# The Riemannian barycentre as a proxy for global optimisation 

Salem Said ${ }^{1}$ and Jonathan H. Manton ${ }^{2}$<br>${ }^{1}$ Laboratoire IMS (CNRS 5218), Université de Bordeaux<br>salem.said@u-bordeaux.fr<br>${ }^{2}$ Department of Electrical and Electronic Engineering, The University of Melbourne<br>j.manton@ieee.org


#### Abstract

Let $M$ be a simply-connected compact Riemannian symmetric space, and $U$ a twice-differentiable function on $M$, with unique global minimum at $x^{*} \in M$. The idea of the present work is to replace the problem of searching for the global minimum of $U$, by the problem of finding the Riemannian barycentre of the Gibbs distribution $P_{T} \propto \exp (-U / T)$. In other words, instead of minimising the function $U$ itself, to minimise $\mathcal{E}_{T}(x)=\frac{1}{2} \int d^{2}(x, z) P_{T}(d z)$, where $d(\cdot, \cdot)$ denotes Riemannian distance. The following original result is proved : if $U$ is invariant by geodesic symmetry about $x^{*}$, then for each $\delta<\frac{1}{2} r_{c x}\left(r_{c x}\right.$ the convexity radius of $\left.M\right)$, there exists $T_{\delta}$ such that $T \leq T_{\delta}$ implies $\mathcal{E}_{T}$ is strongly convex on the geodesic ball $B\left(x^{*}, \delta\right)$, and $x^{*}$ is the unique global minimum of $\mathcal{E}_{T}$. Moreover, this $T_{\delta}$ can be computed explicitly. This result gives rise to a general algorithm for black-box optimisation, which is briefly described, and will be further explored in future work.


Keywords: Riemannian barycentre• black-box optimisation•symmetric space.
It is common knowledge that the Riemannian barycentre $\bar{x}$, of a probability distribution $P$ defined on a Riemannian manifold $M$, may fail to be unique. However, if $P$ is supported inside a geodesic ball $B\left(x^{*}, \delta\right)$ with radius $\delta<\frac{1}{2} r_{c x}$ ( $r_{c x}$ the convexity radius of $M$ ), then $\bar{x}$ is unique and also belongs to $B\left(x^{*}, \delta\right)$. In fact, Afsari has shown this to be true, even when $\delta<r_{c x}$ (see [1] [2]).

Does this statement continue to hold, if $P$ is not supported inside $B\left(x^{*}, \delta\right)$, but merely concentrated on this ball? The answer to this question is positive, assuming that $M$ is a simply-connected compact Riemannian symmetric space, and $P=P_{T} \propto \exp (-U / T)$, where the function $U$ has unique global minimum at $x^{*} \in M$. This is given by Proposition 2, in Section 2 below.

Proposition 2 motivates the main idea of the present work: the Riemannian barycentre $\bar{x}_{T}$ of $P_{T}$ can be used as a proxy for the global minimum $x^{*}$ of $U$. In general, $\bar{x}_{T}$ only provides an approximation of $x^{*}$, but the two are equal if $U$ is invariant by geodesic symmetry about $x^{*}$, as stated in Proposition 3, in Section 4 below.

The following Section 1 introduces Proposition 1, which estimates the Riemannian distance between $\bar{x}_{T}$ and $x^{*}$, as a function of $T$.

## 1 Concentration of the barycentre

Let $P$ be a probability distribution on a complete Riemannian manifold $M$. A (Riemannian) barycentre of $P$ is any global minimiser $\bar{x} \in M$ of the function

$$
\begin{equation*}
\mathcal{E}(x)=\frac{1}{2} \int_{M} d^{2}(x, z) P(d z) \quad \text { for } x \in M \tag{1}
\end{equation*}
$$

The following statement is due to Karcher, and was improved upon by Afsari 1] [2]: if $P$ is supported inside a geodesic ball $B\left(x^{*}, \delta\right)$, where $x^{*} \in M$ and $\delta<\frac{1}{2} r_{c x}\left(r_{c x}\right.$ the convexity radius of $\left.M\right)$, then $\mathcal{E}$ is strongly convex on $B\left(x^{*}, \delta\right)$, and $P$ has a unique barycentre $\bar{x} \in B\left(x^{*}, \delta\right)$.

On the other hand, the present work considers a setting where $P$ is not supported inside $B\left(x^{*}, \delta\right)$, but merely concentrated on this ball. Precisely, assume $P$ is equal to the Gibbs distribution

$$
\begin{equation*}
P_{T}(d z)=(Z(T))^{-1} \exp \left[-\frac{U(z)}{T}\right] \operatorname{vol}(d z) ; T>0 \tag{2}
\end{equation*}
$$

where $Z(T)$ is a normalising constant, $U$ is a $C^{2}$ function with unique global minimum at $x^{*}$, and vol is the Riemannian volume of $M$. Then, let $\mathcal{E}_{T}$ denote the function $\mathcal{E}$ in (1), and let $\bar{x}_{T}$ denote any barycentre of $P_{T}$.

In this new setting, it is not clear whether $\mathcal{E}_{T}$ is differentiable or not. Therefore, statements about convexity of $\mathcal{E}_{T}$ and uniqueness of $\bar{x}_{T}$ are postponed to the following Section 2 For now, it is possible to state the following Proposition 1. In this proposition, $d(\cdot, \cdot)$ denotes Riemannian distance, and $W(\cdot, \cdot)$ denotes the Kantorovich ( $L^{1}$-Wasserstein) distance [3] 4]. Moreover, $\left(\mu_{\min }, \mu_{\max }\right)$ is any open interval which contains the spectrum of the Hessian $\nabla^{2} U\left(x^{*}\right)$, considered as a linear mapping of the tangent space $T_{x^{*}} M$.

Proposition 1. assume $M$ is an n-dimensional compact Riemannian manifold with non-negative sectional curvature. Denote $\delta_{x^{*}}$ the Dirac distribution at $x^{*}$. The following hold,
(i) for any $\eta>0$,

$$
\begin{equation*}
W\left(P_{T}, \delta_{x^{*}}\right)<\frac{\eta^{2}}{(4 \operatorname{diam} M)} \Longrightarrow d\left(\bar{x}_{T}, x^{*}\right)<\eta \tag{3}
\end{equation*}
$$

(ii) for $T \leq T_{o}$ (which can be computed explicitly)

$$
\begin{equation*}
W\left(P_{T}, \delta_{x^{*}}\right) \leq \sqrt{2 \pi}(\pi / 2)^{n-1} B_{n}^{-1}\left(\mu_{\max } / \mu_{\min }\right)^{n / 2}\left(T / \mu_{\min }\right)^{1 / 2} \tag{4}
\end{equation*}
$$

where $B_{n}=B(1 / 2, n / 2)$ in terms of the Beta function.
Proposition 1 is motivated by the idea of using $\bar{x}_{T}$ as an approximation of $x^{*}$. Intuitively, this requires choosing $T$ so small that $P_{T}$ is sufficiently close to $\delta_{x^{*}}$. Just how small a $T$ may be required is indicated by the inequality in (4). This inequality is optimal and explicit, in the following sense.

It is optimal because the dependence on $T^{1 / 2}$ in its right-hand side cannot be improved. Indeed, by the multi-dimensional Laplace approximation (see [5], for example), the left-hand side is equivalent to $\mathrm{L} \cdot T^{1 / 2}$ (in the limit $T \rightarrow 0$ ). While this constant $L$ is not tractable, the constants appearing in Inequality (4) depend explicitly on the manifold $M$ and the function $U$. In fact, this inequality does not follows from the multi-dimensional Laplace approximation, but rather from volume comparison theorems of Riemannian geometry 6].

In spite of these nice properties, Inequality (4) does not escape the curse of dimensionality. Indeed, for fixed $T$, its right-hand side increases exponentially with the dimension $n$ (note that $B_{n}$ decreases like $n^{-1 / 2}$ ). On the other hand, although $T_{o}$ also depends on $n$, it is typically much less affected by dimensionality, and decreases slower that $n^{-1}$ as $n$ increases.

## 2 Convexity and uniqueness

Assume now that $M$ is a simply-connected, compact Riemannian symmetric space. In this case, for any $T$, the function $\mathcal{E}_{T}$ turns out to be $C^{2}$ throughout $M$. This results from the following lemma.
Lemma 1. let $M$ be a simply-connected compact Riemannian symmetric space. Let $\gamma: I \rightarrow M$ be a geodesic defined on a compact interval I. Denote $\operatorname{Cut}(\gamma)$ the union of all cut loci $\operatorname{Cut}(\gamma(t))$ for $t \in I$. Then, the topological dimension of $\operatorname{Cut}(\gamma)$ is strictly less than $n=\operatorname{dim} M$. In particular, $\operatorname{Cut}(\gamma)$ is a set with volume equal to zero.

Remark : the assumption that $M$ is simply-connected cannot be removed, as the conclusion does not hold if $M$ is a real projective space.

The proof of Lemma 1 uses the structure of Riemannian symmetric spaces, as well as some results from topological dimension theory [7] (Chapter VII). The notion of topological dimension arises because it is possible Cut $(\gamma)$ is not a manifold. The lemma immediately implies, for all $t$,

$$
\mathcal{E}_{T}(\gamma(t))=\frac{1}{2} \int_{M} d^{2}(\gamma(t), z) P_{T}(d z)=\frac{1}{2} \int_{M-\operatorname{Cut}(\gamma)} d^{2}(\gamma(t), z) P_{T}(d z)
$$

Then, since the domain of integration avoids the cut loci of all the $\gamma(t)$, it becomes possible to differentiate under the integral. This is used in obtaining the following (the assumptions are the same as in Lemma 1).

Corollary 1. for $x \in M$, let $G_{x}(z)=\nabla f_{z}(x)$ and $H_{x}(z)=\nabla^{2} f_{z}(x)$, where $f_{z}$ is the function $x \mapsto \frac{1}{2} d^{2}(x, z)$. The following integrals converge for any $T$

$$
G_{x}=\int_{M-\operatorname{Cut}(x)} G_{x}(z) P_{T}(d z) \quad ; \quad H_{x}=\int_{M-\operatorname{Cut}(x)} H_{x}(z) P_{T}(d z)
$$

and both depend continuously on $x$. Moreover,

$$
\begin{equation*}
\nabla \mathcal{E}_{T}(x)=G_{x} \quad \text { and } \nabla^{2} \mathcal{E}_{T}(x)=H_{x} \tag{5}
\end{equation*}
$$

so that $\mathcal{E}_{T}$ is $C^{2}$ throughout $M$.

With Corollary 1 at hand, it is possible to obtain Proposition 2, which is concerned with the convexity of $\mathcal{E}_{T}$ and uniqueness of $\bar{x}_{T}$. In this proposition, the following notation is used

$$
\begin{equation*}
f(T)=(2 / \pi)(\pi / 8)^{n / 2}\left(\mu_{\max } / T\right)^{n / 2} \exp \left(-U_{\delta} / T\right) \tag{6}
\end{equation*}
$$

where $U_{\delta}=\inf \left\{U(x)-U\left(x^{*}\right) ; x \notin B\left(x^{*}, \delta\right)\right\}$ for positive $\delta$. The reader may wish to note the fact that $f(T)$ decreases to 0 as $T$ decreases to 0 .
Proposition 2. let $M$ be a simply-connected compact Riemannian symmetric space. Let $\kappa^{2}$ be the maximum sectional curvature of $M$, and $r_{c x}=\kappa^{-1} \frac{\pi}{2}$ its convexity radius. If $T \leq T_{o}$ (see (ii) of Proposition 1), then the following hold for any $\delta<\frac{1}{2} r_{c x}$.
(i) for all $x$ in the geodesic ball $B\left(x^{*}, \delta\right)$,

$$
\begin{equation*}
\nabla^{2} \mathcal{E}_{T}(x) \geq \mathrm{Ct}(2 \delta)(1-\operatorname{vol}(M) f(T))-\pi A_{M} f(T) \tag{7}
\end{equation*}
$$

where $\mathrm{Ct}(2 \delta)=2 \kappa \delta \cot (2 \kappa \delta)>0$ and $A_{M}>0$ is a constant given by the structure of the symmetric space $M$.
(ii) there exists $T_{\delta}$ (which can be computed explicitly), such that $T \leq T_{\delta}$ implies $\mathcal{E}_{T}$ is strongly convex on $B\left(x^{*}, \delta\right)$, and has a unique global minimum $\bar{x}_{T} \in$ $B\left(x^{*}, \delta\right)$. In particular, this means $\bar{x}_{T}$ is the unique barycentre of $P_{T}$.

Note that (ii) of Proposition 2 generalises the statement due to Karcher [1], which was recalled in Section 1 .

## 3 Finding $T_{o}$ and $T_{\delta}$

Propositions 1 and 2 claim that $T_{o}$ and $T_{\delta}$ can be computed explicitly. This means that, with some knowledge of the Riemannian manifold $M$ and the function $U$, $T_{o}$ and $T_{\delta}$ can be found by solving scalar equations. The current section gives the definitions of $T_{o}$ and $T_{\delta}$.

In the notation of Proposition 1, let $\rho>0$ be small enough, so that,

$$
\mu_{\min } d^{2}\left(x, x^{*}\right) \leq 2\left(U(x)-U\left(x^{*}\right)\right) \leq \mu_{\max } d^{2}\left(x, x^{*}\right)
$$

whenever $d\left(x, x^{*}\right) \leq \rho$, and consider the quantity

$$
f(T, m, \rho)=(2 / \pi)^{1 / 2}\left(\mu_{\max } / T\right)^{m / 2} \exp \left(-U_{\rho} / T\right)
$$

where $U_{\rho}$ is defined as in (6). Note that $f(T, m, \rho)$ decreases to 0 as $T$ decreases to 0 , for fixed $m$ and $\rho$. Now, it is possible to define $T_{o}$ as

$$
\begin{gather*}
T_{o}=\min \left\{T_{o}^{1}, T_{o}^{2}\right\} \quad \text { where }  \tag{8}\\
T_{o}^{1}=\inf \left\{T>0: f(T, n-2, \rho)>\rho^{2-n} A_{n-1}\right\} \\
T_{o}^{2}=\inf \left\{T>0: f(T, n+1, \rho)>\left(\mu_{\max } / \mu_{\min }\right)^{n / 2} C_{n}\right\}
\end{gather*}
$$

Here, $A_{n}=E|X|^{n}$ for $X \sim N(0,1)$, and $C_{n}=\omega_{n} A_{n} /(\operatorname{diam} M \times \operatorname{vol} M)$, where $\omega_{n}$ is the surface area of a unit sphere $S^{n-1}$.

With regard to Proposition 2, define $T_{\delta}$ as follows,

$$
\begin{equation*}
T_{\delta}=\min \left\{T_{\delta}^{1}, T_{\delta}^{2}\right\}-\varepsilon \tag{9}
\end{equation*}
$$

for some arbitrary $\varepsilon>0$. Here, in the notation of (4), (6) and (7),

$$
\begin{aligned}
T_{\delta}^{1} & =\inf \left\{T \leq T_{o}: \sqrt{2 \pi}\left(T / \mu_{\min }\right)^{1 / 2}>\delta^{2}\left(\mu_{\min } / \mu_{\max }\right)^{n / 2} D_{n}\right\} \\
T_{\delta}^{2} & =\inf \left\{T \leq T_{o}: f(T)>\operatorname{Ct}(2 \delta)\left(\operatorname{Ct}(2 \delta) \operatorname{vol}(M)+\pi A_{M}\right)^{-1}\right\} \\
\text { where } D_{n} & =(2 / \pi)^{n-1} B_{n} /(4 \operatorname{diam} M)
\end{aligned}
$$

## 4 Black-box optimisation

Consider the problem of searching for the unique global minimum $x^{*}$ of $U$. In black-box optimisation, it is only possible to evaluate $U(x)$ for given $x \in M$, and the cost of this evaluation precludes numerical approximation of derivatives. Then, the problem is to find $x^{*}$ using successive evaluations of $U(x)$ (hopefully, as few of these evaluations as possible).

Here, a new algorithm for solving this problem is described. The idea of this algorithm is to find $\bar{x}_{T}$ using successive evaluations of $U(x)$, in the hope that $\bar{x}_{T}$ will provide a good approximation of $x^{*}$. While the quality of this approximation is controlled by Inequalities (3) and (4) of Proposition 1, in some cases of interest, $\bar{x}_{T}$ is exactly equal to $x^{*}$, for correctly chosen $T$, as in the following proposition 3 .

To state this proposition, let $s_{x^{*}}$ denote geodesic symmetry about $x^{*}$ (see [7]). This is the transformation of $M$, which leaves $x^{*}$ fixed, and reverses the direction of geodesics passing through $x^{*}$.
Proposition 3. assume that $U$ is invariant by geodesic symmetry about $x^{*}$, in the sense that $U \circ s_{x^{*}}=U$. If $T \leq T_{\delta}$ (see (ii) of Proposition 2), then $\bar{x}_{T}=x^{*}$ is the unique barycentre of $P_{T}$.
Proposition 3 follows rather directly from Proposition 2 Precisely, by (ii) of Proposition 2, the condition $T \leq T_{\delta}$ implies $\mathcal{E}_{T}$ is strongly convex on $B\left(x^{*}, \delta\right)$, and $\bar{x}_{T} \in B\left(x^{*}, \delta\right)$. Thus, $\bar{x}_{T}$ is the unique stationary point of $\mathcal{E}_{T}$ in $B\left(x^{*}, \delta\right)$. But, using the fact that $U$ is invariant by geodesic symmetry about $x^{*}$, it is possible to prove that $x^{*}$ is a stationary point of $\mathcal{E}_{T}$, and this implies $\bar{x}_{T}=x^{*}$. The two following examples verify the conditions of Proposition 3 .

Example 1: assume $M=\operatorname{Gr}\left(k, \mathbb{C}^{n}\right)$ is a complex Grassmann manifold. In particular, $M$ is a simply-connected, compact Riemannian symmetric space. Identify $M$ with the set of Hermitian projectors $x: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ such that $\operatorname{tr}(x)=k$, where $\operatorname{tr}$ denotes the trace. Then, define $U(x)=-\operatorname{tr}(C x)$ for $x \in \operatorname{Gr}\left(k, \mathbb{C}^{n}\right)$, where $C$ is a Hermitian positive-definite matrix with distinct eigenvalues. Now, the unique global minimum of $U$ occurs at $x^{*}$, the projector onto the principal $k$-subspace of $C$. Also, the geodesic symmetry $s_{x^{*}}$ is given by $s_{x^{*}} \cdot x=r_{x^{*}} x r_{x^{*}}$, where $r_{x^{*}}: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ denotes reflection through the image space of $x^{*}$. It is elementary to verify that $U$ is invariant by this geodesic symmetry.

Example 2: let $M$ be a simply-connected, compact Riemannian symmetric space, and $U_{o}$ a function on $M$ with unique global minimum at $o \in M$. Assume moreover that $U_{o}$ is invariant by geodesic symmetry about $o$. For each $x^{*} \in M$, there exists an isometry $g$ of $M$, such that $x^{*}=g \cdot o$. Then, $U(x)=U_{o}\left(g^{-1} \cdot x\right)$ has unique global minimum at $x^{*}$, and is invariant by geodesic symmetry about $x^{*}$.

Example 1 describes the standard problem of finding the principal subspace of the covariance matrix $C$. In Example 2, the function $U_{o}$ is a known template, which undergoes an unknown transformation $g$, leading to the observed pattern $U$. This is a typical situation in pattern recognition problems.

Of course, from a mathematical point of view, Example 2 is not really an example, since it describes the completely general setting where the conditions of Proposition 3 are verified. In this setting, consider the following algorithm.

## Description of the algorithm :

$$
\begin{aligned}
- \text { input: } & T \leq T_{\delta} \\
& Q(x, d z)=q(x, z) \operatorname{vol}(d z) \\
& \hat{x}_{0}=z_{0} \in M
\end{aligned}
$$

$\%$ to find such $T$, see Section 3
\% symmetric Markov kernel
$\%$ initial guess for $x^{*}$

- iterate: for $n=1,2, \ldots$
(1) sample $z_{n} \sim q\left(z_{n-1}, z\right)$
(2) compute $r_{n}=1-\min \left\{1, \exp \left[\left(U\left(z_{n-1}\right)-U\left(z_{n}\right)\right) / T\right]\right\}$
(3) reject $z_{n}$ with probability $r_{n} \%$ then, $z_{n}=z_{n-1}$
(4) $\hat{x}_{n}=\hat{x}_{n-1} \#_{\frac{1}{n}} z_{n} \quad \%$ see definition 10 below
- until: $\hat{x}_{n}$ does not change sensibly
- output: $\hat{x}_{n} \quad \%$ approximation of $x^{*}$

The above algorithm recursively computes the Riemannian barycentre $\hat{x}_{n}$ of the samples $z_{n}$ generated by a symmetric Metropolis-Hastings algorithm (see [8]). Here, The Metropolis-Hastings algorithm is implemented in lines (1)--(3). On the other hand, line (4) takes care of the Riemannian barycentre. Precisely, if $\gamma:[0,1] \rightarrow M$ is a length-minimising geodesic connecting $\hat{x}_{n-1}$ to $z_{n}$, let

$$
\begin{equation*}
\hat{x}_{n-1} \#_{\frac{1}{n}} z_{n}=\gamma(1 / n) \tag{10}
\end{equation*}
$$

This geodesic $\gamma$ need not be unique.
The point of using the Metropolis-Hastings algorithm is that the generated $z_{n}$ eventually sample from the Gibbs distribution $P_{T}$. The convergence of the distribution $P_{n}$ of $z_{n}$ to $P_{T}$ takes place exponentially fast. Indeed, it may be inferred from [8] (see Theorem 8, Page 36)

$$
\begin{equation*}
\left\|P_{n}-P_{T}\right\|_{T V} \leq\left(1-p_{T}\right)^{n} \tag{11}
\end{equation*}
$$

where $\|\cdot\|_{T V}$ is the total variation norm, and $p_{T} \in(0,1)$ verifies

$$
p_{T} \leq(\operatorname{vol} M) \inf _{x, z} q(x, z) \exp \left(-\sup _{x} U(x) / T\right)
$$

so the rate of convergence is degraded when $T$ is small.

Accordingly, the intuitive justification of the above algorithm is the following. Since the $z_{n}$ eventually sample from the Gibbs distribution $P_{T}$, and the desired global minimum $x^{*}$ of $U$ is equal to the barycentre $\bar{x}_{T}$ of $P_{T}$ (by Proposition 3), then the barycentre $\hat{x}_{n}$ of the $z_{n}$ is expected to converge to $x^{*}$.

It should be emphasised that, in the present state of the literature, there is no rigorous result which confirms this convergence $z_{n} \rightarrow x^{*}$. It is therefore an open problem, to be confronted in future work.

For a basic computer experiment, consider $M=S^{2} \subset \mathbb{R}^{3}$, and let

$$
\begin{equation*}
U(x)=-P_{9}\left(x^{3}\right) \quad \text { for } x=\left(x^{1}, x^{2}, x^{3}\right) \in S^{2} \tag{12}
\end{equation*}
$$

where $P_{9}$ is the Legendre polynomial of degree 9 [9. The unique global minimiser of $U$ is $x^{*}=(0,0,1)$, and the conditions of Proposition 3 are verified, since $U$ is invariant by reflection in the $x^{3}$ axis, which is geodesic symmetry about $x^{*}$.


Fig. 1. graph of $-P_{9}\left(x^{3}\right)$


Fig. 2. $\hat{x}_{n}^{3}$ versus $n$

Figure 1 shows the dependence of $U(x)$ on $x^{3}$, displaying multiple local minima and maxima. Figure 2 shows the algorithm overcoming these local minima and maxima, and converging to the global minimum $x^{*}=(0,0,1)$, within $n=5000$ iterations. The experiment was conducted with $T=0.2$, and the Markov kernel $Q$ obtained from the von Mises-Fisher distribution (see [10]). The initial guess $\hat{x}_{0}=(0,0,-1)$ is not shown in Figure 2 .

In comparison, a standard simulated annealing method offered less robust performance, which varied considerably with the choice of annealing schedule.

## 5 Proofs

This section is devoted to the proofs of the results stated in previous sections.
As of now, assume that $U\left(x^{*}\right)=0$. There is nos loss of generality in making this assumption.

### 5.1 Proof of Proposition 1

Proof of $(i)$ : denote $f_{x}(z)=\frac{1}{2} d^{2}(x, z)$. By the definition of $\mathcal{E}_{T}$

$$
\begin{equation*}
\mathcal{E}_{T}(x)=\int_{M} f_{x}(z) P_{T}(d z) \tag{13a}
\end{equation*}
$$

Moreover, let $\mathcal{E}_{0}$ be the function

$$
\begin{equation*}
\mathcal{E}_{0}(x)=\int_{M} f_{x}(z) \delta_{x^{*}}(d z)=\frac{1}{2} d^{2}\left(x, x^{*}\right) \tag{13b}
\end{equation*}
$$

For any $x$, it is elementary that $f_{x}(z)$ is Lipschitz continuous, with respect to $z$, with Lipschitz constant diam $M$. Then, from the Kantorovich-Rubinshtein formula [4,

$$
\begin{equation*}
\left|\mathcal{E}_{T}(x)-\mathcal{E}_{0}(x)\right| \leq(\operatorname{diam} M) W\left(P_{T}, \delta_{x^{*}}\right) \tag{13c}
\end{equation*}
$$

a uniform bound in $x \in M$. It now follows that

$$
\begin{align*}
& \inf _{x \in B\left(x^{*}, \eta\right)} \mathcal{E}_{T}(x)-\inf _{x \in B\left(x^{*}, \eta\right)} \mathcal{E}_{0}(x) \leq(\operatorname{diam} M) W\left(P_{T}, \delta_{x^{*}}\right) \text { and }  \tag{13d}\\
& \inf _{x \notin B\left(x^{*}, \eta\right)} \mathcal{E}_{0}(x)-\inf _{x \notin B\left(x^{*}, \eta\right)} \mathcal{E}_{T}(x) \leq(\operatorname{diam} M) W\left(P_{T}, \delta_{x^{*}}\right) \tag{13e}
\end{align*}
$$

However, from 13b), it is clear that

$$
\inf _{x \in B\left(x^{*}, \eta\right)} \mathcal{E}_{0}(x)=0 \quad \text { and } \quad \inf _{x \notin B\left(x^{*}, \eta\right)} \mathcal{E}_{0}(x)=\frac{\eta^{2}}{2}
$$

To complete the proof, replace this into 13 d and 13 e . Then, assuming the condition in $(3)$ is verified,

$$
\begin{equation*}
\inf _{x \in B\left(x^{*}, \eta\right)} \mathcal{E}_{T}(x)<\frac{\eta^{2}}{4}<\inf _{x \notin B\left(x^{*}, \eta\right)} \mathcal{E}_{T}(x) \tag{13f}
\end{equation*}
$$

This means that any global minimum $\bar{x}_{T}$ of $\mathcal{E}_{T}$ must belong to the open ball $B\left(x^{*}, \eta\right)$. In other words, $d\left(\bar{x}_{T}, x^{*}\right)<\eta$. This completes the proof of (3).
Proof of (ii): let $\rho \leq \min \left\{\operatorname{inj} x^{*}, \kappa^{-1} \frac{\pi}{2}\right\}$ where $\operatorname{inj} x^{*}$ is the injectivity radius of $M$ at $x^{*}$, and $\kappa^{2}$ is an upper bound on the sectional curvature of $M$. Assume, in addition, $\rho$ is small enough so

$$
\begin{equation*}
\mu_{\min } d^{2}\left(x, x^{*}\right) \leq 2\left(U(x)-U\left(x^{*}\right)\right) \leq \mu_{\max } d^{2}\left(x, x^{*}\right) \tag{14a}
\end{equation*}
$$

whenever $d\left(x, x^{*}\right) \leq \rho$. Further, consider the truncated distribution

$$
\begin{equation*}
P_{T}^{\rho}(d z)=\frac{\mathbf{1}_{B_{\rho}}(z)}{P_{T}\left(B_{\rho}\right)} \cdot P_{T}(d z) \tag{14b}
\end{equation*}
$$

where $\mathbf{1}$ denotes the indicator function, and $B_{\rho}$ stands for the open ball $B\left(x^{*}, \rho\right)$. Of course, by the triangle inequality,

$$
\begin{equation*}
W\left(P_{T}, \delta_{x^{*}}\right) \leq W\left(P_{T}, P_{T}^{\rho}\right)+W\left(P_{T}^{\rho}, \delta_{x^{*}}\right) \tag{14c}
\end{equation*}
$$

The proof relies on the following estimates, which use the notation of Section 3.
First estimate : if $T \leq T_{o}^{1}$, then
$W\left(P_{T}, P_{T}^{\rho}\right) \leq(\operatorname{diam} M \times \operatorname{vol} M) \frac{2}{\pi}\left(\frac{\pi}{8}\right)^{n / 2}\left(\frac{\mu_{\max }}{T}\right)^{n / 2} \exp \left(-U_{\rho} / T\right)$
Second estimate : if $T \leq T_{o}^{1}$, then

$$
\begin{equation*}
W\left(P_{T}^{\rho}, \delta_{x^{*}}\right) \leq 2 \sqrt{2 \pi}\left(\frac{\pi}{2}\right)^{n-1} B_{n}^{-1}\left(\frac{\mu_{\max }}{\mu_{\min }}\right)^{n / 2}\left(\frac{T}{\mu_{\min }}\right)^{1 / 2} \tag{14e}
\end{equation*}
$$

These two estimates are proved below. Assume now they hold true, and $T \leq T_{o}$. In particular, since $T \leq T_{o}^{2}$, the definition of $T_{o}^{2}$ implies

$$
f(T, n+1, \rho) \leq\left(\mu_{\max } / \mu_{\min }\right)^{n / 2} C_{n}
$$

Recall the definition of $C_{n}$, and express $\omega_{n}$ and $A_{n}$ in terms of the Gamma function [9. The last inequality becomes

$$
(\operatorname{diam} M \times \operatorname{vol} M) f(T, n+1, \rho) \leq 2(2 \pi)^{n / 2} B_{n}^{-1}\left(\mu_{\max } / \mu_{\min }\right)^{n / 2}
$$

This is the same as

$$
(\operatorname{diam} M \times \operatorname{vol} M) \frac{1}{\pi}\left(\frac{\pi}{8}\right)^{n / 2} f(T, n+1, \rho) \leq\left(\frac{\pi}{2}\right)^{n-1} B_{n}^{-1}\left(\mu_{\max } / \mu_{\min }\right)^{n / 2}
$$

By the definition of $f(T, n+1, \rho)$, it now follows the right-hand side of 14 d ) is less than half the right-hand side of (14e). In this case, (4) follows from the triangle inequality (14c).

Proof of first estimate : consider the coupling of $P_{T}$ and $P_{T}^{\rho}$, provided by the probability distribution $K$ on $M \times M$,

$$
\begin{equation*}
K\left(d z_{1} \times d z_{2}\right)=P_{T}^{\rho}\left(d z_{1}\right)\left[P_{T}\left(B_{\rho}\right) \delta_{z_{1}}\left(d z_{2}\right)+\mathbf{1}_{B_{\rho}^{c}}\left(z_{2}\right) P_{T}\left(d z_{2}\right)\right] \tag{15a}
\end{equation*}
$$

where $B_{\rho}^{c}$ denotes the complement of $B_{\rho}$. Recall the definition of the Kantorovich distance (see [4). Replacing (15a) into this definition, it follows that

$$
\begin{equation*}
W\left(P_{T}, P_{T}^{\rho}\right) \leq(\operatorname{diam} M) P_{T}\left(B_{\rho}^{c}\right) \tag{15b}
\end{equation*}
$$

Then, from the definition (2) of $P_{T}$,

$$
\begin{equation*}
P_{T}\left(B_{\rho}^{c}\right) \leq(Z(T))^{-1}(\operatorname{vol} M) \exp \left(-U_{\rho} / T\right) \tag{15c}
\end{equation*}
$$

Now, (14d) follows directly from (15b) and (15c), if the following lower bound on $Z(T)$ can be proved,

$$
\begin{equation*}
Z(T) \geq \frac{\pi}{2}\left(\frac{8}{\pi}\right)^{n / 2}\left(\frac{T}{\mu_{\max }}\right)^{n / 2} \quad \text { for } T \leq T_{o}^{1} \tag{15d}
\end{equation*}
$$

To prove this lower bound, note that

$$
Z(T)=\int_{M} e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \geq \int_{B_{\rho}} e^{-\frac{U(z)}{T}} \operatorname{vol}(d z)
$$

Using this last inequality and 14a, it is possible to write

$$
\begin{equation*}
Z(T) \geq \int_{B_{\rho}} e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \geq \int_{B_{\rho}} e^{-\frac{\mu_{\max }}{2 T} d^{2}\left(x, x^{*}\right)} \operatorname{vol}(d z) \tag{15e}
\end{equation*}
$$

Writing this last integral in Riemannian spherical coordinates,

$$
\begin{equation*}
\int_{B_{\rho}} e^{-\frac{\mu_{\max }}{2 T} d^{2}\left(x, x^{*}\right)} \operatorname{vol}(d z)=\int_{0}^{\rho} \int_{S^{n-1}} e^{-\frac{\mu_{\max }}{2 T} r^{2}} \lambda(r, s) d r \omega_{n}(d s) \tag{15f}
\end{equation*}
$$

where $\lambda(r, s)$ is the volume density in the Riemannian spherical coordinates, $r \geq 0$ and $s \in S^{n-1}$, and where $\omega_{n}(d s)$ is the area element of $S^{n-1}$. From the volume comparison theorem in [6] (see Page 129),

$$
\begin{equation*}
\lambda(r, s) \geq\left(\kappa^{-1} \sin (\kappa r)\right)^{n-1} \geq((2 / \pi) r)^{n-1} \tag{15g}
\end{equation*}
$$

where the second inequality follows since $x \mapsto \sin (x)$ is concave for $x \in(0, \pi)$. Now, it follows from 15e and 15f,

$$
\begin{equation*}
Z(T) \geq \omega_{n}\left(\frac{2}{\pi}\right)^{n-1} \int_{0}^{\rho} e^{-\frac{\mu_{\max }}{2 T} r^{2}} r^{n-1} d r \tag{15h}
\end{equation*}
$$

where $\omega_{n}$ is the surface area of $S^{n-1}$. Thus, the required lower bound 15d follows by noting that

$$
\int_{0}^{\rho} e^{-\frac{\mu_{\max }}{2 T}} r^{2} r^{n-1} d r=(2 \pi)^{1 / 2}\left(\frac{T}{\mu_{\max }}\right)^{n / 2} A_{n-1}-\int_{\rho}^{\infty} e^{-\frac{\mu_{\max }}{2 T} r^{2}} r^{n-1} d r
$$

where $A_{n}=E|X|^{n}$ for $X \sim N(0,1)$, and that

$$
\int_{\rho}^{\infty} e^{-\frac{\mu_{\max }}{2 T} r^{2}} r^{n-1} d r \leq \rho^{n-2} \frac{T}{\mu_{\max }} e^{-\frac{\mu_{\max }}{2 T} \rho^{2}} \leq \rho^{n-2} \frac{T}{\mu_{\max }} e^{-\frac{U_{\rho}}{T}}
$$

Indeed, taken together, these give

$$
Z(T) \geq \omega_{n}\left(\frac{2}{\pi}\right)^{n-1}\left[(2 \pi)^{1 / 2}\left(\frac{T}{\mu_{\max }}\right)^{n / 2} A_{n-1}-\rho^{n-2} \frac{T}{\mu_{\max }} e^{-\frac{U_{\rho}}{T}}\right]
$$

Finally, $(15 \mathrm{~d})$ can be obtained by noting the second term in square brackets is negligeable compared to the first, as $T$ decreases to 0 , and by expressing $\omega_{n}$ and $A_{n-1}$ in terms of the Gamma function (9].

Proof of second estimate: the Kantorovich distance between $P_{T}^{\rho}$ and the Dirac distribution $\delta_{x^{*}}$ is equal to the expectation of the distance to $x^{*}$, with respect to $P_{T}^{\rho}$ 4]. Precisely,

$$
W\left(P_{T}^{\rho}, \delta_{x^{*}}\right)=\int_{M} d\left(x^{*}, z\right) P_{T}^{\rho}(d z)
$$

According to (2) and 14 b , this is

$$
W\left(P_{T}^{\rho}, \delta_{x^{*}}\right)=\left(P_{T}\left(B_{\rho}\right) Z(T)\right)^{-1} \int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z)
$$

Using (2) to express the probability $P_{T}\left(B_{\rho}\right)$, this becomes

$$
\begin{equation*}
W\left(P_{T}^{\rho}, \delta_{x^{*}}\right)=\frac{\int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z)}{\int_{B_{\rho}} e^{-\frac{U(z)}{T}} \operatorname{vol}(d z)} \tag{16a}
\end{equation*}
$$

A lower bound on the denominator can be found from 15 e and subsequent inequalities, which were used to prove 15 d . Precisely, these inequalities provide

$$
\begin{equation*}
\int_{B_{\rho}} e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \geq \frac{1}{2} \omega_{n}\left(\frac{2}{\pi}\right)^{n-1}(2 \pi)^{1 / 2} A_{n-1}\left(\frac{T}{\mu_{\max }}\right)^{n / 2} \tag{16b}
\end{equation*}
$$

whenever $T \leq T_{o}^{1}$. For the numerator in 16 a , it will be shown that, for any $T$,

$$
\begin{equation*}
\int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \leq \omega_{n}(2 \pi)^{1 / 2} A_{n}\left(\frac{T}{\mu_{\min }}\right)^{(n+1) / 2} \tag{16c}
\end{equation*}
$$

Then, $\sqrt{14 \mathrm{e}}$ follows by dividing $\sqrt{16 \mathrm{c}}$ by 16 b , and replacing in 16 a , after noting that $A_{n} / A_{n-1}=\sqrt{2 \pi} B_{n}^{-1}$. Thus, it only remains to prove 16 c . Using (14a), it is seen that

$$
\int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \leq \int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{\mu_{\min }}{2 T} d^{2}\left(x, x^{*}\right)} \operatorname{vol}(d z)
$$

By expressing this last integral in Riemannian spherical coordinates, as in 15f),

$$
\begin{equation*}
\int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \leq \int_{0}^{\rho} \int_{S^{n-1}} r e^{-\frac{\mu_{\min }}{2 T} r^{2}} \lambda(r, s) d r \omega_{n}(d s) \tag{16d}
\end{equation*}
$$

From the volume comparison theorem in [6] (see Page 130), $\lambda(r, s) \leq r^{n-1}$. Therefore, 16 d becomes

$$
\int_{B_{\rho}} d\left(x^{*}, z\right) e^{-\frac{U(z)}{T}} \operatorname{vol}(d z) \leq \omega_{n} \int_{0}^{\rho} e^{-\frac{\mu_{\min }}{2 T} r^{2}} r^{n} d r \leq \omega_{n} \int_{0}^{\infty} e^{-\frac{\mu_{\min }}{2 T} r^{2}} r^{n} d r
$$

The right-hand side is half the $n$th absolute moment of a normal distribution. Expressing this in terms of $A_{n}$, and replacing in 16 d , gives 16 c$)$.

## 6 Proof of Lemma 1

Denote $G$ the connected component at identity of the group of isometries of $M$. It will be assumed that $G$ is simply-connected and semisimple [7]. Any geodesic $\gamma: I \rightarrow M$ is of the form [7] [11],

$$
\begin{equation*}
\gamma(t)=\exp (t Y) \cdot x \tag{17a}
\end{equation*}
$$

for some $x \in M$ and $Y \in \mathfrak{g}$, the Lie algebra of $G$, where $\exp : \mathfrak{g} \rightarrow G$ denotes the Lie group exponential mapping, and the dot denotes the action of $G$ on $M$. For each $t \in I$, the cut locus $\operatorname{Cut}(\gamma(t))$ of $\gamma(t)$ is given by

$$
\begin{equation*}
\operatorname{Cut}(\gamma(t))=\exp (t Y) \cdot \operatorname{Cut}(x) \tag{17b}
\end{equation*}
$$

This is due to a more general result: let $M$ be a Riemannian manifold and $g: M \rightarrow M$ be an isometry of $M$. Then, $\operatorname{Cut}(g \cdot x)=g \cdot \operatorname{Cut}(x)$ for all $x \in M$. This is because $y \in \operatorname{Cut}(x)$ if and only if $y$ is conjugate to $x$ along some geodesic, or there exist two different geodesics connecting $x$ to $y$ [6] [11. Both of these properties are preserved by the isometry $g$.

In order to describe the set $\operatorname{Cut}(x)$, denote $K$ the isotropy group of $x$ in $G$, and $\mathfrak{k}$ the Lie algebra of $K$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be an orthogonal decomposition, with respect to the Killing form of $\mathfrak{g}$, and let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$. Define $\mathcal{S}=K / C_{\mathfrak{a}}\left(C_{\mathfrak{a}}\right.$ the centraliser of $\mathfrak{a}$ in $\left.K\right)$, and consider the mapping

$$
\begin{equation*}
\phi(s, a)=\exp (\operatorname{Ad}(s) a) \cdot x \quad \text { for }(s, a) \in \mathcal{S} \times \mathfrak{a} \tag{17c}
\end{equation*}
$$

The set $\operatorname{Cut}(x)$ is the image under $\phi$ of a certain set $\mathcal{S} \times \partial Q$, which is now described, following [7] [12].

Let $\Delta_{+}$be the set of positive restricted roots associated to the pair $(G, K)$, (each $\lambda \in \Delta_{+}$is a linear form $\lambda: \mathfrak{a} \rightarrow \mathbb{R}$ ). Then, let $Q$ be the set of $a \in \mathfrak{a}$ such that $|\lambda(a)| \leq \pi$ for all $\lambda \in \Delta_{+}$, and $\partial Q$ the boundary of $Q$. Then

$$
\begin{equation*}
\operatorname{Cut}(x)=\phi(\mathcal{S} \times \partial Q) \tag{17d}
\end{equation*}
$$

Recapitulating 17 b and 17 d ,

$$
\begin{equation*}
\operatorname{Cut}(\gamma)=\Phi(I \times \mathcal{S} \times \partial Q) \text { where } \Phi(t, s, a)=\exp (t Y) \cdot \phi(s, a) \tag{17e}
\end{equation*}
$$

Lemma 1 states that the topological dimension of $\operatorname{Cut}(\gamma)$ is strictly less than $\operatorname{dim} M$. This is proved using results from topological dimension theory [7] [13.

Note that both $I$ and $\mathcal{S}$ are compact. Indeed, $\mathcal{S}$ is compact since it is the continuous image of the compact group $K$ under the projection $K \rightarrow K / C_{\mathfrak{a}}$. Also, $\partial Q$ is compact in $\mathfrak{a}$, and $\partial Q=\cup_{\lambda} \partial Q_{\lambda}$ where $\partial Q_{\lambda}=\partial Q \cap\{\lambda(a)= \pm \pi\}$ for $\lambda \in \Delta_{+}$. Since $\{\lambda(a)= \pm \pi\}$ is the union of two (closed) hyperplanes in $\mathfrak{a}$, $\partial Q_{\lambda}$ is compact. Now, each $I \times \mathcal{S} \times \partial Q_{\lambda}$ is compact, and therefore closed. It follows from 17e that (see 13, Page 30),

$$
\begin{equation*}
\operatorname{dim} \operatorname{Cut}(\gamma)=\operatorname{dim} \bigcup_{\lambda} \Phi\left(I \times \mathcal{S} \times \partial Q_{\lambda}\right) \leq \max _{\lambda} \operatorname{dim} \Phi\left(I \times \mathcal{S} \times \partial Q_{\lambda}\right) \tag{17f}
\end{equation*}
$$

But, for each $\lambda$,

$$
\Phi\left(I \times \mathcal{S} \times \partial Q_{\lambda}\right)=\Phi\left(I \times \mathcal{S}_{\lambda} \times \partial Q_{\lambda}\right) \subset \Phi\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right)
$$

where $\mathcal{S}_{\lambda}=K / C_{\lambda}\left(C_{\lambda}\right.$ the centraliser of $\{\lambda(a)= \pm \pi\}$ in $\left.K\right)$. The above inclusion implies (by 13, Page 26),

$$
\begin{equation*}
\operatorname{dim} \Phi\left(I \times \mathcal{S} \times \partial Q_{\lambda}\right) \leq \operatorname{dim} \Phi\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right) \tag{17~g}
\end{equation*}
$$

To conclude, note that the set $\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}$ is a differentiable manifold. It follows that (see [7], Page 345),

$$
\begin{equation*}
\operatorname{dim} \Phi\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right) \leq \operatorname{dim}\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right) \tag{17~h}
\end{equation*}
$$

The right-hand side of this inequality is

$$
\operatorname{dim}\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right)=1+\operatorname{dim} \mathcal{S}_{\lambda}+\operatorname{dim} \mathfrak{a}-1
$$

since the dimension of a hyperplane in $\mathfrak{a}$ is $\operatorname{dim} \mathfrak{a}-1$. In addition, according to [7] (Page 296), $\operatorname{dim} \mathcal{S}_{\lambda}<\operatorname{dim} \mathcal{S}$. Thus,

$$
\operatorname{dim}\left(\mathbb{R} \times \mathcal{S}_{\lambda} \times\{\lambda(a)= \pm \pi\}\right)=\operatorname{dim} \mathcal{S}_{\lambda}+\operatorname{dim} \mathfrak{a}<\operatorname{dim} M
$$

since $\operatorname{dim} M=\operatorname{dim} \mathcal{S}+\operatorname{dim} \mathfrak{a}[7$. Replacing this into 17h , it follows from 17 f . and 17 g that $\operatorname{dim} \operatorname{Cut}(\gamma)<\operatorname{dim} M$, as required.

## 7 Proof of Corollary 1

The corollary can be split into the two following claims, which will be proved separately.

First claim : both integrals $G_{x}$ and $H_{x}$ converge for any value of $T$.
Second claim : $\mathcal{E}_{T}$ is $C^{2}$ throughout $M$, with derivatives given by (5).
The fact that $G_{x}$ and $H_{x}$ depend continuously on $x$ is contained in the second claim, since (5) states that $G_{x}$ and $H_{x}$ are the gradient and Hessian of $\mathcal{E}_{T}$ at $x$.

In the following proofs, the notation $\mathrm{D}(x)=M-\operatorname{Cut}(x)$ will be used, in order to avoid cumbersome expressions.

Proof of first claim : The convergence of the integral $G_{x}$ is straightforward, since the integrand $G_{x}(z)$ is a smooth and bounded function, from $\mathrm{D}(x)$ to $T_{x} M$. This is because, by definition, $G_{x}(z)$ is given by

$$
\begin{equation*}
G_{x}(z)=-\operatorname{Exp}_{x}^{-1}(z) \tag{18}
\end{equation*}
$$

where Exp is the Riemannian exponential mapping [6. Therefore, $G_{x}(z)$ is smooth. In addition, $G_{x}(z)$ is bounded, in Riemannian norm, by diam $M$.

The convergence of the integral $H_{x}$ is more difficult. While the integrand $H_{x}(z)$ is smooth on $\mathrm{D}(x)$, it is not bounded. It will be seen that $H_{x}$ is an absolutely convergent improper integral.

Recall the mapping $\phi$ defined in $\sqrt{17 \mathrm{c}})$. Let $D_{+}$be the set of points $a \in \mathfrak{a}$ which belong to the interior of $Q$, and which verify $\lambda(a) \geq 0$ for each $\lambda \in \Delta_{+}$. Let $D_{+}^{o}$ be the interior of $D_{+}$. Then, $\phi$ maps $\mathcal{S} \times D_{+}$onto $\mathrm{D}(x)$, and is a diffeomorphism of $\mathcal{S} \times D_{+}^{o}$ onto its image in $\mathrm{D}(x)$ [7] [12] (see Chapter VII in [7]). Using Sard's theorem [14], it follows from the definition of $H_{x}$ that

$$
\begin{equation*}
H_{x}=\int_{\mathcal{S}} \int_{D_{+}} H_{x}(\phi(s, a)) p_{T}(\phi(s, a)) J(a) d a \omega(d s) \tag{19a}
\end{equation*}
$$

where $p_{T}$ denotes the density of $P_{T}$ with respect to the Riemannian volume of $M$, and $J(a)$ is the Jacobian determinant of $\phi$, given by [7]

$$
\begin{equation*}
J(a)=\prod_{\lambda \in \Delta_{+}}(\sin \lambda(a))^{m_{\lambda}} \tag{19b}
\end{equation*}
$$

with $m_{\lambda}$ the multiplicity of the restricted root $\lambda$, and where $\omega(d s)$ is the invariant Riemannian volume induced on $\mathcal{S}$ from $K$.

Now, $H_{x}(\phi(s, a))$ can be expressed as follows (cot is the cotangent function)

$$
\begin{equation*}
H_{x}(\phi(s, a))=\Pi_{0}(s)+\sum_{\lambda \in \Delta_{+}} \lambda(a) \cot \lambda(a) \Pi_{\lambda}(s) \tag{19c}
\end{equation*}
$$

where $\Pi_{0}(s)$ and the $\Pi_{\lambda}(s)$ denote orthogonal projectors, onto the respective eigenspaces of $H_{x}(\phi(s, a))$.

According to this expression, $H_{x}(\phi(s, a))$ diverges to $-\infty$ whenever $\lambda(a)=\pi$. However, the product

$$
H_{x}(\phi(s, a)) p_{T}(\phi(s, a)) J(a)
$$

which appears under the integral in 19a, is clearly continuous and bounded on the domain of integration. Thus, the absolute convergence of the integral $H_{x}$ follows immediately from 19a). It now remains to provide a proof of 19 c . This is here only briefly indicated. Expression 19c) is a slight improvement of the one in [15] (see Theorem IV.1, Page 636), where it is enough to note that if $R$ is the curvature tensor of $M$, then the operator $R_{v}(u)=R(v, u) v$ has the eigenvalues 0 and $(\lambda(a))^{2}$ for each $\lambda \in \Delta_{+}$, whenever $v, u \in T_{x} M \simeq \mathfrak{p}$ with $v=\operatorname{Ad}(s) a$ [7] [12]. It is well-known, by properties of the Jacobi equation [6], that $H_{x}(\phi(s, a))$ has the same eigenspace decomposition as $R_{v}$, in this case.

Proof of second claim : the proof of this claim relies in a crucial way on Lemma 1. To compute the gradient and Hessian of the function $\mathcal{E}_{T}$ at $x \in M$, consider any geodesic $\gamma: I \rightarrow M$, defined on a compact interval $I=[-\tau, \tau]$, such that $\gamma(0)=x$. For each $t \in I$, by definition of the function $\mathcal{E}_{T}$,

$$
\begin{equation*}
\mathcal{E}_{T}(\gamma(t))=\frac{1}{2} \int_{M} d^{2}(\gamma(t), z) P_{T}(d z) \tag{20a}
\end{equation*}
$$

However, Lemma 1 states that the set

$$
\operatorname{Cut}(\gamma)=\bigcup_{t \in I} \operatorname{Cut}(\gamma(t))
$$

has Riemannian volume equal to zero. From (2), it is clear that $P_{T}$ is absolutely continuous with respect to Riemannian volume. Therefor, Cut $(\gamma)$ can be removed from the domain of integration in 20a). Then,

$$
\begin{equation*}
\mathcal{E}_{T}(\gamma(t))=\frac{1}{2} \int_{\mathrm{D}(\gamma)} d^{2}(\gamma(t), z) P_{T}(d z) \tag{20b}
\end{equation*}
$$

where $\mathrm{D}(\gamma)=M-\operatorname{Cut}(\gamma)$. Now, for each $z \in \mathrm{D}(\gamma)$, the function

$$
t \mapsto f_{z}(t)=\frac{1}{2} d^{2}(\gamma(t), z)
$$

is twice continuously differentiable with respect to $t \in I$, with

$$
\begin{equation*}
\frac{d f_{z}}{d t}=\left\langle G_{\gamma(t)}(z), \dot{\gamma}\right\rangle \quad \text { and } \quad \frac{d^{2} f_{z}}{d t^{2}}=H_{\gamma(t)}(z)(\dot{\gamma}, \dot{\gamma}) \tag{20c}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ denotes the Riemannian metric of $M$, and $\dot{\gamma}$ the velocity of the geodesic $\gamma$. Indeed, this holds because the geodesic $\gamma$ does not intersect the cut locus $\operatorname{Cut}(z)$ (see [6]).

The claim that $\mathcal{E}_{T}$ is twice differentiable, and has derivatives given by (5), follows from 20 b and 20 c , by differentiation under the integral sign, provided it can be shown that the families of functions

$$
\left\{z \mapsto G_{\gamma(t)}(z) ; t \in I\right\} \quad \text { and } \quad\left\{z \mapsto H_{\gamma(t)}(z) ; t \in I\right\}
$$

which all have the common domain of definition $\mathrm{D}(\gamma)$, are uniformly integrable with respect to $P_{T}$ [14]. Roughly, uniform integrability means that the rate of absolute convergence of the following integrals does not depend on $t$,

$$
G_{\gamma(t)}=\int_{\mathrm{D}(\gamma)} G_{\gamma(t)}(z) P_{T}(d z) \quad ; \quad H_{\gamma(t)}=\int_{\mathrm{D}(\gamma)} H_{\gamma(t)}(z) P_{T}(d z)
$$

This is clear for the integrals $G_{\gamma(t)}$ because $G_{\gamma(t)}(z)$ is bounded in Riemannian norm by $\operatorname{diam} M$, uniformly in $t$ and $z$ (see the proof of the first claim).

Then, consider the integral $H_{x}=H_{\gamma(0)}$, and recall Formulae 19a) and 19c. Each $z \in \mathrm{D}(\gamma)$ can be written under the form $z=\phi(s, a)$ where $(s, a) \in \mathcal{S} \times D_{+}$. Accordingly, it follows from (19c) that

$$
\begin{equation*}
\left\|H_{x}(z)\right\|_{F} \leq(\operatorname{dim} M)^{\frac{1}{2}} \max \{1,|\kappa(a) \cot \kappa(a)|\} \tag{20d}
\end{equation*}
$$

where $\|\cdot\|_{F}$ is the Frobenius norm with respect to the Riemannian metric of $M$, and $\kappa \in \Delta_{+}$is the highest restricted root [7] $\left(\kappa(a) \geq \lambda(a)\right.$ for $\left.\lambda \in \Delta_{+}, a \in D_{+}\right)$.

The required uniform integrability is equivalent to the statement that

$$
\begin{equation*}
\lim _{K \rightarrow \infty} \int_{\mathrm{D}(\gamma)}\left\|H_{x}(z)\right\|_{F} \mathbf{1}\left\{\left\|H_{x}(z)\right\|_{F}>K\right\} P_{T}(d z)=0 \tag{20e}
\end{equation*}
$$

where the rate of convergence to this limit does not depend on $x$. But, according to 20d, if $K>1$, there exists $\epsilon>0$ such that

$$
\left\{\left\|H_{x}(z)\right\|_{F}>K\right\}=\{\kappa(a)>\pi-\epsilon\}
$$

and $\epsilon \rightarrow 0$ as $K \rightarrow \infty$. In this case, the integral in 20 e is less than

$$
\begin{equation*}
(\operatorname{dim} M)^{\frac{1}{2}}\left(\sup _{z} p_{T}(z)\right) \int_{\mathcal{D}(\gamma)}|\kappa(a) \cot \kappa(a)| \mathbf{1}\{\kappa(a)>\pi-\epsilon\} \operatorname{vol}(d z) \tag{20f}
\end{equation*}
$$

Now, using the same integral formula as in 19a, this last integral is equal to

$$
\begin{aligned}
& \int_{\mathcal{S}} \int_{D_{+}}|\kappa(a) \cot \kappa(a)| \mathbf{1}\{\kappa(a)>\pi-\epsilon\} J(a) d a \omega(d s)= \\
& \omega(\mathcal{S}) \int_{D_{+}}[|\kappa(a) \cot \kappa(a)| J(a)] \mathbf{1}\{\kappa(a)>\pi-\epsilon\} d a
\end{aligned}
$$

In view of 19 b , since $\kappa \in \Delta_{+}$, the function in square brackets is bounded on the closure of $D_{+}$. In fact [7], its supremum is $\kappa^{2}=(\kappa, \kappa)$ where $(\cdot, \cdot)$ is the scalar product induced on $\mathfrak{a}^{*}$ (the dual space of $\mathfrak{a}$ ) by the Killing form of $\mathfrak{g}$. Finally, by (20f), the integral in 20e is less than

$$
(\operatorname{dim} M)^{\frac{1}{2}}\left(\sup _{z} p_{T}(z)\right) \omega(\mathcal{S}) \kappa^{2} \int_{D_{+}} \mathbf{1}\{\kappa(a)>\pi-\epsilon\} d a
$$

Since, $\kappa(a) \in[0, \pi)$ for $a \in D_{+}$, this last integral converges to 0 as $\epsilon \rightarrow 0$, at a rate which does not depend on $x$. This proves the required uniform integrability, so the proof is now complete.

## 8 Proof of Proposition 2

Remark : in the statement of Proposition 2 the notation $\kappa^{2}$ is used for the maximum sectional curvature of $M$. In the previous proof of Corollary 1 the same notation $\kappa^{2}$ was used for the squared norm of the highest restricted root. This is not an abuse of notation, since the two quantities are in fact equal [7] (see Page 334).

Proof of (i) : let $x \in B\left(x^{*}, \delta\right)$. By (5) of Corollary 1. $\nabla^{2} \mathcal{E}_{T}(x)$ is equal to $H_{x}$. To obtain (7), decompose $H_{x}$ into two integrals

$$
\begin{equation*}
H_{x}=\int_{B\left(x, r_{c x}\right)} H_{x}(z) P_{T}(d z)+\int_{\mathrm{D}(x)-B\left(x, r_{c x}\right)} H_{x}(z) P_{T}(d z) \tag{21a}
\end{equation*}
$$

This is possible since $B\left(x, r_{c x}\right) \subset \mathrm{D}(x)$, where $\mathrm{D}(x)=M-\operatorname{Cut}(x)$. The first integral in 21a will be denoted $I_{1}$, and the second integral $I_{2}$.

With regard to $I_{1}$, note the inclusions $B\left(x^{*}, \delta\right) \subset B(x, 2 \delta) \subset B\left(x, r_{c x}\right)$, which follow from the triangle inequality. In addition, note that $H_{x}(z) \geq 0$ (in the Loewner order [16]), for $z \in B\left(x, r_{c x}\right)$. Therefore,

$$
\begin{equation*}
I_{1} \geq \int_{B\left(x^{*}, \delta\right)} H_{x}(z) P_{T}(d z) \tag{21b}
\end{equation*}
$$

However, from (19c) and the definition of $\kappa \in \Delta_{+}$,

$$
\begin{equation*}
H_{x}(z) \geq \kappa(a) \cot \kappa(a) \tag{21c}
\end{equation*}
$$

for $z=\phi(s, a) \in \mathrm{D}(x)$. Using the Cauchy-Scwharz inequality, $\kappa(a) \leq \kappa\|a\|$. Moreover, 17 c implies $\|a\|=d(x, z)$, since $\operatorname{Ad}(s)$ is an isometry. Accordingly, if $z \in B(x, 2 \delta)$, it follows from 21 c )

$$
\begin{equation*}
H_{x}(z) \geq \kappa(a) \cot \kappa(a) \geq 2 \kappa \delta \cot (2 \kappa \delta)=\mathrm{Ct}(2 \delta)>0 \tag{21d}
\end{equation*}
$$

where the last inequality is because $2 \delta<r_{c x}=\kappa^{-1} \frac{\pi}{2}$. Replacing in 21b gives

$$
I_{1} \geq \operatorname{Ct}(2 \delta) P_{T}\left(B\left(x^{*}, \delta\right)\right)=\operatorname{Ct}(2 \delta)\left[1-P_{T}\left(B^{c}\left(x^{*}, \delta\right)\right)\right]
$$

Finally, 15 c and 15 d$)$ imply that $P_{T}\left(B^{c}\left(x^{*}, \delta\right)\right) \leq \operatorname{vol}(M) f(T)$, where $f(T)$ was defined in (6) - Precisely, this follows after replacing $\rho$ by $\delta$ in 15c). Thus,

$$
\begin{equation*}
I_{1} \geq \mathrm{Ct}(2 \delta)(1-\operatorname{vol}(M) f(T)) \tag{21e}
\end{equation*}
$$

The proof of $\sqrt[7]{ }$ will be completed by showing

$$
\begin{equation*}
I_{2} \geq-\pi A_{M} f(T) \tag{22a}
\end{equation*}
$$

To show this, note using 21c that

$$
\begin{equation*}
I_{2} \geq \int_{\mathrm{D}(x)-B\left(x, r_{c x}\right)} \kappa(a) \cot \kappa(a) P_{T}(d z) \tag{22~b}
\end{equation*}
$$

Now, $\kappa(\alpha) \cot \kappa(\alpha)$ is negative if and only if $\kappa(\alpha) \geq \frac{\pi}{2}$. However, the set of $z=\phi(s, a)$ where $\kappa(a) \geq \frac{\pi}{2}$ is a subset of $\mathrm{D}(x)-B\left(x, r_{c x}\right)$. Indeed, $\kappa(a) \geq \frac{\pi}{2}$ implies $\|a\| \geq \kappa^{-1} \frac{\pi}{2}=r_{c x}$, by the Cauchy-Schwarz inequality, and this is the same as $d(x, z) \geq r_{c x}$, since $\|a\|=d(x, z)$. Therefore, it follows from 22b,

$$
\begin{equation*}
I_{2} \geq \int_{\mathrm{D}(x)} 1\{\kappa(a) \geq \pi / 2\} \kappa(a) \cot \kappa(a) P_{T}(d z) \tag{22c}
\end{equation*}
$$

Using the same integral formula as in 19a, this last integral is equal to

$$
\begin{aligned}
& \int_{\mathcal{S}} \int_{D_{+}} 1\{\kappa(a) \geq \pi / 2\} \kappa(a) \cot \kappa(a) p_{T}(\phi(s, a)) J(a) d a \omega(d s) \geq \\
& -\int_{\mathcal{S}} \int_{D_{+}} \mathbf{1}\{\kappa(a) \geq \pi / 2\} \kappa(a) p_{T}(\phi(s, a)) d a \omega(d s)
\end{aligned}
$$

because the product $\cot \kappa(a) J(a) \geq-1$ for all $a \in D_{+}$. Using this last inequality, and the fact that $\kappa(a) \leq \pi$ for all $a \in D_{+}$, it follows from 22c ,

$$
\begin{equation*}
I_{2} \geq-\pi \int_{\mathcal{S}} \int_{D_{+}} \mathbf{1}\{\kappa(a) \geq \pi / 2\} p_{T}(\phi(s, a)) d a \omega(d s) \tag{22d}
\end{equation*}
$$

Recall that $\{\kappa(a) \geq \pi / 2\} \subset B^{c}\left(x, r_{c x}\right)$, as discussed before 22 c . In particular, this implies $\{\kappa(a) \geq \pi / 2\} \subset B^{c}\left(x^{*}, \delta\right)$. However, by (2) and 15d), $p_{T}(z) \leq f(T)$ for all $z \in B^{c}\left(x^{*}, \delta\right)$. Returning to 22 d , this gives

$$
\begin{equation*}
I_{2} \geq-\pi f(T) \int_{\mathcal{S}} \int_{D_{+}} d a \omega(d s) \tag{22e}
\end{equation*}
$$

The double integral on the right-hand side is a constant which depends only on the structure of the symmetric space $M$. Denoting this constant by $A_{M}$ gives the required lower bound (22a), and completes the proof of $(7)$.

Proof of (ii) : fix $\delta<\frac{1}{2} r_{c x}$, and let $T_{\delta}$ be given by (9). If $T \leq T_{\delta}$, then $T<T_{\delta}^{2}$, so the definition of $T_{\delta}^{2}$ implies

$$
\begin{equation*}
f(T)<\frac{\mathrm{Ct}(2 \delta)}{\operatorname{Ct}(2 \delta) \operatorname{vol}(M)+\pi A_{M}} \tag{23a}
\end{equation*}
$$

Now, by (7),

$$
\begin{equation*}
\nabla^{2} \mathcal{E}_{T}(x) \geq \operatorname{Ct}(2 \delta)(1-\operatorname{vol}(M) f(T))-\pi A_{M} f(T) \tag{23b}
\end{equation*}
$$

for all $x \in B\left(x^{*}, \delta\right)$. However, it is clear from 23a, that the right-hand side of this inequality is strictly positive. It follows that $\mathcal{E}_{T}$ is strongly convex on $B\left(x^{*}, \delta\right)$. Thus, to complete the proof, it only remains to show that any global minimum $\bar{x}_{T}$ of $\mathcal{E}_{T}$ must belong to $B\left(x^{*}, \delta\right)$. Indeed, since $\mathcal{E}_{T}$ is strongly convex on $B\left(x^{*}, \delta\right)$, it has only one local minimum in $B\left(x^{*}, \delta\right)$. Therefore, $\mathcal{E}_{T}$ can have only one global minimum $\bar{x}_{T}$.

By (i) of Proposition 11 to prove that $\bar{x}_{T} \in B\left(x^{*}, \delta\right)$, it is enough to prove

$$
\begin{equation*}
W\left(P_{T}, \delta_{x^{*}}\right)<\frac{\delta^{2}}{(4 \operatorname{diam} M)} \tag{23c}
\end{equation*}
$$

However, if $T \leq T_{\delta}$, then $T<T_{o}$. Therefore, by (ii) of Proposition $1, W\left(P_{T}, \delta_{x^{*}}\right)$ satisfies inequality (4). Furthermore, because $T<T_{\delta}^{1}$, it follows from the definition of $T_{\delta}^{1}$ that

$$
\sqrt{2 \pi}\left(T / \mu_{\min }\right)^{1 / 2}<\delta^{2}\left(\mu_{\min } / \mu_{\max }\right)^{n / 2} D_{n}
$$

or, by replacing the expression of $D_{n}$, and simplifying

$$
\begin{equation*}
\sqrt{2 \pi}(\pi / 2)^{n-1} B_{n}^{-1}\left(\mu_{\max } / \mu_{\min }\right)^{n / 2}\left(T / \mu_{\min }\right)^{1 / 2}<\frac{\delta^{2}}{(4 \operatorname{diam} M)} \tag{23~d}
\end{equation*}
$$

Thus, 23 c follows from (4) and 23d). This proves that $\bar{x}_{T}$ belongs to $B\left(x^{*}, \delta\right)$, and therefore $\bar{x}_{T}$ is the unique global minimum of $\mathcal{E}_{T}$. But this is equivalent to saying that $\bar{x}_{T}$ is the unique barycentre of $P_{T}$.

## 9 Proof of Proposition 3

fix $\delta<\frac{1}{2} r_{c x}$, and let $T_{\delta}$ be given by (9). By (ii) of Proposition 2 if $T \leq T_{\delta}$, then $\mathcal{E}_{T}$ is strictly convex on $B\left(x^{*}, \delta\right)$, with unique global minimum $\bar{x}_{T} \in B\left(x^{*}, \delta\right)$. By definition, this unique global minimum $\bar{x}_{T}$ is the unique barycentre of $P_{T}$.

Accordingly, to prove that $\bar{x}_{T}=x^{*}$, it is enough to prove that $x^{*}$ is a stationary point of $\mathcal{E}_{T}$. Indeed, as $\mathcal{E}_{T}$ is strictly convex on $B\left(x^{*}, \delta\right)$, it can have only one stationary point in $B\left(x^{*}, \delta\right)$. This stationary point is then identical to $\bar{x}_{T}$.

The fact that $x^{*}$ is a stationary point of $\mathcal{E}_{T}$ will follow because $U$ is invariant by geodesic symmetry about $x^{*}$. This invariance will be seen to imply

$$
\begin{equation*}
d s_{x^{*}} \cdot G_{x^{*}}=G_{x^{*}} \tag{24a}
\end{equation*}
$$

which is equivalent to $G_{x^{*}}=0$, since the derivative $d s_{x^{*}}$ is equal to minus the identity, on the tangent space $T_{x^{*}} M$ (7). By (5) of Corollary 1] this shows that $\nabla \mathcal{E}_{T}\left(x^{*}\right)=0$, so $x^{*}$ is indeed a stationary point of $\mathcal{E}_{T}$.

To obtain 24a), it is possible to write, from the definition of $G_{x^{*}}$,

$$
\begin{equation*}
d s_{x^{*}} \cdot G_{x^{*}}=d s_{x^{*}} \cdot \int_{\mathrm{D}(x)} G_{x^{*}}(z) P_{T}(d z) \tag{24b}
\end{equation*}
$$

where $\mathrm{D}(x)=M-\operatorname{Cut}(x)$. From (18), since $s_{x^{*}}$ is an isometry, and reverses geodesics passing through $x^{*}$,

$$
d s_{x^{*}} \cdot G_{x^{*}}(z)=G_{x^{*}}\left(s_{x^{*}}(z)\right)
$$

Replacing this into (24b), and using $w=s_{x^{*}}(z)$ as a new variable of integration, it follows that

$$
\begin{equation*}
d s_{x^{*}} \cdot G_{x^{*}}=\int_{\mathrm{D}(x)} G_{x^{*}}(w)\left(P_{T} \circ s_{x^{*}}\right)(d w) \tag{24c}
\end{equation*}
$$

because $s_{x^{*}}^{-1}=s_{x^{*}}$ and $s_{x^{*}}$ maps $\mathrm{D}(x)$ onto itself. Now, note that $P_{T} \circ s_{x^{*}}=P_{T}$. This is clear, since from (2),

$$
\left(P_{T} \circ s_{x^{*}}\right)(d w)=(Z(T))^{-1} \exp \left[-\frac{\left(U \circ s_{x^{*}}\right)(w)}{T}\right]\left(\operatorname{vol} \circ s_{x^{*}}\right)(d w)
$$

However, by assumption, $U \circ s_{x^{*}}(w)=U(w)$. Moreover, since $s_{x^{*}}$ is an isometry, it preserves Riemannian volume, so $\left(\operatorname{vol} \circ s_{x^{*}}\right)(d w)=\operatorname{vol}(d w)$. Thus, 24 c ) reads

$$
d s_{x^{*}} \cdot G_{x^{*}}=\int_{\mathrm{D}(x)} G_{x^{*}}(w) P_{T}(d w)
$$

By definition, the right-hand side is $G_{x^{*}}$, so 24a is obtained.

## References

1. Karcher, H.: Riemannian centre of mass and mollifier smoothing. Comm. Pure. Appl. Math. 30(5), 509-541 (1977).
2. Afsari, B.: Riemannian $L^{p}$ center of Mass: existence, uniqueness, and convexity. Proc. Am. Math. Soc. 139(2), 655-673 (2010).
3. Kantorovich, L.V., Akilov, G.P. : Functional Analysis (Second Edition). Pergamon Press, Oxford (1982).
4. Villani, C.: Optimal transport, old and new. 2nd edn. Springer-Verlag, BerlinHeidelberg (2009).
5. Wong, R.: Asymptotic approximations of Integrals. Society for Industrial and Applied Mathematics (2001).
6. Chavel, I.: Riemannian Geometry, a modern introduction. Cambridge University Press, Cambridge (2006).
7. Helgason, S.: Differential geometry, Lie groups, and symmetric spaces. American Mathematical Society (1978).
8. Roberts, G. O., Rosenthal, J. S.: General state space Markov chains and MCMC algorithms. Probab. Surveys. 1, 20-71 (2004).
9. Beals, R., Wong, R.: Special functions, a graduate text. Cambridge University Press, Cambridge (2010).
10. Mardia, K. V., Jupp, P. E.: Directional statistics. Academic Press, Inc., London (1972).
11. Kobayashi, S., Nomizu, K.: Foundations of differential geometry, Vol. II. Interscience Publishers (1969).
12. Crittenden, R.: Minimum and cojugate points in symmetric spaces. Canad. J. Math. 14, 320-328 (1962).
13. Hurewicz, W., Wallman, H.: Dimension Theory. Princeton University Press (1941).
14. Bogachev, V., I.: Measure theory, Vol. I. Springer-Verlag, Berlin-Heidelberg (2007).
15. Ferreira, R., Xavier, J., Costeira, J. P., Barroso, V.: Newton algorithms for Riemannian distance related problems on connected locally-symmetric manifolds. IEEE J. Sel. Topics Signal Process. 7(4), 634-645 (2013).
16. Zhan, X.: Matrix inequalities (Lecture Notes in Mathematics 1790). SpringerVerlag, Berlin-Heidelberg (2008).
