Bayesian Confirmation and Justifications

Hamzeh Mohammadi[1] and Thomas Studer[2]

[1] Department of Mathematical Science,
Isfahan University of Technology, Isfahan, Iran
hamzeh.mohammadi@math.iut.ac.ir
[2] Institute of Computer Science,
University of Bern, Bern, Switzerland
tstuder@inf.unibe.ch

Abstract. We introduce a family of probabilistic justification logics that feature Bayesian confirmations. Our logics include new justification terms representing evidence that make a proposition firm in the sense of making it more probable. We present syntax and semantics of our logic and establish soundness and strong completeness. Moreover, we show how to formalize in our logic the screening-off condition for transitivity of Bayesian confirmations.

Keywords: Epistemic logic · justification logic · Bayesian confirmation

1 Introduction

Justification logic is a type of logic that explicitly includes justifications why something is known or believed [6, 18]. The first justification logic, the Logic of Proofs [2], has been developed to provide a classical provability semantics for intuitionistic logic. In that approach, justifications represent proofs in a formal system like Peano arithmetic [17]. Later justification logic was introduced into formal epistemology where justifications can represent not only proofs but evidence in general [4]. For instance, an agent's knowledge may be justified by direct observation or by communication with another agent. In this context, notions like common knowledge [3, 8] and public announcements [7, 9] have been studied in detail.

Milnikel [19] was the first to investigate uncertain justifications. This lead to several further frameworks that model uncertain reasoning in justification logic: fuzzy justification logics [12, 21], possibilistic justification logics [11, 28], probabilistic justification logics [13, 14, 20], and logics for combining evidence and uncertainty [1, 25].

Having logics that contain justifications for belief as well as operators for conditional probabilities, it is natural to extend them to a framework in which justifications can represent Bayesian confirmations [29]. The main principle of Bayesian confirmation theory says that (for simplicity we do not consider a background theory here) evidence E confirms hypothesis H if the prior probability of H conditional on E is greater than the prior unconditional probability of H, that is if $\mathsf{P}(H|E) > \mathsf{P}(H)$. Carnap [10] calls this condition *confirmation as increase in firmness*.

We aim at a probabilistic justification logic that implements the above idea, that is in which something like

$$\mathsf{P}(H|E) > \mathsf{P}(H) \quad \text{entails} \quad \mathsf{j}(E) : H \tag{1}$$

holds, where $\mathsf{j}(E)$ is a term that represents the evidence E. Hence in this logic we read the formula $e : F$ as *evidence e confirms F*.

In order to model this relationship between conditional probability and evidence, we need a way to consider formulas as evidence terms. In (1) this is the role of the operator j. It takes a formula E and produces an evidence term $\mathsf{j}(E)$ representing the evidence E.

A similar kind of justification operator has been considered in the treatment of public announcements [16] where the operator up transforms formulas to evidence terms. We will use a similar strategy for the j-operator of (1). Further we will employ operators for conditional probabilities $\mathsf{CP}_{\geq s}$ as in [20, 22] and operators for the degree of confirmation $D_{\geq s}$ as in [26]. A formula $\mathsf{CP}_{\geq s}(A, B)$ means that *the conditional probability of A given B is at least s* and a formula $\mathsf{D}_{\geq s}(A, B)$ means that *the difference between the conditional probability of A given B and the probability of A is at least r*.

The paper is organized as follows. In the next section we introduce syntax and semantics of Bayesian justification logic, i.e. we present the deductive system $\mathsf{BJ}_{\mathsf{CS}}$ and we introduce the class of measurable Bayesian models. Then in Section 3 we establish soundness and completeness of $\mathsf{BJ}_{\mathsf{CS}}$ with respect to those models. Section 4 discusses transitivity of Bayesian confirmations in the framework of justification logic. Finally, Section 5 concludes the paper.

2 Bayesian Justification Logic BJ

2.1 Syntax

We start with countably many constants c_i, countably many variables x_i, and countably many atomic propositions p_i. Further, we define $S := \mathbb{Q} \cap [0, 1]$ and $S^* := \mathbb{Q} \cap [-1, 1]$, where $\mathbb{Q}$ is the set of all rational numbers. The (evidence) terms and formulas of the language of BJ are defined by simultaneous induction as follows:

- Evidence terms.
  - Every constant c_i and every variable x_i is an atomic term. If A is a formula, then j_A is an atomic term. Every atomic term is a term.
  - If t and s are terms, then $t \cdot s$ is a term and $!t$ is a term.
- Formulas.
  - Every atomic proposition p_i is a formula.
  - $\perp$ is a formula.
  - If A and B are formulas, t is a term, $s \in S$, and $r \in S^*$, then $A \to B$, $t : A$, $\mathsf{CP}_{\geq s}(A, B)$, and $\mathsf{D}_{\geq r}(A, B)$ are formulas.

The set of all constants is denoted by Con, the set of all terms is Tm, and we use $t, s, u, v, \dots$ to denote terms. The set of atomic propositions and the set of justification formulas are denoted by Prop and Fml, respectively. We use $A, B, \dots$ to denote formulas. The classical Boolean connectives $\neg, \vee, \wedge, \leftrightarrow$ are defined as usual and we set

$$\mathsf{CP}_{\leq s}(B,C) := \mathsf{CP}_{\geq 1-s}(\neg B, C) \qquad \text{and} \qquad \mathsf{D}_{\leq r}(B,C) := \mathsf{D}_{\geq -r}(\neg B, C)$$

for $s \in S$ and $r \in S^*$. Moreover, we use the standard abbreviations, see [22], for the following formulas:

$$\mathsf{CP}_{<s}(A,B) \quad \mathsf{CP}_{>s}(A,B) \quad \mathsf{CP}_{=s}(A,B) \quad \mathsf{P}_{\rho s}(A) \quad \text{for } \rho \in \{\geq, \leq, >, <, =\}$$

and similarly for $\mathsf{D}_{<s}(A,B)$, $\mathsf{D}_{>s}(A,B)$ and $\mathsf{D}_{=s}(A,B)$.

The axiom schemes of BJ are the following where $\bigcirc \in \{\mathsf{CP}, \mathsf{D}\}$:

1. all classical tautologies
2. $t : (A \to B) \to (s : A \to t \cdot s : B)$
3. $\mathsf{CP}_{\geq 0}(A,B)$
4. $\bigcirc_{\leq s}(A,B) \to \bigcirc_{<t}(A,B)$, for $t > s$
5. $\bigcirc_{<s}(A,B) \to \bigcirc_{\leq s}(A,B)$
6. $\mathsf{P}_{\geq 1}(A \leftrightarrow B) \to (\mathsf{P}_{=s}A \to \mathsf{P}_{=s}B)$
7. $\mathsf{P}_{=s}A \wedge \mathsf{P}_{=t}B \wedge \mathsf{P}_{\geq 1}\neg(A \wedge B)) \to \mathsf{P}_{=\min\{1,s+t\}}(A \vee B)$
8. $\mathsf{P}_{=0}B \to \mathsf{CP}_{=1}(A,B)$
9. $\mathsf{P}_{\geq s}(A \wedge B) \wedge \mathsf{P}_{\leq t}B \to \mathsf{CP}_{\geq \frac{s}{t}}(A,B)$, for $t \neq 0$
10. $\mathsf{P}_{\leq s}(A \wedge B) \wedge \mathsf{P}_{\geq t}B \to \mathsf{CP}_{\leq \frac{s}{t}}(A,B)$, for $t \neq 0$
11. $\mathsf{CP}_{\geq s}(A,B) \wedge \mathsf{P}_{\leq t}A \to \mathsf{D}_{\geq s-t}(A,B)$
12. $\mathsf{CP}_{\leq s}(A,B) \wedge \mathsf{P}_{\geq t}A \to \mathsf{D}_{\leq s-t}(A,B)$
13. $j_B : A \leftrightarrow \mathsf{D}_{>0}(A,B)$

Axioms 1 to 10 come from justification logic with conditional probabilities, see [20]. The main difference is that we replaced the axiom

$$\mathsf{P}_{=s}(A \wedge B) \wedge \mathsf{P}_{=t}B \to \mathsf{CP}_{=\frac{s}{t}}(A,B) \quad \text{for } t \neq 0$$

from [20] with our axioms 9 and 10, which yields a slightly stronger system. This additional power is needed to prove Lemma 4. Axioms 11 and 12 formalize the relationship between conditional probabilities and degrees of confirmation as in [26]. Axiom 13 finally states that terms j_B represent Bayesian confirmations.

A *constant specification* is any set CS that satisfies

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an instance of some axiom of } \mathsf{BJ}\}.$$

Let CS be any constant specification. The deductive system $\mathsf{BJ}_{\mathsf{CS}}$ is the Hilbert system obtained by adding to the axioms of BJ the rules (MP), (CE), (ST.1), (ST.2) and (AN!) as given in Figure 1.

Note that (ST.1) and (ST.2) are infinitary rules, which we need to obtain strong completeness. Observe also the difference in the definitions of rules (MP), (ST.1), (ST.2), and (CE) in Figure 1. Rule (CE) can only be applied to theorems of BJ (i.e. formulas that are deducible from the empty set), whereas (MP), (ST.1), and (ST.2) can always be applied.

	axioms of BJ
	$+$
(AN!)	$\vdash\, !^n c : !^{n-1} c : \cdots : !c : c : A$, where $(c, A) \in \mathsf{CS}$ and $n \in \mathbb{N}$
(MP)	if $T \vdash A$ and $T \vdash A \to B$ then $T \vdash B$
(CE)	if $\vdash A$ then $\vdash \mathsf{P}_{\geq 1} A$
(ST.1)	if $T \vdash A \to \mathsf{CP}_{\geq s - \frac{1}{k}}(B, C)$ for every integer $k \geq \frac{1}{s}$ and $s > 0$ then $T \vdash A \to \mathsf{CP}_{\geq s}(B, C)$
(ST.2)	if $T \vdash A \to \mathsf{D}_{\geq r - \frac{1}{k}}(B, C)$ for every integer $k \geq \frac{1}{1+r}$ and $r > -1$ then $T \vdash A \to \mathsf{D}_{\geq r}(B, C)$

Fig. 1. System $\mathsf{BJ}_{\mathsf{CS}}$

2.2 Semantics

To introduce semantics for $\mathsf{BJ}_{\mathsf{CS}}$, we begin with the notion of a basic evaluation, which is the cornerstone for many interpretations of justification logic [5, 15]. In the following we use $\mathcal{P}(X)$ to denote the power set of a set X.

Definition 1 (Basic Evaluation). *Let* CS *be a constant specification. A* basic evaluation *for* CS*, or a basic* CS*-evaluation, is a function* $*$ *that maps atomic propositions to truth values and maps justification terms to subsets of* Fml*, i.e.*

$$* : \mathsf{Prop} \to \{\mathsf{T}, \mathsf{F}\} \quad \textit{and} \quad * : \mathsf{Tm} \to \mathcal{P}(\mathsf{Fml}),$$

such that for $u, v \in \mathsf{Tm}$*, for* $c \in \mathsf{Con}$ *and* $A, B \in \mathsf{Fml}$ *we have:*

1. $\big(A \to B \in u^*$ *and* $A \in v^*\big) \Longrightarrow B \in (u \cdot v)^*$
2. *if* $(c, A) \in \mathsf{CS}$ *then for all* $n \in \mathbb{N}$ *we have*[3]*:*

$$!^{n-1} c : !^{n-2} c : \cdots : !c : c : A \in (!^n c)^*$$

We usually write t^* *and* p^* *instead of* $*(t)$ *and* $*(p)$*, respectively.*

Definition 2 (Algebra over a Set). *Let* W *be a non-empty set and let* H *be a non-empty subset of* $\mathcal{P}(W)$*. We call* H *an* algebra over W *iff the following hold:*

- $W \in H$
- $U, V \in H \Longrightarrow U \cup V \in H$
- $U \in H \Longrightarrow W \setminus U \in H$

Definition 3 (Finitely Additive Measure). *Let* H *be an algebra over* W *and* $\mu : H \to [0, 1]$*. We call* μ *a* finitely additive measure *iff the following hold:*

1. $\mu(W) = 1$

[3] We agree to the convention that the formula $!^{n-1} c : !^{n-2} c : \cdots : !c : c : A$ represents the formula A for $n = 0$.

2. *for all* $U, V \in H$*:*

$$U \cap V = \emptyset \quad \Longrightarrow \quad \mu(U \cup V) = \mu(U) + \mu(V)$$

Definition 4 (Probability Space). *A* probability space *is a triple*

$$P = \langle W, H, \mu \rangle,$$

where:

- W *is a non-empty set*
- H *is an algebra over* W
- $\mu : H \to [0,1]$ *is a finitely additive measure*

Definition 5 (Model). *Let* CS *be a constant specification. A* $\mathsf{BJ}_{\mathsf{CS}}$-model *is a quintuple* $M = \langle U, W, H, \mu, * \rangle$ *where:*

1. U *is a non-empty set of objects called worlds*
2. W, H, μ *and* $*$ *are functions, which have* U *as their domain, such that for every* $w \in U$*:*
   - $\langle W(w), H(w), \mu(w) \rangle$ *is a probability space with* $W(w) \subseteq U$
   - $*_w$ *is a basic* CS*-evaluation*[4]

The ternary satisfaction relation $\models$ is defined between models, worlds, and formulas. We will use μ_w for $\mu(w)$, p^*_w for p^{*_w}, and t^*_w for t^{*_w}.

Definition 6 (Truth in a $\mathsf{BJ}_{\mathsf{CS}}$-model). *Let* CS *be a constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be a* $\mathsf{BJ}_{\mathsf{CS}}$*-model. We define by simultaneous induction*

1. *what it means for a formula to hold in* M *at a world* $w \in U$ *and*
2. *what it means for a formula to be measurable in* M *at a world* $w \in U$

as follows:

- $M, w \models p$ *iff* $p^*_w = \mathsf{T}$ *for* $p \in \mathsf{Prop}$*;*
- $M, w \not\models \bot$*;*
- $M, w \models A \to B$ *iff* $M, w \not\models A$ *or* $M, w \models B$*;*
- $M, w \models t : A$ *iff* $A \in t^*_w$*;*
- $M, w \models \mathsf{CP}_{\geq s}(A, B)$ *iff* $A \wedge B$ *and* B *are measurable at* w *and either* $\mu_w([B]) = 0$*, or* $\mu_w([B]) > 0$ *and* $\frac{\mu_w([A \wedge B])}{\mu_w([B])} \geq s$*;*
- $M, w \models \mathsf{D}_{\geq s}(A, B)$ *iff* A *and* B *are measurable at* w *and either* $\mu_w([B]) = 0$ *and* $1 - \mu_w([A]) \geq s$*, or* $\mu_w([B]) > 0$ *and* $\frac{\mu_w([A \wedge B])}{\mu_w([B])} - \mu_w([A]) \geq s$*.*

We say a formula B *is measurable in* M *at a world* $w \in U$ *if the set*

$$[B]_{M,w} := \{x \in W(w) \mid M, x \models B\}$$

is an element of $H(w)$*.*

[4] We will usually write $*_w$ instead of $*(w)$.

Definition 7 (Measurable Model). *Let* CS *be a constant specification and let* $M = \langle U, W, H, \mu, * \rangle$ *be a* $\mathsf{BJ_{CS}}$*-model.* M *is called measurable iff every formula* A *is measurable at each* $w \in U$*.* $\mathsf{BJ_{CS,Meas}}$ *denotes the class of measurable* $\mathsf{BJ_{CS}}$ *models.*

Finally, we call a model Bayesian if terms of the form j_A represent Bayesian evidence.

Definition 8. *A* $\mathsf{BJ_{CS}}$*-model* $M = \langle U, W, H, \mu, * \rangle$ *is called* Bayesian model *if at each* $w \in U$*,*

$$M, w \models \mathrm{D}_{>0}(A, B) \quad \textit{iff} \quad M, w \models j_B : A.$$

The class of Bayesian $\mathsf{BJ_{CS}}$-models is denoted by $\mathsf{BJ_{CS,Bayes}}$. The class of Bayesian measurable $\mathsf{BJ_{CS}}$-models is denoted by $\mathsf{BJ_{CS,Meas,Bayes}}$.

For a model $M = \langle U, W, H, \mu, * \rangle$, $M \models A$ means that $M, w \models A$ for all $w \in U$. Let $T \subseteq \mathsf{Fml}$. Then $M \models T$ means that $M \models A$ for all $A \in T$. Further $T \models A$ means that for all $M \in \mathsf{BJ_{CS,Meas,Bayes}}$, $M \models T$ implies $M \models A$.

To be precise we should write $T \vdash_{\mathsf{CS}} A$ and $T \models_{\mathsf{CS}} A$ instead of $T \vdash A$ and $T \models A$, respectively, since these two notions depend on a given constant specification CS. However, CS will always be clear from the context and we thus omit it.

3 Soundness and Completeness for Bayesian Justification Logic

Soundness of $\mathsf{BJ_{CS}}$ can be proved by induction on the depth of derivations. To establish completeness, we make use of a canonical model construction. For lack of space, however, we cannot give a detailed completeness proof here. We will only present a series of definitions and lemmas (without proofs) that leads to the completeness result.

Theorem 1 (Soundness). *Let* CS *be a constant specification. The axiomatic system* $\mathsf{BJ_{CS}}$ *is sound with respect to the class of* $\mathsf{BJ_{CS,Meas,Bayes}}$*-models, i.e., for any formula* A *and any set* $T \subseteq \mathsf{Fml}$ *we have*

$$T \vdash A \quad \Longrightarrow \quad T \models A.$$

Now we define the notion of a $\mathsf{BJ_{CS}}$-consistent sets.

Definition 9 ($\mathsf{BJ_{CS}}$-Consistent Sets). *Let* CS *be any constant specification and let* T *be a set of formulas.*

- *T is said to be $\mathsf{BJ_{CS}}$-consistent if and only if $T \nvdash_{\mathsf{BJ_{CS}}} \bot$. Otherwise T is said to be $\mathsf{BJ_{CS}}$-inconsistent.*
- *T is said to be maximal if and only if for every $A \in \mathsf{Fml}$ either $A \in T$ or $\neg A \in T$.*
- *T is said to be maximal $\mathsf{BJ_{CS}}$-consistent if and only if it is maximal and $\mathsf{BJ_{CS}}$-consistent.*

We have the following deduction theorem for $\mathsf{BJ}_{\mathsf{CS}}$. The proof is similar to the one given in [13, 22].

Theorem 2 (Deduction Theorem for $\mathsf{BJ}_{\mathsf{CS}}$). *Let T be a set of formulas and A and B be formulas. We have*

$$T, A \vdash B \quad \textit{iff} \quad T \vdash A \to B.$$

The deduction theorem makes it possible to establish the following property of consistent sets of formulas, see [13, Lemma 27].

Lemma 1. *Let CS be a constant specification and let T be a $\mathsf{BJ}_{\mathsf{CS}}$-consistent set of formulas.*

1. *If $\neg(B \to \mathsf{CP}_{\geq s}(A, C)) \in T$ for $s > 0$, then there is some integer $n \geq \frac{1}{s}$ such that $T, \neg(B \to \mathsf{CP}_{\geq s - \frac{1}{n}}(A, C))$ is $\mathsf{BJ}_{\mathsf{CS}}$-consistent.*
2. *If $\neg(B \to \mathsf{D}_{\geq r}(A, C)) \in T$ for $r > -1$, then there is some integer $n \geq \frac{1}{r+1}$ such that $T, \neg(B \to \mathsf{D}_{\geq r - \frac{1}{n}}(A, C))$ is $\mathsf{BJ}_{\mathsf{CS}}$-consistent.*

The Lindenbaum lemma for probabilistic justification logics has been established in [13]. The proof for $\mathsf{BJ}_{\mathsf{CS}}$ is similar.

Lemma 2 (Lindenbaum). *Let CS be a constant specification. Every $\mathsf{BJ}_{\mathsf{CS}}$-consistent set of formulas can be extended to a maximal $\mathsf{BJ}_{\mathsf{CS}}$-consistent set.*

Definition 10 (Canonical Model). *Let CS be a constant specification. The* canonical model *for $\mathsf{BJ}_{\mathsf{CS}}$ is given by the quintuple $M = \langle U, W, H, \mu, * \rangle$, defined as follows:*

- *$U = \{w \mid w$ is a maximal $\mathsf{BJ}_{\mathsf{CS}}$-consistent set of formulas$\}$*
- *for every $w \in U$ the probability space $\langle W(w), H(w), \mu(w) \rangle$ is defined as follows:*
  1. *$W(w) = U$*
  2. *$H(w) = \{(A)_M \mid A \in \mathsf{Fml}\}$ where $(A)_M = \{x \mid x \in U, A \in x\}$*
  3. *for all $A \in \mathsf{Fml}$, $\mu(w)\big((A)_M\big) = \sup_s \{\mathsf{P}_{\geq s} A \in w\}$*
- *for every $w \in W$ the basic CS-evaluation $*_w$ is defined as follows:*
  1. *for all $p \in \mathsf{Prop}$:*
  $$p^*_w = \begin{cases} \mathsf{T} & \textit{if } p \in w \\ \mathsf{F} & \textit{if } \neg p \in w \end{cases}$$
  2. *for all $t \in \mathsf{Tm}$:*
  $$t^*_w = \{A \mid t : A \in w\}$$

Lemma 3. *Let CS be a constant specification. The canonical model for $\mathsf{BJ}_{\mathsf{CS}}$ is a $\mathsf{BJ}_{\mathsf{CS}}$-model.*

The following lemma is proved by induction on the complexity of the formula A where we make use of a complexity measure such that the complexity of $\mathsf{CP}_{\geq s}(B, C)$ and $\mathsf{D}_{\geq s}(B, C)$ is greater than the complexity of $B \wedge C$.

Lemma 4. *Let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for $\mathsf{BJ}_{\mathsf{CS}}$. Then we have*

$$(\forall A \in \mathsf{Fml})(\forall w \in U)\big[[A]_{M,w} = (A)_M\big].$$

From Lemma 4 we get the following corollary.

Corollary 1. *Let CS be any constant specification. The canonical model for $\mathsf{BJ}_{\mathsf{CS}}$ is a $\mathsf{BJ}_{\mathsf{CS,Meas}}$-model.*

Making use of the properties of maximal consistent sets, we can establish the truth lemma.

Lemma 5 (Truth Lemma). *Let CS be a constant specification and let $M = \langle U, W, H, \mu, * \rangle$ be the canonical model for $\mathsf{BJ}_{\mathsf{CS}}$. For every $A \in \mathsf{Fml}$ and any $w \in U$ we have:*

$$A \in w \iff M, w \models A.$$

Using the truth lemma we find that the canonical model satisfies the condition for Bayesian models, i.e. we have the following corollary.

Corollary 2. *Let CS be any constant specification. The canonical model for $\mathsf{BJ}_{\mathsf{CS}}$ is a $\mathsf{BJ}_{\mathsf{CS,Meas,Bayes}}$-model.*

Finally, we get the completeness theorem as usual.

Theorem 3 (Strong Completeness for BJ). *Let CS be a constant specification, let $T \subseteq \mathsf{Fml}$ and let $A \in \mathsf{Fml}$. Then we have:*

$$T \models A \implies T \vdash A.$$

4 Transitivity

It is well known that Bayesian confirmation is not transitive, i.e., the following principle is not valid

$$\mathsf{P}(B|A) > \mathsf{P}(B) \text{ and } \mathsf{P}(C|B) > \mathsf{P}(C) \implies \mathsf{P}(C|A) > \mathsf{P}(C)\ . \tag{2}$$

We refer to, e.g., [24, 27] for examples where transitivity fails.

It turns out, however, that there are conditions under which (2) holds. Shogenji [27] introduces the following condition, called *screening-off condition*,

$$\mathsf{P}(C|A \wedge B) = \mathsf{P}(C|B) \quad \text{and} \quad \mathsf{P}(C|A \wedge \neg B) = \mathsf{P}(C|\neg B) \tag{3}$$

and shows that transitivity holds under it. Intuitively, (3) means that once truth or falsity of B is known, A is irrelevant to the probability of C. In other words, A affects the probability of C only indirectly through its impact on B [24].

Roche [23] presents the following weakening of (3)

$$\mathsf{P}(C|A \wedge B) \geq \mathsf{P}(C|B) \quad \text{and} \quad \mathsf{P}(C|A \wedge \neg B) \geq \mathsf{P}(C|\neg B)\ . \tag{4}$$

and shows that transitivity also holds under this weaker condition.

We are now going to formalize this result in Bayesian justification logic. We show that we can represent (4) in BJ and that this condition entails transitivity of Bayesian justifications.

Theorem 4. *Let A, B, and C be formulas of* Fml. *Let T be the set of formulas that consists of:*

1. $\mathsf{CP}_{=r}(C, B) \to \mathsf{CP}_{\geq r}(C, A \wedge B)$ *for all* $r \in S$,
2. $\mathsf{CP}_{=r}(C, \neg B) \to \mathsf{CP}_{\geq r}(C, A \wedge \neg B)$ *for all* $r \in S$,
3. $\mathsf{P}_{\neq 0}A$, $\mathsf{P}_{\neq 0}(A \wedge B)$, $\mathsf{P}_{\neq 0}(A \wedge \neg B)$, $\mathsf{P}_{\neq 0}B$, *and* $\mathsf{P}_{\neq 0}\neg B$.

Then we have that

$$T \vdash j_A : B \wedge j_B : C \ \to \ j_A : C \ .$$

Let M be any $\mathsf{BJ}_{\mathsf{CS},\mathsf{Meas},\mathsf{Bayes}}$-model such that $M \models T$. We observe that since M satisfies all formulas in T, the model M also satisfies condition (4). Thus we can show that $M \models j_A : B \wedge j_B : C \ \to \ j_A : C$ by essentially following the original proof that transitivity holds under (4) given in [23]. The theorem follows by strong completeness of $\mathsf{BJ}_{\mathsf{CS}}$.

5 Conclusion

In this paper we have introduced $\mathsf{BJ}_{\mathsf{CS}}$, a family of justification logics that feature Bayesian confirmations. Because the language of Bayesian justification logics includes both probability operators and explicit justifications, we were able to define a class of models that satisfies condition (1). Hence $\mathsf{BJ}_{\mathsf{CS}}$ not only includes justification terms built up from variables and constants, i.e. terms that represent assumptions and logical axioms, but also terms that represent Bayesian confirmations. In particular, a formula $j_A : B$, i.e. j_A justifies B, can be read as evidence A confirms B in the sense of *increase in firmness*.

We have established soundness and completeness of $\mathsf{BJ}_{\mathsf{CS}}$ with respect to Bayesian models. Further we have shown that we can formalize the screening-off condition and that this condition entails transitivity of confirmation in Bayesian models.

Future work includes studying the computational properties of Bayesian justification logic, i.e., establishing decidability and complexity results, as well as developing a corresponding proof theory.

Acknowledgements

Hamzeh Mohammadi has been supported by the Ministry of Science, Research and Technology of Iran and part of the research was carried out during a visit at University of Bern.
Thomas Studer has been supported by the Swiss National Science Foundation grant 200021_165549.

References

1. Artemov, S.: On aggregating probabilistic evidence. In: Artemov, S., Nerode, A. (eds.) LFCS 2016. pp. 27–42. Springer (2016)

2. Artemov, S.N.: Explicit provability and constructive semantics. Bulletin of Symbolic Logic **7**(1), 1–36 (Mar 2001)
3. Artemov, S.N.: Justified common knowledge. Theoretical Computer Science **357**(1–3), 4–22 (Jul 2006). https://doi.org/10.1016/j.tcs.2006.03.009
4. Artemov, S.N.: The logic of justification. The Review of Symbolic Logic **1**(4), 477–513 (Dec 2008). https://doi.org/10.1017/S1755020308090060
5. Artemov, S.N.: The ontology of justifications in the logical setting. Studia Logica **100**(1–2), 17–30 (Apr 2012). https://doi.org/10.1007/s11225-012-9387-x, published online February 2012
6. Artemov, S.N., Fitting, M.: Justification logic. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. Fall 2012 edn. (2012), http://plato.stanford.edu/archives/fall2012/entries/logic-justification/
7. Bucheli, S., Kuznets, R., Renne, B., Sack, J., Studer, T.: Justified belief change. In: Arrazola, X., Ponte, M. (eds.) Proc. LogKCA10, pp. 135–155. University of the Basque Country Press (2010)
8. Bucheli, S., Kuznets, R., Studer, T.: Justifications for common knowledge. Journal of Applied Non-classical Logic **21**(1), 35–60 (Jan–Mar 2011). https://doi.org/10.3166/JANCL.21.35-60
9. Bucheli, S., Kuznets, R., Studer, T.: Realizing public announcements by justifications. J. of Comp. and Sys. Sc. **80**(6), 1046–1066 (2014). https://doi.org/10.1016/j.jcss.2014.04.001
10. Carnap, R.: Logical Foundations of Probability. University of Chicago Press, 2nd edn. (1962)
11. Fan, T., Liau, C.: A logic for reasoning about justified uncertain beliefs. In: Yang, Q., Wooldridge, M. (eds.) Proc. IJCAI 2015. pp. 2948–2954. AAAI Press (2015)
12. Ghari, M.: Pavelka-style fuzzy justification logics. Logic Journal of IGPL **24**(5), 743–773 (2016)
13. Kokkinis, I., Maksimović, P., Ognjanović, Z., Studer, T.: First steps towards probabilistic justification logic. Logic Journal of IGPL **23**(4), 662–687 (2015). https://doi.org/10.1093/jigpal/jzv025
14. Kokkinis, I., Ognjanović, Z., Studer, T.: Probabilistic justification logic. In: Artemov, S., Nerode, A. (eds.) Proceedings of Logical Foundations of Computer Science LFCS'16. Springer (2016)
15. Kuznets, R., Studer, T.: Justifications, ontology, and conservativity. In: Bolander, T., Braüner, T., Ghilardi, S., Moss, L. (eds.) Advances in Modal Logic, Volume 9, pp. 437–458. College Publications (2012)
16. Kuznets, R., Studer, T.: Update as evidence: Belief expansion. In: Artemov, S.N., Nerode, A. (eds.) Proc. LFCS 13, LNCS, vol. 7734, pp. 266–279. Springer (2013). https://doi.org/10.1007/978-3-642-35722-0_19
17. Kuznets, R., Studer, T.: Weak arithmetical interpretations for the logic of proofs. Logic Journal of IGPL **24**(3), 424–440 (2016)
18. Kuznets, R., Studer, T.: Logics of Proofs and Justifications. College Publications (2019)
19. Milnikel, R.S.: The logic of uncertain justifications. APAL **165**(1), 305–315 (2014). https://doi.org/10.1016/j.apal.2013.07.015
20. Ognjanović, Z., Savić, N., Studer, T.: Justification logic with approximate conditional probabilities. In: Baltag, A., Seligman, J., Yamada, T. (eds.) Logic, Rationality, and Interaction. pp. 681–686. Springer (2017)
21. Pischke, N.: A note on strong axiomatization of Gdel-justification logic. E-print 1809.09608, arXiv.org (2018)

22. Rašković, M., Marković, Z., Ognjanović, Z.: A logic with approximate conditional probabilities that can model default reasoning. Int. J. Approx. Reasoning **49**(1), 52–66 (2008). https://doi.org/10.1016/j.ijar.2007.08.006
23. Roche, W.: A weaker condition for transitivity in probabilistic support. European Journal for Philosophy of Science **2**(1), 111–118 (2012). https://doi.org/10.1007/s13194-011-0033-7
24. Roche, W., Shogenji, T.: Confirmation, transitivity, and moore: the screening-off approach. Philosophical Studies **168**(3), 797–817 (2014). https://doi.org/10.1007/s11098-013-0161-3
25. Schechter, L.M.: A logic of plausible justifications. Theor. Comput. Sci. **603**, 132–145 (2015). https://doi.org/10.1016/j.tcs.2015.07.018
26. Schlesinger, G.N.: Measuring degrees of confirmation. Analysis **55**(3), 208–212 (1995). https://doi.org/10.1093/analys/55.3.208
27. Shogenji, T.: A condition for transitivity in probabilistic support. The British Journal for the Philosophy of Science **54**(4), 613–616 (2003)
28. Su, C.P., Fan, T.F., Liau, C.J.: Possibilistic justification logic: Reasoning about justified uncertain beliefs. ACM Trans. Comput. Logic **18**(2), 15:1–15:21 (2017). https://doi.org/10.1145/3091118
29. Talbott, W.: Bayesian epistemology. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. Summer 2015 edn. (2015)