$ denotes the determinant of the positive-definite matrix Σ . The natural parameter consists in a vector part θ_v and a matrix part θ_M : $\theta = (\theta_v, \theta_M) = (\Sigma^{-1}\mu, \Sigma^{-1})$. The conic natural parameter space is $\Theta = \mathbb{R}^d \times S_{++}^d$, where S_{++}^d denotes the cone of positive definite matrices of dimension $d \times d$. The sufficient statistics are (x, xx^\top) . The log-partition function is:

$$F(\theta) = \frac{1}{2} \theta_v^T \theta_M^{-1} \theta_v - \frac{1}{2} \log |\theta_M| + \frac{d}{2} \log 2\pi.$$

- **Wishart family.** The probability density is

$$p(X; n, S) = \frac{|X|^{\frac{n-d-1}{2}} e^{-\frac{1}{2}\text{tr}(S^{-1}X)}}{2^{\frac{nd}{2}} |S|^{\frac{n}{2}} \Gamma_d\left(\frac{n}{2}\right)}, \quad X \in S_{++}^d$$

with $S \succ 0$ denoting the scale matrix and $n > d - 1$ denoting the number of degrees of freedom, where Γ_d is the multivariate Gamma function:

$$\Gamma_d(x) = \pi^{d(d-1)/4} \prod_{j=1}^d \Gamma(x + (1-j)/2).$$

$\text{tr}(X)$ denotes the trace of matrix X . The natural parameter is composed of a scalar θ_s and a matrix part θ_M : $\theta = (\theta_s, \theta_M) = \left(\frac{n-d-1}{2}, S^{-1}\right)$. The conic natural parameter space is $\Theta = \mathbb{R}_+ \times S_{++}^d$. The sufficient statistics are $(\log |X|, X)$. The log-partition function is:

$$F(\theta) = \frac{(2\theta_s + d + 1)d}{2} \log 2 + \left(\theta_s + \frac{d+1}{2}\right) \log |\theta_M| + \log \Gamma_d\left(\theta_s + \frac{d+1}{2}\right).$$

See [17] for an application of Wishart mixtures.

5 Minkowski's diversity index

Informally speaking, a *diversity index* is a quantity that measures the variability of elements in a data set (i.e., the diversity of a population). For example, the (sample) variance of a (finite)

point set is a diversity index. Point sets uniformly filling a large volume have large variance (and a large diversity index) while point sets with points concentrating to their centers of mass have low variance (and a small diversity index).

Recall that the Jensen diversity index [34] of a normalized weighted set $\{p_1 = p_{\theta_1}, \dots, p_n = p_{\theta_n}\}$ of densities belonging to the same exponential family (also called information radius [23] or Bregman information [5, 36]) is defined for a strictly convex generator F by:

$$J_F(\theta_1, \dots, \theta_n; w_1, \dots, w_n) := \sum_{i=1}^n w_i F(\theta_i) - F\left(\sum_{i=1}^n w_i \theta_i\right) \geq 0.$$

When $F(\theta) = \frac{1}{2}\langle \theta, \theta \rangle$, we recover from J_F the variance.

We shall consider finite mixtures [24, 5] with *linearly independent* component densities. Using Minkowski's inequality iteratively for $f_1, \dots, f_n \in L_\alpha(\mu)$, we get:

$$\left(\int \left|\sum_{i=1}^n f_i(x)\right|^\alpha d\mu(x)\right)^{\frac{1}{\alpha}} \leq \sum_{i=1}^n \left(\int |f_i(x)|^\alpha d\mu(x)\right)^{\frac{1}{\alpha}}. \quad (27)$$

When $\alpha > 1$, equality holds when the f_i 's are proportional (a.e. μ). By setting $f_i = w_i p_i$, we define the *Minkowski's diversity index*:

Definition 8 (Minkowski's diversity index). *Define the Minkowski diversity index of n weighted probability densities of $L_\alpha(\mu)$ for $\alpha > 1$ by:*

$$\begin{aligned} J_\alpha^M(p_1, \dots, p_n; w_1, \dots, w_n) &:= \sum_{i=1}^n w_i \left(\int p_i(x)^\alpha d\mu(x)\right)^{\frac{1}{\alpha}} - \left(\int \left|\sum_i w_i p_i(x)\right|^\alpha d\mu(x)\right)^{\frac{1}{\alpha}}, \quad (28) \\ &= \sum_{i=1}^n w_i \|p_i\|_\alpha - \left\| \sum_{i=1}^n w_i p_i \right\|_\alpha \geq 0. \quad (29) \end{aligned}$$

It follows a closed-form formula for the Minkowski's diversity index of a weighted set of distributions (ie., a mixture) belonging to the same CEF:

Corollary 9. *The Minkowski's diversity index of n weighted probability distributions belonging to the same conic exponential family is finite and admits a closed-form formula for any integer $\alpha \geq 2$.*

6 Conclusion and perspectives

Designing novel statistical distances which admit closed-form formula for Gaussian mixture models is important for a wide range of applications in machine learning, computer vision and signal processing [18]. In this paper, we proposed to use the Minkowski's inequality to design novel statistical symmetric Minkowski distances by measuring the tightness of the inequality either as an arithmetic difference or as a log-ratio of the left-hand-side and right-hand-side of the inequality. We showed that these novel statistical Minkowski distances yield closed-form formula for mixtures of exponential families with conic natural parameter spaces whenever the integer exponent $\alpha \geq 2$. In particular, this result holds for Gaussian mixtures, Bernoulli mixtures, Wishart mixtures, etc. We termed those families as Conic Exponential Families (CEFs). We also reported a closed-form

formula for the ordinary statistical Minkowski distance for even positive integer exponents. Finally, we defined the Minkowski’s diversity index of a weighted population of probability distributions (a mixture), and proved that this diversity index admits a closed-form formula when the distributions belong to the same CEF.

Let us conclude by listing the formula of the statistical Minkowski distances for $\alpha = 2$ for comparison with the Cauchy-Schwarz (CS) divergence:

$$\begin{aligned} M_2(m_1, m_2) &:= \|m_1 - m_2\|_2, \\ D_2(m_1, m_2) &:= \|m_1 + m_2\|_2 - (\|m_1\|_2 + \|m_2\|_2), \\ L_2(m_1, m_2) &:= -\log \frac{\|m_1 + m_2\|_2}{\|m_1\|_2 + \|m_2\|_2}, \\ \text{CS}(m_1, m_2) &:= -\log \frac{\|m_1 m_2\|_1}{\|m_1\|_2 \|m_2\|_2} = -\log \frac{\langle m_1, m_2 \rangle_2}{\|m_1\|_2 \|m_2\|_2}, \end{aligned}$$

where $\langle f, g \rangle_2 = \int f(x)g(x)d\mu(x)$ for $f, g \in L_2(\mu)$. Note that for $\alpha = 2$, $L_2(\mu)$ is a Hilbert space when equipped with this inner product. We get closed-form formula for these statistical Minkowski’s distances between mixtures m_1 and m_2 of CEFs, as well as for the Cauchy-Schwarz divergence. All those statistical distances can be computed in quadratic time in the number of mixture components.

Selecting a proper divergence from *a priori* first principles for a given application is a paramount but difficult task [9]. Often one is left by checking experimentally the performances of a few candidate divergences in order to select the *a posteriori* ‘best’ one. We hope that these newly proposed statistical Minkowski’s distances, D_α and scale-invariant L_α , will prove experimentally useful in a number of applications ranging from computer vision to machine learning and signal processing.

Additional material is available from

<https://franknielsen.github.io/MinkowskiStatDist/>

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