

Shortest Reconfiguration of Matchings

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December 14, 2018

Abstract

Imagine that unlabelled tokens are placed on the edges of a graph, such that no two tokens are placed on incident edges. A token can jump to another edge if the edges having tokens remain independent. We study the problem of determining the *distance* between two token configurations (resp., the corresponding matchings), which is given by the length of a shortest transformation. We give a polynomial-time algorithm for the case that at least one of the two configurations is not inclusion-wise maximal and show that otherwise, the problem admits no polynomial-time sublogarithmic-factor approximation unless $P = NP$. Furthermore, we show that the distance of two configurations in bipartite graphs is fixed-parameter tractable parameterized by the size d of the symmetric difference of the source and target configurations, and obtain a d^ε -factor approximation algorithm for every $\varepsilon > 0$ if additionally the configurations correspond to maximum matchings. Our two main technical tools are the Edmonds-Gallai decomposition and a close relation to the DIRECTED STEINER TREE problem. Using the former, we also characterize those graphs whose corresponding configuration graphs are connected. Finally, we show that deciding if the distance between two configurations is equal to a given number ℓ is complete for the class D^P , and deciding if the diameter of the graph of configurations is equal to ℓ is D^P -hard.

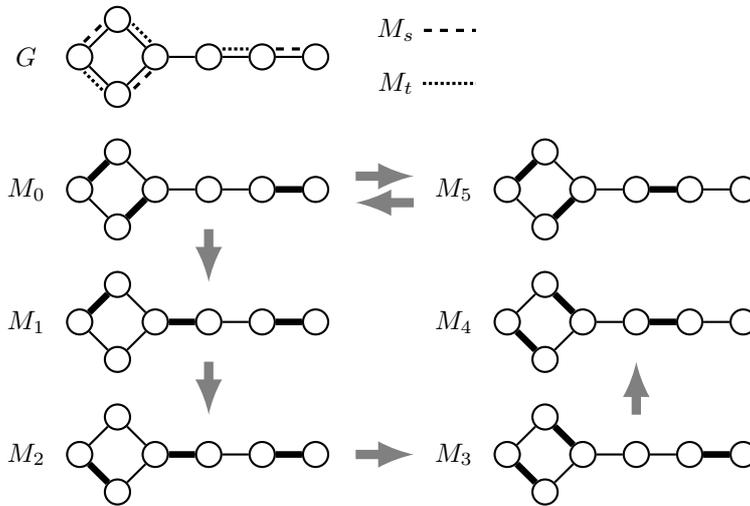


Figure 1: A reconfiguration sequence $M_s = M_0, M_1, \dots, M_4 = M_t$ of matchings in a graph G .

1 Introduction

A reconfiguration problem asks for the existence of a step-by-step transformation between two given configurations, where in each step we apply some simple modification to the current configuration. The set of configurations may for instance be the set of k -colorings [3, 10] or independent sets [13, 14, 16] of a graph, or the set of satisfying assignments of Boolean formulas [12]. A suitable modification may for example alter the color of a single vertex, or the truth value of a variable in a satisfying assignment. For a survey on reconfiguration problems, the reader is referred to [23] or [19].

Recently, there has been considerable interest in the complexity of finding *shortest* transformations between configurations. Examples include finding a shortest transformation between triangulations of planar point sets [21] and simple polygons [1], configurations of the Rubik's cube [6], and satisfying assignments of Boolean formulas [17]. For all of these problems, except the last one, we can decide efficiently if a transformation between two given configurations exists. However, deciding if there is a transformation of at most a given length is NP-complete. In particular, the flip distance of triangulations of planar point sets is known to be APX-hard [21] and, on the positive side, fixed-parameter tractable (FPT) in the length of the transformation [15]. Our reference problem is the task of computing the length of a shortest transformation between matchings of a graph. We show that even in a very restricted setting the problem admits no $o(\log n)$ -factor approximation unless $P = NP$ and we give polynomial-time algorithms in some special cases. Furthermore, we show that the problem is FPT in the size of the symmetric difference of the two given configurations, which implies that it is also FPT in the length of the transformation.

Reconfiguration of matchings. Deciding if there is a transformation between two matchings of a graph is known as an early example of a reconfiguration problem that admits a non-trivial polynomial-time algorithm [13]. Recall

that a *matching* M of a graph is a set of pairwise independent edges. (Figure 1 shows the six different matchings of the graph G .) We may consider a matching as a placement of (unlabeled) *tokens* on independent edges: Then, the *Token Jumping (TJ)* operation provides an adjacency relation on the set of matchings of a graph, all having the same cardinality¹: Two matchings M and M' of a graph G are *adjacent* (under TJ) if one can be obtained from the other by relocating a single token, that is, if $|M \setminus M'| = 1$ and $|M' \setminus M| = 1$. We say that a sequence M_0, M_1, \dots, M_ℓ of matchings of G is a *reconfiguration sequence* of length ℓ from M to M' , if $M_0 = M$, $M_\ell = M'$, and M_{i-1} and M_i are adjacent for each i , $1 \leq i \leq \ell$. (See the sequence M_0, M_1, \dots, M_4 in Figure 1 as an example.) The following question is often referred to as the *reachability variant* of the matching reconfiguration problem:

MATCHING RECONFIGURATION

Input: Graph G and two matchings M_s, M_t of G .

Question: Is there a reconfiguration sequence from M_s to M_t ?

For YES-instances of the above problem the polynomial-time algorithm given in [13] gives a bound of $O(n^2)$ on the length of a transformation. The *distance* between two matchings is the length of a shortest transformation between them (under TJ). If there is no transformation between two matchings, we regard their distance as infinity. In this paper we study the complexity of the following optimization problem related to matching reconfiguration, which is also referred to as the *shortest variant*.

MATCHING DISTANCE

Input: Graph G and two matchings M_s, M_t of G .

Task: Compute the distance between M_s and M_t .

We also study two related *exact* problems. The first is the exact version of MATCHING DISTANCE, which takes as input also the supposed distance ℓ of the given matchings.

EXACT MATCHING DISTANCE

Input: Graph G , matchings M_s, M_t of G , and number $\ell \in \mathbb{N}$.

Question: Is ℓ equal to the distance between M_s and M_t ?

The second decides the maximum distance (*diameter*) of any two matchings of a given cardinality k in a graph.

EXACT MATCHING DIAMETER

Input: Graph G and numbers $k, \ell \in \mathbb{N}$.

Question: Is ℓ equal to the maximum distance between any two matchings of cardinality k of G ?

Related results. Despite recent intensive studies on reconfiguration problems (see, e.g., a survey [19]), most of known algorithmic (positive) results are obtained for reachability variants. However, they sometimes give answers to

¹There is another well-studied operation, called Token Sliding (TS), for reconfiguration of subgraphs having the same cardinality. In this paper, we employ TJ as the default operation. However, some of our results apply also to TS, because TJ and TS are equivalent for maximum-cardinality matchings.

their shortest variants: if the algorithm constructs an actual reconfiguration sequence which, at any step, transforms an edge of the initial matching into an edge of the target one, then the sequence is indeed a shortest one.

Generally speaking, finding shortest transformations is much more difficult if we need a *detour*, which touches an element that is not in the symmetric difference of the source and target configurations. For such a detour-required case, only a few polynomial-time algorithms are known for shortest variants, e.g., satisfying assignments of a certain Boolean formulas by Mouawad et al. [18], and independent sets under the TS operation for caterpillars by Yamada and Uehara [24]. Note that MATCHING RECONFIGURATION belongs to the detour-required case; recall the example in Figure 1, where we need to use the edge in $E(G) \setminus (M_s \cup M_t)$ in any reconfiguration sequence.

The reconfiguration of matchings is a special case of the reconfiguration of independent sets of a graph. To see this, recall that matchings of a graph correspond to independent sets of its line graph. Therefore, by a result of Kamiński et al. [14], we can solve MATCHING DISTANCE in polynomial time if the line graph of a given graph is even-hole free. Note that in this case no detour is required.

Our results. Although the reconfiguration of independent sets is one of the most well-studied reconfiguration problems (see, e.g., a survey [19]), to the best of our knowledge, the shortest variant of independent sets under the TJ operation is known to be solvable only for even-hole-free graphs, as mentioned above. Thus, in this paper, we start a systematic study of the complexity of finding shortest reconfiguration sequences between matchings, and more generally, between independent sets of a graph.

Our first result is the following classification of the complexity of the problem MATCHING DISTANCE. It follows immediately from Theorem 6, Corollary 9, and Lemma 5.

Theorem 1. *MATCHING DISTANCE can be solved in polynomial time if at least one of the two matchings is not inclusion-wise maximal. Furthermore, MATCHING DISTANCE restricted to instances where both matchings are maximal admits no polynomial-time $o(\log n)$ -factor approximation algorithm, unless $P = NP$.*

The hardness part of Theorem 1 holds even for bipartite graphs of maximum degree three. Note that it implies approximation hardness for shortest transformations between b -matchings of a graph and for shortest transformations between independent sets on any graph class containing line graphs.

On the positive side, we show that determining the distance of maximum matchings of *bipartite* graphs is FPT in the size d of the symmetric difference of the input matchings. In our algorithm we consider two cases: either a shortest reconfiguration sequence contains a non-inclusion-wise maximal matching or not. Extending the positive side of Theorem 1, we give a polynomial-time algorithm for the former case. To deal with the latter case, we proceed in two stages. We first generate (many) instances of MATCHING DISTANCE, such that the two input matchings are *maximum*. This allows us to make some additional assumptions based on the Edmonds-Gallai decomposition [22, Ch. 24.4b]. We then further reduce this variant of MATCHING DISTANCE to DIRECTED STEINER TREE with at most $d/2$ terminals and show that optimal Steiner trees are in

correspondence with shortest reconfiguration sequences. Optimal Steiner trees can be computed in FPT time for a constant number of terminals according to the algorithms [2, 9]. By properly combining the reconfiguration sequences obtained from optimal Steiner trees we obtain the following result.

Theorem 2. *MATCHING DISTANCE in bipartite graphs can be solved in time $2^d \cdot n^{O(1)}$, where d is the size of the symmetric difference of two given matchings.*

This result raises hopes for possible generalizations, e.g., an FPT algorithm for finding a shortest transformation between independent sets of claw-free graphs. The reduction from MATCHING DISTANCE restricted to maximum matchings in bipartite graphs to DIRECTED STEINER TREE is approximation-preserving, which implies the following.

Corollary 3. *MATCHING DISTANCE restricted to maximum matchings in bipartite graphs admits a polynomial-time d^ε -factor approximation algorithm for every $\varepsilon > 0$, where d is the size of the symmetric difference of two given matchings.*

We complement Theorem 1 by showing that there is a polynomial-time algorithm that decides if the maximum distance between any two matchings of a graph is *finite*. However, we also show that the problems EXACT MATCHING DISTANCE and EXACT MATCHING DIAMETER are both hard for the class D^P , which contains NP and coNP.

Theorem 4. *The problem EXACT MATCHING DISTANCE is complete for D^P and EXACT MATCHING DIAMETER is D^P -hard.*

The class D^P is a class of decision problems introduced by Papadimitriou and Yannakakis in [20]. It is a natural class for *exact* problems, *critical* problems, and for example for the question, whether a certain inequality is a facet of a polytope [20]. It was proved by Frieze and Teng that the related problem of deciding the diameter of the graph of a polyhedron is also D^P -hard [11].

Notation. We denote by $A \triangle B$, the symmetric difference of two sets A and B . That is, $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Unless stated otherwise, graphs are simple. For standard definitions and notation related to graphs, we refer the reader to [7]. For a graph G , we denote by $V(G)$ (resp., $E(G)$) the set of vertices (resp., edges) of G . We denote by \overline{G} the complement graph of G and by \overline{E} the edge-set $E(\overline{G})$ of the complement graph. Let $G = (V, E)$ be a graph and let $M \subseteq E$ be a matching. A vertex of G that is not incident to any edge in M is called *M -exposed* or *M -free*, otherwise it is *matched* or *covered*. It will sometimes be convenient to work with the *reconfiguration graph* $\mathcal{M}_k(G)$ of matchings of a graph G , which is defined as follows.

$$\begin{aligned} V(\mathcal{M}_k(G)) &:= \{M \subseteq E \mid M \text{ is a matching in } G, |M| = k\} \\ E(\mathcal{M}_k(G)) &:= \{MN \mid M, N \in V(\mathcal{M}_k(G)), |M \triangle N| = 2\} \end{aligned}$$

We denote by $\text{dist}_{\mathcal{M}_k(G)}(M, N)$ the distance of two matchings in $\mathcal{M}_k(G)$ and by $\text{diam}(\mathcal{M}_k(G))$ the maximal distance of any two vertices of $\mathcal{M}_k(G)$.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we provide the FPT algorithm and the approximation algorithm for finding a shortest reconfiguration sequence for two maximum matchings of a bipartite graph. Section 4 contains our hardness results for deciding the exact distance of two matchings and the exact diameter of the graph of matchings.

2 Hardness of Matching Distance

The goal of this section is to prove Theorem 1. The positive part of 1 is a consequence of the following lemma.

Lemma 5 (*). *MATCHING DISTANCE restricted to instances where at least one of the two matchings is not inclusion-wise maximal can be decided in polynomial time.*

To prove Lemma 5, we show that the distance of two matchings M_s and M_t , at least one of which is not inclusion-wise maximal, is either $|M_s \triangle M_t|/2$ or $|M_s \triangle M_t|/2 + 1$. Furthermore, it can be checked in polynomial time, which case applies. To prove the hardness part of Theorem 1, we show the following.

Theorem 6. *MATCHING DISTANCE admits no $o(\log n)$ -factor approximation unless $P = NP$, even when restricted to instances on bipartite graphs of maximum degree three.*

To show approximation hardness we prove in Section 2.2, that a sublogarithmic-factor approximation for MATCHING DISTANCE yields a sublogarithmic-factor approximation of SET COVER, using the construction from Section 2.1. However, the SET COVER problem is not approximable within a sublogarithmic factor, unless $P = NP$ [8].

Remark 1. *The hardness part of Theorem 1 also holds for the Token Sliding operation [14] since M_1 and M_2 are maximum.*

Let us briefly recall some definitions related to the SET COVER problem. An instance $I = (U, \mathcal{S})$ of SET COVER is given by a set U called *items* and a family \mathcal{S} of subsets of U called *hyperedges*. The task is to find the minimal number of sets in \mathcal{S} that are required to cover U . We denote this number by $\text{OPT}(I)$ and let $n := |U|$ and $m := |\mathcal{S}|$. Let $d := \max_{S \in \mathcal{S}} |S|$ be the maximum cardinality of a set in \mathcal{S} and for each $u \in U$ let $f_u = |\{S \in \mathcal{S} \mid u \in S\}|$ be the *frequency* of u . Furthermore, let $f := \max_{u \in U} \{f_u\}$ be frequency of I .

2.1 The Construction

We construct from the SET COVER instance $I = (U, \mathcal{S})$ an instance $I' = (G, M_1, M_2)$ of MATCHING DISTANCE. An illustration of the construction shown in Figure 2.

First, for each item $u \in U$ we create a 4-cycle C_u on the vertices $c_u^1, c_u^2, c_u^3, c_u^4$ and a path P_u on the vertices $p_u^1, p_u^2, \dots, p_u^{2f_u}$. We connect c_u^1 and p_u^1 with an edge for each $u \in U$. Furthermore, for each $S \in \mathcal{S}$, we create a path P_S on the vertices $p_S^1, p_S^2, \dots, p_S^{2|S|}$ and a path Q_S on the vertices $q_S^1, q_S^2, \dots, q_S^L$, where L is

an *odd* number that will be specified later. For each $S \in \mathcal{S}$, we connect P_S to Q_S with the edge $p_S^{2|S|}q_S^1$. Let us now simulate the containment in the hyperedges as follows. For each item $u \in U$, the *terminals* of P_u are the vertices p_u^i where i is even. For each hyperedge $S \in \mathcal{S}$, the *terminals* of P_S are the vertices p_S^i where i is odd. Now, we add edges forming a matching on the terminal vertices such that there is a (unique) edge in the matching between the terminals of P_u and the terminals of P_S if and only if $u \in S$. Note that such a matching exists since there are $|S|$ terminals in P_S and f_u terminals in P_u . Note that G is bipartite, since we create edges between vertices with an even index and vertices with an odd index. Also note that since we assume that $m = \text{poly}(n)$, the size of the instance is polynomial in $n + m$, whenever L is.

It remains to construct two matchings M_1 and M_2 of G . Observe that P_u and P_S (for $u \in U$ and $S \in \mathcal{S}$) are paths with an even number of vertices, they admit a unique perfect matching M_u (M_S) for each $u \in U$ ($S \in \mathcal{S}$). Similarly, the path Q_S admits a matching $N_S := \{q_S^1q_S^2, q_S^3q_S^4, \dots, q_S^{L-2}q_S^{L-1}\}$. Note that Q_S also admits another perfect matching if we push the matching to the right (this will be of importance for the proof of Theorem 6). Now let $M' = \bigcup_{u \in U} M_u \cup \bigcup_{S \in \mathcal{S}} M_S \cup N_S$. The matching M_1 is M' plus edges $c_u^1c_u^2$ and $c_u^3c_u^4$ for each $u \in U$. And the matching M_2 is M' plus the edges $E(C_u) \setminus M_1 = \{c_u^2c_u^3, c_u^1c_u^4\}$ for each $u \in U$. That is, M_1 and M_2 only differ on the 4-cycles in G and the vertices q_S^L are M_1 -free and M_2 -free for each $S \in \mathcal{S}$. This completes the construction of the instance $I' = (G, M_1, M_2)$. Note that the matchings M_1 and M_2 are maximum.

Figure 2 illustrates the above construction. The dashed edges are M_1 -edges and the dotted edges are M_2 edges. The terminal nodes that are used in order to model the incidence relation of the SET COVER instance are indicated by the square shapes. Informally, we think of L as a very large number, so in order to find a short reconfiguration sequence from M_1 to M_2 , it is desirable to minimize the number of times an alternating path from q_S^1 to q_S^L is reconfigured in order to switch the matching edges on each cycle gadget.

2.2 Proof of Theorem 6

Let $I = (U, \mathcal{S})$ be an instance of SET COVER and let $I' = (G, M_1, M_2)$ is an instance of MATCHING DISTANCE, which is constructed from I as described in Section 2.1. In order to prove the hardness-of-approximation result, we need to construct a reconfiguration sequence from M_1 to M_2 from a cover and vice versa. We need two lemmas and let $L := |U|(2 + f + d)$. Observe that since L is polynomial in $|I|$, the instance I' can be constructed in polynomial time.

Lemma 7 (*). *Let $C \subseteq \mathcal{S}$ be a cover of U . Then there is a reconfiguration sequence from M_1 to M_2 of length at most $2L|C| + 2|U|(2 + f + d)$.*

Lemma 8 (*). *There is a polynomial-time algorithm A' that constructs from a reconfiguration sequence τ from M_1 to M_2 of length $|\tau|$ a set cover $C \subseteq \mathcal{S}$ of cardinality at most $|\tau|/2L$.*

We are now ready to prove Theorem 6.

Proof of Theorem 6 (sketch). Suppose we have an $f(n')$ -factor approximation

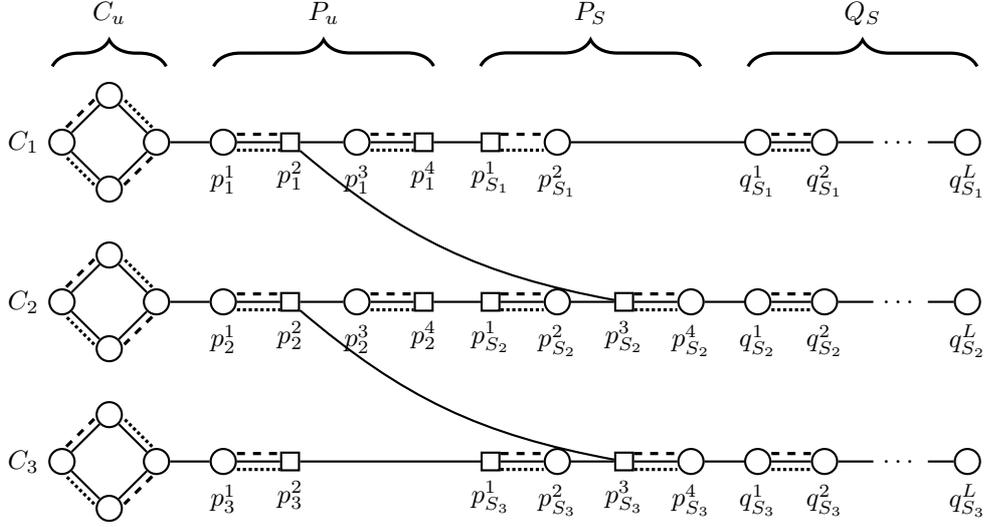


Figure 2: An example of the construction of an instance (G, M_1, M_2) of MATCHING DISTANCE from an instance (U, \mathcal{S}) of SET COVER, where $U = \{1, 2, 3\}$ and $\mathcal{S} = \{S_1, S_2, S_3\}$, $S_1 = \{1\}$, $S_2 = \{1, 2\}$, $S_3 = \{2, 3\}$. The terminals are indicated by square node shapes.

algorithm A for MATCHING DISTANCE, where $n' = |I'|$. Then

$$A'(I) \stackrel{\text{Lemma 8}}{\leq} \frac{A(I')}{2L} \stackrel{\text{Lemma 7}}{\leq} \frac{f(n')(\text{OPT}(I) \cdot 2L + 2|U|(f + d + 2))}{2L} \leq 2f(n') \text{OPT}(I) .$$

That is, we can compute in polynomial time a $2f(n')$ -approximate solution of I . Combining this with the result of Dinur and Steurer [8] completes the proof. \square

With a slight modification of the proof, we can show that the problem MATCHING DISTANCE remains hard, even if the matchings are not maximum.

Corollary 9 (*). MATCHING DISTANCE on bipartite graphs of maximum degree three and matchings that are not maximum does not admit a polynomial-time $o(\log n)$ -factor approximation algorithm, unless $\text{P} = \text{NP}$.

3 Distance of Matchings in Bipartite Graphs is FPT

The main result of this section is Theorem 2, which states that MATCHING DISTANCE for matchings on bipartite graphs is FPT, where the parameter is the size of the symmetric difference of the source and target matchings. In the following, let (G, M_s, M_t) be an instance of MATCHING DISTANCE, where the graph $G = (V, E)$ is bipartite. Due to Lemma 5, we may assume that M_s and M_t are inclusion-wise maximal, since otherwise we may find a shortest transformation in polynomial time. Our FPT algorithm distinguishes two cases. First, it may happen that in a shortest transformation some intermediate matching is

not inclusion-wise maximal. We show that in this case a shortest transformation may be found in polynomial time.

Lemma 10 (*). *There is a polynomial-time algorithm that outputs a shortest transformation from M_s to M_t via a matching M that is not inclusion-wise maximal, or indicates that no such transformation exists.*

Note that Lemma 10 properly generalizes Lemma 5 and therefore gives a positive result beyond the the complexity classification stated in Theorem 1. The proof idea is to find a cheapest transformation into a matching that is not inclusion-wise maximal respect to a certain cost measure that reflects the “progress” we make by performing exchanges along a given augmenting path. The progress is essentially the sum of the length of an augmenting path and the length of the remaining transformation, which can be determined by Lemma 5.

On the other hand, we need to check the length of a shortest transformation that avoids matchings that are not inclusion-wise maximal. For this purpose, we use a reduction to the problem DIRECTED STEINER TREE, which is defined as follows.

DIRECTED STEINER TREE

Input: Directed graph $D = (V, A)$, integral arc weights $c \in \mathbb{Z}_{\geq 0}^A$, root vertex $r \in V$, and terminals $T \subseteq V$.

Task: Find a minimum-cost directed tree in D that connects the root r to each terminal.

It is known that DIRECTED STEINER TREE parameterized by the number of terminals is FPT [2, 9]. For the remainder of this section let $d := |M_s \triangle M_t|$. Our reduction gives at most $d/2$ terminals. As a consequence, we may use the FPT algorithm from [2] for DIRECTED STEINER TREE to obtain the following result.

Lemma 11. *Let M_s and M_t be maximum. Then there is an algorithm that finds in time $2^{d/2} \cdot n^{O(1)}$ a shortest transformation from M_s to M_t , or indicates that no such transformation exists.*

In order to deal with matchings that are not maximum, we do the following. Let (U, W) be a bipartition of the vertex set V . For each cycle C in the symmetric difference of M_s and M_t , we have to guess if C is reconfigured using an alternating path from an exposed vertex in U or in W in a shortest transformation. For paths in the symmetric difference there is no choice. Therefore, the number of different choices for reconfiguring all paths and cycles in the symmetric difference is bounded by $2^{d/4}$, so we may check all possibilities and pick a shortest transformation. Theorem 2 then follows from lemmas 10 and 11. If M_s is maximum, then Lemma 11 only needs to be invoked once. By using the approximation algorithm for DIRECTED STEINER TREE from [5] instead of the exact algorithm from [2] in the proof of Lemma 11, we obtain Corollary 3.

Our techniques are not likely to generalize to matchings in non-bipartite graphs. We leave as an open problem whether finding a shortest transformation between two matchings in non-bipartite graphs is FPT in the size of the symmetric difference of source and target matchings.

In the following, let \mathcal{C} (resp., \mathcal{P}) be the set of (M_s, M_t) -alternating cycles (resp., (M_s, M_t) -alternating paths) in $(V, M_s \triangle M_t)$.

3.1 Reduction to Directed Steiner Tree

Let $I = (G, M_s, M_t)$ be an instance of MATCHING DISTANCE, where the $G = (U \cup W, E)$ is bipartite and the matchings M_s and M_t are maximum. We will reduce the task of finding a shortest transformation from M_s to M_t to the DIRECTED STEINER TREE problem. Note that if some edge is not contained in a maximum matching of G , we cannot use it for reconfiguration. Therefore, we may assume that every edge of G is contained in some maximum matching. Let X_s be the set of M_s -free vertices of G . Since M_s and M_t are maximum, we may assume the following.

Fact 12. *We may assume that $X_s \subseteq U$.*

The main feature of the reduction is that the arc-costs in the auxiliary directed graph will reflect the number of exchanges we need to perform in the corresponding transformation. The terminals are placed each cycle in $M_s \Delta M_t$ to ensure a correspondence between Steiner trees and transformations from M_s to M_t . Without loss of generality, let $X_s \subseteq U$ be the set of M_s -free vertices of G . We construct an instance (D, c, r, T) of DIRECTED STEINER TREE as follows. The digraph $D = (U', A)$ is defined by

$$\begin{aligned} U' &:= \{v \in U \mid \exists \text{ an even-length } M_s\text{-alternating path from } X_s \text{ to } v\} \cup \{r\} \\ A &:= \{uw \mid u, w \in U, \exists v \in W : uv \in E \setminus M_s, vw \in M_s\} \cup R, \end{aligned}$$

where r is a new vertex and $R := \{rv \mid v \in X_s\}$. For an arc $uw \in A$, let the weight c_{uw} be given by

$$c_{uw} := \begin{cases} 0 & \text{if } u = r, \\ 1 & \text{if there are two edges } uv \in M_t \text{ and } vw \text{ in } M_s. \\ 2 & \text{otherwise.} \end{cases}$$

The set T of terminals is given by $T := U' \cap \bigcup_{Z \in \mathcal{C} \cup \mathcal{P}} V(Z)$. Note that since M_s and M_t are matchings of G , any two distinct items in $\mathcal{P} \cup \mathcal{C}$ are vertex-disjoint. The root of the Steiner tree is the vertex r . This completes the construction of the instance (D, c, r, T) .

3.2 Proof of Lemma 11

Let (G, M_s, M_t) is an instance of MATCHING DISTANCE, where the $G = (U \cup W, E)$ is a bipartite graph and the matchings M_s and M_t are maximum. By Fact 12, we may assume that each M_s -free vertex is in U . Furthermore, let $I' = (D, c, r, T)$ be an instance of DIRECTED STEINER TREE, constructed according to Section 3.1, where $D = (U', A)$.

Let us first introduce some notation that will be used throughout this section. An arc of weight one is called *special*. Note that, by definition of A , all the arcs entering in w have to pass through the vertex v which is matched to w by M_s . In particular, for each pair of edges $uv \in M_t$ and $vw \in M_s$, there exists a (unique) special arc uw of D . Finally note that, since a vertex u is incident to at most one edge of M_s and one edge of M_t , the vertex u has at most one incoming special arc and at most one outgoing special arc. For each special arc $uw \in A$, there is some $Z \in \mathcal{C} \cup \mathcal{P}$, such that $u, w \in V(Z)$. Let $Z \in \mathcal{P} \cup \mathcal{C}$. We denote by

$A(Z) \in A$ the set of special arcs with both endpoints in $V(Z)$. If Z is a cycle of length $2k$ then $A(Z)$ is a directed cycle of length k and if Z is a path of length $2k$ then $A(Z)$ is a directed path of length k .

An arc of weight zero is called *artificial*. By definition, any artificial arc is incident to the root r . We first observe some properties of optimal Steiner trees for I' .

Proposition 13 (*). *Any optimal Steiner tree F for I' satisfies:*

- (i) *For each $P \in \mathcal{P}$, the tree F contains all arcs in $A(P)$.*
- (ii) *For each $C \in \mathcal{C}$, the tree F misses exactly one arc of $A(C)$.*
- (iii) *For each $P \in \mathcal{P}$, the root r is joined to the M_s -free vertex of P .*

The next lemma shows how to construct from an optimal Steiner tree F of cost $c(F)$ a reconfiguration sequence of length $c(F)$.

Lemma 14. *Let F be an optimal Steiner tree for I' . Then there is a reconfiguration sequence of length $c(F)$ that transforms M_s to M_t .*

Proof. Let A' be the arc-set of F . Let us consider a depth-first-search (DFS) traversal of the tree F starting at r , where at each vertex v , we select a successor that is joined to v by an arc of largest weight among all successors of v in F that have not been visited previously by the DFS. The traversal of F yields a sequence a_1, a_2, \dots, a_m of arcs of A' of length $m := 2|A'|$. This is the case since each arc is visited twice: once when going “down” in the tree F and a second time when going “up”, that is, when we are backtracking. Note that the preference for arcs of weight two will be important to prove the correctness of the transformation.

For $1 \leq i \leq m$, let $u_i, w_i \in U$, such that $a_i = u_i w_i$. If a_i is not artificial, let $P_i = u_i, v_i, w_i$ be the unique path of length two in G , such that $v_i w_i \in M_s$. Furthermore, if a_i is not artificial, let $\bar{e}_i := u_i v_i$ and $e_i := v_i w_i$. Note that for $1 \leq i \leq m$, if a_i not artificial, then $e_i \in M_s$ and $\bar{e}_i \in E \setminus M_s$. By Proposition 13, for each $C \in \mathcal{C}$, the tree F misses exactly one arc a_C of $A(C)$. By definition, the arc a_C is a shortcut for a path u, v, w of length two in G , such that $uv \in M_t$ and $vw \in M_s$. We denote by e_C the unique M_t -edge incident to the source of a_C . We will use the edge e_C when backtracking to complete the reconfiguration of the cycle C .

We now specify the reconfiguration sequence M_0, M_1, \dots, M_m , where $M_0 := M_s$. For each $1 \leq i \leq m$, let M_{i+1} be given as follows.

$$M_{i+1} := \begin{cases} M_i & \text{if } a_{i+1} \text{ is artificial} \\ M_i - e_{i+1} + \bar{e}_{i+1} & \text{otherwise, if we traverse } a_{i+1} \text{ downwards,} \\ M_i & \text{otherwise, if } a_{i+1} \text{ is special,} \\ M_i - \bar{e}_{i+1} + e_C & \text{otherwise, if } a_i \in A(C) \text{ for } C \in \mathcal{C}, \\ M_i - \bar{e}_{i+1} + e_{i+1} & \text{otherwise.} \end{cases} \quad (1)$$

Note that if several cases apply, then we choose the first one in the given order. The next two claims establish that M_0, M_2, \dots, M_m is a reconfiguration sequence from M_s to M_t . Claim 1 is omitted due to space limitations.

Claim 1. Let $1 \leq j \leq m$ and suppose we have processed arcs a_1, a_2, \dots, a_j , so M_j is our current set of edges. Let $a_i \in A$, such that a_i is not artificial. Then the following holds:

- If a_i has been traversed downwards but not upwards, then $\bar{e}_i = u_i v_i$ is in $M_i \setminus M_j$.
- If a_i has not been traversed so far, then $e_i = v_i w_i$ is in $M_i \cap M_j$.
- If an artificial arc rv has not been visited so far, then v is M_j -free.
- Let C be a cycle of \mathcal{C} and let $a_C = uv$ be the arc of C not in F and $a = u'u$ the arc before a_C in C and $a' = ww'$ the arc after it in C . Then if at step j , a has been visited upwards and a' has not then M_j is u -free.

Claim 2. $M_m = M_t$.

From the definition of (1) it is straightforward to see that all the artificial arcs lead to no exchange, all the special arcs lead to one exchange and all the other arcs lead to two exchanges. Therefore, the total number of exchanges is $c(F)$, as claimed. \square

We show how to obtain from a reconfiguration sequence \mathcal{S} that transforms M_s to M_t a subgraph of $D_{\mathcal{S}}$ of D , such that the root r is connected to each terminal in D . Let $\mathcal{S} := M_0, M_1, \dots, M_m$ be a reconfiguration sequence of matchings of G . Let $A(\mathcal{S})$ be the subsets of the arcs of D that correspond to some exchange in \mathcal{S} . That is, $A(\mathcal{S})$ is given by

$$A(\mathcal{S}) := \{uv \in A \mid \exists 1 \leq i \leq m, v \in U' : \{uv, vw\} = M_{i-1} \Delta M_i\}$$

Furthermore, let $D_{\mathcal{S}} := (U', A(\mathcal{S}) \cup R) \subseteq D$ be the subgraph of D given by the arcs $A(\mathcal{S})$ and the artificial arcs R .

Lemma 15 (*). *Let $\mathcal{S} := M_0, M_1, \dots, M_m$ be a reconfiguration sequence from M_s to M_t . Then, for each terminal $w \in T$, there is a rw -path in $D_{\mathcal{S}}$.*

The main idea in the proof of Theorem 11 is as follows. Given some reconfiguration sequence that transforms M_s into M_t , we obtain a subgraph of D that contains a Steiner tree F . Let F^* be an optimal solution to I' . By Lemma 14, we obtain from F^* a reconfiguration sequence of length $m^* = c(F^*)$ that transforms M_s to M_t . Hence, we have $m^* = c(F^*) < c(F) \leq m$. Therefore, the directed Steiner tree F is optimal if and only if \mathcal{S} is a shortest reconfiguration sequence. Using the exact DIRECTED STEINER TREE algorithm from [2] to compute F^* yields Theorem 2, while using the approximation algorithm from [5] yields Corollary 3.

3.3 Proof of Theorem 2

By Lemma 10, we may compute a shortest transformation from M_s to M_t via a matching that is not inclusion-wise maximal in polynomial time. Hence it remains to deal with the case that such a transformation is expensive (or does not exist). For this purpose, we reduce the problem to the case where both matchings are maximum and repeatedly use the algorithm from Lemma 11. Let us fix some bipartition (U, W) of the vertex set V . If M_s is not maximum, then

there is an M_s -augmenting path, so there are M_s -free vertices on both sides of the bipartition (U, W) . In particular, it may happen that M_s -free vertices on both sides can be used in order to reconfigure a cycle in $M_s \Delta M_t$, as explained below. Hence, for each cycle in $|M_s \Delta M_t|$, we have to check if in a shortest transformation it is reconfigured using an exposed vertex from U or from W . Since the number of choices is bounded by a function of the size of the symmetric difference, we just enumerate all possibilities.

The reduction to the restriction of MATCHING DISTANCE where both matchings are maximum is as follows. We recall the following lemma from [13].

Lemma 16 ([13, Lemma 1]). *Suppose that M_s and M_t are maximum matchings of G . Then, there is a transformation from M_s to M_t if and only if, for each cycle $C \in \mathcal{C}$, there is an M_s -alternating path in G connecting an M_s -free vertex to C .*

In the light of this lemma we say that a cycle $C \in \mathcal{C}$ is *reconfigurable from U* if there is an M_s -alternating path that connects an M_s -free vertex in U to C . If a cycle $C \in \mathcal{C}$ is neither reconfigurable from U nor from W , then there is no transformation from M_s to M_t . Furthermore, we say that a path $P \in \mathcal{P}$ is reconfigurable from U , if P contains an M_s -free vertex in U . Let us fix for each $F \in \mathcal{C} \cup \mathcal{P}$ a choice $S_F \in \{U, W\}$, such that F is reconfigurable from S_F . Let $S \in \{U, W\}^{\mathcal{C} \cup \mathcal{P}}$ be a tuple corresponding to feasible choice of the side of the bipartition for each item in $\mathcal{C} \cup \mathcal{P}$. Furthermore, let X_s be the set of M_s -free vertices of G . Based on the choice S , we construct two instances $I_U(S)$ and $I_W(S)$ of MATCHING DISTANCE on maximum matchings as follows. Let

$$M_U(S) := \bigcup_{F \in \mathcal{C} \cup \mathcal{P}: S_F=W} (E(F) \cap M_s) \cup \bigcup_{F \in \mathcal{C} \cup \mathcal{P}: S_F=U} (E(F) \cap M_t) ,$$

and let $I_U(S) := (G - (X_s \cap W), M_s, M_U(S))$ and $I_W(S) := (G - (X_s \cap U), M_U(S), M_t)$. Note that M_s and $M_U(S)$ are maximum in $G - (X_s \cap W)$ and $M_U(S)$ and M_t are maximum in $G - (X_s \cap U)$.

Let us now describe the algorithm that outputs a shortest transformation from M_s to M_t if it exists. For each feasible choice $S \subseteq \{U, W\}^{\mathcal{C} \cup \mathcal{P}}$ we invoke Lemma 11 on I_U (resp., I_W) to obtain a transformation α_1 (resp., α_2) from M_s to $M_U(S)$ (resp., $M_U(S)$ to M_t) if it exists. Clearly, by combining α_1 and α_2 we obtain a transformation from M_s to M_t . Let us denote this transformation by $\alpha(S)$ and by $|\alpha(S)|$ its length. Let $S^* \in \{U, W\}^{\mathcal{P} \cup \mathcal{C}}$ such that $|\alpha(S^*)|$ is minimal. If there is no transformation for any choice S , then $|\alpha(S^*)| = \infty$. According to Lemma 10 we determine in polynomial time the length $|\beta^*|$ of a shortest transformation from M_s to M_t via a matching that is not inclusion-wise maximal. If $|\alpha(S^*)|$ and $|\beta^*|$ are both not finite, then (G, M_s, M_t) must be a NO-instance. Otherwise, we output either the transformation of length $|\beta^*|$ via a matching that is not inclusion-wise maximal, or a transformation of length $|\alpha(S^*)|$ via $M_U(S^*)$, depending on which is shorter.

The running time of the algorithm is dominated by the computation of $|\alpha(S^*)|$ and S^* . Both $|\alpha(S^*)|$ and S^* , as well as a corresponding transformation if it exists can be computed in time $2^{d/2} \cdot 2^{d/2} \cdot n^{O(1)}$ by Lemma 11. The leading factor of $2^{d/2}$ bounds the number of feasible choices of $S \in \{U, W\}^{\mathcal{P} \cup \mathcal{C}}$. The overall running time of the algorithm is therefore bounded by $2^d \cdot n^{O(1)}$ as claimed. The correctness of the algorithm is a consequence of the following

lemma, which ensures that in the case that no optimal transformation visits a matching that is not inclusion-wise maximal, we have that an optimal choice of $S \subseteq \{U, W\}^{\mathcal{C} \cup \mathcal{P}}$ also results in an optimal transformation from M_s to M_t .

Lemma 17 (*). *Suppose that no shortest transformation from M_s to M_t has an intermediate matching that is not inclusion-wise maximal. Let $\tau := M_0, M_1, \dots, M_m$ be a shortest transformation from M_s to M_t of length m . Then $|\alpha(S^*)| \leq m$.*

4 Exact Distance and Diameter

We will consider the exact versions of the problems of determining the distance of two matchings and the diameter of the reconfiguration graph. Before presenting our hardness result for these problems, let us first prove that we can determine in polynomial if the diameter of the reconfiguration graph of matchings is finite. In other words, for any $k \geq 0$, we can determine in polynomial time if $\mathcal{M}_k(G)$ is connected.

Note that proof of [13, Proposition 2] provides a polynomial-time algorithm for deciding, whether the distance of two given matchings is finite. We show that there is a polynomial-time algorithm that decides if the diameter of $\mathcal{M}_k(G)$ is finite. To this end we need a condition that characterizes YES-instances of MATCHING RECONFIGURATION, which was given in [13, Lemma 1], as well as the *Edmonds-Gallai decomposition* of the vertex set of G [22, Ch. 24.4b]. Using these two ingredients, we can check efficiently if the diameter of $\mathcal{M}_k(G)$ is finite.

Theorem 18 (*). *There is a polynomial-time algorithm that, given a graph G and a number $k \in \mathbb{N}$, decides if $\mathcal{M}_k(G)$ is connected.*

For the remainder of this section we will consider maximum matchings. That is, we restrict our attention to instances of EXACT MATCHING DISTANCE and EXACT MATCHING DIAMETER, where the number k is the equal to the size of a maximum matching of the input graph G . Using a similar construction to the one from Section 2.1, we show that the problem of EXACT MATCHING DISTANCE is complete for the class D^{P} , which was introduced in [20] as the following class of languages

$$\text{D}^{\text{P}} := \{L_1 \cap L_2 \mid L_1 \in \text{NP}, L_2 \in \text{coNP}\}$$

From the D^{P} -completeness of EXACT MATCHING DISTANCE, the D^{P} -hardness of EXACT MATCHING DIAMETER will follow in a relatively straightforward manner. To show that EXACT MATCHING DISTANCE is D^{P} -hard we use a reduction from EXACT VERTEX COVER, which is given as follows.

EXACT VERTEX COVER

Input: Graph G and number ℓ .

Question: Is ℓ the minimum size of a vertex cover?

The Construction. Let $H = (V, E)$ be a graph. Starting with the empty graph, we construct a graph G and two matchings M and N of G as follows. We first create a vertex t . For each $e \in E$, we create a 4-cycle C_e on the vertices $c_e^1, c_e^2, c_e^3, c_e^4$. Then, for each $v \in V$, we create two vertices p_v^1 and p_v^2 and create a path P_v $p_v^1 p_v^2 t$. Now, for $e \in E$ and $v \in V$, we add an edge $c_e^1 p_v^1$ whenever

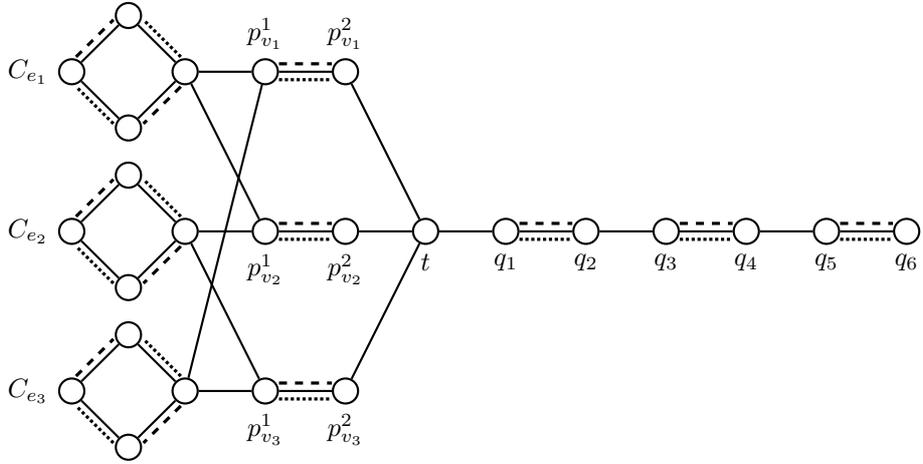


Figure 3: Example of the construction of G , M , N from the graph $H = C_3$.

e is incident to v . In a final step we create new vertices q_1, \dots, q_6 and add a path t, q_1, q_2, \dots, q_6 to G . Note that we added this path in order to avoid a case analysis when we determine the maximal distance of any two matchings. Observe that G is bipartite and that a maximum matching in G has precisely one unmatched vertex. Let M and N be two maximum matchings that both leave t exposed and are disjoint on C_e for each $e \in E$. This completes the construction of G , M , and N . An illustration of the construction where H is a cycle on three vertices is shown in Figure 3. The dashed edges are in M and the dotted edges are in N .

Proof of Theorem 4 (sketch) By [4, Theorem 5.4], the problem EXACT VERTEX COVER is complete for the class D^{P} . Theorem 4 now follows directly from this result and the next two lemmas. The key insight is that if we know the size of a smallest vertex cover, then we know the distance of the two matchings and the diameter of the reconfiguration graph after performing the construction above.

Lemma 19 (*). *Let H be a graph and let G , M , N be the graph and the two matchings obtained according to the construction above. Then $\text{dist}_{\mathcal{M}_k(G)}(M, N) = 3|E(H)| + 2\tau(H)$, where $\tau(H)$ is the size of a smallest vertex cover of H .*

Lemma 20 (*). *Let H be a graph and let G , M , N be the graph and the two matchings obtained according to the construction above. Then $\text{diam}_{\mathcal{M}_k(G)} = \text{dist}_{\mathcal{M}_k(G)}(M, N) + 6$.*

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A Edmonds-Gallai Decomposition

A matching of a bipartite graph $(A \cup B, E)$ is called *A-perfect*, if it matches each vertex of A . Let $G = (V, E)$ be a graph, let M be a maximum matching of G and let $X := \{v \in V(G) \mid v \text{ is } M\text{-free}\}$. Consider the following partition of the vertex set $V(G)$ into $D(G), A(G), C(G) \subseteq V(G)$.

- $D(G) := \{v \in V(G) \mid \text{there is an } M\text{-alternating path of even length from } X \text{ to } v\}$
- $A(G) := \{v \in V(G) \setminus D(G) \mid \text{there is an } M\text{-alternating path from } X \text{ to } v\}$
- $C(G) := \{v \in V(G) \mid \text{there is no } M\text{-alternating path from } X \text{ to } v\}$

The following classical theorem states that $D(G)$, $A(G)$, and $C(G)$ depend only on G , but not on the choice of the maximum matching.

Theorem 21 (Edmonds-Gallai decomposition, see [22, Ch. 24.4b]). *Let G be a graph and $D(G)$, $A(G)$, $C(G)$ be given as above. Then, a maximum matching of G can be partitioned into*

1. *a perfect matching of $G[C(G)]$,*
2. *a matching that leaves precisely one vertex unmatched in each component of $G[D(G)]$,*
3. *and an A -perfect matching from $A(G)$ to $D(G)$.*

Furthermore, the partition of G into $D(G), A(G), C(G)$ can be found in polynomial time.

B Proofs Omitted from Section 2

Proof of Lemma 5. Let $I = (G, M_1, M_2)$ be an instance of MATCHING DISTANCE where at least one of M_1 and M_2 is not inclusion-wise maximal. Without loss of generality, we assume that M_1 is not inclusion-wise maximal. Since M_1 and M_2 are matchings, the graph on the edges $M_1 \triangle M_2$ is composed of disjoint paths and cycles. Let $\text{OPT}(I)$ be the shortest length of a reconfiguration sequence that transforms M_1 into M_2 . We show that $\text{OPT}(I)$ is either $|M_1 \triangle M_2|/2$ or $|M_1 \triangle M_2|/2 + 1$. The exact value can be determined as follows. The shortest reconfiguration sequence is equal to $|M_1 \triangle M_2|/2$ if and only if one of the following statements is true.

- All the components of $M_1 \triangle M_2$ are paths;
- One of the paths in $M_1 \triangle M_2$ has an odd number of edges.

Note that $|M_1 \triangle M_2|/2$ is indeed a lower bound on the length of the reconfiguration sequence. Assume that at least one of the above statements is true. If the symmetric difference of M_1 and M_2 only contains paths, then we can move tokens from M_1 to M_2 in a greedy fashion until the target configuration M_2 is reached. Assume now that $M_1 \triangle M_2$ contains a path with an odd number of vertices. Let us prove that there is a reconfiguration sequence that transforms M_1 into M_2 by induction on the number of cycles in $M_1 \triangle M_2$. Since

$|M_1| = |M_2|$, there is a path in $M_1 \triangle M_2$ containing more edges of M_2 than edges of M_1 . We slide tokens on this path as in the previous case, until a path P consisting of a single M_2 -edge remains in the symmetric difference. Let C be a cycle in $M_1 \triangle M_2$. We can transform C into a path, by moving a token from C to P . This decreases the number of cycles in the symmetric difference by one and leaves a path that contains more M_2 -edges than M_1 -edges. We invoke the induction hypothesis and the statement follows.

Assume now that $M_1 \triangle M_2$ contains a cycle and does not contain a path with an odd number of edges. If there is a transformation of length $|M_1 \triangle M_2|/2$, then, at each step, we must move a token from M_1 to M_2 . We can do so, again in a greedy fashion, by moving tokens on paths in $M_1 \triangle M_2$. At some point however, only cycles remain the symmetric difference. But then, we cannot move any token, such that the number of tokens on M_2 is increased. Therefore, there is no transformation from M_1 to M_2 of length $|M_1 \triangle M_2|/2$.

Let us finally prove that a transformation of length $|M_1 \triangle M_2|/2 + 1$ always exists. Since M_1 is not inclusion-wise maximal, it is possible to move any token of M_1 to some edge e , which is not incident to any edge of M_1 . Let f be an edge of M_1 in a cycle of $M_1 \triangle M_2$. We can move the token of f to e . Let M'_1 the resulting matching. Note that the cycle on which f appeared is now a path of odd length in the symmetric difference. Moreover, we have that $|M'_1 \triangle M_2| \leq |M_1 \triangle M_2|$. By our previous arguments, there is a transformation of length at most $|M_1 \triangle M_2|/2$ between M'_1 and M_2 . It follows that there is a reconfiguration sequence of length $|M_1 \triangle M_2|/2 + 1$ that transforms M_1 into M_2 .

All the steps can indeed be performed in polynomial time, which concludes the proof. \square

Proof of Lemma 7. As usual for the token sliding operation, we will frequently say that we *reconfigure the alternating path P* by “sliding” the matching edges one-by-one, until the other end-vertex of the path P is M_1 -free. In this procedure, one exchange is performed per matching-edge.

Let $C = \{S_1, S_2, \dots, S_k\}$ be a cover of U of size $k = \text{OPT}(I)$. We construct a transformation from M_1 to M_2 of the desired size. For $1 \leq i \leq k$, let $T_i \subseteq U$ be the items that are covered by S_i but not by S_j , $1 \leq j < i$. Note that since C is a set cover, $U = \cup_{i=1}^k T_i$.

Let us present a way to reconfigure $C_u \cap M_1$ to $C_u \cap M_2$ for each $u \in T_i$ as follows in such a way, after these operations the resulting matching still contains the edges of M' (recall that M' is the matching without the edges of the C_4 's). First, we reconfigure the M_1 -alternating path from $q_{S_i}^1$ to the M_1 -free vertex $q_{S_i}^L$, which takes L steps and leaves vertex $q_{S_i}^1$ exposed. Now, for each $u \in T_i$, we have the following path P' in G : the subpath of P_u from p_u^1 until the terminal vertex p_u^j that is connected to some vertex $p_{S_i}^{j'}$ and then the subpath of P_S from $p_{S_i}^{j'}$ to $q_{S_i}^1$. Note that by definition of M' this path is an alternating path. We reconfigure the alternating path P' , which leaves p_u^1 exposed and takes at most $2(|S_i| + f_u)$ steps. Since p_u^1 is exposed, we can reconfigure the cycle $C_u \cap M_1$ to $C_u \cap M_2$ with three exchanges such that q_u^1 is again exposed. We finally undo the changes on P' and have again $p_{S_i}^1$ exposed. We repeat this operation for every $u \in T_i$. We finally reconfigure back the path Q_{S_i} . Note that after all these steps, the resulting matching still contains the edges of M' .

Let us now count the number of steps we performed to reconfigure $C_u \cap M_1$ to $C_u \cap M_2$ for each $u \in T_i$. The reconfiguration of Q_{S_i} costs $2L$ steps (L steps at the beginning and L steps at the end to put it back). Moreover, for each $u \in T_i$ we perform at most $3 + 2(|S_i| + f_u)$ exchanges.

So if we repeat this operation for every S_i , we know that we will reconfigure the matching on every C_4 since $\bigcup S_i = U$. So the total number of exchanges we performed is at most

$$2L|C| + \sum_{1 \leq i \leq k} |T_i|(3 + 2(f + |S_i|)) \leq 2L|C| + 2|U|(2 + f + d)$$

as claimed. \square

Proof of Lemma 8. Since M_1 and M_2 are maximum, any $u \in U$ a transformation from M_1 to M_2 must have p_u^1 exposed at some point. Indeed if a matching edge on the cycle C_u is moved, it has to be moved an edge incident to it and then, by construction on an edge incident to p_u^1 .

Let us consider a transformation from M_1 to M_2 . Given a matching M , we say that a set $S \in \mathcal{S}$ is *active* if q_S^L is covered by M . Let $C \subseteq \mathcal{S}$ be the sets that are active with respect to some matching of the transformation τ . Let us prove that the sets C cover the items U . Let $u \in U$. As we already mentioned, for each $u \in U$, the vertex p_u^1 has to be exposed at some step M of the transformation. Let us prove that a set $S \in \mathcal{S}$ containing u is active with respect to M . We can then define an M -alternating path starting on p_u^1 that alternates between edges in $E \setminus M$ and edges in M . Since P_u is of even length and $p_u^1 p_u^2$ is not in M either we have an augmenting path included in P_u (a contradiction to the maximality of M) or this alternating path must leave P_u . By the construction of G , such an edge is of the form $e := p_u^i p_S^j$ where i is even and j is odd. Note moreover, that Let us follow the M -alternating path along P_S and Q_S until it is no longer possible. Three cases may occur: (i) the alternating path ends at q_S^L . In this case S is active and the conclusion holds since by construction $u \in S$. (ii) the alternating path ends at some other vertex w of P_S or Q_S . Then w is exposed and we have thus discovered an augmenting path, a contradiction with the maximality of M . (iii) the alternating path leaves P_S via some vertex p_S^r (since vertices of Q_S have degree 2 (their precessors and successors in $Q_S \cup P_S$)). Then there is some $v \in U$ such that M contains an edge $p_v^s p_S^r$, where s is even. Since it is no longer possible to follow P_S in p_v^s , it means that p_v^s is incident to an edge of M (otherwise we would take $p_S^r p_S^{r+1}$ in the alternating path). Since $e := p_u^i p_S^j$ where j is odd is in M then r must be even. By construction of G , p_S^r has degree 2 and then there is no edge $p_v^s p_S^r$, a contradiction. So only case (i) can occur, and then a set S containing u is active for M .

Let $C \subseteq \mathcal{S}$ be the subset of sets that are active in some step of the transformation τ . Clearly, there is a polynomial-time algorithm A' that outputs C given τ . By the discussion above, the set C is a set cover. Let us finally prove that the size of C is at most $|\tau|/2L$. For each set $S \in C$ we have to reconfigure an alternating path from q_S^1 to q_S^L twice (once to expose q_S^1 and once to expose again q_S^L), the output C of A' has cardinality at most $|\tau|/2L$. \square

Proof of Theorem 6. Suppose we have an $f(n')$ -factor approximation algorithm A for MATCHING DISTANCE, where $n' = |I'|$. That is, A computes a reconfiguration sequence from M_1 to M_2 of length at most $f(n') \cdot \text{OPT}(I')$. Then we

can use algorithm A' from Lemma 8 to compute a $2f(n')$ -approximate solution of I :

$$\begin{aligned} A'(I) &\leq \frac{A(I')}{2L} \leq \frac{f(n') \cdot \text{OPT}(I')}{2L} \\ &\leq \frac{f(n')(\text{OPT}(I) \cdot 2L + 2|U|(f + d + 2))}{2L} \\ &\leq f(n') \text{OPT}(I) + 1 \\ &\leq 2f(n') \text{OPT}(I) , \end{aligned}$$

where $A'(I)$ is the size of the cover A' computes. The first inequality follows from Lemma 8, the third inequality from Lemma 7 and the fourth one from the definition of L . Suppose for a contradiction that there is a polynomial-time $o(\log n')$ -approximation algorithm for MATCHING DISTANCE, so we let $f(n') = o(\log n')$. By the construction of G and $m = \text{poly}(n)$, we have $n' = \text{poly}(n)$. Then, by the reasoning above, we have a $2f(n^k)$ -approximation algorithm for SET COVER for some constant k . But

$$\limsup_{n \rightarrow \infty} \frac{2f(n^k)}{\log n} = \limsup_{n \rightarrow \infty} \frac{2f(n)}{\log n^{1/k}} = \limsup_{n \rightarrow \infty} \frac{2kf(n)}{\log n} = \limsup_{n' \rightarrow \infty} \frac{2kf(n')}{\log n'} = 0$$

Therefore, a sublogarithmic-factor approximation guarantee for MATCHING DISTANCE implies a sublogarithmic-factor approximation guarantee for SET COVER. But there is no polynomial-time $(1 - \varepsilon) \log n$ -factor approximation algorithm for SET COVER for any $\varepsilon > 0$ unless $\text{P} = \text{NP}$ [8]. \square

Proof of Corollary 9. We modify the construction from Section 2.1 slightly, by adding a path R $r_1, r_2, \dots, r_{L'}$ to the graph G and joining q_S^1 to r_1 , where $L' \geq 3|U| + (L + d + f)|S|$ be odd. Furthermore, we add the edges $r_1r_2, r_3r_4, \dots, r_{L'-2}r_{L'-1}$ to the two matchings M_1 and M_2 . Clearly, the distance from M_1 or M_2 to a matching that is not inclusion-wise maximal is larger than the distance between M_1 and M_2 . Hence, it is easily verified that lemmas 7 and 8 as well as Theorem 6 hold also for the modified construction. \square

C Proofs Omitted from Section 3

C.1 Proof of Lemma 10

We may check in polynomial time if a transformation from M_s to M_t exists [13, Proposion 1]. So let us assume that such a transformation exists. Then there is a transformation from M_s to M_t via a matching that is not inclusion-wise maximal if and only if M_s and M_t are not maximum. Again, we can check this condition in polynomial time and may assume in the following that M_s and M_t are not maximum.

Our first claim follows from the first part of the proof of Lemma 5.

Claim 1. If $M_s \triangle M_t$ contains a path of odd length, then there is a polynomial-time algorithm that outputs a shortest transformation from M_s to M_t .

Hence we may assume that $M_s \triangle M_t$ contains only even cycles and paths.

Observe that by using any M_s -augmenting path, we can reconfigure M_s into a matching that is not inclusion-wise maximal. Hence, our task is to find a

cheapest augmenting path with respect to a certain cost measure that reflects the “progress” we make by performing exchanges along a given augmenting path. The progress is essentially the sum of the length of the remaining transformation and the length of the augmenting path. Let us fix a bipartition (U, W) of the vertex set of G and let X be the set of M_s -free vertices in U . Consider the following auxiliary digraph $D = (U', A)$ given by

$$\begin{aligned} U' &:= \{v \in U \mid \exists \text{ an even-length } M_s\text{-alternating path from } X \text{ to } v\} \\ A &:= \{uw \mid u, w \in U, v \in W : uv \in E \setminus M_s, vw \in M_s\} . \end{aligned}$$

Let $Y := \{v \in U \mid v \text{ has a neighbor in } G \text{ that is } M_s\text{-free}\}$. Directed XY -paths in D are in 1-to-1 correspondence with the M_s -augmenting paths in G . For an XY -path P in D , we denote by P_G the corresponding M_s -alternating path in G . Let us define arc-costs $c \in \mathbb{Z}^A$. For an arc $uw \in A$ that corresponds to a path u, v, w in G , let

$$c_{uw} := \begin{cases} 0 & \text{if } uv \in M_t \setminus M_s \text{ and } vw \in M_s \setminus M_t, \\ 0 & \text{otherwise, if } w \in V(F) \text{ for some } F \in \mathcal{P}, \text{ such that } V(P) \cap Y \neq \emptyset, \\ 1 & \text{otherwise, if } uv \in E \setminus (M_s \cup M_t) \text{ and } vw \in M_s \setminus M_t, \\ 2 & \text{otherwise, if } uv \in E \setminus (M_s \cup M_t) \text{ and } vw \in M_s \cap M_t . \end{cases}$$

For an XY -path P in D , let

$$\delta_P = \begin{cases} 1 & \text{if } (M_s \triangle E(P_G)) \triangle M_t \text{ contains a cycle, and} \\ 0 & \text{otherwise .} \end{cases}$$

Lemma 10 follows from the next two claims.

Claim 2. Let P be an XY -path of cost $c(P)$ in D . Then there is a transformation from M_s to M_t of length at most $c(P) + |M_s \triangle M_t|/2 + \delta_P$.

Proof of Claim 2. Our goal is to transform M_s into a matching that is not inclusion-wise maximal by using exchanges along P . Let $\ell(P)$ be the number of arcs of P . For an arc a of D , let u_a, v_a, w_a be the path in G that corresponds to a . Starting from the matching M_s , for each arc a on P , in forward direction, we swap $v_a w_a$ for $u_a v_a$ to obtain the matching M . This amounts to $\ell(P)$ exchanges. Observe that M is not inclusion-wise maximal. By using the algorithm from Lemma 5, a transformation from M_s to M_t via M has length $\ell(P) + |M \triangle M_t|/2 + \delta_P$.

In order to obtain a bound in terms of $c(P)$, we first perform some preprocessing. Let x (resp., y) be the starting vertex (resp., end vertex) of P . Suppose that P meets an (M_s, M_t) -alternating path containing a vertex in Y . Let F be the first such path and let $z \in U$ be the first vertex of F that is visited by P . Then we can reconfigure M_s into a matching M'_s , such that z has a neighbor that is M'_s -free and the symmetric difference between the current matching and M_t decreases by two in each step. To this end, we perform exchanges along F as follows. Let y' be the vertex of F in Y . Then there is a path Q from z to y' in D that visits only vertices of F . We process each arc a of Q in reverse order (from y' to z). Then w_a is matched to the vertex v_a by M_s . Furthermore, the vertex w_a must be matched to a vertex w' by M_t , such that w' is free with respect to the current matching. Note that if a is the first arc of Q we process, then w_a

must be covered by M_t , since otherwise M_t is not maximal. We exchange $v_a w_a$ for $w_a w'$, which decreases the size of the symmetric difference of the current matching and M_t by two. Let M'_s be the resulting matching.

Now we have that $|M'_s \triangle M_t| = |M_s \triangle M_t| - |M_s \triangle M'_s|$. Let $|M_s \triangle M'_s| =: 2k$. It now remains to consider the xz -subpath P' of P in order to reach a matching that is not inclusion-wise maximal. In the case that P does not meet an (M_s, M_t) -alternating path, we let $P' := P$, $M'_s = M_s$, $k = 0$, and $z := y$. Either way, we have that z has an M'_s -free neighbor and we have performed k exchanges to reach M'_s . Recall that an arc into an (M_s, M_t) -alternating cycle has cost one. It follows that if $\delta_P = 0$ and $\delta_{P'} = 1$, then $c(P) \geq c(P') + 1$. Hence, we have that

$$c(P) + |M_s \triangle M_t|/2 + \delta_P \geq c(P') + |M'_s \triangle M_t|/2 + \delta_{P'} + k .$$

We show that $\ell(P') + |(M'_s \triangle E(P'_G)) \triangle M_t|/2 = c(P') + |M'_s \triangle M_t|/2$. We start with the matching M'_s and process the arcs of P' in forward direction. We exchange for each arc a of P' the edges $v_a w_a$ and $u_a v_a$. To prove the equality, consider the relative changes to the size of the symmetric difference between the current matching and M_t . If a has cost two, then it corresponds to an exchange that increases the size of the symmetric difference by two. If a has cost one, then it corresponds to an exchange that does not alter the size of the symmetric difference. If a has cost zero we distinguish two cases. First, suppose that w_a is not on some (M_s, M_t) -alternating path that contains a vertex in Y . Then the corresponding exchange decreases the size of the symmetric difference by two. Otherwise, due to the preprocessing, we know that a is the final arc of P' and its target vertex w_a is matched by M'_s to v_a . Furthermore, since M_t is maximal, we have that w_a has an M'_s -free neighbor w' , such that $ww' \in M_t \setminus M_s$. So we may exchange $v_a w_a$ for $w_a w'$ to turn the current matching into a non-inclusion-wise maximal one. At the same time, we reduce the size of the symmetric difference with M_t by two. By counting the number of exchanges and keeping track of the size of the symmetric difference of the current matching and M_t , the claimed equation follows.

Putting things together, there is a transformation from M'_s to M_t of length

$$\begin{aligned} \ell(P') + |(M'_s \triangle E(P'_G)) \triangle M_t|/2 + \delta_{P'} &= c(P') + |M'_s \triangle M_t|/2 + \delta_{P'} \\ &\leq c(P) + |M_s \triangle M_t|/2 + \delta_P . \end{aligned}$$

□

Claim 3. Let P^* be a minimum-cost XY -path in D . Then a transformation from M_s to M_t via a matching M that is not inclusion-wise maximal has length at least $c(P^*) + |M_s \triangle M_t|/2 + \delta_{P^*}$.

Proof of Claim 3. Let N^* be a matching that is not inclusion-wise maximal, such that there is a shortest transformation of length L from M_s to M_t via N^* . So there is an edge $e \in E(G)$, such that $N^* + e$ is a matching. Since $N^* + e$ contains more edges than M_s , the symmetric difference of $N^* + e$ and M_s contains an augmenting path, which corresponds to an xy -path Q^* in D , where $x \in X$ and $y \in Y$. Then

$$L \geq \ell(Q^*) + |(M_s \triangle E(Q^*_G)) \triangle M_t|/2 + \delta_{Q^*_G} = c(Q^*) + |M_s \triangle M_t|/2 + \delta_{Q^*_G} . \quad (2)$$

The first inequality follows, since the shortest transformation performs at least $\ell(Q^*)$ exchanges to reach the matching N^* . By Lemma 5, a shortest transformation from M to M_t has length $|M \triangle M_t|/2 + \delta_{Q^*} = |(M_s \triangle E(Q_G^*)) \triangle M_t| + \delta_{Q^*}$.

Let us prove the final equality of (2). The argument is similar to the proof of Claim 2. For an arc a of D , let u_a, v_a, w_a be the path in G that corresponds to a . Note that Q^* visits an (M_s, M_t) -alternating path that contains a vertex in Y at most once; otherwise there is a shorter transformation. If Q^* visits such a path F , we split Q^* into two paths R and R' at a vertex z , such that z is the last vertex of Q^* not on F . If Q^* visits no such path, then we let R' be empty and $R := Q^*$. For each arc a of R' in reverse order (from y to z), we do the following. The vertex w_a is matched to the vertex v_a by M_s . Furthermore, w_a must be matched to a vertex w' by M_t . Note that if a is the first arc of R' we process, then w_a must be covered by M_t , otherwise M_t is not maximal. In each step, we exchange $v_a w_a$ for $w_a w'$, which decreases the size of the symmetric difference of the current matching and M_t by two. Let M be the matching obtained by these exchanges. Then $|M \triangle M_t| = |M_s \triangle M_t| - 2\ell(R')$. Now, we perform an exchange for each arc of R in forward direction. Note that an arc of cost two corresponds to an exchange that increases the size of the symmetric difference by two. Similarly, an arc of cost one (zero) corresponds to an exchange that does not change the size of the symmetric difference (decreases the size of the symmetric difference by two). By keeping track of the size of the symmetric difference with respect to M_t , the final equality of (2) follows.

Recall that P^* is a shortest XY -path in D . By Claim 2, there is a transformation from M_s to M_t of length at most $c(P^*) + |M_s \triangle M_t|/2 + \delta_{P_G^*}$. By the optimality of P_G^* and observing that visiting a cycle $C \in \mathcal{C}$ incurs a cost of at least one for an XY -path, we obtain

$$c(Q^*) + |M_s \triangle M_t|/2 + \delta_{Q_G^*} \geq c(P^*) + |M_s \triangle M_t|/2 + \delta_{P_G^*} . \quad (3)$$

We combine (2) and (3) to prove the claim. \square

Note that we can find in polynomial-time a minimum-cost XY -path P^* in D , for example by Dijkstra's algorithm. Hence, it suffices to transform M_s into the matching $M_s \triangle E(P_G^*)$, which is not inclusion-wise maximal, and then use the algorithm from Lemma 5 to transform the resulting matching into M_t . By Claim 3, the transformation has minimal length with respect to all transformations from M_s to M_t via a matching that is not inclusion-wise maximal.

C.2 Remaining Proofs omitted from Section 3

Proof of Fact 12. If an edge does not occur in some maximum matching of the input graph G then it is useless for reconfiguration. Therefore, we may assume that every edge of G occurs in some maximum matching. Also, we may assume that G is connected, otherwise we consider each component separately. Now consider the Edmonds-Gallai decomposition D, A, C of the graph G . The odd components of the decomposition are just single vertices since they are factor-critical, and no factor-critical graph is bipartite. By our assumptions above, A is an independent set and C is empty (no edge between A and C occurs in a maximum matching; G is connected). Therefore, (D, A) is a bipartition of $V(G)$ with all exposed vertices on the same side, D . \square

Proof of Proposition 13. Observe that since F is a directed tree, the indegree of each vertex of F except r is precisely one. Let A' be the arc-set of F .

Proof of (i). Let $P \in \mathcal{P}$. Suppose that F is missing at least one arc of $A(P)$. Since the indegree of each vertex of F except r is precisely one (they are terminals), there must be two nodes $x, y \in V(P)$, such that $xy \in A(P) \setminus A'$ and the predecessor x' of y is not in $V(P)$. Then $F - x'y + xy$ connects r to each terminal at cost $c(F - x'y + xy) < c(F)$ since we replace an arc of weight two by an arc of weight one. Indeed, as we observed, only one arc entering in y is special. This contradicts the optimality of F .

Proof of (ii). Let $C \in \mathcal{C}$. If A' contains each arc of $A(C)$, then F is not a tree. So F misses at least one arc. Suppose that F is missing at least two arcs of $A(C)$. Since the indegree of each vertex of F except r is precisely one, there must be two vertices $x, y \in V(C)$ as follows. The arc xy is in $A(C)$, but not in A' and the predecessor of y is not in $V(C)$. Let a be the in-arc of y in F . Then $F - a + xy$ connects r to each terminal at cost $c(F - a + xy) < c(F)$. This contradicts the optimality of F .

Proof of (iii). Let $P \in \mathcal{P}$. Let $v \in V(P)$ be the M_s -free vertex of P . By (i), the tree F contains the arcs $A(P)$ and each vertex of F except r has indegree precisely one. Therefore, there is only a single in-arc pv , where $v \in V(P)$ and $p \notin V(P)$. If $p \neq r$, then we may replace pv by rv and obtain a directed tree $F - pv + rv$ that connects r to each terminal at cost $c(F - pv + rv) < c(F)$. Again, this is a contradiction to the optimality of F . \square

Proof of Claim 1 of Lemma 14. The three first statements follow directly from the definition of the transformation (1).

Let us prove the fourth point. First note that a_C exists and all the arcs of $A(C)$ but a_C are in F by Proposition 13. Since a has been visited backwards, it implies that a' has been visited forwards (since a' is an arc of the unique path from r to a).

$a_j \in A'$ by Proposition 13. We first show that the statement holds in step j . If $a_{j-1} = a_j = u'u$ then u is M_{j-1} -free (by the second rule of (1)) and $M_j = M_{j-1}$ (by the third rule of (1)). So we can assume that a_{j-1} is visited backwards. Since a_C is not in F and a_C is the only out-arc of u which is special, the arc $a_{j-1} = uv$ is neither special nor artificial. By definition of rules four and five of (1), we have that $\bar{a}_{j-1} = vu \in M_{j-2}$ is moved to an edge not incident to u in M_{j-1} . Since $M_j = M_{j-1}$, we have that M_j is u -free.

So we can assume that $a_j \neq a$. Let j' be the step where a is visited backwards. Since F is a tree and we consider a DFS traversal, no arc incident to w is considered between steps j' and j since a' has not been visited upwards. So if an edge e incident to u is added in M_j , it can only be by the fourth rule of (1). So u is incident to an edge $e_{C'}$ for $C' \in \mathcal{C}$. Since a' has not been visited backwards, neither is the arc entering in C . So $C \neq C'$, a contradiction since it would mean that u belongs to both cycles C and C' , that must be disjoint. \square

Claim 2. For $0 \leq i \leq m$, the set $M_i \subseteq E$ is a matching of G .

Proof of Claim 2 of Lemma 14. The statement holds for $i = 0$, so let us assume that for some fixed $i < m$ and $1 \leq j \leq i$, we have that M_j is a matching of G . We show that so is M_{i+1} . Note that $|M_{i+1}| = |M_i|$ by the definition of (1). Let us prove that M_{i+1} is a matching. If $M_{i+1} = M_i$ then the conclusion indeed

holds, so we assume that $M_{i+1} \neq M_i$. We distinguish two cases depending on whether the arc $a_{i+1} = u_{i+1}w_{i+1}$ is visited for the first or second time.

Case 1. The arc a_{i+1} is visited for the first time (i.e., a_{i+1} is traversed downwards).

We can assume that a_{i+1} is not artificial, since otherwise $M_i = M_{i+1}$. Let $a_i = u_iw_i$ be the arc visited before a_{i+1} . We distinguish several subcases depending on which rule of (1) was applied to obtain M_i from M_{i-1} . If the first rule of (1) was applied to obtain M_i , the arc a_i is artificial. Then $w_i = u_{i+1}$ is M_{i-1} -free by Claim 1. Since $M_{i-1} = M_i$ by the first rule of (1), u_{i+1} also is M_i -free. So M_{i+1} is a matching of G . If the second rule of (1) was applied, then we have that $u_{i+1} = w_i$ and $M_i = M_{i-1} - e_i + \bar{e}_i$, so w_i is M_i -free as required. So the third, fourth or fifth rule of (1) was applied, and then a_{i+1} is traversed downwards and a_i is traversed upwards. Since arcs of larger weight are traversed first and at most one outgoing arc of u_i has weight one, we have that a_i has weight two. Therefore, the arc a_i cannot be special and rule three of (1) is not applicable. Finally, if the fourth or fifth rule of (1) were applied, then $u_i = u_{i+1}$ and, by definition of M_i , in both cases the vertex u_i is M_i -free. Therefore, the set $M_{i+1} = M_i - e_i + \bar{e}_i$ is a matching of G .

Case 2. The arc a_{i+1} is visited for the second time (i.e., a_{i+1} is traversed upwards).

If M_{i+1} is obtained via the first or the third rule, then $M_{i+1} = M_i$ and the conclusion holds. Furthermore, rule two is not applicable, since a_{i+1} is traversed upwards by assumption. So M_{i+1} is obtained by rules four or five of (1). As in Case 1, we distinguish several subcases, depending on which rule of (1) was applied to obtain M_i from M_{i-1} when processing $a_i = u_iw_i$.

If the first rule of (1) was applied, then $u_i = w_{i+1} = r$, which contradicts our construction of the instance. If rule two of (1) was applied, then $M_i = M_{i-1} - e_i + \bar{e}_i$ and $a_{i+1} = a_i$. So in particular, the fourth rule of (1) cannot be used to obtain M_{i+1} . Indeed $a_i = a_{i+1}$ cannot be both, special (a_i is because of rule four for a_{i+1}) and not special (a_{i+1} is not since rule three is not applied). So we have either $M_{i+1} = M_{i-1}$ or $M_{i+1} = M_i$ and the conclusion holds. We may therefore assume that a_i is traversed for the second time and hence, we have $u_i = w_{i+1}$. So M_{i+1} is the result of rules four or five of (1) and M_i from rules three, four, or five of (1).

Assume first, that the fifth rule of (1) was applied to obtain M_{i+1} . We first prove by contradiction that M_i cannot be obtained using rule three of (1). Indeed otherwise, a_i would be special and belong to Z and a_{i+1} would not be special. Since rule four is not applied to obtain M_{i+1} and a_{i+1} would be the only arc entering $V(Z)$. By Proposition 13 Z would be a path. But then u_i would incident to r , so $a_{i+1} = ru$ by Proposition 13, a contradiction. If rules four or five of (1) were applied to obtain M_i , then we have that $u_i = w_{i+1}$ is M_i -free, so $M_{i+1} = M_i - \bar{e}_{i+1} + e_{i+1}$ is a matching of G .

Finally assume that the fourth rule of (1) was applied to obtain M_{i+1} . By definition of rule four, rule three of (1) was applied to a_i . So $M_{i+1} = M_i - \bar{e}_{i+1} + e_C$ for some $C \in \mathcal{C}$. Note that by definition of a_C , we have that $w_{i+1} = u_i$ is a vertex of C . Let us denote by u the other endpoint of a_C . By Proposition 13, all the arcs of C but a_C are in F and since F is traversed in DFS order, they were already traversed backwards. The last point of Claim 1 ensures that u is

M_{i-1} -free. Since $M_i = M_{i-1}$, it is also M_i -free. Thus the exchange of \bar{e}_{i+1} for e_C results in a matching.

This concludes the proof of Claim 2. \square

Proof of Claim 2 of Lemma 14. By the definition of (1), an edge $vw \in M_s$ is exchanged at most twice during the algorithm, once when the arc a of F ending in w is traversed downwards and once when it is traversed backwards.

Let us first prove that at the end of the algorithm, if an edge $e = vw \in M_s$ is exchanged, then it is exchanged with an edge in M_t . Let $a = uw$ be the unique in-arc of w in F . When a is traversed forwards, the edge vw is exchanged with uv (rule two of (1)). If a is special, then uv is in M_t (by the definition of special edge) and uv is not exchanged when a is visited backwards (rule three of (1)), so the conclusion holds in this case. Hence we can assume that a is not special. If $vw \notin M_s \triangle M_t$, then w is not in a cycle C of \mathcal{C} , and then it has no out-arc that is special. Thus, when a is traversed backwards, rule five of (1) applies, and then $vw \in M_t$. Let us finally assume that a is not special but $vw \in M_t - M_s$. By Proposition 13, the arc $a = uw$ is the unique arc entering in some item $Z \in \mathcal{C} \cup \mathcal{P}$. Since a is not artificial, Proposition 13 ensures that $Z = C \in \mathcal{C}$. By construction, when a is traversed backwards, the edge uv is exchanged for e_C which is, by definition, in M_t .

To conclude, we simply have to prove that each edge $M_s \setminus M_t$ has been exchanged at some point. This is indeed the case since, for each arc $vw \in M_s - M_t$, the vertex w is a terminal. Since F is a Steiner tree, it connects r to each terminal. \square

Proof of Lemma 15. Let $w \in T$ be a terminal. Then, by the definition of the arc-set A , there is some edge $vw \in M_s$. By the definition of the Edmonds-Gallai decomposition in Section 4, there is an even-length alternating path from an M_s -free vertex to w . Therefore, $w \in D(G)$. Let vw be the M_s -edge incident to w . Then $v \in A(G)$. Since, by Theorem 21, any maximum matching of G induces an $A(G)$ -perfect matching into $D(G)$ and $C(G)$ is completely matched by any maximum matching, it is only possible to exchange vw for some edge vz , where z is a neighbor of v . Since $vw \notin M_t$, there is some step i , such that $M_i = M_{i-1} - vw + vz$, for some neighbor z of v . Therefore, the indegree of w in $D_{\mathcal{S}}$ is at least one. If z is M_s -free, then there is some rw -path in $D_{\mathcal{S}}$ and we are done. Suppose this is not the case, so M_i does not cover z , but M_s does. Then, there is some index $i' < i$, such that $M_{i'}$ does not cover z , but $M_{i'-1}$ does. Among all such indices, let i' be the minimal one. So in particular the edge of $M_{i'-1}$ incident to z is the same as in M_s . That is, there is some neighbor v' of z and some neighbor z' of v' , such that $M_{i'} = M_{i'-1} - zv' + v'z'$. Since $z \in D(G)$, we have that the indegree of z in $D_{\mathcal{S}}$ is at least one. Again, if z is M_s -free then we are done. We can repeat this argument until we reach either an M_s -free vertex or either M_s or M_t are not maximum, or to \mathcal{S} is not a reconfiguration sequence transforming M_s to M_t . We conclude that there is an rw -path in $D_{\mathcal{S}}$ for each $w \in T$. \square

Proof of Lemma 17. We show that we can decompose τ into two transformations τ_1 and τ_2 , such that τ_1 is a transformation from M_s to $M_U(S)$ for some $S \in \{U, W\}^{\mathcal{P} \cup \mathcal{C}}$ and τ_2 transforms $M_U(S)$ into M_t . Our claim then follows, since $\alpha(S^*)$ is a shortest such transformation.

We consider τ as a sequence $((e_i, f_i))_{1 \leq i \leq m}$ of exchanges, where $e_i = v_i w_i$ and $f_i = u_i v_i$. For each $1 \leq i \leq m$, we have that $M_i = M_{i-1} - e_i + f_i$. Note that for each exchange (e_i, f_i) , the vertex v_i is M_i -free. Consider the transformations $\tau_1 := ((e_i, f_i))_{1 \leq i \leq m: u_i \in U}$ and $\tau_2 := ((e_i, f_i))_{1 \leq i \leq m: u_i \in W}$. We show that we can apply τ_1 to M_s to obtain an intermediate matching M and that we can apply τ_2 to M to obtain M_t . For $j \in \{1, 2\}$, let $|\tau_j|$ be the length of τ_j and re-index the exchanges from 1 to $|\tau_j|$.

Suppose for a contradiction that for some index i , the exchange $(v_i w_i, u_i v_i)$ of τ_1 cannot be performed and let i be the smallest such index. This means, that $v_i w_i$ is not in the current matching because some exchange $(v'_j w'_j, u'_j v'_j)$ of τ_2 needs to be performed before $(v_i w_i, u_i v_i)$. Again, we assume that j is the smallest such index. We distinguish two cases. If $(v_i w_i, u_i v_i)$ cannot be performed since u_i is not exposed, then $w'_j = u_i$, so $u'_j \in U$, which contradicts the construction of τ_2 . On the other hand, if the exchange $(v_i w_i, u_i v_i)$ cannot be performed since $v_i w_i$ is not in the current matching, then $u'_j v'_j = v_i w_i$. Then we can perform exchanges $1, 2, \dots, j-1$ of τ_2 and $1, 2, \dots, i-1$ of τ_1 to obtain a matching M' . But $u_i v_i$ is an edge of G and u_i and v_i are both M' -free. Therefore, M' is not inclusion-wise maximal. Since τ contains all the exchanges we performed so far to obtain M' , there is a transformation from M_s to M_t via M' that has length at most m by Lemma 5, a contradiction.

Let us now prove that τ_2 is a transformation from M' to M_t . The argument is similar to the one above. Suppose for a contradiction that for some index i , the exchange $(v_i w_i, u_i v_i)$ of τ_2 cannot be performed and let i be the smallest such index. If $(v_i w_i, u_i v_i)$ cannot be performed since u_i is not exposed, then $w'_j = u_i$, so $u'_j \in V$, which contradicts the construction of τ_1 . On the other hand, if the exchange $(v_i w_i, u_i v_i)$ cannot be performed, because $v_i w_i$ is not in the current matching, then $v'_j w'_j = v_i w_i$. Then we can perform exchanges $1, 2, \dots, j$ of τ_1 and $1, 2, \dots, i-1$ of τ_2 to obtain a matching M' . But $u_i v_i$ is an edge of G and u_i and v_i are both M' -free. Therefore M' is not inclusion-wise maximal. Since τ contains all the exchanges we performed so far to obtain M' , there is a transformation from M_s to M_t via M' that has length at most m by Lemma 5, a contradiction.

We show that if (e_i, f_i) is the last exchange of τ_1 that involves the edge f_i , then f_i cannot be moved by τ_2 , so f_i is in the target matching M_t . Suppose for a contradiction, that there is some index j , such that the (e_j, f_j) of τ_2 has $f_i = e_j$. Let j be the smallest such index. Then we can apply τ_1 to M_s and the perform all exchanges of τ_2 up to index j . Let M' be the resulting matching. Then M' is not inclusion-wise maximal, since $M' + e_i$ is a matching of G . Therefore, by Lemma 5, there is a transformation from M_s to M_t via M' of length at most m , a contradiction.

By swapping τ_1 and τ_2 and using an analogous argument, we may conclude that if (e_i, f_i) is the last exchange of τ_2 that involves f_i , then f_i is in M_t . Therefore, each exchange (e, f) of τ , such f is involved for the last time occurs as a final exchange involving f either in τ_1 or τ_2 . Therefore, we obtain M_t by applying τ_1 followed by τ_2 to M_s .

Let M be the matching obtained by applying τ_1 to M_s and consider the set \mathcal{P} of paths and the set \mathcal{C} of cycles in $M_s \Delta M_t$. We claim that for each $F \in \mathcal{C} \cup \mathcal{P}$, when reaching M , we have either completely reconfigured F or performed no change at all on F . That is, for each $F \in \mathcal{C} \cup \mathcal{P}$, we have that $M \cap E(F)$ is either $M_s \cap E(F)$ or $M_t \cap E(F)$. Suppose for a contradiction, that this is not

the case for some $F \in \mathcal{C} \cup \mathcal{P}$. If some edge $f \notin M_s \cup M_t$ is incident to F in M , then, by the argument above, the edge f cannot be involved in any exchange of τ_2 and will remain in the final matching, a contradiction. If the matching M contains at least one edge in $M_s \cap E(F)$ and at least one edge in $M_t \cap E(F)$, then there is an M -free vertex $u \in V(F)$. Due to the construction of τ_1 , we have that $u \in U$. But each exchange in τ_2 only swaps two edges incident to a common vertex in U , so u is M_t -free. Therefore, F must be a path and u is one of its end vertices. Since all edges of M incident to F are in $M_s \cup M_t$ and u is M_t -free, we have that $M \cap E(F) = M \cap E(M_t)$, a contradiction.

Finally, we choose $S \in \{U, W\}^{\mathcal{C} \cup \mathcal{P}}$, such that for $F \in \mathcal{C} \cup \mathcal{P}$, we have that $S_F = U$ if $M_t \cap E(F) = M \cap E(F)$ and $S_F = W$ otherwise. By the definition of τ_1 and our arguments above, we have that $M = M_U(S)$. By the optimality of the choice of S^* , we have that $|\alpha(S^*)| \leq |\alpha(S)| = m$. \square

D Proofs Omitted from Section 4

Proof of Theorem 18. Let $\nu(G)$ be the size of a maximum matching in G and let $A := A(G)$, $D := D(G)$ and $C := C(G)$ the partition of $V(G)$ according to the Edmonds-Gallai decomposition. If $k < \nu(G)$ then $\mathcal{M}_k(G)$ is connected according to the proof of [13, Proposition 1], so let $k = \nu(G)$. From Theorem 21, we have that $G[C]$ is perfectly matched by any maximum matching of G . Let us prove that $\mathcal{M}_k(G)$ is connected if and only if $G[C]$ admits a unique perfect matching.

First consider the case that $G[C]$ admits two distinct perfect matchings. Let M_1 and M_2 be extensions of the two distinct perfect matchings on $G[C]$ to maximum matchings of G . Assume by contradiction that there is a transformation and let M be the matching just before the first modification of an edge in $G[C]$. By definition of C , there is no M -alternating path from an M -free vertex to C for any maximum matching M of G . In particular, there is no exchange $M - e + f$, where $e \in E(G[C]) \cap M$ and $f \in E(G) \setminus M$ for any maximum matching M of G (since otherwise a maximum matching would not be perfect on $G[C]$). And since M is C -perfect one cannot replace an edge in $G[C]$ by another. Therefore, all the matchings in the connected component of M_1 in $\mathcal{M}_k(G)$ agree on $G[C]$. Thus M_1 and M_2 cannot be connected in $\mathcal{M}_k(G)$, and then $\mathcal{M}_k(G)$ is not connected.

On the other hand, suppose that all the maximum matchings agree on $G[C]$. Let M_1 and M_2 be two maximum matchings of G and suppose that $G[M_1 \triangle M_2]$ contains an even-length (M_1, M_2) -alternating cycle C . Then C must be contained in $G[D \cup A]$. By the definition of D and A in Theorem 21, there is an M_1 -alternating path from an M_1 -free vertex to C . Therefore, by Lemma 16, (G, M_1, M_2) is a YES instances of MATCHING RECONFIGURATION. It follows that if $G[C]$ has a unique perfect matching, then $\mathcal{M}_k(G)$ is connected, so $\text{diam}(\mathcal{M}_k(G))$ is finite. \square

Proof of Lemma 19. Let $e \in E$. Reconfiguring $M \cap E(C_e)$ to $N \cap E(C_e)$ requires that p_v^1 is exposed for some $v \in V$ such that e is incident to v . If this is the case, then the reconfiguration requires precisely three exchanges. So we need $3|E|$ exchanges to reconfigure the cycles on four vertices.

To reconfigure a cycle, we need that one of its neighbors is exposed. By definition of $\tau(H)$ and by construction at least $\tau(H)$ vertices have to be exposed at some step of the algorithm. Let us prove that two steps are needed to expose a vertex p_u^1 if p_v^1 is exposed. Since all but one vertex are covered by the matching, we have perfect matchings on all the C_4 . Thus to expose p_u^1 we need to push the edge $p_u^1 p_u^2$ on $p_u^2 t$. Since there is an edge incident to t (otherwise two vertices are exposed), we need to push this edge. Thus at least two steps are needed to expose p_u^1 . Finally we need one step to expose one vertex p_u^1 at the beginning and one step to expose t at the end. Therefore, $\text{dist}(M, N) \geq 3|E(H)| + 2\tau(H)$. And one can easily prove that a transformation of this length exists. \square

Proof of Lemma 20. Let M^* and N^* be two maximum matchings of G of maximal distance in $\mathcal{M}_k(G)$. Note that due to the maximality of $\text{dist}(M, N)$, for each $e \in E$ such that M^* and N^* leave no vertex of C_e exposed, we have that either $M^* \cap E(C_e) = \{c_e^1 c_e^2, c_e^3 c_e^4\}$ and $N^* \cap C_e = \{c_e^2 c_e^3, c_e^4 c_e^1\}$ or vice versa (indeed a transformation where they are different can immediately be adapted into a transformation where they are the same). So we may assume that M^* agrees with M and N^* agrees with N on each $e \in E$ such that C_e contains no exposed vertex of M^* or N^* . Now, due to the construction of G and since M and N leave t exposed, $\text{dist}(M^*, M) \leq 3$ and $\text{dist}(N^*, N) \leq 3$. Indeed if some p_u^1 or q_i is exposed, we can expose t in at most 3 steps. Similarly, if a vertex of a C_4 is exposed, then we can expose t in two steps.

Hence we have

$$\text{diam}(\mathcal{M}_k(G)) = \text{dist}(M^*, N^*) \leq \text{dist}(M, N) + 6 \quad (4)$$

Now let M' (N') be a maximum matching of G that leaves q_6 exposed and agrees with M (N) on each C_e for each $e \in E$. Then, any reconfiguration sequence from M' to N' must include M and N and since $\text{dist}(M', M) = \text{dist}(N', N) = 3$, we have that $\text{dist}(M', N') = \text{dist}(M, N) + 6$. By (4), the matchings M' and N' have maximal distance in $\mathcal{M}_k(G)$ and the lemma follows. \square

Proof of Theorem 4. Note that the length of a shortest transformation between two matchings of G is bounded by $O(|V^2|)$ [13]. Due to the polynomial size of a certificate, the question “is the distance of two matchings in $\mathcal{M}_k(G)$ at most ℓ ” is an NP-question. Therefore, EXACT MATCHING DISTANCE is in D^P . In order to show that EXACT MATCHING DISTANCE and EXACT MATCHING DIAMETER are D^P -hard, we give a polynomial-time reduction from SAT-UNSAT, which is complete for D^P [20] and defined as follows.

SAT-UNSAT

Input: 3-CNF formulas F and F'

Output: YES if and only if F is satisfiable and F' is not.

Note that we may as well reduce from EXACT VERTEX COVER, which is also known to be D^P -complete [4, Theorem 5.4]. However, to make the proof more self-contained, we include the simple reduction step from SAT-UNSAT to EXACT VERTEX COVER. Let F be a 3-CNF formula with m clauses and n variables. Recall that H has a ℓ -clique if and only if \overline{H} a ℓ -stable set if and only if \overline{H} has a vertex cover of size $|V| - \ell$. Using the standard reduction from 3-SAT to

CLIQUE and adding a clique of size $m - 1$ to the resulting graph and we have

$$\begin{aligned}
F \in 3\text{-SAT} &\Leftrightarrow H \text{ has a maximum clique of size } m \\
&\Leftrightarrow \overline{H} \text{ has a minimum vertex cover of size } |V| - m \\
&\stackrel{\text{Lemma 19}}{\Leftrightarrow} \text{dist}_{\mathcal{M}_k(G)}(M, N) = 3|E(\overline{H})| + 2(|V(\overline{H})| - m) \\
&\stackrel{\text{Lemma 20}}{\Leftrightarrow} \text{diam}_{\mathcal{M}_k(G)} = 3|\overline{H}| + 2(|V(\overline{H})| - m + 3)
\end{aligned} \tag{5}$$

On the other hand, using similar arguments we have

$$\begin{aligned}
F \notin 3\text{-SAT} &\Leftrightarrow \text{dist}_{\mathcal{M}_k(G)}(M, N) = 3|E(\overline{H})| + 2(|V(\overline{H})| - m + 1) \\
&\Leftrightarrow \text{diam}_{\mathcal{M}_k(G)} = 3|E(\overline{H})| + 2(|V(\overline{H})| - m + 4)
\end{aligned} \tag{6}$$

Let (F_1, F_2) be an instance of SAT-UNSAT. Let H_1 (H_2) be the graph constructed from F_1 (F_2) in the reduction from SAT to CLIQUE. Furthermore, let G_1 , M_1 , and N_1 (G_2 , M_2 and N_2) be the graph and the two matchings obtained from $\overline{H_1}$ ($\overline{H_2}$) according to the construction given above. We may obtain in polynomial time from (F_1, F_2) an instance (G, M, N, k, ℓ) of EXACT MATCHING DISTANCE as follows: Let G consist of two copies of G_1 and one copy of G_2 and let M (N) be the obvious matching in G created from two copies of M_1 (M_2) and one copy of N_1 (N_2). Finally, let k be the size of a maximum matching of G and $\ell := 6|E(G_1)| + 4(|V(G_1)| - m_1) + 3|E(G_2)| + 2(|V(G_2)| - m_2) + 2$. Clearly, the instance I can be constructed from (F_1, F_2) in polynomial time. Using (5) and (6) it is readily verified that

$$(F_1, F_2) \in \text{SAT-UNSAT} \Leftrightarrow \text{dist}_{\mathcal{M}_k(G)}(M, N) = \ell$$

and that

$$(F_1, F_2) \in \text{SAT-UNSAT} \Leftrightarrow \text{diam}_{\mathcal{M}_k(G)} = \ell + 6$$

□