# Conjunction of Conditional Events and T-norms

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**Abstract.** We study the relationship between a notion of conjunction among conditional events, introduced in recent papers, and the notion of Frank t-norm. By examining different cases, in the setting of coherence, we show each time that the conjunction coincides with a suitable Frank t-norm. In particular, the conjunction may coincide with the Product t-norm, the Minimum t-norm, and Lukasiewicz t-norm. We show by a counterexample, that the prevision assessments obtained by Lukasiewicz t-norm may be not coherent. Then, we give some conditions of coherence when using Lukasiewicz t-norm.

**Keywords:** Coherence, Conditional Event, Conjunction, Frank t-norm.

## 1 Introduction

In this paper we use the coherence-based approach to probability of de Finetti ([1,2,7,9,10,13,14,16,17,18,22,25]). We use a notion of conjunction which, differently from other authors, is defined as a suitable conditional random quantity with values in the unit interval (see, e.g. [20,21,23,24,36]). We study the relationship between our notion of conjunction and the notion of Frank t-norm. For some aspects which relate probability and Frank t-norm see, e.g., [5,6,8,11,15,34]. We show that, under the hypothesis of logical independence, if the prevision assessments involved with the conjunction $(A|H) \wedge (B|K)$ of two conditional events are coherent, then the prevision of the conjunction coincides, for a suitable $\lambda \in [0, +\infty]$, with the Frank t-norm $T_\lambda(x, y)$, where $x = P(A|H), y = P(B|K)$. Moreover, $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. Then, we consider the case $A = B$, by determining the set of all coherent assessment $(x, y, z)$ on $\{A|H, A|K, (A|H) \wedge (A|K)\}$. We show that, under coherence, it holds that $(A|H) \wedge (A|K) = T_\lambda(A|H, A|K)$, where $\lambda \in [0, 1]$. We also study the particular case where $A = B$ and $HK = \varnothing$. Then, we consider conjunctions of three conditional events and we show that to make prevision assignments by means of the Product t-norm, or the Minimum t-norm, is coherent. Finally, we examine the Lukasiewicz t-norm and we show by a counterexample that coherence is in general not assured. We give some conditions for coherence when the prevision assessments are made by using the Lukasiewicz t-norm.

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## 2 Preliminary Notions and Results

In our approach, given two events $A$ and $H$, with $H \neq \emptyset$, the conditional event $A|H$ is looked at as a three-valued logical entity which is true, or false, or void, according to whether $AH$ is true, or $\bar{A}H$ is true, or $\bar{H}$ is true. We observe that the conditional probability and/or conditional prevision values are assessed in the setting of coherence-based probabilistic approach. In numerical terms $A|H$ assumes one of the values 1, or 0, or $x$, where $x = P(A|H)$ represents the assessed degree of belief on $A|H$. Then, $A|H = AH + x\bar{H}$. Given a family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, for each $i \in \{1, \ldots, n\}$ we denote by $\{x_{i1}, \ldots, x_{ir_i}\}$ the set of possible values of $X_i$ when $H_i$ is true; then, for each $i$ and $j = 1, \ldots, r_i$, we set $A_{ij} = (X_i = x_{ij})$. We set $C_0 = \bar{H}_1 \cdots \bar{H}_n$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $H_1 \vee \cdots \vee H_n$. Hence $\bigwedge_{i=1}^{n}(A_{i1} \vee \cdots \vee A_{ir_i} \vee \bar{H}_i) = \bigvee_{h=0}^{m} C_h$. With each $C_h$, $h \in \{1, \ldots, m\}$, we associate a vector $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hi} = x_{ij}$ if $C_h \subseteq A_{ij}$, $j = 1, \ldots, r_i$, while $q_{hi} = \mu_i$ if $C_h \subseteq \bar{H}_i$; with $C_0$ it is associated $Q_0 = \mathcal{M} = (\mu_1, \ldots, \mu_n)$. Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, the condition $\mathcal{M} \in \mathcal{I}$ amounts to the existence of a vector $(\lambda_1, \ldots, \lambda_m)$ such that: $\sum_{h=1}^{m} \lambda_h Q_h = \mathcal{M}$, $\sum_{h=1}^{m} \lambda_h = 1$, $\lambda_h \geqslant 0$, $\forall\, h$; in other words, $\mathcal{M} \in \mathcal{I}$ is equivalent to the solvability of the system $(\varSigma)$, associated with $(\mathcal{F}, \mathcal{M})$,

$$(\varSigma) \quad \sum_{h=1}^{m} \lambda_h q_{hi} = \mu_i\,,\ i \in \{1, \ldots, n\}\,, \sum_{h=1}^{m} \lambda_h = 1,\ \ \lambda_h \geqslant 0\,,\ h \in \{1, \ldots, m\}\,. \tag{1}$$

Given the assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$, let $S$ be the set of solutions $\varLambda = (\lambda_1, \ldots, \lambda_m)$ of system $(\varSigma)$. We point out that the solvability of system $(\varSigma)$ is a necessary (but not sufficient) condition for coherence of $\mathcal{M}$ on $\mathcal{F}$. When $(\varSigma)$ is solvable, that is $S \neq \emptyset$, we define:

$$I_0 = \{i : \max_{\varLambda \in S} \sum_{h: C_h \subseteq H_i} \lambda_h = 0\},\ \mathcal{F}_0 = \{X_i|H_i\,, i \in I_0\},\ \ \mathcal{M}_0 = (\mu_i,\, i \in I_0)\,. \tag{2}$$

For what concerns the probabilistic meaning of $I_0$, it holds that $i \in I_0$ if and only if the (unique) coherent extension of $\mathcal{M}$ to $H_i|(\bigvee_{j=1}^{n} H_j)$ is zero. Then, the following theorem can be proved ([3, Theorem 3])

**Theorem 1.** [*Operative characterization of coherence*] A conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ is coherent if and only if the following conditions are satisfied:
(i) the system $(\varSigma)$ defined in (1) is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{M}_0$ is coherent.

Coherence can be related to proper scoring rules ([4,19,30,31,32]).

**Definition 1.** *Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, their conjunction is the conditional random quantity $(A|H) \wedge (B|K)$, with $\mathbb{P}[(A|H) \wedge (B|K)] = z$, defined as*

$$(A|H) \wedge (B|K) = \begin{cases} 1, \text{ if } AHBK \text{ is true,} \\ 0, \text{ if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, \text{ if } \bar{H}BK \text{ is true,} \\ y, \text{ if } AH\bar{K} \text{ is true,} \\ z, \text{ if } \bar{H}\,\bar{K} \text{ is true.} \end{cases} \tag{3}$$

In betting terms, the prevision $z$ represents the amount you agree to pay, with the proviso that you will receive the quantity $(A|H) \wedge (B|K)$. Different approaches to compounded conditionals, not based on coherence, have been developed by other authors (see, e.g., [27,33]). We recall a result which shows that Fréchet-Hoeffding bounds still hold for the conjunction of conditional events ([23, Theorem 7]).

**Theorem 2.** Given any coherent assessment $(x, y)$ on $\{A|H, B|K\}$, with $A, H, B, K$ logically independent, $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:

$$\max\{x + y - 1, 0\} = z' \leqslant z \leqslant z'' = \min\{x, y\}\,. \tag{4}$$

*Remark 1.* From Theorem 2, as the assessment $(x, y)$ on $\{A|H, B|K\}$ is coherent for every $(x, y) \in [0, 1]^2$, the set $\Pi$ of coherent assessments $(x, y, z)$ on $\{A|H, B|K, (A|H) \wedge (B|K)\}$ is

$$\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, \max\{x + y - 1, 0\} \leqslant z \leqslant \min\{x, y\}\}. \tag{5}$$

The set $\Pi$ is the tetrahedron with vertices the points $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 0)$. For other definition of conjunctions, where the conjunction is a conditional event, some results on lower and upper bounds have been given in [35].

**Definition 2.** *Let be given $n$ conditional events $E_1|H_1, \ldots, E_n|H_n$. For each subset $S$, with $\emptyset \neq S \subseteq \{1, \ldots, n\}$, let $x_S$ be a prevision assessment on $\bigwedge_{i\in S}(E_i|H_i)$. The conjunction $\mathcal{C}_{1\cdots n} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ is defined as*

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^{n} E_iH_i, \text{ is true} \\ 0, & \text{if } \bigvee_{i=1}^{n} \bar{E}_iH_i, \text{ is true}, \\ x_S, & \text{if } \bigwedge_{i\in S} \bar{H}_i \bigwedge_{i\notin S} E_iH_i \text{ is true}, \ \emptyset \neq S \subseteq \{1, 2 \ldots, n\}. \end{cases} \tag{6}$$

In particular, $\mathcal{C}_1 = E_1|H_1$; moreover, for $\mathcal{S} = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, the conjunction $\bigwedge_{i\in S}(E_i|H_i)$ is denoted by $\mathcal{C}_{i_1\cdots i_k}$ and $x_{\mathcal{S}}$ is also denoted by $x_{i_1\cdots i_k}$. Moreover, if $\mathcal{S} = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, the conjunction $\bigwedge_{i\in S}(E_i|H_i)$ is denoted by $\mathcal{C}_{i_1\cdots i_k}$ and $x_{\mathcal{S}}$ is also denoted by $x_{i_1\cdots i_k}$. In the betting framework, you agree to pay $x_{1\cdots n} = \mathbb{P}(\mathcal{C}_{1\cdots n})$ with the proviso that you will receive: 1, if all conditional events are true; 0, if at least one of the conditional events is false; the prevision of the conjunction of that conditional events which are void, otherwise. The operation of conjunction is associative and commutative. We observe that, based on Definition 2, when $n = 3$ we obtain

$$\mathcal{C}_{123} = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true}, \\ 0, & \text{if } \bar{E}_1H_1 \vee \bar{E}_2H_2 \vee \bar{E}_3H_3 \text{ is true}, \\ x_1, & \text{if } \bar{H}_1E_2H_2E_3H_3 \text{ is true}, \\ x_2, & \text{if } \bar{H}_2E_1H_1E_3H_3 \text{ is true}, \\ x_3, & \text{if } \bar{H}_3E_1H_1E_2H_2 \text{ is true}, \\ x_{12}, & \text{if } \bar{H}_1\bar{H}_2E_3H_3 \text{ is true}, \\ x_{13}, & \text{if } \bar{H}_1\bar{H}_3E_2H_2 \text{ is true}, \\ x_{23}, & \text{if } \bar{H}_2\bar{H}_3E_1H_1 \text{ is true}, \\ x_{123}, & \text{if } \bar{H}_1\bar{H}_2\bar{H}_3 \text{ is true}. \end{cases} \tag{7}$$

We recall the following result ([24, Theorem 15]).

**Theorem 3.** *Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. Then, the set $\Pi$ of all coherent assessments $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is the set of points $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ which satisfy the following conditions*

$$\begin{cases} (x_1, x_2, x_3) \in [0,1]^3, \\ \max\{x_1 + x_2 - 1, x_{13} + x_{23} - x_3, 0\} \leqslant x_{12} \leqslant \min\{x_1, x_2\}, \\ \max\{x_1 + x_3 - 1, x_{12} + x_{23} - x_2, 0\} \leqslant x_{13} \leqslant \min\{x_1, x_3\}, \\ \max\{x_2 + x_3 - 1, x_{12} + x_{13} - x_1, 0\} \leqslant x_{23} \leqslant \min\{x_2, x_3\}, \\ 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23} \geqslant 0, \\ x_{123} \geqslant \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x_{123} \leqslant \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{cases} \tag{8}$$

*Remark 2.* As shown in (8), the coherence of $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ amounts to the condition

$$\begin{aligned} &\max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\} \leqslant x_{123} \leqslant \\ &\leqslant \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{aligned} \tag{9}$$

Then, in particular, the extension $x_{123}$ on $\mathcal{C}_{123}$ is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where $x'_{123} = \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}$, $x''_{123} = \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}$.

Then, by Theorem 3 it follows [24, Corollary 1]

**Corollary 1.** *For any coherent assessment $(x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$ on $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$ the extension $x_{123}$ on $\mathcal{C}_{123}$ is coherent if and only if $x_{123} \in [x'_{123}, x''_{123}]$, where*

$$\begin{aligned} x'_{123} &= \max\{0, x_{12} + x_{13} - x_1, x_{12} + x_{23} - x_2, x_{13} + x_{23} - x_3\}, \\ x''_{123} &= \min\{x_{12}, x_{13}, x_{23}, 1 - x_1 - x_2 - x_3 + x_{12} + x_{13} + x_{23}\}. \end{aligned} \tag{10}$$

We recall that in case of logical dependencies, the set of all coherent assessments may be smaller than that one associated with the case of logical independence. However (see [24, Theorem 16]) the set of coherent assessments is the same when $H_1 = H_2 = H_3 = H$ (where possibly $H = \Omega$; see also [26, p. 232]) and a corollary similar to Corollary 1 also holds in this case. For a similar result based on copulas see [12].

## 3 Representation by Frank t-norms for $(A|H) \wedge (B|K)$

We recall that for every $\lambda \in [0, +\infty]$ the Frank t-norm $T_\lambda : [0,1]^2 \to [0,1]$ with parameter $\lambda$ is defined as

$$T_\lambda(u,v) = \begin{cases} T_M(u,v) = \min\{u,v\}, & \text{if } \lambda = 0, \\ T_P(u,v) = uv, & \text{if } \lambda = 1, \\ T_L(u,v) = \max\{u+v-1, 0\}, & \text{if } \lambda = +\infty, \\ \log_\lambda(1 + \frac{(\lambda^u - 1)(\lambda^v - 1)}{\lambda - 1}), & \text{otherwise.} \end{cases} \tag{11}$$

We recall that $T_\lambda$ is continuous with respect to $\lambda$; moreover, for every $\lambda \in [0,+\infty]$, it holds that $T_L(u,v) \leqslant T_\lambda(u,v) \leqslant T_M(u,v)$, for every $(u,v) \in [0,1]^2$ (see, e.g., [28],[29]). In the next result we study the relation between our notion of conjunction and t-norms.

**Theorem 4.** *Let us consider the conjunction $(A|H) \wedge (B|K)$, with $A,B,H,K$ logically independent and with $P(A|H) = x$, $P(B|K) = y$. Moreover, given any $\lambda \in [0,+\infty]$, let $T_\lambda$ be the Frank t-norm with parameter $\lambda$. Then, the assessment $z = T_\lambda(x,y)$ on $(A|H) \wedge (B|K)$ is a coherent extension of $(x,y)$ on $\{A|H, B|K\}$; moreover $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. Conversely, given any coherent extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ of $(x,y)$, there exists $\lambda \in [0,+\infty]$ such that $z = T_\lambda(x,y)$.*

*Proof.* We observe that from Theorem 2, for any given $\lambda$, the assessment $z = T_\lambda(x,y)$ is a coherent extension of $(x,y)$ on $\{A|H, B|K\}$. Moreover, from (11) it holds that $T_\lambda(1,1) = 1$, $T_\lambda(u,0) = T_\lambda(0,v) = 0$, $T_\lambda(u,1) = u$, $T_\lambda(1,v) = v$. Hence,

$$T_\lambda(A|H, B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \text{ is true or } \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } \bar{K}AH \text{ is true,} \\ T_\lambda(x,y), & \text{if } \bar{H}\,\bar{K} \text{ is true,} \end{cases} \tag{12}$$

and, if we choose $z = T_\lambda(x,y)$, from (3) and (12) it follows that $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$.
Conversely, given any coherent extension $z$ of $(x,y)$, there exists $\lambda$ such that $z = T_\lambda(x,y)$. Indeed, if $z = \min\{x,y\}$, then $\lambda = 0$; if $z = \max\{x+y-1, 0\}$, then $\lambda = +\infty$; if $\max\{x+y-1,0\} < z < \min\{x,y\}$, then by continuity of $T_\lambda$ with respect to $\lambda$ it holds that $z = T_\lambda(x,y)$ for some $\lambda \in ]0,\infty[$ (for instance, if $z = xy$, then $z = T_1(x,y)$) and hence $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. □

*Remark 3.* As we can see from (3) and Theorem 4, in case of logically independent events, if the assessed values $x,y,z$ are such that $z = T_\lambda(x,y)$ for a given $\lambda$, then the conjunction $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. For instance, if $z = T_1(x,y) = xy$, then $(A|H) \wedge (B|K) = T_1(A|H, B|K) = (A|H) \cdot (B|K)$. Conversely, if $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$ for a given $\lambda$, then $z = T_\lambda(x,y)$. Then, the set $\Pi$ given in (5) can be written as $\Pi = \{(x,y,z) : (x,y) \in [0,1]^2, z = T_\lambda(x,y), \lambda \in [0,+\infty]\}$.

## 4 Conjunction of $(A|H)$ and $(A|K)$

In this section we examine the conjunction of two conditional events in the particular case when $A = B$, that is $(A|H) \wedge (A|K)$. By setting $P(A|H) = x$, $P(A|K) = y$ and $\mathbb{P}[(A|H) \wedge (A|K)] = z$, it holds that

$$(A|H) \wedge (A|K) = AHK + x\bar{H}AK + y\bar{K}AH + z\bar{H}\,\bar{K} \in \{1,0,x,y,z\}.$$

**Theorem 5.** *Let $A, H, K$ be three logically independent events, with $H \neq \emptyset$, $K \neq \emptyset$. The set $\Pi$ of all coherent assessments $(x, y, z)$ on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is given by*

$$\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, T_P(x, y) = xy \leqslant z \leqslant \min\{x, y\} = T_M(x, y)\}. \tag{13}$$

*Proof.* Let $\mathcal{M} = (x, y, z)$ be a prevision assessment on $\mathcal{F}$. The constituents associated with the pair $(\mathcal{F}, \mathcal{M})$ and contained in $H \vee K$ are: $C_1 = AHK$, $C_2 = \bar{A}HK$, $C_3 = \bar{A}\bar{H}K$, $C_4 = \bar{A}H\bar{K}$, $C_5 = A\bar{H}K$, $C_6 = AH\bar{K}$. The associated points $Q_h$'s are $Q_1 = (1, 1, 1), Q_2 = (0, 0, 0), Q_3 = (x, 0, 0), Q_4 = (0, y, 0), Q_5 = (x, 1, x), Q_6 = (1, y, y)$. With the further constituent $C_0 = \bar{H}\bar{K}$ it is associated the point $Q_0 = \mathcal{M} = (x, y, z)$. Considering the convex hull $\mathcal{I}$ (see Figure 1) of $Q_1, \ldots, Q_6$, a necessary condition for the coherence of the prevision assessment $\mathcal{M} = (x, y, z)$ on $\mathcal{F}$ is that $\mathcal{M} \in \mathcal{I}$, that is the following system must be solvable

$$(\Sigma) \begin{cases} \lambda_1 + x\lambda_3 + x\lambda_5 + \lambda_6 = x, \;\; \lambda_1 + y\lambda_4 + \lambda_5 + y\lambda_6 = y, \;\; \lambda_1 + x\lambda_5 + y\lambda_6 = z, \\ \sum_{h=1}^{6} \lambda_h = 1, \;\; \lambda_h \geqslant 0, \; h = 1, \ldots, 6. \end{cases}$$

First of all, we observe that solvability of $(\Sigma)$ requires that $z \leqslant x$ and $z \leqslant y$, that is $z \leqslant \min\{x, y\}$. We now verify that $(x, y, z)$, with $(x, y) \in [0, 1]^2$ and $z = \min\{x, y\}$, is coherent. We distinguish two cases: $(i)$ $x \leqslant y$ and $(ii)$ $x > y$.
Case $(i)$. In this case $z = \min\{x, y\} = x$. If $y = 0$ the system $(\Sigma)$ becomes

$$\lambda_1 + \lambda_6 = 0, \;\; \lambda_1 + \lambda_5 = 0, \;\; \lambda_1 = 0, \; \lambda_2 + \lambda_3 + \lambda_4 = 1, \;\; \lambda_h \geqslant 0, \;\; h = 1, \ldots, 6.$$

which is clearly solvable. In particular there exist solutions with $\lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$, by Theorem 1, as the set $I_0$ is empty the solvability of $(\Sigma)$ is sufficient for coherence of the assessment $(0, 0, 0)$. If $y > 0$ the system $(\Sigma)$ is solvable and a solution is $\Lambda = (\lambda_1, \ldots, \lambda_6) = (x, \frac{x(1-y)}{y}, 0, \frac{y-x}{y}, 0, 0)$. We observe that, if $x > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $\mathcal{C}_1 = HK \subseteq H \vee K$, so that $\mathcal{M} = (x, y, x)$ is coherent. If $x = 0$ (and hence $z = 0$), then $\lambda_4 = 1$ and $I_0 \subseteq \{2\}$. Then, as the sub-assessment $P(A|K) = y$ is coherent, it follows that the assessment $\mathcal{M} = (0, y, 0)$ is coherent too.
Case $(ii)$. The system is solvable and a solution is $\Lambda = (\lambda_1, \ldots, \lambda_6) = (y, \frac{y(1-x)}{x}, \frac{x-y}{x}, 0, 0, 0)$. We observe that, if $y > 0$, then $\lambda_1 > 0$ and $I_0 = \emptyset$ because $\mathcal{C}_1 = HK \subseteq H \vee K$, so that $\mathcal{M} = (x, y, y)$ is coherent. If $y = 0$ (and hence $z = 0$), then $\lambda_3 = 1$ and $I_0 \subseteq \{1\}$. Then, as the sub-assessment $P(A|H) = x$ is coherent, it follows that the assessment $\mathcal{M} = (x, 0, 0)$ is coherent too. Thus, for every $(x, y) \in [0, 1]^2$, the assessment $(x, y, \min\{x, y\})$ is coherent and, as $z \leqslant \min\{x, y\}$, the upper bound on $z$ is $\min\{x, y\} = T_M(x, y)$.
We now verify that $(x, y, xy)$, with $(x, y) \in [0, 1]^2$ is coherent; moreover we will show that $(x, y, z)$, with $z < xy$, is not coherent, in other words the lower bound for $z$ is $xy$. First of all, we observe that $\mathcal{M} = (1-x)Q_4 + xQ_6$, so that a solution of $(\Sigma)$ is $\Lambda_1 = (0, 0, 0, 1-x, 0, x)$. Moreover, $\mathcal{M} = (1-y)Q_3 + yQ_5$, so that another solution is $\Lambda_2 = (0, 0, 1-y, 0, y, 0)$. Then $\Lambda = \frac{\Lambda_1 + \Lambda_2}{2} = (0, 0, \frac{1-y}{2}, \frac{1-x}{2}, \frac{y}{2}, \frac{x}{2})$

is a solution of $(\Sigma)$ such that $I_0 = \emptyset$. Thus the assessment $(x, y, xy)$ is coherent for every $(x, y) \in [0,1]^2$. In order to verify that $xy$ is the lower bound on $z$ we observe that the points $Q_3, Q_4, Q_5, Q_6$ belong to a plane $\pi$ of equation: $yX + xY - Z = xy$, where $X, Y, Z$ are the axis' coordinates. Now, by considering the function $f(X,Y,Z) = yX + xY - Z$, we observe that for each constant $k$ the equation $f(X,Y,Z) = k$ represents a plane which is parallel to $\pi$ and coincides with $\pi$ when $k = xy$. We also observe that $f(Q_1) = f(1,1,1) = x + y - 1 = T_L(x,y) \leqslant xy = T_P(x,y)$, $f(Q_2) = f(0,0,0) = 0 \leqslant xy = T_P(x,y)$, and $f(Q_3) = f(Q_4) = f(Q_5) = f(Q_6) = xy = T_P(x,y)$. Then, for every $\mathcal{P} = \sum_{h=1}^{6} \lambda_h Q_h$, with $\lambda_h \geqslant 0$ and $\sum_{h=1}^{6} \lambda_h = 1$, that is $\mathcal{P} \in \mathcal{I}$, it holds that $f(\mathcal{P}) = f\big(\sum_{h=1}^{6} \lambda_h Q_h\big) = \sum_{h=1}^{6} \lambda_h f(Q_h) \leqslant xy$. On the other hand, given any $a > 0$, by considering $\mathcal{P} = (x, y, xy - a)$ it holds that $f(\mathcal{P}) = f(x, y, xy - a) = xy + xy - xy + a = xy + a > xy$. Therefore, for any given $a > 0$ the assessment $(x, y, xy - a)$ is not coherent because $(x, y, xy - a) \notin \mathcal{I}$. Then, the lower bound on $z$ is $xy = T_P(x,y)$. Finally, the set of all coherent assessments $(x, y, z)$ on $\mathcal{F}$ is the set $\Pi$ in (13). □

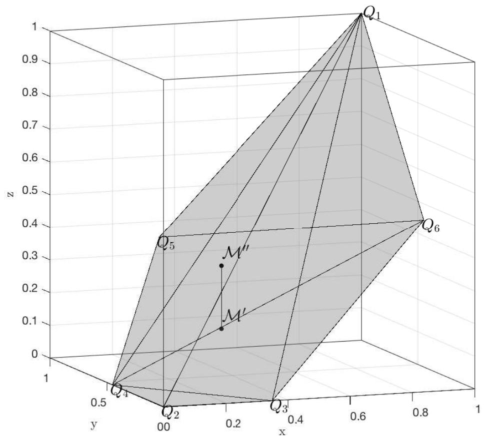


**Fig. 1.** Convex hull $\mathcal{I}$ of the points $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$. $\mathcal{M}' = (x, y, z'), \mathcal{M}'' = (x, y, z'')$, where $(x,y) \in [0,1]^2$, $z' = xy$, $z'' = \min\{x, y\}$. In the figure the numerical values are: $x = 0.35$, $y = 0.45$, $z' = 0.1575$, and $z'' = 0.35$.

Based on Theorem 5, we can give an analogous version for the Theorem 4 (when $A = B$).

**Theorem 6.** *Let us consider the conjunction $(A|H) \wedge (A|K)$, with $A, H, K$ logically independent and with $P(A|H) = x$, $P(A|K) = y$. Moreover, given any $\lambda \in [0,1]$, let $T_\lambda$ be the Frank t-norm with parameter $\lambda$. Then, the assessment $z = T_\lambda(x,y)$ on $(A|H) \wedge (A|K)$ is a coherent extension of $(x,y)$ on $\{A|H, A|K\}$; moreover $(A|H) \wedge (A|K) = T_\lambda(A|H, A|K)$. Conversely, given any coherent extension $z = \mathbb{P}[(A|H) \wedge (A|K)]$ of $(x,y)$, there exists $\lambda \in [0,1]$ such that $z = T_\lambda(x,y)$.*

The next result follows from Theorem 5 when $H$, $K$ are incompatible.

**Theorem 7.** *Let $A, H, K$ be three events, with $A$ logically independent from both $H$ and $K$, with $H \neq \emptyset$, $K \neq \emptyset$, $HK = \emptyset$. The set $\Pi$ of all coherent assessments $(x, y, z)$ on the family $\mathcal{F} = \{A|H, A|K, (A|H) \wedge (A|K)\}$ is given by $\Pi = \{(x, y, z) : (x, y) \in [0, 1]^2, z = xy = T_P(x, y)\}$.*

*Proof.* We observe that

$$(A|H) \wedge (A|K) = \begin{cases} 0, & \text{if } \bar{A}\bar{H}K \vee \bar{A}H\bar{K} \text{ is true,} \\ x, & \text{if } \bar{H}AK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true.} \end{cases}$$

Moreover, as $HK = \emptyset$, the points $Q_h$'s are $(x, 0, 0), (0, y, 0), (x, 1, x), (1, y, y)$, which coincide with the points $Q_3, \ldots, Q_6$ of the case $HK \neq \emptyset$. Then, as shown in the proof of Theorem 5, the condition $\mathcal{M} = (x, y, z)$ belongs to the convex hull of $(x, 0, 0), (0, y, 0), (x, 1, x), (1, y, y)$ amounts to the condition $z = xy$. □

*Remark 4.* From Theorem 7, when $HK = \emptyset$ it holds that $(A|H) \wedge (A|K) = (A|H) \cdot (A|K) = T_P(A|H, A|K)$, where $x = P(A|H)$ and $y = P(A|K)$.

## 5 Further Results on Frank t-norms

In this section we give some results which concern Frank t-norms and the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$. We recall that, given any t-norm $T(x_1, x_2)$ it holds that $T(x_1, x_2, x_3) = T(T(x_1, x_2), x_3)$.

### 5.1 On the Product t-norm

**Theorem 8.** *Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. If the assessment $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is such that $(x_1, x_2, x_3) \in [0, 1]^3$, $x_{ij} = T_1(x_i, x_j) = x_i x_j$, $i \neq j$, and $x_{123} = T_1(x_1, x_2, x_3) = x_1 x_2 x_3$, then $\mathcal{M}$ is coherent. Moreover, $\mathcal{C}_{ij} = T_1(\mathcal{C}_i, \mathcal{C}_j) = \mathcal{C}_i \mathcal{C}_j$, $i \neq j$, and $\mathcal{C}_{123} = T_1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3$.*

*Proof.* From Remark 2, the coherence of $\mathcal{M}$ amounts to the inequalities in (9). As $x_{ij} = T_1(x_i, x_j) = x_i x_j$, $i \neq j$, and $x_{123} = T_1(x_1, x_2, x_3) = x_1 x_2 x_3$, the inequalities (9) become

$$\begin{aligned} &\max\{0, x_1(x_2 + x_3 - 1), x_2(x_1 + x_3 - 1), x_3(x_1 + x_2 - 1)\} \leqslant x_1 x_2 x_3 \leqslant \\ &\leqslant \min\{x_1 x_2, x_1 x_3, x_2 x_3, (1 - x_1)(1 - x_2)(1 - x_3) + x_1 x_2 x_3\}. \end{aligned} \tag{14}$$

Thus, by recalling that $x_i + x_j - 1 \leqslant x_i x_j$, the inequalities are satisfied and hence $\mathcal{M}$ is coherent. Moreover, from (3) and (7) it follows that $\mathcal{C}_{ij} = T_1(\mathcal{C}_i, \mathcal{C}_j) = \mathcal{C}_i \mathcal{C}_j$, $i \neq j$, and $\mathcal{C}_{123} = T_1(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3$. □

### 5.2 On the Minimum t-norm

**Theorem 9.** *Assume that the events $E_1, E_2, E_3, H_1, H_2, H_3$ are logically independent, with $H_1 \neq \emptyset, H_2 \neq \emptyset, H_3 \neq \emptyset$. If the assessment $\mathcal{M} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123})$ on $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$ is such that $(x_1, x_2, x_3) \in [0,1]^3$, $x_{ij} = T_M(x_i, x_j) = \min\{x_i, x_j\}$, $i \neq j$, and $x_{123} = T_M(x_1, x_2, x_3) = \min\{x_1, x_2, x_3\}$, then $\mathcal{M}$ is coherent. Moreover, $\mathcal{C}_{ij} = T_M(\mathcal{C}_i, \mathcal{C}_j) = \min\{\mathcal{C}_i, \mathcal{C}_j\}$, $i \neq j$, and $\mathcal{C}_{123} = T_M(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \min\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$.*

*Proof.* From Remark 2, the coherence of $\mathcal{M}$ amounts to the inequalities in (9). Without loss of generality, we assume that $x_1 \leqslant x_2 \leqslant x_3$. Then $x_{12} = T_M(x_1, x_2) = x_1$, $x_{13} = T_M(x_1, x_3) = x_1$, $x_{23} = T_M(x_2, x_3) = x_2$, and $x_{123} = T_M(x_1, x_2, x_3) = x_1$. The inequalities (9) become

$$\max\{0, x_1, x_1 + x_2 - x_3\} = x_1 \ \leqslant\ x_1 \ \leqslant x_1 = \min\{x_1, x_2, 1 - x_3 + x_1\}. \quad (15)$$

Thus, the inequalities are satisfied and hence $\mathcal{M}$ is coherent. Moreover, from (3) and (7) it follows that $\mathcal{C}_{ij} = T_M(\mathcal{C}_i, \mathcal{C}_j) = \min\{\mathcal{C}_i, \mathcal{C}_j\}$, $i \neq j$, and $\mathcal{C}_{123} = T_M(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = \min\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$. □

*Remark 5.* As we can see from (15) and Corollary 1, the assessment $x_{123} = \min\{x_1, x_2, x_3\}$ is the unique coherent extension on $\mathcal{C}_{123}$ of the assessment $(x_1, x_2, x_3, \min\{x_1, x_2\}, \min\{x_1, x_3\}, \min\{x_2, x_3\})$ on $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$. We also notice that, if $\mathcal{C}_1 \leqslant \mathcal{C}_2 \leqslant \mathcal{C}_3$, then $\mathcal{C}_{12} = \mathcal{C}_1$, $\mathcal{C}_{13} = \mathcal{C}_1$, $\mathcal{C}_{23} = \mathcal{C}_2$, and $\mathcal{C}_{123} = \mathcal{C}_1$. Moreover, $x_{12} = x_1$, $x_{13} = x_1$, $x_{23} = x_2$, and $x_{123} = x_1$.

### 5.3 On Lukasiewicz t-norm

We observe that in general the results of Theorems 8 and 9 do not hold for the Lukasiewicz t-norm (and hence for any given Frank t-norm), as shown in the example below. We recall that $T_L(x_1, x_2, x_3) = \max\{x_1 + x_2 + x_3 - 2, 0\}$.

*Example 1.* The assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3), T_L(x_1, x_2, x_3))$ on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}, \mathcal{C}_{123}\}$, with $(x_1, x_2, x_3) = (0.5, 0.6, 0.7)$ is not coherent. Indeed, by observing that $T_L(x_1, x_2) = 0.1$ $T_L(x_1, x_3) = 0.2$, $T_L(x_2, x_3) = 0.3$, and $T_L(x_1, x_2, x_3) = 0$, formula (9) becomes $\max\{0, 0.1 + 0.2 - 0.5, 0.1 + 0.3 - 0.6, 0.2 + 0.3 - 0.7\} \leqslant 0 \leqslant \min\{0.1, 0.2, 0.3, 1 - 0.5 - 0.6 - 0.7 + 0.1 + 0.2 + 0.3\}$, that is: $\max\{0, -0.2\} \leqslant 0 \leqslant \min\{0.1, 0.2, 0.3, -0.2\}$; thus the inequalities are not satisfied and the assessment is not coherent.

More in general we have

**Theorem 10.** *The assessment $(x_1, x_2, x_3, T_L(x_1, x_2), T_L(x_1, x_3), T_L(x_2, x_3))$ on the family $\mathcal{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_{12}, \mathcal{C}_{13}, \mathcal{C}_{23}\}$, with $T_L(x_1, x_2) > 0$, $T_L(x_1, x_3) > 0$, $T_L(x_2, x_3) > 0$ is coherent if and only if $x_1 + x_2 + x_3 - 2 \geqslant 0$. Moreover, when $x_1 + x_2 + x_3 - 2 \geqslant 0$ the unique coherent extension $x_{123}$ on $\mathcal{C}_{123}$ is $x_{123} = T_L(x_1, x_2, x_3)$.*

*Proof.* We distinguish two cases: ($i$) $x_1+x_2+x_3-2<0$; ($ii$) $x_1+x_2+x_3-2 \geqslant 0$. Case ($i$). From (8) the inequality $1-x_1-x_2-x_3+x_{12}+x_{13}+x_{23} \geqslant 0$ is not satisfied because $1-x_1-x_2-x_3+x_{12}+x_{13}+x_{23} = x_1+x_2+x_3-2 < 0$. Therefore the assessment is not coherent.
Case ($ii$). We set $x_{123} = T_L(x_1,x_2,x_3) = x_1+x_2+x_3-2$. Then, by observing that $0 < x_i+x_j-1 \leqslant x_1+x_2+x_3-2$, $i \neq j$, formula (9) becomes $\max\{0, x_1+x_2+x_3-2\} \leqslant x_1+x_2+x_3-2 \leqslant \min\{x_1+x_2-1, x_1+x_3-1, x_2+x_3-1, x_1+x_2+x_3-2\}$, that is: $x_1+x_2+x_3-2 \leqslant x_1+x_2+x_3-2 \leqslant x_1+x_2+x_3-2$. Thus, the inequalities are satisfied and the assessment $(x_1,x_2,x_3,T_L(x_1,x_2),T_L(x_1,x_3),T_L(x_2,x_3),T_L(x_1,x_2,x_3))$ on $\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_{12},\mathcal{C}_{13},\mathcal{C}_{23},\mathcal{C}_{123}\}$ is coherent and the sub-assessment $(x_1,x_2,x_3,T_L(x_1,x_2),T_L(x_1,x_3),T_L(x_2,x_3))$ on $\mathcal{F}$ is coherent too. □

A result related with Theorem 10 is given below.

**Theorem 11.** *If the assessment* $(x_1,x_2,x_3,T_L(x_1,x_2),T_L(x_1,x_3),T_L(x_2,x_3), T_L(x_1,x_2,x_3))$ *on the family* $\mathcal{F} = \{\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_{12},\mathcal{C}_{13},\mathcal{C}_{23},\mathcal{C}_{123}\}$*, is such that* $T_L(x_1,x_2,x_3) > 0$*, then the assessment is coherent.*

*Proof.* We observe that $T_L(x_1,x_2,x_3) = x_1+x_2+x_3-2 > 0$; then $x_i > 0$, $i=1,2,3$, and $0 < x_i+x_j-1 \leqslant x_1+x_2+x_3-2$, $i \neq j$. Then formula (9)) becomes: $\max\{0, x_1+x_2+x_3-2\} \leqslant x_1+x_2+x_3-2 \leqslant$
$\leqslant \min\{x_1+x_2-1, x_1+x_3-1, x_2+x_3-1, x_1+x_2+x_3-2\}$, that is:
$x_1+x_2+x_3-2 \leqslant x_1+x_2+x_3-2 \leqslant x_1+x_2+x_3-2$.
Thus, the inequalities are satisfied and the assessment is coherent. □

## 6 Conclusions

We have studied the relationship between the notions of conjunction and of Frank t-norms. We have shown that, under logical independence of events and coherence of prevision assessments, for a suitable $\lambda \in [0,+\infty]$ it holds that $\mathbb{P}((A|H) \wedge (B|K)) = T_\lambda(x,y)$ and $(A|H) \wedge (B|K) = T_\lambda(A|H, B|K)$. Then, we have considered the case $A = B$, by determining the set of all coherent assessment $(x,y,z)$ on $(A|H, B|K, (A|H) \wedge (A|K))$. We have shown that, under coherence, for a suitable $\lambda \in [0,1]$ it holds that $(A|H) \wedge (A|K) = T_\lambda(A|H, A|K)$. We have also studied the particular case where $A = B$ and $HK = \varnothing$. Then, we have considered the conjunction of three conditional events and we have shown that the prevision assessments produced by the Product t-norm, or the Minimum t-norm, are coherent. Finally, we have examined the Lukasiewicz t-norm and we have shown, by a counterexample, that coherence in general is not assured. We have given some conditions for coherence when the prevision assessments are based on the Lukasiewicz t-norm. Future work should concern the deepening and generalization of the results of this paper.

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