Covariant Momentum Map Thermodynamics for Parametrized Field Theories

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Abstract

Inspired by Souriau's symplectic generalization of the Maxwell-Boltzmann-Gibbs equilibrium in Lie group thermodynamics, we investigate a *spacetime-covariant* formalism for statistical mechanics and thermodynamics in the multi-symplectic framework for relativistic field theories. A general-covariant Gibbs state is derived, via a maximal entropy principle approach, in terms of the *covariant momentum map* associated with the lifted action of the diffeomorphisms group on the extended phase space of the fields. Such an equilibrium distribution induces a canonical spacetime foliation, with a Lie algebra-valued generalized notion of temperature associated to the covariant choice of a reference frame, and it describes a system of fields allowed to have non-vanishing probabilities of occupying states different from the diffeomorphism invariant configuration. We focus on the case of parametrized first order field theories, as a concrete simplified model for fully constrained field theories sharing fundamental general covariant features with canonical general relativity. In this setting, we investigate how *physical* equilibrium, hence time evolution, emerge from such a state via a gauge-fixing of the diffeomorphism symmetry.

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Contents

1 Introduction

In the attempt to shed light on the nature of gravity at the quantum scale, and to understand how continuum spacetime and its general-relativistic dynamics emerge in the classical regime, both string theory [1] and non-perturbative approaches to quantum gravity [2] have incorporated a pletora of concepts and tools from statistical mechanics, condensed matter physics and information theory, with an exceptional interdisciplinary effort. Nevertheless, a general covariant framework for describing the statistical fluctuations of the gravitational field, hence of spacetime geometry, is still missing, the very definition of a *spacetime covariant* formulation for statistical mechanics being a thorny open issue [3]. At the heart of the problem lies the conceptual clash of Einstein's general covariant scheme, characterised by systems with vanishing canonical Hamiltonian, with statistical mechanics, whose classical formulation grounds on the notion of Hamiltonian time flow and energy [4].

Important insights towards a general covariant formulation of statistical mechanics came from the *thermal time hypothesis* framework [5], inspired by the Tomita-Takasaki theorem in algebraic quantum field theory [6]. In this framework, timeless evolution is realised by the modular one-parameter flow of automorphisms of the *covariant space* algebra of the system, which can be induced by any modular thermal state in the algebra of the gauge invariant observables of the theory. The thermal time hypothesis then reinterprets the relation between Gibbs states and time flow generated by H form a covariant phase space viewpoint, by considering any equilibrium state as generating its own time flow instead of being determined by the time flow. This leaves open the problem of characterizing the states that are in *physical* equilibrium, as any state is stationary with respect to its own flow. In [7], a physical characterisation of such general covariant Gibbs states has been proposed to be associated with states whose thermal time is a *flow in spacetime*.

Most work in this direction [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] grounded on these ideas, seeking for new physical characterizations of equilibrium, by means of a covariant extension of the fundamental postulates of ordinary statistical physics. These results have been so far limited to simple low dimensional parametrized mechanical systems, and a complete field-theoretic formulation of the problem is missing [7, 17].

In this work, we consider the problem of the physical characterisation of a general covariant Gibbs' equilibrium within a multi-symplectic framework for relativistic field theories, by introducing an extended *off-shell*, epistemic notion of equilibrium [20], built in terms of the conserved covariant momentum map induced by the Hamiltonian flow of the diffeomorphism symmetry group on the extended phase space of the theory.

In Einstein's theory of gravity, as in any parametrized classical field theory, dynamics is fully characterized by the initial value constraints and inherently related to the gauge freedom of the theory [21, 22, 23, 24, 25, 26, 27]. Such an intimate relation between gauge symmetry and dynamics, between covariant and canonical formulation, is encoded in the notion of covariant momentum map in a multi-symplectic formalism [21, 22, 23, 28, 29]. For (first order) parametrized field theories, canonical initial value constraints coincide with the vanishing of the *instantaneous* reduction of the covariant momentum map [21, 22, 23], associated to the action of the gauge group of the theory on its *extended phase-space*. Such induced energy-momentum map appears to encode all the dynamical information carried by the theory [21, 22, 23]. In the canonical formalism, first class constraints generating the diffeomorphism symmetry are encoded in the normal and tangential components of the instantaneous reduction of the covariant momentum map so that the constraint surface is identified with its zero level set [22, 23, 26, 27]. In the ADM formulation of gravity, for instance, the super-hamiltonian and super-momenta are the components of such energy-momentum map, reflecting the gauge symmetry (diffeomorphism covariance) of the full theory in the instantaneous setting [30, 31].

In general, group-theoretic terms, the notion of momentum map is associated to a natural generalization of the Hamiltonian function, which comprises all conserved charges associated to the symplectic action of some dynamical group on a given phase space [32]. In this terms, the momentum map can be naturally used to generalize the Maxwell-Boltzmann-Gibbs approach to thermodynamics [33] to the case where the energy function is vector-valued and not just restricted to time translation symmetry. Along this line, in the 70's, Souriau first proposes a geometric Lie group-covariant formulation of thermodynamics, building on the theory of symplectic momenta, cohomology and distribution tensors. The idea goes like follows: Consider a connected symplectic manifold $(\mathcal{M}, \omega)$ (the manifold of motions in most cases) and a connected Lie group G acting on $\mathcal{M}$ by a Hamiltonian action Ψ[1]. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}^*$ be its dual space, and $J : \mathcal{M} \to \mathfrak{g}^*$ be a *momentum map* of the G-action Ψ, that is for any $\xi \in \mathfrak{g}$ the function $J(\xi) : \mathcal{M} \to \mathbb{R}$ by $J(\xi)(m) = \langle J(m), \xi \rangle$ is the Hamiltonian function associated to the vector field $\xi_{\mathcal{M}} = \psi(\xi) \in \mathfrak{X}(\mathcal{M})$ generating the action ψ of $\mathfrak{g}$ on $\mathcal{M}$. A *statistical state* on $(\mathcal{M}, \omega)$ is simply a probability law μ on $\mathcal{M}$ defined by the product of the Liouville density of $\mathcal{M}$ with a classical distribution function [32]

$$\mu(\mathcal{A}) = \int_{\mathcal{A}} \rho(m)\, \omega^n(m) \tag{1.1}$$

for each Borel subset $\mathcal{A}$ of $\mathcal{M}$, with $\rho : \mathcal{M} \to \mathbb{R}\,([0, +\infty[)$ being a continuous density function, such that $\int_{\mathcal{M}} \rho(m)\, \omega^n(m) = 1$. The entropy of the statistical state μ is defined to be the averaged

[1]The action $\Psi : G \times \mathcal{M} \to \mathcal{M}$ of a Lie group G on a symplectic manifold $(\mathcal{M}, \omega)$ is said to be symplectic if, for any $g \in G$, $\Psi_g : \mathcal{M} \to \mathcal{M}$ is a symplectomorphism, i.e. a smooth diffeomorphism of $\mathcal{M}$ such that $\Psi_g^* \omega = \omega$. The associated action $\psi : \mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ of the Lie algebra $\mathfrak{g}$ of G on $\mathcal{M}$ is the one parameter group action induced by Ψ via the fundamental vector fields associated to each Lie algebra element $\xi \in \mathfrak{g}$ satisfying

$$\psi(\xi)(m) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\Psi_{\exp(\lambda \xi)} \cdot m \right) \Big|_{\lambda=0}$$

where $\psi(\xi) \in \mathfrak{X}(\mathcal{M})$ is a vector field on $\mathcal{M}$ and $m \in \mathcal{M}$. A symplectic G-action Ψ is said to be Hamiltonian if the associated Lie algebra action ψ is Hamiltonian, that is there exists a smooth real function f on $\mathcal{M}$ such that $\mathrm{i}_{\psi(\xi)} \omega = -\mathrm{d} f$.

value S of $-\ln\rho$ with respect ρ,

$$S(\rho) = -\int_{\mathcal{M}} \rho(m)\, \log\left(\rho(m)\right) \omega^n(m)\,, \tag{1.2}$$

with $m\,\log(m) = 0$ if $m = 0$, and it measures the amount of uncertainty (or broadness) represented by the probability density ρ. For a given constant mean value of the momentum map J, thermodynamic equilibria μ_{eq} are states which maximize the entropy [20, 33], and such that $S(\rho_{eq})$ is stationary with respect to all infinitesimal smooth variations of the probability density. The quantity

$$Z(\xi) = \int_{\mathcal{M}} e^{-\langle J(m),\xi\rangle}\, \omega^n, \tag{1.3}$$

defines a generalized partition function, associated to a Gibbs distribution

$$\rho(\xi) = \frac{1}{Z(\xi)} e^{-\langle J(m),\xi\rangle}, \tag{1.4}$$

which is invariant under the action of any one-parameter subgroup of G on $\mathcal{M}$. The Hamiltonian function $\langle J, \xi\rangle : \mathcal{M} \to \mathbb{R}$ is the so-called *comomentum map*. As for the standard case of time translations, generalised free energy F and internal energy Q can be defined as smooth functions of the variable $\xi \in \mathfrak{g}$, taking value in $\mathbb{R}$ and in $\mathfrak{g}^*$, respectively. In this sense, the (co)momentum map provides a natural vector-valued generalisation of the Hamiltonian function, retaining the operational information of all conserved charges associated to the symplectic action of a dynamical group on a given system's phase space. In particular, this leads to a remarkable *covariant* generalization of Gibbs's equilibrium as soon as we consider the action of the symmetry group of the system (e.g. Galileo, Poincaré) on its manifold of motions [34].

Souriau's geometric generalisation of Gibbs equilibrium essentially relies on the symplectic character of the phase space and on the Hamiltonian nature of the group action. Such a phase space coincides with the *fully constrained* phase space of the system, thereby providing an on-shell definition of covariant equilibrium state, analogous to the one later developed by the thermal time hypothesis.

In this work, we explore a radical conceptual extension of Souriau's Lie group thermodynamics to the framework of general covariant, or reparametrization invariant, field theories, where the covariant symmetry group of the system is *gauge* and the symplectic phase space consists of the full extended phase space of fields. We focus on parametrized field theories considered as paradigmatic models for diffeomorphism-covariant theories sharing important features with general relativity. In particular, to emphasise on the key geometric ingredients for our analysis, we work in the multi-symplectic framework for first order parametrized field theories, where a notion of covariant (multi)momentum map associated to the action of spacetime diffeomorphisms can be constructed. In absence of boundaries in the underlying spacetime base manifold, the pre-symplectic nature of both the constraint surface and the space of solutions of the field equations prevents us from representing the diffeomorphism group via a Poisson algebra of functions on the on-shell field configurations as the corresponding co-momentum Hamiltonian would be trivial [35]. This issue requires to move the entire analysis at the off-shell level. Here, building on the notion of covariant (multi)momentum map and on the geometric characterization of the parametrization procedure introduced in [36, 37], where diffeomorphisms are promoted to dynamical fields themselves (*covariance fields*), we are able to define a spacetime covariant notion of equilibrium state for fully constrained field theories. The representation of the algebra of diffeomorphisms on the parametrized phase space of fields is achieved by taking into account the lifted action of spacetime diffeomorphisms on the embeddings naturally induced by that on the covariance fields. As a side result, this also allows us to recover the procedure introduced by Isham and Kuchař in [38] for the case of a scalar field.

The paper is organized as follows. In Sec. 2, we present the main geometric setup on which our analysis is based. After recalling the multi-symplectic formulation for classical field theories (Sec. 2.1) and the main steps of the parametrization procedure (Sec. 2.2), we elaborate this framework for first order parametrized field theories in Sec. 2.3 where the notion of covariant multi-momentum map is discussed. Sec. 2.4 focuses on the corresponding canonical formalism and the appearance of constraints in such a framework, thus providing the necessary preliminaries for the representation of spacetime diffeomorphisms via a covariant momentum map on the parametrized phase space of fields discussed in Sec. 2.5. With this framework at our disposal, the extension of Souriau's Lie group themodynamics to parametrized field theories – that we call *covariant gauge group thermodynamics* – is then discussed in Sec. 3. The covariant notion of Gibbs equilibrium state as well as the corresponding generalized thermodynamic functions are constructed in Sec. 3.1 and 3.2, respectively. Furthermore, in Sec. 3.3 we explore the possibility of a microcanonical imposition of the constraints from a statistical perspective by considering the thermodynamic limit, and discuss a generalized second law in Sec. 3.4. Finally, in Sec. 4 we consider some explicit check for our formalism. In particular, in Sec. 4.2 we investigate how time evolution equilibrium emerges from a suitable gauge-fixing of diffeomorphism symmetry, while in Sec. 4.3 we discuss analogies and differences with thermal time hypothesis and elaborate on the spacetime interpretation of the thermal flow associated to the covariant Gibbs state. Some future perspectives are reported in Sec. 5.

2 Multisymplectic Formulation for Generally Covariant Field Theories

The first step in our construction consists in defining a suitable covariant Hamiltonian framework for our approach, where the key momentum map construction can be generalized in spacetime covariant terms, for the case of first order parametrized field theories. The content is based on the seminal works [21, 22, 23] and references within to which we refer for a more detailed exposition. Other works on this topic which include also different starting points are [39, 40, 41, 42].

2.1 Setup

Let $\mathcal{X}$ be an oriented $(n+1)$-dimensional manifold, which in many examples is spacetime, and let $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ be a finite-dimensional fiber bundle over $\mathcal{X}$ whose fibers $\mathcal{Y}_x$ over $x \in \mathcal{X}$ have dimension N. This is called the **configuration bundle** and is the field theoretic analogue of the configuration space in classical mechanics. Physical fields correspond to sections of this bundle. A set of local coordinates (x^μ, y^A) on $\mathcal{Y}$ is provided by the $n+1$ local coordinates x^μ, $\mu = 0, \ldots, n$, on $\mathcal{X}$ and the N fiber coordinates y^A, $A = 1, \ldots, N$, which represent the field components at a given point $x \in \mathcal{X}$. As discussed in [43, 44], this notion of extended configuration space for field theories has a nice operational motivation based on the observation that coordinates of $\mathcal{Y}$ (i.e., field values and spacetime positions) are the partial observables of the theory [45]. Indeed, one needs N measuring devices to measure the components of the field at a given point $x \in \mathcal{X}$, and $n+1$ devices to determine x[2] thus resulting in a $(n+N+1)$-dimensional configuration space. A point in $\mathcal{Y}$ represents a correlation between these observables, that is, a possible outcome of a simultaneous measurement of the partial observables.

The Lagrangian density for a first order classical field theory is given by

$$\mathscr{L} : J^1(\mathcal{Y}) \longrightarrow \Lambda^{n+1}(\mathcal{X}) \,, \tag{2.1}$$

where $J^1(\mathcal{Y})$ is the first jet bundle[3] of $\mathcal{Y}$ and $\Lambda^{n+1}(\mathcal{X})$ is the space of $(n+1)$-forms on $\mathcal{X}$. The first jet bundle $J^1(\mathcal{Y})$ of $\mathcal{Y}$ here plays the role of the field-theoretic analogue of the tangent bundle of classical mechanics[4]. Local coordinates (x^μ, y^A) on $\mathcal{Y}$ induce coordinates v^A_μ on the

[2] To determine an event in a $n+1$-dimensional spacetime we need one clock and n devices giving us the distance from n reference objects.

[3] For higher order theories, a k^{th}-order Lagrangian density will be defined on the k^{th} jet bundle $J^k(\mathcal{Y})$ of $\mathcal{Y}$ [46].

[4] In this case $\mathcal{Y} = \mathbb{R} \times \mathcal{Q}$ is the extended configuration space regarded as an $\mathbb{R}$-bundle over $\mathcal{Q}$, and $J^1(\mathcal{Q} \times \mathbb{R})$ is isomorphic to the bundle $T\mathcal{Q} \times T\mathbb{R}$.

fibers of $J^1(\mathcal{Y})$ so that the first jet prolongation $j^1\phi$ of a section ϕ of the bundle $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ reads as

$$j^1\phi : x^\mu \longmapsto (x^\mu, y^A, v^A_\mu) = (x^\mu, y^A(x), y^A_{,\mu}(x)) \ , \tag{2.2}$$

where $y^A_{,\mu} = \partial_\mu y^A$ and $\partial_\mu = \partial/\partial x^\mu$. The Lagrangian then reads

$$\mathscr{L}(j^1\phi) = L\left(x^\mu, y^A(x), y^A_{,\mu}(x)\right) \mathrm{d}^{n+1}x \ , \tag{2.3}$$

where $\mathrm{d}^{n+1}x = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$ is the volume form on $\mathcal{X}$.

By introducing the *multimomenta* p^μ_A and the *covariant Hamiltonian* p defined via Legendre transformation as

$$p^\mu_A = \frac{\partial L}{\partial v^A_\mu} \qquad , \qquad p = L - \frac{\partial L}{\partial v^A_\mu} v^A_\mu \ , \tag{2.4}$$

the field-theoretic analogue of the phase space of classical mechanics is provided by the so-called **multiphase space** $\mathcal{Z}$ defined as the sub-bundle of 2-horizontal $(n+1)$-forms on $\mathcal{Y}$ whose elements can be uniquely written in terms of the fiber coordinates (p, p^A_μ) as

$$z = p\,\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu \ , \tag{2.5}$$

with

$$\mathrm{d}^n x_\mu = \mathrm{i}_{\partial_\mu} \mathrm{d}^{n+1}x \ , \tag{2.6}$$

so that the contraction $\mathrm{i}_V \mathrm{i}_W z$ with any two vertical vector fields $V = V^A\partial/\partial y^A$, $W = W^A\partial/\partial y^A$ on $\mathcal{Y}$ vanishes. As proved in [21], the space $\mathcal{Z}$ is canonically isomorphic to the dual jet bundle $J^1(\mathcal{Y})^*$, the latter playing the role of the field-theoretic analogue of the cotangent bundle.

In complete analogy to standard symplectic mechanics [47, 48], the **canonical Poincaré-Cartan** $(n+1)$**-form** Θ on $\mathcal{Z}$ is given by

$$\Theta = p\,\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu \ , \tag{2.7}$$

and the **canonical** $(n+2)$**-form** Ω on $\mathcal{Z}$ is then defined by

$$\Omega = -\mathrm{d}\Theta = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x \ . \tag{2.8}$$

A remarkable feature of this formalism is that for each field component y^A there are multiple momenta p^μ_A, spatial in addition to temporal. The pair $(\mathcal{Z}, \Omega)$ is an example of multisymplectic manifold[5] and the usual definitions of classical mechanics on the extended phase space are

[5]According to [21, 49, 50], a multisymplectic manifold $(\mathcal{M}, \Omega)$ is a manifold endowed with a closed non-degenerate k-form Ω ($k = n+2$ in our case), i.e., such that $\mathrm{d}\Omega = 0$ and $\mathrm{i}_V\Omega \neq 0$ for any nonzero tangent vector V on $\mathcal{M}$.

recovered when $\mathcal{X}$ is one-dimensional (i.e., $n = 0$) as reported into the following table which summarizes the analogies between classical symplectic mechanics and the multisymplectic formulation of classical field theories:

Classical Mechanics $(n = 0, \mathcal{X} \equiv \mathbb{R})$	**Field Theory** $(n > 0, \dim \mathcal{X} = n + 1)$
extended configuration space $\mathcal{Y} = \mathbb{R} \times \mathcal{Q}$	configuration bundle over spacetime $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$
local coordinates on $\mathcal{Y}$ (t, q^A)	local coordinates on $\mathcal{Y}$ (x^μ, y^A)
extended phase space $\mathcal{P} = T^*\mathcal{Y} = T^*\mathbb{R} \times T^*\mathcal{Q}$	multiphase space $J^1(\mathcal{Y})^* \cong \mathcal{Z} \subset \Lambda^{n+1}(\mathcal{Y})$
local coordinates on $\mathcal{P}$ (t, q^A, E, p_A)	local coordinates on $\mathcal{Z}$ (x^μ, y^A, p, p^μ_A)
Poincaré-Cartan 1-form on $\mathcal{P}$ $\Theta = p_A \mathrm{d}q^A + E\mathrm{d}t$	Poincaré-Cartan $(n+1)$-form on $\mathcal{Z}$ $\Theta = p\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu$
symplectic 2-form on $\mathcal{P}$ $\Omega = \mathrm{d}q^A \wedge \mathrm{d}p_A - \mathrm{d}E \wedge \mathrm{d}t$	multisymplectic $(n+2)$-form on $\mathcal{Z}$ $\Omega = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x$

2.2 Parametrization and Covariance Fields

As it is well known from the pioneering work of Dirac [51], further developed by Kuchař and Isham [38, 52, 53], field theories with a fixed background metric can be made generally covariant, i.e., with the spacetime diffeomorphism group as symmetry group, by means of the so-called **parametrization** procedure. Roughly speaking, this amounts to introduce the diffeomorphisms themselves as new dynamical fields so that the covariance group of the theory can be enlarged while leaving the solution space unchanged. A precise geometric reformulation of the parametrization procedure within the context of multi-symplectic field theories, which provides an intrinsic extension and refinement of the idea of treating coordinate changes as fields ("parametrization"), was developed by Castrillón López, Gotay and Marsden in [36, 37]. The main steps of the construction are the following:

1) One introduces the so-called **covariance fields** which are just (oriented) diffeomorphisms of $\mathcal{X}$ reinterpreted as sections $\eta : \mathcal{X} \to \tilde{\mathcal{X}}$ of the bundle $\tilde{\mathcal{X}} \times \mathcal{X} \xrightarrow{\tilde{\pi}} \mathcal{X}$ where $(\tilde{\mathcal{X}}, g)$ is a copy the (spacetime) base manifold.

2) Regarding η as new dynamical fields, the configuration bundle $\mathcal{Y}$ is then replaced by the fibered product $\tilde{\mathcal{Y}} = \mathcal{Y} \times_{\mathcal{X}} (\tilde{\mathcal{X}} \times \mathcal{X})$ whose sections are thought of as pairs (ϕ, η) according to the diagram

$$\begin{tikzcd}
\mathcal{Y} \times_{\mathcal{X}} (\tilde{\mathcal{X}} \times \mathcal{X}) \arrow[d, "\pi_{\mathcal{X}\tilde{\mathcal{Y}}}"] \\
\mathcal{X} \arrow[u, dashed, bend left=70, "\phi"] \arrow[u, dashed, bend right=70, "\eta"']
\end{tikzcd}
\qquad \text{i.e.} \qquad
\begin{tikzcd}
& \mathcal{Y} \times_{\mathcal{X}} (\tilde{\mathcal{X}} \times \mathcal{X}) \arrow[dl, "\pi_\ell"'] \arrow[dr, "\pi_r"] \arrow[dd, "\pi_{\mathcal{X}\tilde{\mathcal{Y}}}"] & \\
\mathcal{Y} \arrow[dr, bend right, "\pi_{\mathcal{X}\mathcal{Y}}"'] & & \tilde{\mathcal{X}} \times \mathcal{X} \arrow[dl, bend left, "\tilde{\pi}"] \\
& \mathcal{X} \arrow[ul, dashed, "\phi"] \arrow[ur, dashed, "\eta"'] &
\end{tikzcd}$$

where, denoting coordinates on $\tilde{\mathcal{X}}$ by u^a $(a = 0, \ldots, n)$, $\pi_\ell : \tilde{\mathcal{Y}} \to \mathcal{Y}$ and $\pi_r : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}} \times \mathcal{X}$ are the projections on the first and second factor of $\tilde{\mathcal{Y}}$ respectively acting as $\pi_\ell : (x^\mu, y^A, u^a) \mapsto (x^\mu, y^A)$ and $\pi_r : (x^\mu, y^A, u^a) \mapsto (x^\mu, u^\alpha)$, and $\pi_{\mathcal{X}\mathcal{Y}}(y) = \tilde{\pi}(u)$.

3) The Lagrangian density $\mathscr{L}$ of the starting theory is modified by introducing the new Lagrangian density

$$\tilde{\mathscr{L}} : J^1(\tilde{\mathcal{Y}}) \longrightarrow \Lambda^{n+1}(\mathcal{X}) \,, \tag{2.9}$$

defined as

$$\tilde{\mathscr{L}}(j^1\phi, j^1\eta) := \mathscr{L}(j^1\phi, \eta^* g) \,. \tag{2.10}$$

which, denoting coordinates on $J^1(\tilde{\mathcal{Y}})$ by $(x^\mu, y^A, v^A_\mu, u^a, u^a_\mu)$ with u^a_μ the jet coordinates associated to u^a, reads

$$\tilde{\mathscr{L}}(x^\mu, y^A, v^A_\mu, u^a, u^a_\mu) = \mathscr{L}(x^\mu, y^A, v^A_\mu \,; G_{\mu\nu}) \,, \tag{2.11}$$

where

$$G_{\mu\nu} \equiv (\eta^* g)_{\mu\nu} = \eta^a_{,\mu}\, \eta^b_{,\nu}\, g_{ab} \circ \eta = u^a_\mu\, u^b_\nu\, g_{ab} \circ \eta \,. \tag{2.12}$$

Let then $\alpha_{\mathcal{X}} \in \mathsf{Diff}(\mathcal{X})$, we denote by $\alpha_{\mathcal{Y}} \in \mathsf{Aut}(\mathcal{Y})$ its lift to $\mathcal{Y}$. This can be extended to an action by bundle automorphisms on $\tilde{\mathcal{Y}}$ by requiring that $\mathsf{Diff}(\mathcal{X})$ acts trivially on $\tilde{\mathcal{X}}$, i.e.

$$\alpha_{\tilde{\mathcal{X}}} : \tilde{\mathcal{X}} \times \mathcal{X} \longrightarrow \tilde{\mathcal{X}} \times \mathcal{X} \qquad \text{by} \qquad (u, x) \longmapsto (u, \alpha_{\mathcal{X}}(x)) \,. \tag{2.13}$$

The induced action on the space $\tilde{\mathscr{Y}} \equiv \Gamma(\mathcal{X}, \tilde{\mathcal{Y}})$ of sections of $\tilde{\mathcal{Y}}$ is then given by

$$\alpha_{\tilde{\mathscr{Y}}}(\phi, \eta) = (\alpha_{\mathscr{Y}}(\phi), \alpha_{\tilde{\mathscr{X}}}(\eta)) \,, \tag{2.14}$$

where

$$\alpha_{\mathscr{Y}}(\phi) = \alpha_{\mathcal{Y}} \circ \phi \circ \alpha_{\mathcal{X}}^{-1} \qquad , \qquad \phi \in \mathscr{Y} \equiv \Gamma(\mathcal{X}, \mathcal{Y}) \tag{2.15}$$

generalizes the usual push-forward action on tensor fields, and

$$\alpha_{\tilde{\mathscr{X}}}(\eta) = \eta \circ \alpha_{\mathcal{X}}^{-1} , \tag{2.16}$$

is the (left) action by composition on sections of the trivial bundle $\tilde{\mathcal{X}} \times \mathcal{X}$. The modified field theory on $J^1(\tilde{\mathcal{Y}})$ with Lagrangian density (2.10) is **Diff($\mathcal{X}$)-covariant**. Indeed, the Lagrangian density (2.10) is **Diff($\mathcal{X}$)-equivariant**, i.e. [36, 37]

$$\tilde{\mathscr{L}}\left(j^1(\alpha_{\mathscr{Y}}(\phi)), j^1(\alpha_{\tilde{\mathscr{X}}}(\eta))\right) = (\alpha_{\mathcal{X}}^{-1})^* \left[\tilde{\mathscr{L}}(j^1\phi, j^1\eta)\right] . \tag{2.17}$$

To sum up, the key point of the construction is that now the fixed background metric g is no longer thought of as living on $\mathcal{X}$, but rather just as a geometric object on the copy $\tilde{\mathcal{X}}$ in the fiber of the extended configuration bundle $\tilde{\mathcal{Y}}$. On the other hand, the metric variable $G = \eta^* g$ on $\mathcal{X}$ inherits a dynamical character via the covariance field η. The true dynamical fields of the parametrized theory are thus provided by ϕ and η. As discussed in [36, 37], the Euler-Lagrange equations for the fields ϕ remains unchanged while those for the covariance fields η give the conservation of the stress-energy-momentum tensor. These fields do not modify then the physical content of the original theory and provide an efficient way of parametrizing it. The construction outlined above therefore provides us with a diffeomorphism-covariant reformulation of field theories where all fields are treated as variational entities, thus satisfying all the requirements of general covariance[6] [54].

2.3 Covariant Momentum Map

Along the same steps discussed in Sec. 2.1, we can now develop a covariant Hamiltonian formalism for (first order) parametrized field theories by considering the **covariant** or **parametrized multiphase space** $\tilde{\mathcal{Z}} \cong J^1(\tilde{\mathcal{Y}})^*$ equipped with a canonical Poincaré-Cartan $(n+1)$-form

$$\tilde{\Theta} = \tilde{p}\, \mathrm{d}^{n+1}x + p^{\mu}_{A} \mathrm{d}y^A \wedge \mathrm{d}^n x_{\mu} + \varrho^{\mu}_{a} \mathrm{d}u^a \wedge \mathrm{d}^n x_{\mu} , \tag{2.18}$$

where, taking into account that both ϕ and η are dynamical variables for the extended theory, the covariant Hamiltonian $\tilde{p}$ and the multimomenta $p^{\mu}_{A}, \varrho^{\mu}_{a}$ (respectively conjugate to the

[6] This has to be contrasted with considering the metric g directly as a genuine field on $\mathcal{X}$ which will then make the Lagrangian Diff($\mathcal{X}$)-equivariant but g itself cannot be considered variational unless one adds a source term (e.g., the Einstein-Hilbert Lagrangian) to the Lagrangian density.

multivelocities v^A_μ and u^a_μ) are defined w.r.t. the Lagrangian (2.11), i.e.

$$\tilde{p} = \tilde{L} - \frac{\partial \tilde{L}}{\partial v^A_\mu} v^A_\mu - \frac{\partial \tilde{L}}{\partial u^a_\mu} u^a_\mu, \qquad p^\mu_A = \frac{\partial \tilde{L}}{\partial v^A_\mu} = \frac{\partial L}{\partial v^A_\mu}, \qquad \varrho^\mu_a = \frac{\partial \tilde{L}}{\partial u^a_\mu} = \mathcal{T}^{\mu\nu} u^b_\nu g_{ab} \; , \tag{2.19}$$

with $\mathcal{T}^{\mu\nu} = 2\frac{\partial L}{\partial G_{\mu\nu}}$ the so-called *Piola-Kirchhoff stress-energy-momentum tensor density* [36]. The multisymplectic $(n+2)$-form $\tilde{\Omega} = -\mathrm{d}\tilde{\Theta}$ on $\tilde{\mathcal{Z}}$ then reads

$$\tilde{\Omega} = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^n x_\mu + \mathrm{d}u^a \wedge \mathrm{d}\varrho^\mu_a \wedge \mathrm{d}^n x_\mu - \mathrm{d}\tilde{p} \wedge \mathrm{d}^{n+1} x \; . \tag{2.20}$$

Let now $\mathcal{G}$ be a Lie group (perhaps infinite-dimensional) realizing the gauge group of the theory and denote by $\mathfrak{g}$ its Lie algebra. In the case of generally covariant field theories, $\mathcal{G}$ is a subgroup of $\mathsf{Aut}(\tilde{\mathcal{Y}})$ covering diffeomorphisms on $\mathcal{X}$. Given an element $\xi \in \mathfrak{g}$, we denote by $\xi_\mathcal{X}, \xi_\mathcal{Y}, \xi_{\tilde{\mathcal{Y}}}$, and $\xi_{\tilde{\mathcal{Z}}}$ the infinitesimal generators of the corresponding transformations on $\mathcal{X}, \mathcal{Y}, \tilde{\mathcal{Y}}$, and $\tilde{\mathcal{Z}}$, i.e., the infinitesimal generators on $\mathcal{X}, \mathcal{Y}, \tilde{\mathcal{Y}}$, and $\tilde{\mathcal{Z}}$ of the one-parameter group generated by ξ. The group $\mathcal{G}$ is said to act on $\tilde{\mathcal{Z}}$ by **covariant canonical transformation** if the $\mathcal{G}$-action corresponds to an infinitesimal multi-symplectomorphism, i.e.

$$\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}} \tilde{\Omega} = 0 \; , \tag{2.21}$$

where $\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}}$ denotes the Lie derivative along $\xi_{\tilde{\mathcal{Z}}}$, while it is said to act by **special covariant canonical transformations** if $\tilde{\Theta}$ is $\mathcal{G}$-invariant, that is

$$\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}} \tilde{\Theta} = 0 \; . \tag{2.22}$$

This is the Hamiltonian counterpart of the $\mathcal{G}$-equivariance property (2.17) of the Lagrangian which in turn amounts to $\mathcal{G}$-invariance of the Lagrangian form $\Theta_{\tilde{\mathscr{L}}}$ on $J^1(\tilde{\mathcal{Y}})$ defined as the pull-back of $\tilde{\Theta}$ along the covariant Legendre transform.

In analogy to the definition of momentum maps in symplectic geometry [47, 48, 55], a **covariant momentum map** (or a **multimomentum map**) associated to the action of $\mathcal{G}$ on $\tilde{\mathcal{Z}}$ by covariant canonical transformations is a map

$$\tilde{\mathcal{J}} : \tilde{\mathcal{Z}} \longrightarrow \mathfrak{g}^* \otimes \Lambda^n(\tilde{\mathcal{Z}}) \; , \tag{2.23}$$

given by

$$\mathrm{d}\tilde{\mathcal{J}}(\xi) = \mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}} \tilde{\Omega}, \tag{2.24}$$

where $\tilde{\mathcal{J}}(\xi)$ is the n-form on $\tilde{\mathcal{Z}}$ whose value at $\tilde{z} \in \tilde{\mathcal{Z}}$ is $\langle \tilde{\mathcal{J}}(\tilde{z}), \xi \rangle$ with $\langle \cdot, \cdot \rangle$ being the pairing between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

Let then $\alpha \in \mathcal{G}$ be the transformation associated to $\xi \in \mathfrak{g}$, the covariant momentum map $\tilde{\mathcal{J}}$ is said to be Ad^*-**equivariant** if

$$\tilde{\mathcal{J}}(\mathrm{Ad}_\alpha^{-1}\xi) = \alpha^*_{\tilde{\mathcal{Z}}}[\tilde{\mathcal{J}}(\xi)] \ . \tag{2.25}$$

When $\mathcal{G}$ acts by special covariant canonical transformations, the (special) covariant momentum map admits an explicit expression given by

$$\tilde{\mathcal{J}}(\xi) = \mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}}\tilde{\Theta} \ , \tag{2.26}$$

so that $\mathrm{d}\tilde{\mathcal{J}}(\xi) = \mathrm{d}\mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}}\tilde{\Theta} = (\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}} - \mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}}\mathrm{d})\tilde{\Theta} = \mathrm{i}_{\xi_{\tilde{\mathcal{Z}}}}\tilde{\Omega}$. In particular – and this is the case of interest for parametrized field theories – if the $\mathcal{G}$-action on $\tilde{\mathcal{Z}}$ is the lift of an action of $\mathcal{G}$ on $\tilde{\mathcal{Y}}$, then for any $\xi \in \mathfrak{g}$ realized as a (complete) vector field $\xi_{\mathcal{X}} = \xi^\mu(x)\frac{\partial}{\partial x^\mu}$ on $\mathcal{X}$, we have

$$\xi_{\tilde{\mathcal{Y}}} = \xi^\mu(x)\frac{\partial}{\partial x^\mu} + \xi^A(x,y,u,[\xi])\frac{\partial}{\partial y^A} \ , \tag{2.27}$$

where in general $\xi^A(x,y,u,[\xi])$ is a smooth functional of ξ^ν which for tensor field theories reads

$$\xi^A(x,y,u,[\xi]) = C^{A\rho}_\nu(x,y,u)\xi^\nu_{,\rho}(x) + C^A_\nu(x,y,u)\xi^\nu(x) \ , \tag{2.28}$$

with coefficients C depending on x^μ, y^B and u^b, and reduces to $\xi^A(x,y,u,[\xi]) = C^A_\nu(x,y,u)\xi^\nu(x)$ in the case of a scalar field. Note that in Eq. (2.27) there are no components in the "u-directions" since, as discussed in the previous section (cfr. Eq. (2.13)), for any $\alpha_{\mathcal{X}} \in \mathsf{Diff}(\mathcal{X})$, there is a lift $\alpha_{\mathcal{Y}} \in \mathsf{Aut}(\mathcal{Y})$ which is trivially extended to $\tilde{\mathcal{Y}}$ by $\alpha_{\tilde{\mathcal{Y}}} : (y,u,x) \mapsto (\alpha_{\mathcal{Y}}(y), u, \alpha_{\mathcal{X}}(x))$. In coordinates, Eq. (2.26) then reads

$$\langle \tilde{\mathcal{J}}(\tilde{z}), \xi\rangle = \left(\tilde{p}\,\xi^\mu + p^\mu_A\xi^A\right)\mathrm{d}^n x_\mu - p^\mu_A\xi^\nu \mathrm{d}y^A \wedge \mathrm{d}^{n-1}x_{\mu\nu} - \varrho^\mu_a\xi^\nu\mathrm{d}u^a \wedge \mathrm{d}^{n-1}x_{\mu\nu}, \tag{2.29}$$

where $\mathrm{d}^{n-1}x_{\mu\nu} = \mathrm{i}_{\partial_\nu}\mathrm{i}_{\partial_\mu}\mathrm{d}^{n+1}x$.

2.4 Canonical Phase Space and Energy-Momentum Map

In the covariant formulation fields are defined as sections of bundles over spacetime. The canonical formalism instead relies on fields defined as "time-evolving" cross sections of bundles over a Cauchy surface. Therefore, in order to construct the canonical formulation of a field theory we need to introduce a foliation of spacetime and consequently of the bundles over it. The base manifold $\mathcal{X}$ can be thus decomposed into a smooth disjoint union of space-like hypersurfaces[7].

[7] In what follows, we assume the spacetime manifold to be globally hyperbolic, i.e., $\mathcal{X} \cong \Sigma \times \mathbb{R}$, so that the foliation introduced to construct the canonical formalism covers the whole manifold thus avoiding technicalities concerning the possibility of defining the canonical formalism only locally.

Let then Σ be a compact, oriented, connected, boundaryless 3-manifold and let $\mathsf{Emb}_G(\Sigma,\mathcal{X})$ be the set of all space-like embeddings of Σ in $\mathcal{X}$. A foliation $\mathfrak{s}_{\mathcal{X}}$ of $\mathcal{X}$ then corresponds to a 1-parameter family $\lambda \mapsto \tau_\lambda$ of space-like embeddings $\tau_\lambda \in \mathsf{Emb}_G(\Sigma,\mathcal{X})$ of Σ in $\mathcal{X}$, i.e.

$$\mathfrak{s}_{\mathcal{X}} : \Sigma \times \mathbb{R} \to \mathcal{X} \quad \text{by} \quad (\vec{x},\lambda) \mapsto \mathfrak{s}_{\mathcal{X}}(\vec{x},\lambda)\,, \tag{2.30}$$

such that

$$\tau \equiv \tau_\lambda : \Sigma \to \mathcal{X} \quad \text{by} \quad \tau(\vec{x}) \equiv \tau_\lambda(\vec{x}) := \mathfrak{s}_{\mathcal{X}}(\vec{x},\lambda)\ , \tag{2.31}$$

where $\vec{x}$ is a shorthand notation for the spatial coordinates x^i, $i = 1,\ldots,n$, on the space-like hypersurface $\Sigma_\tau = \tau(\Sigma)$. The generator of $\mathfrak{s}_{\mathcal{X}}$ is a complete vector field $\zeta_{\mathcal{X}}$ on $\mathcal{X}$ everywhere transverse to the slices defined by

$$\dot{\tau}(\vec{x}) = \frac{\partial}{\partial\lambda}\mathfrak{s}_{\mathcal{X}}(\vec{x},\lambda) = \zeta_{\mathcal{X}}\left(\mathfrak{s}_{\mathcal{X}}(\vec{x},\lambda)\right)\ . \tag{2.32}$$

A foliation of $\mathcal{X}$ induces a *compatible slicing* of bundles over it whose generating vector fields project to $\zeta_{\mathcal{X}}$. The flow of such a generating vector field defines a one-parameter group of bundle automorphisms. For parametrized field theories we are interested in a so-called **$\mathcal{G}$-slicing** in which case the one-parameter group of automorphisms of the extended configuration bundle is induced by a one-parameter subgroup of the gauge group $\mathcal{G}$, i.e., $\zeta_{\tilde{\mathcal{Y}}} = \xi_{\tilde{\mathcal{Y}}}$ for some $\xi \in \mathfrak{g}$. The corresponding slicing $\mathfrak{s}_{\tilde{\mathcal{Z}}}$ of $\tilde{\mathcal{Z}}$ is then generated by the canonical lift $\zeta_{\tilde{\mathcal{Z}}} = \xi_{\tilde{\mathcal{Z}}}$ of $\xi_{\tilde{\mathcal{Y}}}$ to $\tilde{\mathcal{Z}}$ whose flow defines a one-parameter group of bundle automorphisms by special canonical transformations on $\tilde{\mathcal{Z}}$ (i.e., $\mathcal{L}_{\xi_{\tilde{\mathcal{Z}}}}\tilde{\Theta} = 0$)[8].

Spatial fields will then be identified with smooth sections of the pull-back bundle $\mathcal{Y}_\tau \to \Sigma_\tau$ over a Cauchy surface given by $\varphi := \phi_\tau = \tau^*\phi$. Note that, as the subscript τ is meant to recall, the spatial fields $\varphi(\vec{x}) = \phi_\tau(\vec{x})$ are functionals of the embedding τ and at the same time functions of the point $\vec{x}$ on the spatial slice. Moreover, according to the parametrization procedure discussed in Sec. 2.2, the space-like embedding $\tau \in \mathsf{Emb}_G(\Sigma,\mathcal{X})$ acquires a dynamical character through the covariance fields η. Indeed, we have $\tau = \eta^{-1} \circ \tilde{\tau}$ for a given space-like embedding $\tilde{\tau} \in \mathsf{Emb}_g(\Sigma,\tilde{\mathcal{X}})$ of Σ into $\tilde{\mathcal{X}}$ associated to the slicing of $\tilde{\mathcal{X}}$ w.r.t. the fixed metric g. The canonical parametrized configuration space then consists of the pairs (φ,τ) of spatial fields defined over a Cauchy slice and the space-like embeddings identifying a $\mathcal{G}$-slicing of spacetime w.r.t. one-parameter subgroups of diffeomorphisms. Let then $(x^0,x^1,\ldots,x^n)$ be a chart on $\mathcal{X}$ adapted to τ, i.e. such that Σ_τ is locally a level set of x^0. Denoting by (φ,Π,τ,P) a point in the canonical parametrized phase space $T^*\tilde{\mathscr{Y}}_\tau = T^*\mathscr{Y}_\tau \times T^*\mathsf{Emb}_G(\Sigma,\mathcal{X})$, the canonical symplectic

[8]As already stressed before, this essentially reflects the equivariance property of the Lagrangian density w.r.t. to the one-parameter groups of automorphisms associated to the induced slicings of $J^1(\tilde{\mathcal{Y}})$ and $\Lambda^{n+1}(\mathcal{X})$.

structure $\tilde{\omega}_\tau$ on $T^*\tilde{\mathscr{Y}}_\tau$ reads as [56]

$$\tilde{\omega}_\tau(\varphi,\Pi,\tau,P) = \int_{\Sigma_\tau} \left(\mathrm{d}\varphi^A\wedge \mathrm{d}\Pi_A + \mathrm{d}\tau^\mu \wedge \mathrm{d}P_\mu\right)\otimes \mathrm{d}^n x_0 \ . \tag{2.33}$$

Following the construction of [22] (cfr. Ch. 5), the multisymplectic structure on $\tilde{\mathcal{Z}}$ induces a presymplectic structure on the space $\tilde{\mathscr{Z}}_\tau$ of sections of the bundle $\tilde{\mathcal{Z}}_\tau \to \Sigma_\tau$ given by

$$\tilde{\Omega}_\tau(\sigma)(V,W) = \int_{\Sigma_\tau}\sigma^*(\mathrm{i}_W \mathrm{i}_V \tilde{\Omega}) \qquad , \qquad \sigma\in\tilde{\mathscr{Z}}_\tau \ , \ V,W\in T_\sigma\tilde{\mathscr{Z}}_\tau \tag{2.34}$$

which in turn is related to $\tilde{\omega}_\tau$ via $\tilde{\Omega}_\tau = R_\tau^*\tilde{\omega}_\tau$, where R_τ is the bundle map $R_\tau : \tilde{\mathscr{Z}}_\tau \to T^*\tilde{\mathscr{Y}}_\tau$ relating in adopted coordinates the momenta Π_A and P_a respectively to the temporal components of the multimomenta p^μ_A and ϱ^μ_a as

$$\Pi_A = p^0_A\circ\sigma \qquad , \qquad P_a = \varrho^0_a\circ\sigma \ . \tag{2.35}$$

In particular [22], $\ker T_\sigma R_\tau = \ker\tilde{\Omega}_\tau(\sigma)$ and the canonical parametrized phase space $T^*\tilde{\mathscr{Y}}_\tau$ is thus isomorphic to the quotient $\tilde{\mathscr{Z}}_\tau/\ker\tilde{\Omega}_\tau$.

Let now $\sigma\in\tilde{\mathscr{Z}}\equiv\Gamma(\mathcal{X},\tilde{\mathcal{Z}})$ be a section of the bundle $\tilde{\mathcal{Z}}$ over $\mathcal{X}$, and let $\alpha_{\tilde{\mathcal{Z}}} : \tilde{\mathcal{Z}}\to\tilde{\mathcal{Z}}$ be a covariant canonical transformation covering a diffeomorphism $\alpha_{\mathcal{X}} : \mathcal{X}\to\mathcal{X}$ whose induced action on sections is given by $\alpha_{\tilde{\mathscr{Z}}}(\sigma) = \alpha_{\tilde{\mathcal{Z}}}\circ\sigma\circ\alpha_{\mathcal{X}}^{-1}$ (cfr. Eq. (2.15)). The corresponding transformation on $\tilde{\mathscr{Z}}_\tau\equiv\Gamma(\Sigma_\tau,\tilde{\mathscr{Z}})$ given by

$$\begin{aligned} \alpha_{\tilde{\mathscr{Z}}_\tau} : \tilde{\mathscr{Z}}_{\eta^{-1}\circ\tilde{\tau}} &\longrightarrow \tilde{\mathscr{Z}}_{\alpha_{\tilde{\mathscr{X}}}(\eta)^{-1}\circ\tilde{\tau}} \\ \sigma &\longmapsto \alpha_{\tilde{\mathscr{Z}}_\tau}(\sigma) = \alpha_{\tilde{\mathcal{Z}}}\circ\sigma\circ\alpha_\tau^{-1} \ , \end{aligned} \tag{2.36}$$

with $\alpha_{\tilde{\mathscr{X}}}(\eta)$ defined in (2.16) and $\alpha_\tau := \alpha_{\mathcal{X}}|_{\Sigma_\tau}$ is a (special) covariant canonical transformation relative to the presymplectic 2-form (2.34) if $\alpha_{\tilde{\mathcal{Z}}}$ is a (special) covariant canonical transformation [22].

The covariant multimomentum map (2.24) associated to the $\mathcal{G}$-action on $\tilde{\mathcal{Z}}$ will then induce a so-called (**parametrized**) **energy-momentum map** on $\tilde{\mathscr{Z}}_\tau$ defined by

$$\tilde{\mathcal{E}}_\tau : \tilde{\mathscr{Z}}_\tau \longrightarrow \mathfrak{g}^* \qquad , \qquad \tilde{\mathcal{E}}_\tau(\sigma,\eta) = \tilde{\mathcal{E}}_{\eta^{-1}\circ\tilde{\tau}}(\sigma) := \int_{\Sigma_\tau}\sigma^*\langle\tilde{\mathcal{J}},\xi\rangle \ , \tag{2.37}$$

which is Ad^*-equivariant w.r.t. the action (2.36), namely

$$\langle\tilde{\mathcal{E}}_\tau(\sigma,\eta),\mathrm{Ad}_\alpha^{-1}\xi\rangle = \langle\alpha^*_{\tilde{\mathscr{Z}}_\tau}\big[\tilde{\mathcal{E}}_\tau(\sigma,\eta)\big],\xi\rangle \ . \tag{2.38}$$

Indeed

$$
\begin{aligned}
\langle \alpha^*_{\tilde{\mathscr{Z}}_\tau}\big[\tilde{\mathcal{E}}_\tau(\sigma,\eta)\big],\xi\rangle &= \langle \tilde{\mathcal{E}}_{\alpha_{\tilde{\mathscr{X}}}(\eta)^{-1}\circ\tilde{\tau}}(\alpha_{\tilde{\mathscr{Z}}_\tau}(\sigma))\rangle \\
&= \int_{(\eta\circ\alpha_{\mathcal{X}}^{-1})^{-1}\circ\tilde{\tau}(\Sigma)} (\alpha_{\tilde{\mathcal{Z}}}\circ\sigma\circ\alpha_\tau^{-1})^*\,\langle\tilde{\mathcal{J}},\xi\rangle \\
&= \int_{\alpha_{\mathcal{X}}\circ(\eta^{-1}\circ\tilde{\tau})(\Sigma)} (\alpha_\tau^{-1})^*\,\sigma^*\,\alpha^*_{\tilde{\mathcal{Z}}}\,\langle\tilde{\mathcal{J}},\xi\rangle \\
&= \int_{\eta^{-1}\circ\tilde{\tau}(\Sigma)} \sigma^*\,\alpha^*_{\tilde{\mathcal{Z}}}\,\langle\tilde{\mathcal{J}},\xi\rangle \qquad\qquad \text{(change of variables)} \\
&= \int_{\eta^{-1}\circ\tilde{\tau}(\Sigma)} \sigma^*\,\langle\tilde{\mathcal{J}},\mathrm{Ad}_\alpha^{-1}\xi\rangle \qquad \text{(Ad}^*\text{-equivariance of } \tilde{\mathcal{J}}) \\
&= \langle\tilde{\mathcal{E}}_{\eta^{-1}\circ\tilde{\tau}}(\sigma),\mathrm{Ad}_\alpha^{-1}\xi\rangle \\
&= \langle\tilde{\mathcal{E}}_\tau(\sigma,\eta),\mathrm{Ad}_\alpha^{-1}\xi\rangle\ .
\end{aligned} \tag{2.39}
$$

The energy-momentum map is intimately related to the initial value constraints which generate the covariant gauge freedom thus providing on the one hand a fundamental link between dynamics and the gauge group, and on the other hand encoding in a single geometrical object all the physically relevant information about a given classical field theory [22].

To see this, let us denote by $\tilde{\mathscr{P}}_\tau$ the primary constraint submanifold in $T^*\tilde{\mathscr{Y}}_\tau$ defined as $\tilde{\mathscr{P}}_\tau = R_\tau(\tilde{\mathscr{N}_\tau}) \subset T^*\tilde{\mathscr{Y}}_\tau$ with $\tilde{\mathscr{N}_\tau} = \mathbb{F}\mathcal{L}((j^1\tilde{\mathscr{Y}})_\tau) \subset \tilde{\mathscr{Z}}_\tau$, $\mathbb{F}\mathscr{L}$ being the Legendre transform. For lifted actions – and this is the case for a $\mathcal{G}$-slicing discussed before – the projection $\tilde{\mathcal{J}}_H$: $\tilde{\mathscr{P}}_\tau \to \mathfrak{g}^*$ on $\tilde{\mathscr{P}}_\tau$ of the parametrized energy-momentum map (2.37) encodes the first class secondary constraints respectively as its components in the transversal and tangential directions to the spatial slice. Indeed at the level of densities, using adapted coordinates and recalling the expressions (2.26), (2.29), we have

$$
\begin{aligned}
\sigma^*(\mathrm{i}_{\zeta_{\tilde{\mathcal{Z}}}}\tilde{\Theta}) = \Big[(p_A^0\circ\sigma)(\zeta^A\circ\sigma \quad &- \quad \zeta^\mu\sigma^A_{,\mu}) - (\varrho_a^0\circ\sigma)\zeta^\mu\sigma^a_{,\mu} \\
&+ \quad \Big(\tilde{p}\circ\sigma + (p^\mu_A\circ\sigma)\sigma^A_{,\mu} + (\varrho^\mu_a\circ\sigma)\sigma^a_{,\mu}\Big)\zeta^0\Big]\mathrm{d}^n x_0
\end{aligned} \tag{2.40}
$$

for any $\sigma\in\tilde{\mathscr{Z}}_\tau$ and $\zeta_{\tilde{\mathcal{Z}}}$ the canonical lift of the generator of the $\mathcal{G}$-slicing $\zeta_{\mathcal{X}} = \xi_{\mathcal{X}}$ to $\tilde{\mathcal{Z}}$. Now, for any σ *holonomic lift* of (φ,Π,τ,P) to $\tilde{\mathscr{N}_\tau}$, that is $\sigma\in R_\tau^{-1}\{(\phi,\Pi,\tau,P)\}\cap\tilde{\mathscr{N}_\tau}$, we have $\sigma^A = \phi^A\big|_{\Sigma_\tau} = \varphi^A$ and $\sigma^a = \eta^a\big|_{\Sigma_\tau}$ so that

$$
\zeta^A\circ\sigma - \zeta^\mu\sigma^A_{,\mu} = (\zeta^A\circ\phi - \zeta^\mu\phi^A_{,\mu})\big|_{\Sigma_\tau} = -\left(\mathcal{L}_\zeta\phi\right)^A\big|_{\Sigma_\tau} =: -\dot{\varphi}^A\ , \tag{2.41}
$$

and

$$
\tilde{p}\circ\sigma + (p^\mu_A\circ\sigma)\sigma^A_{,\mu} + (\varrho^\mu_a\circ\sigma)\sigma^a_{,\mu} = \tilde{L}(\sigma)\ , \tag{2.42}
$$

where we used the expressions for the covariant Hamiltonian and the multimomenta (Eqs. (2.19)) and for the canonical momenta (Eq. (2.35)). Hence, Eq. (2.40) yields

$$\sigma^*(\mathrm{i}_{\zeta_{\tilde{\mathcal{Z}}}}\tilde{\Theta}) = -\Big(\Pi_A\dot{\varphi}^A + P_a\zeta^\mu\eta^a_{,\mu} - \tilde{L}(\sigma)\zeta^0\Big)\,\mathrm{d}^n x_0\ , \tag{2.43}$$

from which, by using the fact that

$$\tilde{L}(\sigma)\zeta^0\mathrm{d}^n x_0 = \tau^* i_{\zeta_{\mathcal{X}}}\tilde{\mathscr{L}}(j^1\phi, j^1\eta) = i_{\zeta_{\mathcal{X}}}\tilde{\mathscr{L}}(j^1\varphi, \dot{\varphi}, j^1\eta_\tau, \dot{\eta}_\tau),$$

it follows that the parametrized energy-momentum map (2.37) induces a functional on $\tilde{\mathscr{P}}_\tau$

$$\tilde{\mathcal{J}}_H : \tilde{\mathscr{P}}_\tau \longmapsto \mathfrak{g}^*\ , \tag{2.44}$$

given by

$$\begin{aligned} \langle\tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P),\zeta\rangle &= -\int_{\Sigma_\tau}\mathrm{d}^n x_0(\zeta^\mu\mathcal{H}^{(\varphi)}_\mu + \zeta^\mu P_\mu) \\ &= -\Big(H^{(\varphi)}(\zeta)(\varphi,\Pi,\tau) + P(\zeta)(\tau,P)\Big)\ , \end{aligned} \tag{2.45}$$

where $P_\mu = \eta^a_{,\mu}P_a$ is the pull-back of P_a to Σ along η. The functional (2.45) on $\tilde{\mathscr{P}}_\tau$ is nothing but the total Hamiltonian whose components in the tangential and transversal direction to the spatial slice yields the super-momenta and Hamiltonian constraints.

As it will be discussed in the next subsection, this functional is a equivariant moment map w.r.t. the $\mathcal{G}$-action on the canonical parametrized phase space and it provides us with a representation of the Lie algebra of $\mathsf{Diff}(\mathcal{X})$ on the parametrized phase space. Differently from the standard case of instantaneous canonical formalism where the equivariance of the corresponding functional is spoiled by the algebra of constraints, the key ingredient which give rise to a Lie algebra (anti)homomorphism between the algebra $\mathsf{diff}(\mathcal{X})$ and the Poisson bracket algebra of observable functionals on the parametrized phase space relies on the introduction of the covariance fields which in turn allows to induce a corresponding $\mathcal{G}$-action on the space of embeddings of Σ into $\mathcal{X}$. The action of the diffeomorphism group on the embeddings τ induced via the action on η was actually what ensured the equivariance of the parametrized energy-momentum map (cfr. Eq. (2.39)).

2.5 Representation of Spacetime Diffeomorphisms: $\mathsf{Diff}(\mathcal{X})$-equivariant Momentum Map

Let $\mathcal{G} = \mathsf{Diff}(\mathcal{X})$ be the group of diffeomorphisms (i.e., smooth and inventible active point transformations[9]) of the spacetime manifold $\mathcal{X}$. The Lie algebra $\mathfrak{g} = \mathsf{diff}(\mathcal{X})$ can be realized as the set of all (complete) vector fields on $\mathcal{X}$. Indeed, for any element $\xi \in \mathsf{diff}(\mathcal{X})$, we can associate to it a vector field $\xi_{\mathcal{X}} \in \mathfrak{X}(\mathcal{X})$ generating a one-parameter group α^{ξ}_{λ} of spacetime diffeomorphisms by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\alpha^{\xi}_{\lambda}(x) = \xi_{\mathcal{X}}(\alpha^{\xi}_{\lambda}(x)) \qquad \forall\, x \in \mathcal{X}\ . \tag{2.46}$$

In the standard setting of instantaneous canonical formalism, the generating vector field is decomposed into its "lapse" and "shift" components. i.e. it can be written as the sum of two vectors which are respectively normal and tangent vectors to the space-like hypersurface, say

$$\xi^{\mu}_{\mathcal{X}}(\vec{x}) = N(\vec{x})n^{\mu}(\tau(\vec{x})) + N^{k}(\vec{x})\tau^{\mu}_{,k}(\vec{x}) \tag{2.47}$$

where $n^{\mu} = G^{\mu\nu}n_{\nu}$ is the future-pointing normal such that $\tau^{*}n = 0$ for any $\vec{x} \in \Sigma_{\tau}$ and $G^{\mu\nu}n_{\nu}n_{\nu} = -1$, and $N \in C^{\infty}(\Sigma, \mathcal{X})$, $\vec{N} \in T\Sigma$ are respectively called the *lapse function* (N) and the *shift vector* ($\vec{N}$) of the foliation [26, 31]. They respectively specify the magnitude of the normal and tangential deformation at every point on a spatial hypersurface. The location of a neighbouring slice (i.e. how the slices are embedded in a given spacetime) is determined by specifying lapse and shift which play the role of arbitrary Lagrange multipliers in the action implementing the first-class constraints of the theory.

The algebra $\mathsf{diff}(\mathcal{X})$ of the spacetime diffeomorphism group is however "deformed" by the decomposition into perpendicular and tangential directions to the spatial hypersurfaces. Indeed, the projected constraint functions do not form a genuine Lie algebra but rather a Lie algebroid structure known as the *Dirac hypersurface-deformation algebra* [51, 58, 59] which, after smearing with lapse and shift, reads[10]

$$\begin{aligned}
\{H[\vec{N}], H[\vec{M}]\} &= H[\mathcal{L}_{\vec{N}}\vec{M}]\,,\\
\{H[\vec{N}], H[M]\} &= H[\mathcal{L}_{\vec{N}}M]\,,\\
\{H[N], H[M]\} &= H[\vec{K}]\,,
\end{aligned} \tag{2.48}$$

where $\mathcal{L}_{\vec{N}}$ is the Lie derivative along the vector field $\vec{N}$ (i.e., $\mathcal{L}_{\vec{N}}\vec{M} = [\vec{N}, \vec{M}]$ and $\mathcal{L}_{\vec{N}}M =$

[9] This should not be confused with the pseudo-group of passive transformations which describes the relations between overlapping pairs of coordinate charts. For details on the notion of passive and active diffeomorphisms and their connection we refer to [57] and references therein.

[10] We refer to [38] Sec. 3.3 for the detailed calculation of the constraint algebra for the case of a canonical parametrized scalar field theory.

$N^k M_{,k}$), and $\vec{K}$ is such that

$$K^j = Q^{jk}\left(NM_{,k} - MN_{,k}\right) \ . \tag{2.49}$$

Therefore, the Poisson brackets between two super-Hamiltonians $H[N]$, $H[M]$ depend not only on the pair of shift functions N and M but also explicitly on the canonical embedding variable through the induced (inverse) 3-metric Q^{ab}. This implies that the full algebra of spacetime diffeomorphisms cannot be represented on the canonical parametrized phase space as it is not homomorphically mapped into a Poisson bracket algebra on the parametrized phase space. Only the subalgebra $\mathsf{diff}(\Sigma)$ of spatial diffeomorphisms can be represented in the canonical formalism, as the map $\vec{N} \mapsto H[\vec{N}]$ provides a Lie algebra homomorphism of $\mathsf{diff}(\Sigma)$ into a Poisson bracket algebra on the phase space of the system according to the first equation in (2.48). This reflects into the fact that the (*instantaneous*) *energy-momentum map* or *super-momentum map*

$$\mathcal{E}_\tau : T^*\mathsf{Emb}_G(\Sigma, \mathcal{X}) \times T^*\mathscr{Y}_\tau \to \Lambda^0_d \times \Lambda^1_d \,, \tag{2.50}$$

given by

$$\begin{aligned} \mathcal{E}_\tau[N, \vec{N}] &= \int_\Sigma \mathrm{d}^3x \, \langle (N, \vec{N}), \mathcal{E}_\tau \rangle \\ &= \int_\Sigma \mathrm{d}^n x_0 \, \left(N\mathcal{H} + N^k \mathcal{H}_k \right) = H[N] + H[\vec{N}] \end{aligned} \tag{2.51}$$

with $\Lambda^0_d \times \Lambda^1_d$ the dual of the space of lapses and shifts, i.e., Λ^0_d and Λ^1_d respectively denote the spaces of function densities and 1-form densities on Σ, is not a true momentum map. Indeed, as can be checked by direct computation, from the Dirac algebra (2.48) it follows that

$$\{\mathcal{E}_\tau[N, \vec{N}], \mathcal{E}_\tau[M, \vec{M}]\} = \mathcal{E}_\tau[\mathcal{L}_{\vec{N}} M - \mathcal{L}_{\vec{M}} N, \mathcal{L}_{\vec{N}} \vec{M} + \vec{K}] \,, \tag{2.52}$$

i.e., $\mathcal{E}_\tau$ is not (infinitesimally) equivariant. Only if we restrict to the spatial diffeomorphisms, i.e., the subgroup $\mathcal{G}_\tau = \mathsf{Diff}(\Sigma_\tau)$ of transformations which stabilize the image of τ, the energy-momentum map (2.51) induces a momentum map w.r.t. the $\mathcal{G}_\tau$-action, say

$$J_\tau := \mathcal{E}_\tau\big|_{\mathfrak{g}_\tau} : T^*\mathsf{Emb}_G(\Sigma, \mathcal{X}) \times T^*\mathscr{Y}_\tau \to \mathfrak{g}^*_\tau \tag{2.53}$$

such that

$$J_\tau[\vec{N}] = \int_\Sigma \mathrm{d}^n x_0 \, N^k \mathcal{H}_k = H[\vec{N}] \ , \tag{2.54}$$

which, as expected from the first of equations (2.48), is equivariant under the $\mathcal{G}_\tau$-action, thus providing a representation of the Lie algebra $\mathfrak{g}_\tau = \mathsf{diff}(\Sigma_\tau)$ of spatial diffeomorphisms in terms of a Poisson bracket algebra of functionals on the extended phase space.

However, the action of the full group of diffeomorphisms on the parametrized phase space of the theory can be recovered by considering the action of $\mathsf{Diff}(\mathcal{X})$ on the embedding themselves.

Specifically, the left action of $\mathsf{Diff}(\mathcal{X})$ on $\mathcal{X}$ induces a natural left action of $\mathsf{Diff}(\mathcal{X})$ on the space $\mathsf{Emb}(\Sigma,\mathcal{X})$ of all embeddings of Σ in $\mathcal{X}$

$$\Psi : \mathsf{Emb}(\Sigma,\mathcal{X}) \times \mathsf{Diff}(\mathcal{X}) \longrightarrow \mathsf{Emb}(\Sigma,\mathcal{X}) \qquad \text{by} \qquad (\tau,\alpha_{\mathcal{X}}) \longmapsto \alpha_{\mathcal{X}} \circ \tau \ , \tag{2.55}$$

which carries the points $\tau(\vec{x},\lambda)$ of the hypersurface $\Sigma_\tau = \tau(\Sigma)$ into new spacetime positions $\alpha_{\mathcal{X}}(\tau(\vec{x},\lambda))$ forming a new hypersurface. Indeed, according to the action (2.16) of the diffeomorphism group on η, we have

$$\begin{aligned} \alpha\cdot\tau &= \alpha_{\tilde{\mathscr{X}}}(\eta)^{-1}\circ\tilde{\tau} \\ &= \left(\eta\circ\alpha_{\mathcal{X}}^{-1}\right)^{-1}\circ\tilde{\tau} \\ &= \alpha_{\mathcal{X}}\circ\eta^{-1}\circ\tilde{\tau} \\ &= \alpha_{\mathcal{X}}\circ\tau \ . \end{aligned} \tag{2.56}$$

The corresponding generating vector field,

$$\xi_\tau(\vec{x}) = \xi_{\mathcal{X}}(\tau(\vec{x})) = \xi^\mu_{\mathcal{X}}(\tau(\vec{x}))\frac{\partial}{\partial x^\mu}\bigg|_{\tau(\vec{x},\lambda)} \ , \tag{2.57}$$

yields a representation of the algebra $\mathsf{diff}(\mathcal{X})$ by vector fields on $\mathsf{Emb}_G(\Sigma,\mathcal{X})$[11]. Indeed, although a generic diffeomorphism $\alpha\in\mathsf{Diff}(\mathcal{X})$ in general would not preserve the space-like nature of the embeddings, for any $\tau\in\mathsf{Emb}_G(\Sigma,\mathcal{X})$ there exists a open neighborhood of the identity in $\mathsf{Diff}(\mathcal{X})$ such that the transformed embedding $\alpha\circ\tau$ is still space-like. An element $\xi\in\mathsf{diff}(\mathcal{X})$, realized as a complete vector field $\xi_{\mathcal{X}}$ on $\mathcal{X}$, thus yields a vector field on $\mathsf{Emb}_G(\Sigma,\mathcal{X})$ by the prescription (2.57). Such a vector field restricted to the embeddings can be then decomposed into the corresponding lapse and shift components which now are not freely specifiable but are some definite functionals of τ. Taking this dependence into account, for any $\xi\in\mathsf{diff}(\mathcal{X})$ it is

[11]Following [38] and references therein, the set $\mathsf{Emb}_G(\Sigma,\mathcal{X})$ of space-like embeddings of Σ into $\mathcal{X}$ is thought of as an infinite-dimensional manifold. Indeed, $\mathsf{Emb}_G(\Sigma,\mathcal{X})$ is a open subset of the set $\mathsf{Emb}(\Sigma,\mathcal{X})$ of all embeddings (not necessarily space-like) of Σ into $\mathcal{X}$ which in turn is an open subset of the infinite-dimensional manifold $C^\infty(\Sigma,\mathcal{X})$ of smooth functions from Σ into $\mathcal{X}$ equipped with the compact-open topology, thus inheriting its differential structure. The tangent space $T_\tau\mathsf{Emb}_G(\Sigma,\mathcal{X})$ at $\tau\in\mathsf{Emb}_G(\Sigma,\mathcal{X})$ is then defined as

$$T_\tau\mathsf{Emb}_G(\Sigma,\mathcal{X}) := \{\xi_\tau : \Sigma\to T\mathcal{X} \mid \xi_\tau(\vec{x})\in T_{\tau(\vec{x})}\mathcal{X} \ , \ \forall\,\vec{x}\in\Sigma\},$$

and similarly

$$T^*_\tau\mathsf{Emb}_G(\Sigma,\mathcal{X}) := \{\gamma_\tau : \Sigma\to T^*\mathcal{X} \mid \gamma_\tau(\vec{x})\in T^*_{\tau(\vec{x})}\mathcal{X} \ , \ \forall\,\vec{x}\in\Sigma\},$$

with L^2-dual pairing given by

$$\langle\gamma_\tau,\xi_\tau\rangle := \int_\Sigma \mathrm{d}^n x_0\,\sqrt{\det Q(\vec{x})}\,\gamma_\mu(\tau(\vec{x}))\xi^\mu(\tau(\vec{x})) \ ,$$

where $Q=\tau^*G$ is the induced metric on Σ.

possible to define a new Hamiltonian functional on the parametrized phase space related to the equivariant momentum map (2.45) via

$$
\begin{aligned}
H(\xi)(\varphi,\Pi,\tau,P) :&= -\langle \xi, \tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P)\rangle \\
&= \int_\Sigma \mathrm{d}^n x_0\, \xi^\mu_{\mathcal{X}}(\tau(\vec{x}))\mathcal{H}_\mu(\varphi,\Pi,\tau,P) \\
&= \int_\Sigma \mathrm{d}^n x_0\, \xi^\mu_{\mathcal{X}}(\tau(\vec{x}))\Big(\mathcal{H}^{(\varphi)}_\mu + P_\mu\Big)\,.
\end{aligned}
\tag{2.58}
$$

This result is compatible with the procedure introduced by Isham and Kuchař in [38] for the case of a parametrized scalar field theory where the key step in representing spacetime diffeomorphisms was the observation that the embedding variables (which tells us how the model Cauchy surface Σ lies in the spacetime manifold) provide a link between the spatial and spatiotemporal pictures.

At the infinitesimal level, the equivariance property of the Hamiltonian (2.58) can be seen as follows. Using the fact that the Poisson bracket between the unprojected constraint functions vanishes strongly and the canonical Poisson brackets $\{\tau^\mu(\vec{x}), P_\nu(\vec{x}\,')\} = \delta^\mu_\nu \delta(\vec{x},\vec{x}\,')$ between the embeddings and their conjugate momenta as well as that the embedding commutes with $\mathcal{H}^{(\varphi)}$, for any two Lie algebra elements $\xi,\zeta \in \mathsf{diff}(\mathcal{X})$ whose corresponding vector fields $\xi_{\mathcal{X}}(\tau(\vec{x})), \zeta_{\mathcal{X}}(\tau(\vec{x}))$ generate one-parameter groups of spacetime diffeomorphisms, we have

$$
\begin{aligned}
\{H(\xi), H(\zeta)\} &= -\int_\Sigma \mathrm{d}^n x_0\, [\xi_{\mathcal{X}},\zeta_{\mathcal{X}}]^\mu\big|_{\tau(\vec{x})}\mathcal{H}_\mu(\vec{x}) \\
&= -H([\xi_{\mathcal{X}},\zeta_{\mathcal{X}}]) \\
&= H([\xi,\zeta])\,,
\end{aligned}
\tag{2.59}
$$

where the Lie bracket $[\xi,\zeta]$ is defined as the opposite of the commutator between the corresponding vector fields, say

$$
[\xi,\zeta] = -[\xi_{\mathcal{X}},\zeta_{\mathcal{X}}] = -\left(\xi^\nu_{\mathcal{X}}\zeta^\mu_{\mathcal{X},\nu} - \zeta^\nu_{\mathcal{X}}\xi^\mu_{\mathcal{X},\nu}\right)\frac{\partial}{\partial x^\mu}\,. \tag{2.60}
$$

Thus, the mapping $\xi \mapsto H(\xi)$ is a (anti)homomorphism between the Lie algebra $\mathsf{diff}(\mathcal{X})$ and the Poisson bracket algebra of observable functionals on the parametrized phase space. This shows that for any $\xi \in \mathsf{diff}(\mathcal{X})$ the functional $H(\xi)$ (resp. $\tilde{\mathcal{J}}_H(\xi)$) defines a equivariant momentum map w.r.t. the action of the spacetime diffeomorphism group on the parametrized phase space. Such property reflects the equivariance of the parametrized energy-momentum map (2.37), which in turn was provided by considering the action (2.55) of diffeomorphisms on the embeddings induced via the covariance fields.

Finally, the total Hamiltonian (2.58) is constructed in such a way that the constraints are preserved along the flow generated by $H(\xi)$, that is

$$\dot{\mathcal{H}}_\alpha(\vec{x}) = \int_\Sigma \mathrm{d}^n x_0' \{\mathcal{H}_\mu(\vec{x}), \xi^\nu_{\mathcal{X}}(\tau(\vec{x}\,'))\}\mathcal{H}_\nu(\vec{x}\,') \tag{2.61}$$

vanishes on the constraint surface. Moreover, as any functional of the embedding commutes with $H^{(\varphi)}$, we have

$$\dot{\tau}^\mu(\vec{x}) = \int_\Sigma \mathrm{d}^n x_0'\, \xi^\nu_{\mathcal{X}}(\tau(\vec{x}\,'))\{\tau^\mu(\vec{x}), P_\nu(\vec{x}\,')\} = \xi^\mu_{\mathcal{X}}(\tau(\vec{x}))\ , \tag{2.62}$$

i.e., $\xi_{\mathcal{X}}(\tau(\vec{x}))$ is the deformation vector of the foliation which can be decomposed into its transversal $\vec{\xi}_{\|}(\tau(\vec{x}))$ and normal $\xi_\perp(\tau(\vec{x}))$ components which, unlike the Lagrange multipliers N and $\vec{N}$ entering the parametrized action, are now specific functionals of the embedding. On the other hand, since $P(\xi)$ commutes with the field variables, the rates of change of the field φ and its conjugate momentum Π yield the Hamiltonian field equations with deformation vector $\xi_{\mathcal{X}}(\tau(\vec{x}))$.

Therefore, the co-momentum map $H(\xi)$ defined in Eq. (2.58) generates the deformation of the embedding induced by the vector field $\xi_{\mathcal{X}}$ on $\mathcal{X}$ together with the dynamical evolution of the field variables and, since the constraints are preserved along this flow, on-shell field configurations are compatibly evolved along the constraint surface. In other words, the canonical action of $\mathsf{Diff}(\mathcal{X})$ represented by $H(\xi)$ generates a displacement of the spatial hypersurface embedded in spacetime and also set the correctly evolved Cauchy data for fields on the deformed hypersurface. Note that the explicit embedding-dependence of the induced vector field $\xi_{\mathcal{X}}$ on $\mathcal{X}$ entering (2.58) implies that $H(\xi)$ comes to be the Hamiltonian function along the flow lines generated by $\xi_{\mathcal{X}}$ which correspond to a one-parameter family of emebeddings which in turn identifies a foliation with deformation vector $\xi_{\mathcal{X}}(\tau(\vec{x}))$. Thus, for any $\xi \in \mathsf{diff}(\mathcal{X})$, realized as a complete vector field on $\mathcal{X}$, the induced vector field on $\mathsf{Emb}_G(\Sigma, \mathcal{X})$ is the tangent vector field to a curve of embeddings which identifies a foliation of spacetime. Different Lie algebra elements would identify different foliations whose corresponding deformation vectors are given by the induced vector fields on $\mathcal{X}$ restricted to the embedding. In this sense the momentum map $H(\xi)$ (resp. $\tilde{\mathcal{J}}_H(\xi)$) provides a faithful representation of the Lie algebra of $\mathsf{Diff}(\mathcal{X})$ on the parametrized phase space.

3 Covariant Gauge Group Thermodynamics

We can finally proceed to extend the generalized notion of thermodynamic equilibrium states á la Souriau to parametrized field theories in which the Hamiltonian action we are interested in is that of the spacetime diffeomorphism group $\mathsf{Diff}(\mathcal{X})$ or, more precisely, the lifted action to the parametrized phase space of the automorphisms of the extended configuration bundle covering diffeomorphisms on the base manifold $\mathcal{X}$.

3.1 Generally Covariant Gibbs State

To avoid weighing the notation down, in what follows we generically denote the canonical parametrized phase space of the theory by Υ, i.e., $\Upsilon \equiv T^*\mathscr{Y}_\tau \times T^*\mathsf{Emb}_G(\Sigma, \mathcal{X})$.

A statistical state $\rho : \Upsilon \to \mathbb{R}([0, +\infty[)$ on the parametrized phase space is a smooth probability density on Υ such that, for any Borel subset $\mathscr{A}$ of Υ, the integral

$$\mu(\mathscr{A}) = \int_{\mathscr{A}} \mathcal{D}[\varphi, \Pi, \tau, P]\, \rho(\varphi, \Pi, \tau, P) \tag{3.1}$$

defines a probability measure on Υ with the normalization condition

$$Z(\rho) = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P]\, \rho(\varphi, \Pi, \tau, P) = 1 \ , \tag{3.2}$$

where $\mathcal{D}[\varphi, \Pi, \tau, P]$ formally denotes the integration measure on Υ. To such a statistical state, we can associate a entropy functional

$$S(\rho) = -\int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P]\, \rho(\varphi, \Pi, \tau, P) \log \rho(\varphi, \Pi, \tau, P) \ , \tag{3.3}$$

with the convention that $\rho \log \rho = 0$ for $\rho = 0$. Given a functional on Υ, say $f \in \mathcal{F}(\Upsilon)$, the mean value of f w.r.t. ρ is then defined as

$$\mathbb{E}_\rho(f) = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P] \rho(\varphi, \Pi, \tau, P) f(\varphi, \Pi, \tau, P). \tag{3.4}$$

In particular, for deriving our Gibbs state, we shall consider the mean value $\mathbb{E}_\rho(\tilde{\mathcal{J}}_H)$ of the momentum map (2.45)

$$\mathbb{E}_\rho(\tilde{\mathcal{J}}_H) : \mathcal{F}(\Upsilon) \longrightarrow \mathfrak{g}^* \quad , \quad \mathbb{E}_\rho(\tilde{\mathcal{J}}_H) = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P] \rho(\varphi, \Pi, \tau, P) \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) \tag{3.5}$$

such that, for $\xi \in \mathfrak{g} = \mathsf{diff}(\mathcal{X})$, it yields

$$\langle \xi, \mathbb{E}_\rho(\tilde{\mathcal{J}}_H) \rangle : \mathcal{F}(\Upsilon) \longrightarrow \mathbb{R} \tag{3.6}$$

by

$$\langle \xi, \mathbb{E}_\rho(\tilde{\mathcal{J}}_H) \rangle = \int_{\Upsilon} \mathcal{D}[\varphi, \Pi, \tau, P] \rho(\varphi, \Pi, \tau, P) \, \langle \xi, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) \rangle = \mathbb{E}_\rho(\tilde{\mathcal{J}}_H(\xi)) \ . \tag{3.7}$$

Stationarity of the entropy functional (3.3) under an infinitesimal smooth variation $\rho_s(\varphi, \Pi, \tau, P)$ with $s \in]-\varepsilon, \varepsilon[$, $\varepsilon > 0$ of the statistical state ρ with fixed mean value of $\tilde{\mathcal{J}}_H$ can be therefore implemented by introducing two Lagrange multipliers $b \in \mathfrak{g} = \mathsf{diff}(\mathcal{X}), a \in \mathbb{R}$ respectively associated to the constraint $\mathbb{E}_\rho(\tilde{\mathcal{J}}_H) = const.$ and the normalization condition (3.2) via

$$\mathcal{S}(\rho_s) = S(\rho_s) - \langle b, \mathbb{E}_{\rho_s}(\tilde{\mathcal{J}}_H)\rangle - aZ(\rho_s) \ , \tag{3.8}$$

such that

$$\left.\frac{\delta \mathcal{S}(\rho_s)}{\delta s}\right|_{s=0} = 0 \qquad \forall\ \rho_s \ . \tag{3.9}$$

Hence, we get

$$\begin{aligned} 0 &= \left.\frac{\delta \mathcal{S}(\rho_s)}{\delta s}\right|_{s=0} \\ &= -\int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P]\Big(1 + \log\left(\rho(\varphi, \Pi, \tau, P)\right) + \langle b, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle + a\Big)\left.\frac{\delta \rho_s}{\delta s}\right|_{s=0} \ , \end{aligned} \tag{3.10}$$

for any ρ_s, from which it follows that

$$\rho_{a,b}(\varphi, \Pi, \tau, P) = \exp\Big(-1 - a - \langle b, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\Big). \tag{3.11}$$

The normalization condition (3.2) then implies

$$Z(b) = \exp\left(1 + a\right) = \int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P] \exp\Big(- \langle b, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\Big) \ , \tag{3.12}$$

where we limit b to the subset $\Omega \subset \mathfrak{g}$ such that the above integral *converge*. The generally covariant Gibbs statistical state is given by

$$\begin{aligned} \rho_b^{(\mathrm{eq})}(\varphi, \Pi, \tau, P) &= \frac{1}{Z(b)} \exp\Big(- \langle b, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\Big) \\ &= \frac{1}{Z(b)} \exp\Big(\int_\Sigma \mathrm{d}^n x_0\, \xi^\mu_{(b)}(\tau(\vec{x}))\, \Big(\mathcal{H}^{(\varphi)}_\mu(\vec{x}) + P_\mu(\vec{x})\Big)\Big) \end{aligned} \tag{3.13}$$

where $\xi_{(b)}$ denotes the vector field on $\mathcal{X}$ associated to $b \in \mathsf{diff}(\mathcal{X})$ generating a one-parameter family of spacetime diffeomorphisms. Note that the statistical state (3.13) is now a functional of the fields (φ, Π, τ, P) through the comomentum map functional $\langle b, \tilde{\mathcal{J}}_H\rangle$. In particular, being a functional on the parametrized phase space, the dependence from the spacetime coordinates occurs only through the dynamical variables thus respecting the coordinate-independence of relativistic theories. Moreover, as a functional of the embeddings, the statistical state (3.13) is covariant in the sense that the momentum map is evaluated on any space-like hyper-surface without fixing the slicing a priori. The one-parameter group of automorphisms of the extended configuration space generated by $b \in \mathsf{diff}(\mathcal{X})$ identifies a generalized concept of "time evolution" w.r.t. which the Gibbs state is of equilibrium.

3.2 Generalized Thermodynamic Potentials

Let us assume that the Hamiltonian action of the – $\mathcal{G}$-slicing preserving – one-parameter subgroup of the diffeomorphisms, and its covariant moment map $\tilde{\mathcal{J}}_H : \tilde{\mathscr{P}}_\tau \longmapsto \mathfrak{g}^*$ are such that the ensemble of generalised (inverse) temperatures $\Omega \subset \mathfrak{g}$ is non-empty.

Whenever the generalized Gibbs states (3.13) can be defined, the infinite differentiability of the associated partition function $Z : \Omega \to \mathbb{R}$ in (3.12) allows us to define generalized macroscopic functions over Ω, in terms of differentials of $Z(b)$ [32].

The thermodynamic free energy potential, $F(b) \equiv -\log Z(b)$, encodes complete thermodynamic information about the system. The *equilibrium* internal energy $Q : \Omega \to \mathfrak{g}^*$ is given by the first differential of $F(b)$, taken as a smooth map from $\mathfrak{g}$ in $\mathbb{R}$,

$$Q(b) \equiv DF(b) = -D(\log Z(b)), \tag{3.14}$$

corresponding to the average momentum map in the generalised Gibbs ensemble. Exactly as for standard equilibrium thermodynamics, to a given Q corresponds at most one value of b, so that $F(b)$ and the equilibrium probability density ρ_b are uniquely determined.

The entropy function $S : \Omega \to \mathbb{R}$, as defined in (3.3), has a strict maximum $S(b)$ at equilibrium, given by

$$\begin{aligned} S(b) &= \log Z(b) - \langle D(\log Z(b)), b\rangle \\ &= -F(b) + \langle Q(b), b\rangle \quad . \end{aligned} \tag{3.15}$$

From the second differential of $F(b)$, for $Y, Z \in \mathfrak{g}$, $b \in \Omega$, one gets a generalised geometric expression for the covariance matrix,

$$\langle DQ(b)(Y), Z\rangle = \langle Q(b), Y\rangle \, \langle Q(b), Z\rangle - \mathbb{E}_{\rho_b}\left[\langle \tilde{\mathcal{J}}_H, Y\rangle \langle \tilde{\mathcal{J}}_H, Z\rangle\right] \quad . \tag{3.16}$$

In particular, $\forall b \in \Omega$, $\forall Y \in \mathfrak{g}$, the contracted differential

$$\langle DQ(b)(Y), Y\rangle = -\mathbb{E}_{\rho_b}\left[\left(\langle \tilde{\mathcal{J}}_H - Q(b), Y\rangle\right)^2\right] \leq 0 \tag{3.17}$$

defines the *concavity* of the free energy. On the diagonal, $\forall\, Y \in \mathfrak{g}, Y \neq 0$, such that $\langle \tilde{\mathcal{J}}_H, Y\rangle \neq$ const on $\tilde{\mathscr{P}}_\tau$, the above formula defines a *positive definite* bilinear symmetric form,

$$D(D(\log Z(b)))(Y, Y) = -\langle DQ(b)(Y), Y\rangle > 0 \quad , \tag{3.18}$$

which corresponds, in thermodynamic terms [34], to a geometric notion of heat capacity. [12]

[12] In information geometry [60, 61], the negative of the second derivative (Hessian) of the free energy in (3.17)

3.3 Canonical vs Microcanonical Imposition of the Constraints

The state (3.13), together with the associated thermodynamic potentials define an equilibrium thermodynamic (or thermostatic) characterisation for the system of fields on $\tilde{\mathscr{P}}_\tau$. At any fixed temperature, the system is governed by the principle of minimum free energy. Instead of occupying a single definite state, given by the Dirac measure $\delta(\tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P))$ which imposes the diffeomorphisms symmetry constraint on Υ, the system of fields is allowed to have different probabilities of occupying different states, and these probabilities will be chosen to minimize the generalized free energy

$$F(b) = \langle Q(b), b\rangle - S(b) \quad . \tag{3.19}$$

In standard thermodynamics, the principle of minimum free energy reduces to the principle of minimum energy when $b \to \infty$. In our generalised statistical framework, despite $\tilde{\mathcal{J}}_H$ being not necessarily bounded below and the measure $\mathcal{D}$ not being finite, the free energy function $F(b)$ is strictly concave and analytic on the open interval $\Omega \subset \mathfrak{g}$. As $b \to 0$, $\rho_b^{(\mathrm{eq})}$ tends to the *uniform* distribution over Υ, for $\lim_{b\to 0} e^{-\langle b, \tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P)\rangle} = 1$. On the other hand, as $b \to \pm\infty$ we have that $\lim_{b\to\infty} e^{-\langle b, \tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P)\rangle} = 1$ if $\tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) = 0$, and 0 otherwise. Therefore, the Gibbs distribution converges to the uniform distribution defined over the reduced support with vanishing (though not necessarily minimal) momentum map,[13] namely

$$\rho_b^{(\mathrm{eq})} \to \delta(\bar{\sigma}) \tag{3.20}$$

with $\bar{\sigma} \in \tilde{\mathcal{J}}_H^{-1}(0) \subset \tilde{\mathscr{P}}_\tau$, corresponding to field configurations satisfying both primary and secondary first class constraints. From the second Noether theorem, we know that field configurations $\bar{\sigma}$ such that $\tilde{\mathcal{J}}_H(\sigma) = 0$, represent *solutions* to the Euler-Lagrange equations [21]. In this sense, we can reformulate symplectic reduction problem for our first order parametrized field theory in thermodynamic terms, as a limiting case of a general principle of maximum entropy[14] [20], and in accordance with the *geometrodynamics regained* program of Kuchař [63]. To the idea that the geometry and symmetry together determine the field theory, here we add a further statistical characterisation, showing that from symmetry assumptions and the energy-momentum

defines the *Fisher information* matrix, measuring the variance of the equilibrium distribution with respect to the given b. The positive definite form in (3.18) defines a *pairing* on $\mathfrak{g}$. In finite dimensions, for e.g. $\mathfrak{g} \simeq \mathbb{R}^N$, this gives a Riemannian metric on the N-dimensional parameter space, the Fisher metric of Information Geometry [60]. This defines an interesting connection between Lie group thermodynamics and Information Geometry first remarked in [62].

[13] Indeed, for (3.17), the derivative of the expected value of the momentum map with respect to the b parameter is always negative, so that the expected value decreases monotonically to zero as $b \to \pm\infty$.

[14] For b such that $\langle Q(b), b\rangle > 0$, minimizing the free energy is the same as maximizing S.

map it is possible to recover the information on the solutions of the Euler-Lagrange equations in some thermodynamic limit, corresponding to the passage from a canonical description to a microcanonical description of the constrained theory space.

In a sense, it is natural to understand the generalized Gibbs state in (3.13) as an *off-shell* generalisation corresponding to a canonical statistical distribution in which a non-zero weight is assigned to configurations which do not solve the constraint equations, while at the same time weighting more such solutions compared to generic configurations in the thermodynamic limit. From a statistical viewpoint, the passage from our canonical description to the microcanonical definition of the constrained theory space requires to *coarse-grain* the information encoded in the algebra element b.[15]

3.4 Generalized Second Law

In the equilibrium, thermostatic setting described above, different $b \in \Omega$ identifies different *slicings* of the base manifold (spacetime), corresponding to different Cauchy problems. Each b corresponds to a choice of frame, which we can identify with a virtual observer, as well as a virtual reservoir for the system. This allows to recast relations among different equilibrium Cauchy slicings in terms of thermodynamic relations.

Transformations among equilibrium states induced by the Hamiltonian action of the group under concern (e.g. whenever two temperatures are related by the adjoint action of the one-d subgroup of the diffeomorphisms) are trivial due to the equivariance of the multi-momentum map.[16] Actual thermodynamical relations require infinitesimal departure from equilibrium. In the following, we shall then consider how the equilibrium thermodynamic potentials vary due to infinitesimal perturbation our equilibrium state, by introducing an externally controlled parameter s, as shown in 3.1. We can think of s as a work parameter, as we perform work on the system by varying it. We can then denote an equilibrium state of the system $\rho_{s,b}$ by a temperature and a work parameter (see e.g. [64]).

Following a standard procedure, let us evolve the system from one equilibrium state to another, while generally driving it away from equilibrium in the interim. We start with a system at equilibrium $\rho_{i,b}$, for fixed $s = i$, and we perturb it by varying the work parameter to a value $s = f$, while moving along the one-dimensional diffeos flow parameter. Finally, we allow

[15] Indeed, we can see the uniform distribution partition function as a weighted superposition of canonical partition functions over various values of the intensive b parameter.

[16] See [34] for an explicit derivation of the variations of the thermodynamic potentials under adjoint action of the group, in the generic case

the system to somehow re-equilibrate with the equilibrium observer–reservoir and relax to $\rho_{f,b}$.

We can measure the difference between initial and final states due to the work done while deviating from equilibrium via Kullback-Leibler divergence (KL) [60]

$$D(\rho_{i,b}|\rho_{f,b}) \quad = \quad \int_{\Upsilon} \mathcal{D}[\sigma]\rho_{i,b}(\sigma)\log\left(\frac{e^{-\langle b,\tilde{\mathcal{J}}_{H_i}\rangle}/Z_i(b)}{e^{-\langle b,\tilde{\mathcal{J}}_{H_f}\rangle}/Z_f(b)}\right) \geq 0 \tag{3.21}$$

The KL predicts that the external work performed on the system is no less than the free energy difference between the final state. Using our notation for the generalised thermodynamic functions, we have

$$\begin{aligned} D(\rho_{i,b}|\rho_{f,b}) &= -\log Z_i(b) - \langle \mathbb{E}_{i,b}(\tilde{\mathcal{J}}_{H_i}), b\rangle + \log Z_f(b) - \langle \mathbb{E}_{i,b}(\tilde{\mathcal{J}}_{H_f}), b\rangle & (3.22)\\ &= \log Z_f(b) - \log Z_i(b) + \langle \mathbb{E}_{i,b}(\tilde{\mathcal{J}}_{H_f} - \tilde{\mathcal{J}}_{H_i}), b\rangle \geq 0 & (3.23) \end{aligned}$$

By interpreting $W \equiv \tilde{\mathcal{J}}_{H_f} - \tilde{\mathcal{J}}_{H_i}$ as the work associated to the jump in the energy-momentum map $\tilde{\mathcal{J}}_{H_i} \to \tilde{\mathcal{J}}_{H_f}$, we have

$$\langle \mathbb{E}_{i,b}(W), b\rangle \geq \Delta F \equiv F_f(b) - F_i(b) \tag{3.24}$$

where $F_s(b)$ is the free energy of the state $\rho_{s,b}$ as defined above. This is the Clausius inequality of classical thermodynamics, which express the essential statement of the second law of thermodynamics for an *isothermal* transformation.

When the parameter is varied slowly enough that the system remains in equilibrium along the flow, then the process is reversible and isothermal, and $\mathbb{E}_{i,b}(W) = \Delta F$.[17] The work performed during a reversible, isothermal process depends only on the initial and final states and not on the sequence of equilibrium states that mark the journey from $[i, b]$ to $[f, b]$.[18]

Via KL divergence, we can describe the second law also for the case of an *adiabathic* transformation, where we imagine to keep $\tilde{\mathcal{J}}$ fixed, while changing the temperature $b_i \to b_f$. In this case, starting from the general expression in (3.23), we use the general definition of the entropy,

[17] The case where the same two states at $[i, b]$ and $[f, b]$ are related via the action of a $\mathcal{G}$-slicing preserving one-d diffeomorphism subgroup is apparent: moving along an adjoint orbit in $\mathfrak{g}^*$, by definition, we have $\Delta F = 0$.

[18] The nonequilibrium work relation [64] further extends this statement to irreversible processes

$$\log\left\langle e^{-\langle W,b\rangle}\right\rangle = -\Delta F. \tag{3.25}$$

The value of the nonlinear average on the left depends only on equilibrium states $[i, b]$ and $[f, b]$, and not on the intermediate, out-of-equilibrium states visited by the system. This implies that we can determine an equilibrium free energy difference of the system from its statistical fluctuations, by observing it away from equilibrium, provided we repeat the process many times.

$S(b) = \log Z(b) + \langle \mathbb{E}_b(\tilde{\mathcal{J}}_H), b \rangle$ to get

$$
\begin{aligned}
S(b_f) - S(b_i) &\geq \langle \mathbb{E}_{b_f}(\tilde{\mathcal{J}}_H), b_f \rangle - \langle \mathbb{E}_{b_i}(\tilde{\mathcal{J}}_H), b_f \rangle \\
&= \langle \mathbb{E}_{b_f}(\tilde{\mathcal{J}}_H) - \mathbb{E}_{b_i}(\tilde{\mathcal{J}}_H), b_f \rangle \\
&= \langle \Delta Q, b_f \rangle
\end{aligned} \tag{3.26}
$$

By changing abruptly the temperature, hence the reference observer–reservoir, the entropy of the system increase, while the reservoir entropy decrease by an amount $\langle \Delta Q, b_f \rangle$, with an overall change in entropy

$$
\Delta S - \langle \Delta Q, b_f \rangle \geq 0 \tag{3.27}
$$

in accordance with the second law of thermodynamics.

A slow variation of the state realised via a series of equilibrium states defines a reversible *adiabathic* transformation, with $\Delta S = \langle \Delta Q, b_f \rangle$. Again, this quantity would be zero if measured along an adjoint orbit in $\mathfrak{g}^*$.

Differently form the isothermal process, here we perform a change on the slicing of the manifold, which results in an increase of the entropy of the state of the fields.

4 Equilibrium and Dynamical Evolution

Once our formalism has been settled down and the relevant thermodynamical quantities have been generalized to the fully covariant framework of parametrized field theories, it might be useful to check whether previous standard notions can be recovered out of it. In this section we then focus on the one hand on the relation between spacetime and spatial diffeomorphisms and the generalized Gibbs states w.r.t. the corresponding one-parameter subgroups. On the other hand we analyze the possibility of extracting a equilibrium state w.r.t. dynamical evolution by gauge-fixing the diffeomorphism symmetry to disentangle the dynamics encoded in the constraints. This not only provides us with a consistency check for our formalism, but also gives some insight on the physical interpretation of the Lie algebra valued temperature in such a framework. In particular, we elaborate on its inbuilt observer dependence and its relation with the thermal time hypothesis.

4.1 Spacetime vs. Spatial Diff-Equilibrium State

As discussed in Sec. 2.5, the action of the diffeomorphism group on the covariance fields induces a natural action of $\mathsf{Diff}(\mathcal{X})$ on the space of embeddings by left action (2.55). The natural action of the group $\mathsf{Diff}(\Sigma)$ of spatial diffeomorphisms on the space of embeddings is instead the right action, namely:

$$\mathsf{Emb}(\Sigma,\mathcal{X})\times\mathsf{Diff}(\Sigma)\longrightarrow\mathsf{Emb}(\Sigma,\mathcal{X})\qquad\text{by}\qquad(\tau,\alpha_\Sigma)\longmapsto\tau\circ\alpha_\Sigma\ . \tag{4.1}$$

according to which a point $\vec{x}\in\Sigma$ is mapped by $\alpha_\Sigma\in\mathsf{Diff}(\Sigma)$ into a new point $\vec{x}'=\alpha_\Sigma(\vec{x})\in\Sigma$, and then this point is mapped by the embedding τ into the spacetime point $\tau(\vec{x}')\in\mathcal{X}$.

The right action (4.1) of $\mathsf{Diff}(\Sigma)$ induces a left action of $\mathsf{Diff}(\mathcal{X})$ on a given hypersurface $\tau(\Sigma)$ which preserves the hypersurface fixed, i.e., for a given $\tau\in\mathsf{Emb}(\Sigma,\mathcal{X})$, we have

$$\tau\circ\alpha_\Sigma=\Phi_{\alpha_\Sigma}\circ\tau\ , \tag{4.2}$$

such that the following diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Phi_{\alpha_\Sigma}"] & \mathcal{X} \\ \Sigma \arrow[u, "\tau"] \arrow[r, "\alpha_\Sigma"'] & \Sigma \arrow[u, "\tau"'] \end{tikzcd} \tag{4.3}$$

commutes. In particular, considering a one-parameter subgroup of $\mathsf{Diff}(\Sigma)$ which induces a one-parameter subgroup of hypersurface preserving spacetime diffeomorphisms, the corresponding generating vector fields $\xi_\Sigma\in\mathfrak{X}(\Sigma)$ and $\xi_\mathcal{X}\in\mathfrak{X}(\mathcal{X})$ are related by

$$\xi_\mathcal{X}=\tau_*\xi_\Sigma\qquad\text{i.e.}\qquad\xi^\mu_\mathcal{X}(\tau(\vec{x}))=\tau^\mu_{,k}(\vec{x})\xi^k_\Sigma(\vec{x})\ . \tag{4.4}$$

Note that, unlike the fully spacetime covariant case discussed in the previous sections, the embedding τ is now fixed thus restricting ourselves to a given spatial hypersurface $\Sigma_\tau=\tau(\Sigma)$. Correspondingly, the restriction of the covariant momentum map $\tilde{\mathcal{J}}_H$ to the subalgebra $\mathfrak{g}_\tau=\mathsf{diff}_{\Sigma_\tau}(\mathcal{X})$ of spatial diffeomorphisms which preserves the image of τ identifies an equivariant momentum map w.r.t. the action of the subgroup $\mathcal{G}_\tau=\mathsf{Diff}(\Sigma_\tau)$. Indeed, using the relation (4.4), for $b_\tau\in\mathfrak{g}_\tau=\mathsf{diff}_{\Sigma_\tau}(\mathcal{X})$ we have

$$\begin{aligned}\langle\tilde{\mathcal{J}}_H(\varphi,\Pi,\tau,P),b_\tau\rangle&=-\int_{\Sigma_\tau}\mathrm{d}^nx_0\,\xi^\mu_{(b_\tau)}(\tau(\vec{x}))\mathcal{H}_\mu(\vec{x})\\&=-\int_{\Sigma_\tau}\mathrm{d}^nx_0\,\tau^\mu_{,k}(\vec{x})\zeta^k_{(b_\tau)}(\vec{x})\mathcal{H}_\mu(\vec{x})\\&=-\int_{\Sigma_\tau}\mathrm{d}^nx_0\,\zeta^k_{(b_\tau)}(\vec{x})\mathcal{H}_k(\vec{x})\\&=-(\vec{P}(\zeta)+\vec{H}(\zeta))\\&=:\langle\mathcal{J}_\tau(\varphi,\Pi,\tau,P),\zeta\rangle\quad,\quad\zeta\in\mathsf{diff}(\Sigma_\tau)\end{aligned} \tag{4.5}$$

so that, for $\zeta,\zeta'\in\mathsf{diff}(\Sigma_\tau)$, we have

$$\{\mathcal{J}_\tau(\zeta), \mathcal{J}_\tau(\zeta')\} = \{\vec{H}(\zeta), \vec{H}(\zeta')\} = \vec{H}([\zeta, \zeta']) \ , \tag{4.6}$$

as can be checked by direct computation using the fact that $\zeta^j(\vec{x}), \zeta'^k(\vec{x}')$ do not depend on the embeddings and the Poisson bracket for the spatial diffeomorphism constraint. Thus, the map $\mathcal{J}_\tau = \tilde{\mathcal{J}}_H\big|_{\Sigma_\tau}$ provides a representation of the algebra of spatial diffeomorphisms on the extended phase space. Consistently, the map (4.5) gives the expected lift of the shift vector into the deformation vector tangential to the spatial slice, i.e.

$$\begin{aligned}\dot{\tau}^\mu(\vec{x}) &= \{\tau^\mu(\vec{x}), \mathcal{J}_\tau(\zeta)\} = \{\tau^\mu(\vec{x}), \vec{P}(\zeta)\} \\ &= \int_{\Sigma_\tau} \mathrm{d}^n x'_0 \, \zeta^k(\vec{x}')\tau^\nu_{,k}(\vec{x}')\{\tau^\mu(\vec{x}), P_\nu(\vec{x}')\} \\ &= \tau^\mu_{,k}(\vec{x})\zeta^k(\vec{x}) \ .\end{aligned} \tag{4.7}$$

In other words, being the embedding fixed, we are now "constraining" the lifted action of the diffeomorphism group on the space of embeddings in such a way that the induced spacetime diffeomorphism preserves the spatial hypersurface thus yielding an equivariant momentum map under spatial diffeomorphisms only and not under the full spacetime diffeomorphisms group. Accordingly, restricting ourselves to the subgroup of spatial diffeomorphisms, the covariant Gibbs state (3.13) reduces to

$$\begin{aligned}\rho^{(\mathrm{eq})}_{b_\tau}(\varphi, \Pi, \tau, P) &= \frac{1}{Z(b_\tau)} \exp\Big(-\langle b_\tau, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\Big) \\ &= \frac{1}{Z(\zeta)} \exp\Big(-\langle \zeta, \mathcal{J}_\tau(\varphi, \Pi, \tau, P)\rangle\Big) =: \rho^{(\mathrm{eq})}_\zeta(\varphi, \Pi, \tau, P) \ ,\end{aligned} \tag{4.8}$$

with

$$\begin{aligned}Z(b_\tau) &= \int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P] \exp\Big(-\langle b_\tau, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P)\rangle\Big) \\ &= \int_\Upsilon \mathcal{D}[\varphi, \Pi, \tau, P] \exp\Big(-\langle \zeta, \mathcal{J}_\tau(\varphi, \Pi, \tau, P)\rangle\Big) = Z(\zeta) \ ,\end{aligned} \tag{4.9}$$

and we limit ζ to the subset $\Omega_\tau \subset \mathfrak{g}_\tau = \mathsf{diff}(\Sigma_\tau)$ such that the above integral *converge*. Due to the equivariance of the momentum map $\mathcal{J}_\tau$ under the action (4.2), the Gibbs statistical state (4.8) is then an equilibrium state w.r.t. the one-parameter family of spatial diffeomorphisms generated by the vector field (4.4) associated to $\zeta \in \mathsf{diff}(\Sigma_\tau)$.

4.2 Time Evolution Gibbs State via Gauge Fixing

As already discussed, the momentum map (2.58) provides on the one hand a representation of the algebra of spacetime diffeomorphisms on the parametrized phase space of the field theory under consideration. On the other hand it also generates the displacement of the spatial hypersurface embedded in spacetime setting at the same time the correctly evolved Cauchy data for fields and the constraints on the deformed hypersurface. The corresponding one-parameter subgroup of diffeomorphisms associated with the Lie algebra element b thus identifies a generalized notion of "dynamical evolution" w.r.t. which the covariant Gibbs state (3.13) is an equilibrium state. By this we mean that the vector field $\xi_{(b)}$ associated with $b \in \mathsf{diff}(\mathcal{X})$ comes to be the generating vector field of a foliation of spacetime and the corresponding flow defines a generalized concept of time evolution.

As expected for generally covariant systems, dynamics and gauge symmetry are deeply intertwined so that no preferred notion of time is available and the Hamiltonian is a combination of constraints. This makes the connection between the off- and on-shell levels of our covariant statistical analysis quite subtle. In Sec. 3.3 we discussed the possibility of recovering the micro-canonical description out of the canonical ensemble defined by the generalized Gibbs state which reduces to the usual delta-like distribution in the thermodynamic limit. However, a still remaining open question is to understand if a standard "time evolution" equilibrium state can be defined on-shell on the constraint surface.

At first sight, due to the longstanding puzzle of gauge versus dynamics in generally covariant theories, it might seem difficult to extract the dynamical evolution of the physical fields as the Hamiltonian vanishes after imposing the constraints. This issue may be solved by introducing suitable gauge-fixing conditions such that the gauge evolution is frozen and the dynamics of fields can be disentangled. Indeed, as discussed in [65, 66], in reparametrization invariant systems such as relativistic particles, general relativity or any diffeomorphism covariant field theory, at least one of the gauge-fixing conditions must depend on the time variable. This makes the application of the Dirac algorithm of constraints and gauge-fixing more delicate. A detailed study of Hamiltonian formalism for systems with explicitly time-dependent second-class constraints has been carried out in [67, 68, 69] and reference within.

The key point is that in the case of time-dependent gauge-fixing constraints the correct dynamical evolution for physical fields on the reduced phase space is not generated by the restriction of the original Hamiltonian on the extended phase space – which for reparametrization-invariant theories vanishes by imposing the constraints – but rather by a new Hamiltonian which in general would differ from the starting one as determined by the time-dependent Hamilton's equations of motions written in terms of the brackets on the reduced phase space induced by the Dirac bracket. The explicit form of this Hamiltonian of course depends on the details of the gauge-fixing conditions.

Following the strategy of [67, 68, 69], let us specialize the main steps of the construction to the general setting of parametrized field theories we are interested in the present paper. Let then consider the parametrized phase space of our theory $\Upsilon = T^*\mathscr{Y}_\tau \times T^*\mathsf{Emb}_G(\Sigma, \mathcal{X})$ with canonical variables $(\varphi^A(\vec{x}), \Pi_A(\vec{x}), \tau^\mu(\vec{x}), P_\mu(\vec{x}))$, $A = 1, \dots, N$, $\mu = 0, \dots, n$, first-class constraints $\mathcal{H}_\mu = P_\mu + \mathcal{H}_\mu^{(\varphi)} \approx 0$ generating gauge symmetries, and Hamiltonian $H(\xi)$ given in (2.58). The canonical variables on Υ can be split into two disjoint sets (φ^A, Π_A) and (τ^μ, P_μ), the former being the physical field content of the theory (and their conjugate momenta) while the latter are the auxiliary field variables resulting from the parametrization procedure at the canonical level. These are the canonical variables we would like to eliminate by gauge-fixing. To this aim, let us introduce the gauge-fixing conditions of the form

$$\chi^\mu = \tau^\mu - F^\mu(\varphi^A, \lambda) \qquad , \qquad \mu = 0, \dots, n \tag{4.10}$$

where the F^μ are certain functions depending only on the configuration field variables φ^A as well as on the time parameter[19] such that the conditions $\chi^\mu = 0$ provide us with a complete set of gauge-fixing constraints. This means that the set of all constraints that we will collectively denote as $\psi_I = (\mathcal{H}_\mu, \chi^\nu)$ is now a second-class set (i.e., $\{\mathcal{H}_\mu, \chi^\nu\} \not\approx 0$), and the conditions $\chi^\mu = 0$ eliminate all the gauge freedom, namely

$$\{\chi^\mu, \int \mathrm{d}^n x_0\, \epsilon^\nu \mathcal{H}_\nu\} = 0 \quad , \quad \forall\lambda \quad \Rightarrow \quad \epsilon^\mu = 0\ . \tag{4.11}$$

The new set of constraints $\psi_I = (\mathcal{H}_\mu, \chi^\nu)$ identifies a reduced phase space $\overline{\Upsilon} \subset \Upsilon$ defined by $\psi_I = 0$. Stability of time dependent constraints amounts to require that

$$\frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda} = \frac{\partial\chi^\mu}{\partial\lambda} + \{\chi^\mu, H\} \approx 0\ , \tag{4.12}$$

where now the weak equality $\approx$ denotes equality up to terms which vanish on $\overline{\Upsilon}$. Requiring (4.12) to vanish no longer freezes the dynamics. Indeed, denoting by $(\bar{\varphi}^A, \bar{\Pi}_A)$ a set of canonical variables on $\overline{\Upsilon}$, i.e., certain smooth functions on Υ which provides a set of coordinates on $\overline{\Upsilon}$ when $\psi_I = 0$, to determine the restriction of the dynamical evolution to $\overline{\Upsilon} \subset \Upsilon$ amounts to seek for a Hamiltonian $\bar{H}$ such that an equation on Υ of the kind

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} \approx \frac{\partial f}{\partial\lambda} + \{f, \bar{H}\}_* \tag{4.13}$$

holds for any function $f(\bar{\varphi}^A, \bar{\Pi}_A, \lambda)$ of some chosen set of variables $(\bar{\varphi}^A, \bar{\Pi}_A)$. Here, $\{\cdot, \cdot\}_*$ denotes the Dirac bracket defined by

$$\{\cdot, \cdot\}_* := \{\cdot, \cdot\} - \{\cdot, \psi_I\}\mathcal{C}^{IJ}\{\psi_J, \cdot\} \tag{4.14}$$

[19] Here by time we mean the evolution parameter entering the canonical formulation of the theory.

with $\mathcal{C}^{IJ} = (\mathcal{C})^{-1}_{IJ}$ the inverse of the matrix $\mathcal{C}$ whose entries are given by the Poisson brackets of all constraints, say $\mathcal{C}_{IJ} = \{\psi_I, \psi_J\}$[20]. As by construction $\{f, \psi_I\}_* = 0$ holds strongly for any second class constraint ψ_I and any function f on Υ, all constraints (now second class) can be imposed strongly inside the bracket to eliminate the corresponding number of variables and obtain a well-defined induced bracket on $\overline{\Upsilon}$.

Therefore, the crucial point is to look for the Hamiltonian $\bar{H}$ which generates dynamical evolution equations written in terms of the Dirac bracket. The corresponding trajectories in Υ project onto trajectories in $\overline{\Upsilon}$ lying in $\overline{\Upsilon}$ for all time as implied by the stability of the constraints $\frac{\mathrm{d}\psi_I}{\mathrm{d}\lambda} \approx 0$. As discussed in [67, 68, 69], the Hamiltonian $\bar{H}$ is given by

$$\bar{H} = H - \int_\Sigma \mathrm{d}^n x_0 \, \frac{\partial F^\mu}{\partial \lambda} P_\mu \; , \tag{4.15}$$

i.e., in general, it would be different from just H restricted on $\overline{\Upsilon}$ which in the case of generally covariant systems under consideration would actually vanish by restricting on $\overline{\Upsilon}$ where $\psi_I = 0$ and hence it does not give the correct dynamics for the variables $(\bar{\varphi}^A, \bar{\Pi}_A)$. To show (4.15) let us notice that since the gauge-fixing conditions (4.10) commute (strongly) among themselves, there exists a canonical transformation $(\varphi^A, \Pi_A, \tau^\mu, P_\mu) \mapsto (\bar{\varphi}^A, \bar{\Pi}_A, \bar{Q}^\mu, \bar{P}_\mu)$ such that $\bar{Q}^\mu = \chi^\mu$. The generating function of such a transformation is given by

$$\mathfrak{F}(\varphi^A, \tau^\mu, \bar{\Pi}_A, \bar{P}_\mu, \lambda) = \varphi^A \bar{\Pi}_A + \Big(\tau^\mu - F^\mu(\varphi^A, \lambda)\Big) \bar{P}_\mu \; , \tag{4.16}$$

so that we have

$$\begin{cases} \bar{Q}^\mu = \frac{\partial \mathfrak{F}}{\partial \bar{P}_\mu} = \tau^\mu - F^\mu(\varphi^A, \lambda) = \chi^\mu \\ \bar{\varphi}^A = \frac{\partial \mathfrak{F}}{\partial \bar{\Pi}_A} = \varphi^A \\ P_\mu = \frac{\partial \mathfrak{F}}{\partial \tau^\mu} = \bar{P}_\mu \\ \Pi_A = \frac{\partial \mathfrak{F}}{\partial \varphi^A} = \bar{\Pi}_A - \frac{\partial F^\mu}{\partial \varphi^A} \bar{P}_\mu \end{cases} \tag{4.17}$$

and Hamiltonian given by

$$\bar{H} = H + \int_\Sigma \mathrm{d}^n x_0 \, \frac{\partial \mathfrak{F}}{\partial \lambda} = H - \int_\Sigma \mathrm{d}^n x_0 \, \frac{\partial F^\mu}{\partial \lambda} \bar{P}_\mu \; . \tag{4.18}$$

In the new variables the gauge-fixing conditions now are part of the field configuration variables and have no explicit λ-dependence anymore, while the constraints $\mathcal{H}_\mu$ will now become λ-dependent in general. Hence, stability of the gauge-fixing constraints simply amounts to require that

$$\dot{\bar{Q}}^\mu = \frac{\mathrm{d}\chi^\mu}{\mathrm{d}\lambda} = \{\chi^\mu, \bar{H}\} \approx 0 \; . \tag{4.19}$$

[20] As $\psi_I = (\mathcal{H}_\mu, \chi^\nu)$ is a second class set, the matrix $\mathcal{C}$ is non-degenerate that is $\det \|\{\psi_I, \psi_J\}\| \neq 0$.

As can be checked by direct computation, from the constraint algebra it follows that the matrix $\mathcal{C} = \|\{\psi_I, \psi_J\}\|$ takes the block form

$$\mathcal{C}_{IJ} = \left(\begin{array}{c|c} 0 & \mathcal{A}_{\mu\nu} \\ \hline -\mathcal{A}_{\mu\nu} & 0 \end{array}\right) \qquad , \qquad \mathcal{A}_{\mu\nu} = \{\mathcal{H}_\mu, \chi_\nu\} \tag{4.20}$$

so that for any function f on Υ the Dirac bracket yields

$$\begin{aligned} \{f, \bar{\mathcal{H}}\}_* &= \{f, \bar{\mathcal{H}}\} + \sum_{\mu,\nu} \Big(\{f, \mathcal{H}_\mu\}(\mathcal{A}^{-1})_{\mu\nu}\{\chi_\nu, \bar{\mathcal{H}}\} - \{f, \chi_\mu\}(\mathcal{A}^{-1})_{\mu\nu}\{\mathcal{H}_\nu, \bar{\mathcal{H}}\} \Big) \\ &\approx \{f, \bar{\mathcal{H}}\} - \sum_{\mu,\nu} \{f, \chi_\mu\}(\mathcal{A}^{-1})_{\mu\nu}\{\mathcal{H}_\nu, \bar{\mathcal{H}}\} \; , \end{aligned} \tag{4.21}$$

where in the last line we used Eq. (4.19). Now, as $\bar{Q}^\mu = \chi^\mu$, the second term on the r.h.s. of (4.21) vanishes if we restrict $f = f(\bar{\varphi}^A, \bar{\Pi}_A, \lambda)$. Thus, for any function $f(\bar{\varphi}^A, \bar{\Pi}_A, \lambda)$ we get

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\partial f}{\partial \lambda} + \{f, \bar{H}\} \approx \frac{\partial f}{\partial \lambda} + \{f, \bar{H}\}_* \; , \tag{4.22}$$

which is the desired form (4.13) of the dynamical evolution.

Let us then consider the case in which the functions F^μ do not depend on the fields φ^A but only on λ, say $\chi^\mu = \tau^\mu - F^\mu(\lambda)$[21]. According to the expressions (4.17), in this case the canonical variables on $\overline{\Upsilon}$ are just given by the physical fields and their conjugate momenta, i.e.:

$$\bar{\varphi}^A = \varphi^A \qquad , \qquad \bar{\Pi}_A = \Pi_A \; . \tag{4.23}$$

The constraints $\mathcal{H}_\mu = 0$ can be then solved to express the momenta $P_\mu = \bar{P}_\mu$ in terms of the variables $(\bar{\varphi}^A, \bar{\Pi}_A)$ yielding $P_\mu = -\mathcal{H}^{(\varphi)}_\mu$. The new Hamiltonian (4.15) then reads

$$\bar{H} = H - \int_\Sigma \mathrm{d}^n x_0 \, \dot{F}^\mu \bar{P}_\mu \; , \tag{4.24}$$

from which, restricting on $\overline{\Upsilon}$ and taking into account that $H = 0$ imposing the constraints, it follows that

$$\bar{H} = \int_\Sigma \mathrm{d}^n x_0 \, \dot{F}^\mu \mathcal{H}^{(\varphi)}_\mu \; . \tag{4.25}$$

Note that $\dot{F}^0$ and $\dot{\vec{F}}$ respectively play the role of lapse and shift consistently with $\xi^\mu(\tau(\vec{x})) = \dot{\tau}^\mu$ being the deformation vector field of the foliation which after gauge-fixing yields $\dot{\tau}^\mu = \dot{F}^\mu$. In particular, choosing the gauge

$$x^0 = \lambda \qquad , \qquad x^k = 0 \quad k = 1, \ldots, n \tag{4.26}$$

[21] A specific realization of this situation is provided for instance by the gauge choice $F^\mu = x^\mu$ in which the functions F^μ depend only on λ through the spacetime coordinates x^μ.

the new Hamiltonian (4.25) is nothing but the usual field Hamiltonian

$$\bar{H} = \int_\Sigma \mathrm{d}^n x_0 \, \mathcal{H}_0^{(\varphi)} = H^{(\varphi)} \ , \tag{4.27}$$

which as such generates the correct dynamics for the field variables (φ^A, Π_A) after gauge-fixing.

Finally, we can define a Gibbs state on $\overline{\Upsilon}$ given by

$$\rho_F^{(\mathrm{eq})}(\varphi, \Pi) = \frac{1}{Z(F)} \, e^{-\bar{H}(\varphi,\Pi)} = \frac{1}{Z(F)} \, \exp\left(- \int_\Sigma \mathrm{d}^n x_0 \, \dot{F}^\mu \mathcal{H}_\mu^{(\varphi)} \right) \ , \tag{4.28}$$

with

$$Z(F) = \int_{\overline{\Upsilon}} \mathcal{D}[\varphi, \Pi] \, e^{-\bar{H}(\varphi,\Pi)} \ . \tag{4.29}$$

This is an equilibrium state w.r.t. the dynamical evolution generated by the Hamiltonian $\bar{H}$. In particular, choosing the temporal gauge $F^0 = x^0$, $\vec{F} = \vec{0}$, we get the standard relativistic equilibrium state with temperature given by the inverse of the lapse.

4.3 On the Thermodynamic Characterization of Covariant Equilibrium

In this section we comment on the relation between the generalized covariant Gibbs state for parametrized field theories presented in this work and the thermodynamic characterization of covariant statistical equilibrium arising in previous investigations based on the thermal time hypothesis. Inspired by the Tomita-Takasaki theorem in algebraic quantum field theory [6], thermal time hypothesis [3, 5] states that timeless evolution is given by the modular one-parameter flow of automorphisms of the covariant space algebra of the system, induced by any modular thermal state in the algebra of the gauge invariant observables of the theory. The essence of thermal time thus relies on a fundamental reinterpretation of the relation between equilibrium states and time flow according to which any statistical state ρ is in equilibrium w.r.t. its own modular flow. The modular Hamiltonian $H = -\log \rho$ generates a one-parameter group of transformations which defines the time flow associated to the state ρ. In particular, physical equilibrium states are those whose thermal time identifies a flow in spacetime [7], thus providing a thermodynamical characterization of the notion of time experienced by an observer [3].

Similarly, for any Lie algebra element ξ the co-momentum map $H(\xi)$ defined in (2.58) can be thought of as the thermal modular Hamiltonian associated with the covariant Gibbs state (3.13), namely $H(\xi)(\varphi, \Pi, \tau, P) = -\langle \xi, \tilde{\mathcal{J}}_H(\varphi, \Pi, \tau, P) \rangle = \log \rho_\xi^{(\mathrm{eq})}(\varphi, \Pi, \tau, P)$. The corresponding thermal flow identifies a one-parameter group on $\Upsilon \equiv T^*\mathscr{Y}_\tau \times T^*\mathsf{Emb}_G(\Sigma, \mathcal{X})$ generated by the Hamiltonian vector field X_H satisfying

$$\rho_\xi^{(\mathrm{eq})} \mathrm{i}_{X_H} \tilde{\omega} = \mathrm{d}\rho_\xi^{(\mathrm{eq})} \qquad \text{or equivalently} \qquad \mathrm{i}_{X_H} \tilde{\omega} = \mathrm{d} \log \rho_\xi^{(\mathrm{eq})} = \mathrm{d}H(\xi) \ , \tag{4.30}$$

where $\tilde{\omega}$ is the symplectic structure on Υ given in (2.33) and, with a slight abuse of notation, we still denote by d the exterior differential on the extended parametrized phase space of fields although it should not be confused with that on spacetime. More explicitly, as already anticipated in Sec. 2.5, the equations of motion associated to the Hamiltonian $H(\xi)$ are given by

$$\dot{\varphi}^A(\vec{x}) = \{\varphi^A(\vec{x}), H(\xi)\} = \int_\Sigma \mathrm{d}^n x_0' \xi^\nu(\tau(\vec{x}\,')) \{\varphi^A(\vec{x}), \mathcal{H}_\nu^{(\varphi)}(\vec{x}\,')\} = \xi^\nu(\tau(\vec{x})) \frac{\delta \mathcal{H}_\nu^{(\varphi)}}{\delta \Pi_A} \,, \tag{4.31}$$

$$\dot{\Pi}_A(\vec{x}) = \{\Pi_A(\vec{x}), H(\xi)\} = \int_\Sigma \mathrm{d}^n x_0' \xi^\nu(\tau(\vec{x}\,')) \{\Pi_A(\vec{x}), \mathcal{H}_\nu^{(\varphi)}(\vec{x}\,')\} = -\xi^\nu(\tau(\vec{x})) \frac{\delta \mathcal{H}_\nu^{(\varphi)}}{\delta \varphi^A} \,, \tag{4.32}$$

$$\dot{\tau}^\mu(\vec{x}) = \{\tau^\mu(\vec{x}), H(\xi)\} = \int_\Sigma \mathrm{d}^n x_0' \xi^\nu(\tau(\vec{x}\,')) \{\tau^\mu(\vec{x}), P_\nu(\vec{x}\,')\} = \xi^\mu(\tau(\vec{x})) \,, \tag{4.33}$$

$$\dot{P}_\mu(\vec{x}) = \{P_\mu(\vec{x}), H(\xi)\} \approx \int_\Sigma \mathrm{d}^n x_0' \xi^\nu(\tau(\vec{x}\,')) \{P_\mu(\vec{x}), P_\nu(\vec{x}\,')\} = 0 \,, \tag{4.34}$$

so that the vector field

$$\begin{aligned} X_H &= \dot{\varphi}^A \frac{\delta}{\delta \varphi^A} + \dot{\tau}^\mu \frac{\delta}{\delta \tau^\mu} + \dot{\Pi}_A \frac{\delta}{\delta \Pi_A} + \dot{P}_\mu \frac{\delta}{\delta P_\mu} \\ &= \xi^\mu(\tau(\vec{x})) \left[\left(\frac{\delta \mathcal{H}_\mu^{(\varphi)}}{\delta \Pi_A} \right) \frac{\delta}{\delta \varphi^A} - \left(\frac{\delta \mathcal{H}_\mu^{(\varphi)}}{\delta \varphi^A} \right) \frac{\delta}{\delta \Pi_A} + \frac{\delta}{\delta \tau^\mu} \right] \,, \end{aligned} \tag{4.35}$$

generates the correct dynamical evolution for the fields as well as the constraints (see Eq. (2.61)). In particular, regarding Υ as a bundle over $\mathsf{Emb}(\Sigma, \mathcal{X})$ whose fiber above $\tau \in \mathsf{Emb}(\Sigma, \mathcal{X})$ is the field phase space $T^*\mathscr{Y}_\tau$ coordinatized by the spatial matter fields and their conjugate momenta, we see from (4.33) that the above vector field restricted to $\mathsf{Emb}(\Sigma, \mathcal{X})$ generates a one-parameter curve of embeddings $c : \mathbb{R} \to \mathsf{Emb}(\Sigma, \mathcal{X})$ by $c(\lambda) = \tau(\lambda)$ which in turn identifies a slicing $\mathfrak{s}_{\mathcal{X}}(\vec{x}, \lambda) = \tau(\lambda)(\vec{x})$ in spacetime generated by the vector field $\xi_{\mathcal{X}}(\tau(\vec{x}))$. The corresponding curve $c_\tau(\lambda) = (\varphi(\lambda), \Pi(\lambda))$ in $T^*\mathscr{Y}_\tau$ identifies then a compatible $\mathcal{G}$-slicing of the field space. Therefore, as schematically reported in Fig. 1, the thermal flow associated to the Gibbs state (3.13) with thermal time parameter λ defines a one-parameter group of bundle automorphisms as well as a compatible slicing of spacetime so that the Lie algebra-valued temperature ξ identifies a direction in spacetime along which geometry and matter fields evolve.

Moreover, along the line of [7], we can define a local temperature function, i.e. a map $T : \Upsilon \times \Sigma \mapsto \mathbb{R}$, given by

$$T((\varphi, \Pi, \tau, P), \vec{x}) = T(\varphi(\vec{x}), \Pi(\vec{x}), \tau(\vec{x}), P(\vec{x})) := |\xi(\tau(\vec{x}))|^{-1} \,, \tag{4.36}$$

with the vector field $\xi^\mu(\tau(\vec{x}))$ playing the role of *multi-fingered time*. Consistently, with the gauge choice $F^0 = x^0$ discussed in the previous section, the above temperature function reduces on-shell to the inverse of the lapse function, say $T = \frac{1}{\dot{x}^0}$, thus locally yielding the rate of change of the thermal time parameter λ and the coordinate time x^0.

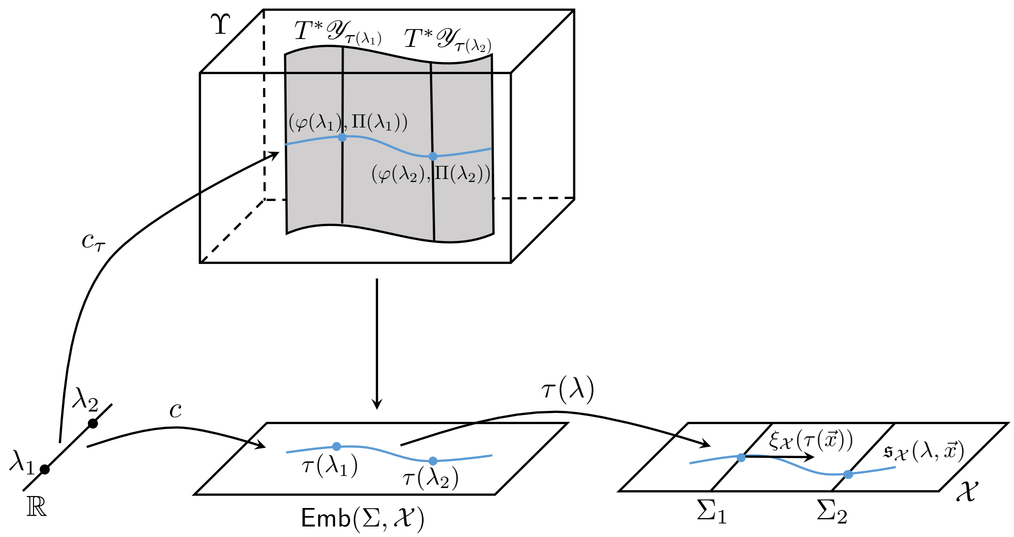

Figure 1: Slicing of spacetime and of the field bundles over it generated by the thermal flow associated to the Gibbs state (3.13). The Lie algebra-valued temperature ξ thus defines a generalized notion of direction of dynamical evolution characterized by a "thermal time parameter" λ.

Another similarity with thermal time hypothesis relies on the inbuilt observer dependence of the corresponding notion of equilibrium. In our case, in fact, the Lie algebra element ξ identifies a foliation and the associated one-parameter flow in spacetime characterizes a direction of evolution associated with the corresponding canonical observer, as can be seen in adapted coordinates to the foliation. Changing element in the Lie algebra would then correspond to change observer by picking up a different foliation identified by the corresponding one-parameter subgroup of diffeomorphisms. Two main differences can be pointed out however in comparing the above analysis with the framework of thermal time hypothesis. First of all, the generalized Gibbs state (3.13) is defined on the parametrized canonical phase space and not on the space of solutions of the field equations. Only after introducing suitable gauge choices the thermal flow associated with the gauge-fixed Hamiltonian generates the on-shell dynamics for the matter fields (see Sec. 4.2). In this sense, our framework can be thought of as an off-shell generalization of the previous setting which allows us to construct an equilibrium state w.r.t. (one-parameter subgroups of) spacetime diffeomorphisms which in turn identify a physical thermal flow in spacetime. Moreover, in the present setting the Lie algebra-valued temperature – which plays the role of a global temperature – is intrinsically related with the notion of local temperature as the corresponding Lie algebra element is realized as the deformation vector field of the spacetime foliation induced by the generating vector field of the $\mathcal{G}$-action on the parametrized phase space, where as expected the dependence on the embeddings plays a fundamental role in connecting the phase space and spacetime pictures.

5 Conclusions and Outlook

In this work, we consider the problem of defining statistical mechanics and thermodynamics for generally covariant field theories, where both dynamics and gauge symmetry are encoded in (first-class) constraints, hence no standard notions of time and energy are available. The key issue consists in the definition of statistical equilibrium, beyond time translations, for a symmetry group flow given by spacetime diffeomorphisms. Souriau's symplectic reformulation of statistical equilibrium for Hamiltonian Lie group actions provides a useful conceptual and formal setting in this sense, generalizing the standard notion of time-translation equilibrium.

A straightforward application of Souriau's Lie group thermodynamics to the case of a fully constrained system leads to define a Gibbs-like state with respect to the gauge group action generated by the first-class constraints. In this framework, the Hamiltonian Lie group action is characterized by the vanishing of the associated momentum map. The vanishing momentum map is equivalent to a first class constraint, reflecting the presence of a gauge symmetry for the system, and the constraint surface is identified with the zero-level set of the associated momentum map [70, 23, 26].

However, in generally covariant theories the super-Hamiltonian and super-momentum constraints do not close a genuine Lie algebra which reflects into the non-equivariance of the energy-momentum map preventing a straightforward representation of the algebra of the spacetime diffeomorphism group. The application of Jaynes' entropy maximization principle [20] with constant mean values of the first-class constraints necessarily leads to a Gibbs-like state which, due to the non-equivariance of the energy-momentum map, can not be an equilibrium state under the gauge flow generated by the constraints. In terms of the classification of different kinds of Gibbs states given in [71], this state would correspond to a *thermodynamical* rather than *dynamical* statistical state. In particular, it would not be of equilibrium with respect to a one-parameter group of spacetime diffeomorphisms, but at most only with respect to spatial diffeomorphisms for which there is a momentum map induced by the energy-momentum map restricted to the Lie algebra of spatial diffeomorphisms (cfr. Eqs. (2.54) and (4.5)).

We overcome this issue by *extending* our multi-symplectic phase space via the introduction of covariance fields. This allows us to recover the action of the spacetime diffeomorphism group by means of the observable Hamiltonian functional $H(\xi)$ defined in (2.58) which identifies a equivariant momentum map homomorphically relating the diffeomorphism algebra to the Poisson bracket algebra of functionals on the parametrized phase space. The derivation of a Gibbs state associated to this equivariant momentum map via the prescriptions of Lie group thermodynamics then provides us with a *dynamical* (in the sense of [71]) equilibrium state on the parametrized phase space. Such a state is of equilibrium with respect to the one-parameter group of diffeomorphisms generated by the vector field $\xi_{\mathcal{X}}$ associated to $\xi \in \mathfrak{g} = \mathsf{diff}(\mathcal{X})$. In this sense, it defines a spacetime covariant notion of thermodynamical equilibrium.

The gauge character of spacetime diffeomorphisms implies a radical conceptual shift in the definition of equilibrium state with respect to Souriau's work. By replacing a dynamical symmetry with a gauge one, we move our analysis from the fully reduced symplectic space of motions (on-shell) to the unconstrained extended phase space of the system (off-shell). While being defined off-shell, the covariant Gibbs state is by construction an observable of the theory, and it encodes, via the covariant momentum map functional, all the dynamical information carried by the given parametrized field theory: its canonical Hamiltonian, its initial value constraints, its gauge freedom, and its stress energy-momentum tensor [21]. Therefore, we expect the off-shell equilibrium to play a role similar to a generating functional partition function in field theory, with a Gibbs state corresponding to a "soft" imposition of the constraints of the theory. This might allow for a *probabilistic* approach to symplectic reduction, a further interesting perspective which we leave open for future investigation.

The proposed result points to a deep connection between geometrical methods, information theory and field theories. We expect our approach to open the road for a spacetime covariant formulation of statistical mechanics, possibly capable of describing the fluctuations of the gravitational field in a general relativistic context (see e.g. [17, 72]). From an information-geometric viewpoint, we further expect the derived covariant Gibbs state functional to provide a useful tool for exploring a statistical generalization of symplectic reduction in field theory, as well as a further support to the use of momentum map and Lie group formalism in the study of covariant generative models in machine learning [73].

As our original motivation is general relativity, the present analysis needs to be extended to gravity for which a covariant Hamiltonian formalism has been extensively studied in the last years, see e.g. [44, 30, 74, 75, 76, 77] and references within. However, the intrinsic parametrized nature of gravity [78] makes the achievement of a phase space representation of spacetime diffeomorphisms a non-trivial task. Useful insights in this respect may be found in the work by Isham and Kuchař [38].

A further interesting and closely related aspect to explore consists in including boundaries in our covariant description (see e.g. [79, 80, 81] for the analysis of boundaries in the framework of multi-symplectic field theories). The inclusion of boundaries would be needed to identify a

notion of subsystems which is crucial in any thermodynamical analysis. As remarked above, in this covariant setting the constraints are beautifully encoded in the momentum map associated with the gauge group action on the field space. The latter plays a key role in the construction of the covariant Gibbs state. As shown in [82], in presence of boundaries, the kinematical constraint algebra can be written as conservation laws for boundary charges. These charges would then enter the boundary momentum map out of which a statistical mechanic framework for the on-shell boundary modes can be constructed. This may open fruitful connections with recent work on boundary modes in quantum gravity [83]. Of particular interest would be the case in which the finite boundaries describe horizons, with potential application to black hole thermodynamics.
From a broader point of view, the application of our covariant statistical mechanics formalism to discrete gravity models may provide interesting insights to the study of coarse-graining approaches and the continuum limit from a thermodynamical perspective, as initiated for instance in [84, 85].

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