A Turing Kernelization Dichotomy for Structural Parameterizations of \mathcal{F} -Minor-Free Deletion

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- Abstract -

For a fixed finite family of graphs \mathcal{F} , the \mathcal{F} -Minor-Free Deletion problem takes as input a graph G and an integer ℓ and asks whether there exists a set $X \subseteq V(G)$ of size at most ℓ such that G - X is \mathcal{F} -minor-free. For $\mathcal{F} = \{K_2\}$ and $\mathcal{F} = \{K_3\}$ this encodes Vertex Cover and Feedback Vertex Set respectively. When parameterized by the feedback vertex number of G these two problems are known to admit a polynomial kernelization. Such a polynomial kernelization also exists for any \mathcal{F} containing a planar graph but no forests. In this paper we show that \mathcal{F} -Minor-Free Deletion parameterized by the feedback vertex number is MK[2]-hard for $\mathcal{F} = \{P_3\}$. This rules out the existence of a polynomial kernel assuming NP $\not\subseteq$ coNP/poly, and also gives evidence that the problem does not admit a polynomial Turing kernel. Our hardness result generalizes to any \mathcal{F} not containing a P_3 -subgraph-free graph, using as parameter the vertex-deletion distance to treewidth min tw(\mathcal{F}), where min tw(\mathcal{F}) denotes the minimum treewidth of the graphs in \mathcal{F} . For the other case, where \mathcal{F} contains a P_3 -subgraph-free graph, we present a polynomial Turing kernelization. Our results extend to \mathcal{F} -Subgraph-Free Deletion.

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1 Introduction

Background and motivation Kernelization is a framework for the scientific investigation of provably effective preprocessing procedures for NP-hard problems. It uses the notion of a parameterized (decision) problem to capture meaningful performance guarantees for preprocessing. In a parameterized problem, every problem input x has an associated integer k called the parameter, which captures the difficulty of the input in some way. A *kernelization* for a parameterized problem is a polynomial-time algorithm that transforms any parameterized instance (x, k) into an instance (x', k') with the same answer, such that |x'| and k' are both bounded by f(k) for some computable function f. The function f is the *size* of the kernel. Of

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particular interest are kernels of polynomial size. Determining which parameterized problems admit kernels of polynomial size has become a rich area of algorithmic research [7, 17, 32].

A common approach in kernelization [1, 16, 29, 23] is to take the solution size as the parameter k, with the aim of showing that large inputs that ask for a small solution can be efficiently reduced in size. However, this method does not give any nontrivial guarantees when the solution size is known to be proportional to the total size of the input. For that reason, there is an alternative line of research [9, 10, 13, 21, 25, 26, 27, 38] that focuses on parameterizations based on a measure of nontriviality of the instance (cf. [33]). One formal way to capture nontriviality of a graph problem, is to measure how many vertex-deletions are needed to reduce the input graph to a graph class in which the problem can be solved in polynomial time. Since many graph problems can be solved in polynomial time on trees and forests, the structural graph parameter feedback vertex number (the minimum number of vertex deletions needed to make the graph acyclic, i.e., a forest) is a relevant measure of the distance of the input to a trivially solvable one.

Previous research has shown that for the Vertex Cover problem, there is a polynomial kernel parameterized by the feedback vertex number [25]. This preprocessing algorithm guarantees that inputs which are large with respect to their feedback vertex number, can be efficiently reduced. The VERTEX COVER problem is the simplest in a family of so-called minor-free deletion problems. For a fixed finite family of graphs \mathcal{F} , an input to \mathcal{F} -MINOR-Free Deletion consists of a graph G and an integer ℓ . The question is whether there is a set S of at most ℓ vertices in G, such that the graph G-S obtained by removing these vertices does not contain any graph from \mathcal{F} as a minor. Various classic graph optimization problems such as Vertex Cover, Feedback Vertex Set, and Vertex Planarization fit this framework by a suitable choice of \mathcal{F} . The investigation of minor-free deletion problems has led to numerous advances in the study of kernelization and parameterized algorithmics [3, 14, 15, 16, 19]. Motivated by the fact that VERTEX COVER and FEEDBACK VERTEX SET, arguably the simplest \mathcal{F} -Minor-Free Deletion problems, admit polynomial kernels when parameterized by the feedback vertex number, we set out to resolve the following question: Do all F-MINOR-FREE DELETION problems admit a polynomial kernel when parameterized by the feedback vertex number?

Results To our initial surprise, we prove that the answer to this question is *no*. While the parameterization by feedback vertex number admits polynomial kernels for $\mathcal{F} = \{K_2\}$ [25], for $\mathcal{F} = \{K_3\}$ [8, 37, 23], and for any set \mathcal{F} containing a planar graph² but no forests [16], there are also cases that do not admit polynomial kernels (under the assumption that NP $\not\subseteq$ coNP/poly, which we tacitly assume throughout the informal discussion in this introduction). For example, the case of \mathcal{F} consisting of a single graph P_3 that forms a path on three vertices does not admit a polynomial kernel. This lower bound for $\mathcal{F} = \{P_3\}$ follows from a more general theorem that we state below.

Recall that a graph is a forest if and only if its treewidth is one [6]. Hence the feedback vertex number is exactly the minimum number of vertex deletions needed to obtain a graph of treewidth one. Let tw(G) denote the treewidth of graph G, and define $min tw(\mathcal{F}) := min_{H \in \mathcal{F}} tw(H)$. Our lower bound also holds for \mathcal{F} -SUBGRAPH-FREE DELETION, which is the related problem that asks whether there is a vertex set S of size at most k such that G - S

 $^{^2}$ If \mathcal{F} contains no forests, then any acyclic graph is \mathcal{F} -minor free, implying the size of an optimal solution is at most the size of a feedback vertex set. Hence the kernelization for the solution-size parameterization yields a kernel of size bounded polynomially in the feedback vertex number.

contains no graph $H \in \mathcal{F}$ as a *subgraph*. We prove the following.

▶ **Theorem 1.** Let \mathcal{F} be a finite set of graphs, such that each graph in \mathcal{F} has a connected component on at least three vertices. Then \mathcal{F} -MINOR-FREE DELETION and \mathcal{F} -SUBGRAPH-FREE DELETION do not admit polynomial kernels when parameterized by the vertex-deletion distance to a graph of treewidth min $\operatorname{tw}(\mathcal{F})$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

To see that Theorem 1 implies the claimed lower bound for $\mathcal{F} = \{P_3\}$, observe that whenever \mathcal{F} contains an acyclic graph with at least one edge we have $\min \operatorname{tw}(\mathcal{F}) = 1$ and therefore the vertex-deletion distance to treewidth $\min \operatorname{tw}(\mathcal{F})$ equals the feedback vertex number. The theorem also generalizes earlier results of Cygan et al. [10, Theorem 13], who investigated the problem of losing treewidth. They proved that for each fixed $1 \leq \eta < \rho$, the η -Transversal problem (delete at most ℓ vertices to get a graph of treewidth at most η) does not have a polynomial kernel when parameterized by the vertex-deletion distance to treewidth ρ . Since the treewidth of a graph does not increase when taking minors, there is a finite set \mathcal{F}_{η} of forbidden minors (cf. [35]) that characterize the graphs of treewidth at most η . As the members of the obstruction set for $\eta \geq 1$ are easily seen to be connected, have treewidth $\eta + 1$, and at least three vertices, the lower bound of Theorem 1 encompasses the theorem of Cygan et al. and generalizes it to arbitrary \mathcal{F} -Minor-Free Deletion problems.

Theorem 1 is obtained through a polynomial-parameter transformation from the CNF-SAT problem parameterized by the number of variables, for which a superpolynomial kernelization lower bound is known [11, 18]. The main technical contribution in the hardness proof consists of the design of a gadget that acts as a clause checker. A certain budget of vertex deletions is available to break all \mathcal{F} -minors present in the gadget, and this is possible if and only if one of the neighboring vertices in a variable gadget is removed by the solution. This removal encodes that the variable is set in a way that satisfies the clause. The intricate part of the construction is to design the gadget knowing only that \mathcal{F} has a graph with a connected component of at least three vertices. Here we extensively rely on the fact that minimal minor models of biconnected graphs live in biconnected subgraphs, together with the fact the treewidth of a graph does not increase when attaching structures along cut vertices in a tree-like manner.

Using the framework of Hermelin et al. [22], our polynomial-parameter transformation from CNF-SAT parameterized by the number of variables to the structural parameterization of \mathcal{F} -MINOR-FREE DELETION, also rules out the existence of polynomial-size Turing kernelizations under a certain hardness assumption. Turing kernelization [12] is a relaxation of the traditional form of kernelization. Intuitively, it investigates whether inputs (x, k) can be solved efficiently using the answers to subproblems of size f(k) which are provided by an oracle, which models an external computation cluster. Note that a parameterized problem that has a kernel of size $\mathcal{O}(k^c)$ can be solved by a polynomial-time algorithm that first spends polynomial time to prepare a query of size $\mathcal{O}(k^c)$, and then queries an oracle for its answer. Turing kernelization investigates if and how polynomial-time algorithms can solve NP-hard parameterized problems by querying an oracle for the answers to instances of size $k^{\mathcal{O}(1)}$, potentially multiple times. Some problems that do *not* admit polynomial kernelizations, do admit polynomial-size Turing kernelizations [5, 31, 24, 28, 39].

Formally, a Turing kernelization of size f for a parameterized problem $\mathcal Q$ is an algorithm that can query an oracle to obtain the answer to any instance of problem $\mathcal Q$ of size and parameter bounded by f(k) in a single step, and using this power solves any instance (x,k) in time polynomial in |x|+k. The reduction proving Theorem 1 also proves the non-existence of polynomial-size Turing kernelizations, unless all parameterized problems in the complexity class $\mathsf{MK}[2]$ defined by Hermelin et al. [22] have polynomial Turing kernels. (The CNF-SAT

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problem with clauses of unbounded length, parameterized by the number of variables, is MK[2]-complete [22, Thm. 1, cf. Thm. 10] and widely believed *not* to admit polynomial-size Turing kernels.)

Motivated by the general form of the lower bound statement in Theorem 1, we also investigate upper bounds and derive a complexity dichotomy. For any \mathcal{F} that does not meet the criterion of Theorem 1, we obtain a polynomial Turing kernel.

▶ Theorem 2. Let \mathcal{F} be a finite set of graphs, such that some $H \in \mathcal{F}$ has no connected component of three or more vertices. Then \mathcal{F} -MINOR-FREE DELETION and \mathcal{F} -SUBGRAPH-FREE DELETION admit polynomial Turing kernels when parameterized by the vertex-deletion distance to a graph of treewidth min tw(\mathcal{F}).

The main insight in the Turing kernelization is the following. If $H \in \mathcal{F}$ has no connected component of three or more vertices, then H consists of disjoint edges and isolated vertices. If H only has isolated vertices, then \mathcal{F} -minor-free deletion is polynomial-time solvable because the leftover graph has less than $|V(H)| \in \mathcal{O}(1)$ vertices, for which we can search by brute force. Otherwise, H is a matching of size $t \ge 1$ plus potentially some isolated vertices. The isolated vertices turn out only to make a difference if the solution \mathcal{F} -free graph has constant size. In the interesting case, we can focus on $H \in \mathcal{F}$ being a matching of size t. Then a graph that is \mathcal{F} -minor-free does not admit a matching of size t, and therefore has a vertex cover of size at most t. Hence a solution to \mathcal{F} -MINOR-FREE DELETION can be extended to a vertex cover by including $\mathcal{O}(1)$ additional vertices. Using the Tutte-Berge formula, we can make the relation between \mathcal{F} -MINOR-FREE DELETION and the vertex cover precise, and use it to reduce an instance of \mathcal{F} -MINOR-FREE DELETION parameterized by deletion distance to min $tw(\mathcal{F})$, to the logical OR of a polynomial number of instances of Vertex Cover parameterized by deletion distance to $\min \operatorname{tw}(\mathcal{F})$. If \mathcal{F} has a graph with no component of size at least three, then $\min \operatorname{tw}(\mathcal{F}) = 1$, implying that the parameter is the feedback vertex set size. This allows us to use the polynomial kernel for VERTEX COVER parameterized by feedback vertex set on each generated instance. We query the resulting instances of size $k^{\mathcal{O}(1)}$ to the oracle to find the answer.

Organization We present preliminaries on graphs and kernelization in Section 2. Section 3 develops the lower bounds on (Turing) kernelization when all graphs in \mathcal{F} have a connected component with at least three vertices. In Section 4 we show that in all other cases, a polynomial Turing kernelization exists.

2 Preliminaries

All graphs we consider in this paper are simple, finite and undirected. We denote the vertex set and edge set of a graph G by V(G) and E(G) respectively. For a vertex set $S \subseteq V(G)$ let G[S] be the subgraph of G induced by S, and let G - S denote the subgraph of G induced by S. For a vertex S we use S we use S and a vertex S shorthand for S and a vertex S and S are constant S are constant S and S are constant S are constant S are constant S and S are constant S are constant S and S are constant S are consta

A graph H is a minor of graph G, denoted by $H \leq G$, if H can be obtained from G by a series of edge contractions, edge deletions, and vertex deletions. An H-model in G is a function $\varphi \colon V(H) \to 2^{V(G)}$ such that (i) for every vertex $v \in V(H)$, the graph $G[\varphi(v)]$

is connected, (ii) for every edge $\{u,v\} \in E(H)$ there exists an edge $\{u',v'\} \in E(G)$ with $u' \in \varphi(u)$ and $v' \in \varphi(v)$, and (iii) for distinct $u,v \in V(H)$ we have $\varphi(v) \cap \varphi(u) = \emptyset$. The sets $\varphi(v)$ are called branch sets. Clearly, $H \leq G$ if and only if there is an H-model in G. For any function $f \colon A \to B$ and set $A' \subseteq A$ we use f(A') as a shorthand for $\bigcup_{a \in A'} f(a)$. Specifically in the case of a minor-model φ and graph G, we use $\varphi(G)$ to denote $\bigcup_{v \in V(G)} \varphi(v)$. We say a graph H is a component-wise minor of a graph G, denoted as $H \lesssim G$, when every connected component of H is a minor of G.

- ▶ Observation 3. For graphs G, H, if $H \subseteq G$ and $G \subseteq H$ then H and G are isomorphic.
- ▶ **Observation 4.** For graphs H, G, if $H \subset G$ then $\operatorname{tw}(H) \leq \operatorname{tw}(G)$ since the treewidth of H is the maximum treewidth of its connected components and each connected component of H is a minor of G, so its treewidth it at most $\operatorname{tw}(H)$.
- ▶ **Definition 5.** Let \mathcal{F} be a family of graphs and let $G \in \mathcal{F}$. For every relation $\subseteq \{ \leq, \lesssim \}$ we define minimal and maximal elements as follows:
- \blacksquare G is said to be \triangleleft -minimal in \mathcal{F} when for all graphs $H \in \mathcal{F}$ we have $H \triangleleft G \Rightarrow G \triangleleft H$.
- G is said to be \unlhd -maximal in \mathcal{F} when for all graphs $H \in \mathcal{F}$ we have $G \subseteq H \Rightarrow H \subseteq G$.
- ▶ **Definition 6.** We call a connected component C of a graph G a \leq -maximal component of G when C is \leq -maximal in the set of graphs that form the connected components of G.

For $type \in \{minor, subgraph\}$ and a finite family of graphs \mathcal{F} , we define:

 \mathcal{F} -type-Free Deletion

Input: A graph G and an integer ℓ .

Parameter: vertex-deletion distance to a graph of treewidth min $tw(\mathcal{F})$.

Question: Is there a set $X \subseteq V(G)$ of at most ℓ vertices such that G - X does not

contain any $H \in \mathcal{F}$ as a type?

A vertex $v \in V(G)$ is a *cut vertex* when its removal from G increases the number of connected components. A graph is called *biconnected* when it is connected and contains no cut vertex. A *biconnected component* of a graph G is a maximal biconnected subgraph of G. For any integer α , a graph G is called α -robust when $|V(G)| \ge \alpha$ and no vertex $v \in V(G)$ exists such that G - v contains a connected component with less than $\alpha - 1$ vertices.

▶ **Proposition 7.** Any graph G has a unique maximal α -robust subgraph. Any α -robust subgraph of G is a subgraph of the maximal α -robust subgraph of G.

Proof. The proposition follows straightforwardly from the fact that if G[A] and G[B] are α -robust, then so is $G[A \cup B]$. We now prove this fact.

Consider two vertex sets $A, B \subseteq V(G)$, such that G[A] and G[B] are α -robust. We show that $G[A \cup B]$ is α -robust. Since G[A] is α -robust we have $|A| \geq \alpha$ so then $|A \cup B| \geq \alpha$. Suppose for contradiction that there exists a vertex $v \in A \cup B$ such that $G[A \cup B] - v$ contains a connected component of size smaller than $\alpha - 1$. Let C be the vertices of this connected component. We know C contains vertices of at least one of A and B. Assume w.l.o.g. $A \cap C \neq \emptyset$, then $G[A \cap C]$ is a connected component of size less than $\alpha - 1$ in G[A] - v. If $v \in A$ this directly contradicts α -robustness of G[A], so assume $v \notin A$. Now G[A] contains a connected component with less than $\alpha - 1$ vertices. Since $|C| < \alpha \leq |A|$ there exists a vertex $u \in A \setminus C$, so then G[A] - u contains a connected component with less than $\alpha - 1$ vertices, which contradicts α -robustness of G[A].

For any graph G and integer α , let α -prune(G) denote the unique maximal α -robust subgraph of G, which may be empty. We define a leaf-block of a graph G as a biconnected component of G that contains at most one cut vertex of G. The size of a leaf-block G is G is denoted as G is denoted as G is G observe that G is G-robust if and only if G is G.

A polynomial-parameter transformation from parameterized problem \mathcal{P} to parameterized problem \mathcal{Q} is a polynomial-time algorithm that, given an instance (x, k) of \mathcal{P} , outputs an instance (x', k') of \mathcal{Q} such that all of the following are true:

- 1. $(x,k) \in \mathcal{P} \Leftrightarrow (x',k') \in \mathcal{Q}$,
- **2.** k' is upper-bounded by a polynomial in k.

3 Lower bound

In this section we consider the case where all graphs in \mathcal{F} contain a connected component of at least three vertices and give a polynomial-parameter transformation from CNF-SAT parameterized by the number of variables. In this construction we make use of the way biconnected components of graphs G and H restrict the options for an H-model to exist in G.

▶ Proposition 8. Let H be an α -robust graph and let φ be a minimal H-model in a graph G, then $G[\varphi(H)]$ is α -robust.

Proof. Take an arbitrary vertex $v \in \varphi(H)$ and let $u \in V(H)$ be such that $v \in \varphi(u)$. Since H-u does not have connected components smaller than $\alpha-1$, $G[\varphi(H)]-\varphi(u)$ cannot have connected components smaller than $\alpha-1$. Consider a spanning tree of $G[\varphi(u)]$. Each leaf of this spanning tree must be connected to a vertex in a different branch set, otherwise φ is not minimal. We know every connected component in $G[\varphi(u)]-v$ contains at least one leaf of this spanning tree, hence every connected component of $G[\varphi(u)]-v$ is connected to $G[\varphi(H)]-\varphi(u)$. So $G[\varphi(H)]-v$ does not contain a connected component smaller than $\alpha-1$. Since v was arbitrary, $G[\varphi(H)]$ is α -robust.

▶ Proposition 9. Let φ be an H-model in G, and B a biconnected component of H. Then $G[\varphi(B)]$ contains a biconnected subgraph on at least |B| vertices.

Proof. Let φ' a minimal B-model in G such that $\varphi'(v) \subseteq \varphi(v)$ for all $v \in V(B)$. Hence $G[\varphi'(B)]$ is a subgraph of $G[\varphi(B)]$. It suffices to show that $G[\varphi'(B)]$ contains a biconnected component on at least |V(B)| vertices. Since B is biconnected, it is |V(B)|-robust so by Proposition 8 we know $G[\varphi'(B)]$ is |V(B)|-robust. Hence $G[\varphi'(B)]$ contains a biconnected component on at least |V(B)| vertices.

- ▶ **Observation 10.** For any graph H, which may be an empty graph on vertex set \emptyset , and integers $\alpha \geq \beta$ we have α -prune(β -prune(H)) = α -prune(H).
- ▶ **Proposition 11.** For graphs H and G we have $H \leq G \Rightarrow \alpha$ -prune $(H) \leq \alpha$ -prune(G) for any integer α .

Proof. Clearly α -prune $(H) \leq H$ so since $H \leq G$ we have α -prune $(H) \leq G$. For simplicity let $H' := \alpha$ -prune(H). Let φ be a minimal H'-model in G and observe $H' \leq G[\varphi(H')]$. From Proposition 8 we know $G[\varphi(H')]$ is α -robust. Since $G[\varphi(H')]$ is an α -robust subgraph of G it is also a subgraph of α -prune(G) by Proposition 7. Hence α -prune $(H) \leq G[\varphi(H')] \leq \alpha$ -prune(G).

▶ **Proposition 12.** For any $\leq \in \{\leq, \lesssim\}$, two integers $\alpha \geq \beta$, and graphs H and G we have that $H \leq G \Rightarrow \alpha$ -prune $(H) \leq \beta$ -prune(G).

Proof. Suppose $H \leq G$, then

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\alpha-prune(H) \leq \alpha-prune(G) by Proposition 11
= \alpha-prune(\beta-prune(G)) by Observation 10
\leq \beta-prune(G).
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Alternatively, suppose $H \preceq G$. Let H' be a connected component of α -prune(H), then there exists a connected component H'' of H such that $H' \preceq H''$. Since $H \preceq G$ we have $H'' \preceq G$ so then $H' \preceq G$. As shown above, this implies α -prune $(H') \preceq \beta$ -prune(G). Note that α -prune(H') = H' since H' is a connected component of α -prune(H), hence α -prune $(H) \preceq \beta$ -prune(G).

We proceed to construct a clause gadget to be used in the polynomial-parameter transformation from CNF-SAT.

- ▶ **Lemma 13.** For any connected graph H with at least three vertices there exists a polynomial-time algorithm that, given an integer $n \ge 1$, outputs a graph G and a vertex set $S \subseteq V(G)$ of size n such that all of the following are true:
- 1. $tw(G) \leq tw(H)$,
- **2.** G contains a packing of 3n-1 vertex-disjoint H-subgraphs,
- 3. G-S contains a packing of 3n-2 vertex-disjoint H-subgraphs, and
- **4.** $\forall v \in S$ there exists $X \subseteq V(G)$ of size 3n-1 s.t. all of the following are true:
 - a. $v \in X$,
 - **b.** G X is H-minor-free,
 - c. $\lambda(H)$ -prune $(G-X) \lesssim H$, and
 - **d.** for all connected components G_c of G-X that contain a vertex of S we have $|V(G_c)| < \lambda(H)$ and G_c contains exactly one vertex of S.

Proof. Consider a subgraph L of H such that L is a smallest leaf-block of H. Let R be the graph obtained from H by removing all vertices of L that are not a cut vertex in H. Note that when H is biconnected, L = H and R is an empty graph. We distinguish three distinct vertices a, b, c in H. Vertices c and b are both part of L, where c is the cut vertex (if there is one) and b is any other vertex in L. Finally vertex a is any vertex in H that is not c or b. See Fig. 1a. In the construction of G we will combine copies of H such that a, b, and c form cut vertices in G and are part of two different H-subgraphs. Vertices b and c are chosen such that removing either one from a copy of H in G means no vertex from the L-subgraph of this copy of H can be used in a minimal H-model in G. In the remainder of this proof we use $f_{K \to K'} \colon V(K) \to V(K')$ for isomorphic graphs K and K' to denote a fixed isomorphism.

Take two copies of H, call them H_1 and H_2 . Let R_1 and L_1 denote the subgraphs of H_1 related to R and L, respectively, by the isomorphism between H and H_1 . Similarly let R_2 and L_2 denote the subgraphs of H_2 . Take a copy of L which we call L_3 . Let M be the graph obtained from the disjoint union of H_1 , H_2 , and L_3 by identifying the pair $f_{H\to H_1}(c)$ and $f_{H\to H_2}(b)$ into a single vertex s, and identifying the pair $f_{H\to H_2}(c)$ and $f_{L\to L_3}(c)$ into a single vertex t. We label $f_{H\to H_1}(a)$, $f_{H\to H_1}(b)$, and $f_{L\to L_3}(b)$ as u, w, and v respectively.

This construction is motivated by the fact that the graphs $M - \{v, s\}$, $M - \{u, t\}$, and $M - \{w, t\}$ are all H-minor-free, which we will exploit in the formal correctness argument

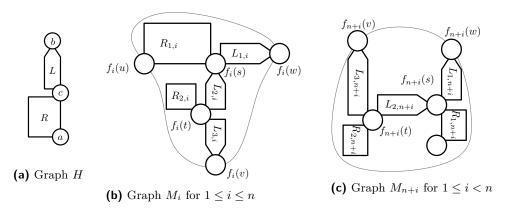


Figure 1 We show the situation where a is contained in R. Note that a can always be chosen such that it is contained in R when H is not biconnected. Note that the graphs in Fig. 1b and 1c are isomorphic but drawn differently.

later. We will connect copies of M to each other via the vertices u, v, and w so that, although two vertices need to be removed in every copy of M, one such vertex can always be in two copies of M at the same time.

Now take 2n-1 copies of M, call them M_1, \ldots, M_{2n-1} . For readability we denote $f_{M \to M_i}$ as f_i for all $1 \le i \le 2n-1$. For all $1 \le i < n$ we identify $f_i(w)$ and $f_{n+i}(v)$, and we identify $f_{n+i}(w)$ and $f_{i+1}(u)$. Let this graph be G, and let S be the set of vertices $f_i(v)$ for all $1 \le i \le n$. Let $H_{1,i}, H_{2,i}, R_{1,i}, R_{2,i}, L_{1,i}, L_{2,i}$, and $L_{3,i}$ denote the subgraphs in M_i that correspond to the subgraphs $H_1, H_2, R_1, R_2, L_1, L_2$, and L_3 in M. See Fig. 1b and 1c.

This concludes the description of graph G and set S. It is easily seen that these can be constructed in polynomial time. It remains to verify that all conditions of the lemma statement are met.

- (1) Since we connected copies of L and R in a treelike fashion along cut vertices, we did not introduce any new biconnected components. Since the treewidth of a graph is equal to the maximum treewidth over all its biconnected components we know that $tw(G) \le max\{tw(R), tw(L)\} = tw(H)$.
- (2) For each $1 \leq i \leq n$ we can distinguish two H-subgraphs in M_i , namely $H_{1,i}$ and $L_{3,i} \cup R_{2,i}$. This gives us 2n H-subgraphs in G. Note that since all M_1, \ldots, M_n are vertex-disjoint, these 2n H-subgraphs are also vertex-disjoint in G. For each $n < i \leq 2n 1$ we distinguish one H-subgraph, namely $H_{2,i}$. Note that since $H_{2,i}$ is vertex-disjoint from all $M_1, \ldots, M_{i-1}, M_{i+1}, \ldots, M_{2n-1}$ we have a total of 2n + n 1 = 3n 1 vertex-disjoint H-subgraphs in G. This packing is shown in Fig. 2a.
- (3) Alternatively, for each $1 \le i \le n$ we can distinguish one H-subgraph in M_i , namely $H_{2,i}$. For each $n < i \le 2n-1$ we distinguish two H-subgraphs in M_i , namely $H_{1,i}$ and $L_{3,i} \cup R_{2,i}$. Again these H-subgraphs are vertex-disjoint, and since they also do not contain any vertices of S, they form a packing of n + 2(n-1) = 3n-2 vertex-disjoint H-subgraphs in G S. See Fig. 2b.
- (4) Finally we prove that for all $v \in S$ there exists a set $Z \subseteq V(G)$ of size 3n-1 such that the four conditions listed in condition 4 are true. For this purpose we first identify a family \mathcal{Q} of vertex sets such that any H-model in G spans at least one vertex set in \mathcal{Q} . Let

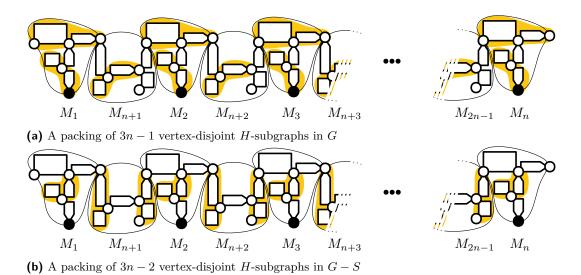


Figure 2 Two packings of vertex-disjoint H-subgraphs in G and G-S. Vertices in S are marked black.

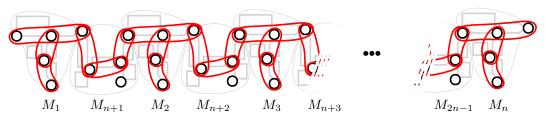


Figure 3 Vertex sets in Q are encircled.

Q be defined as follows: (see Fig. 3)

$$Q = \{ \{ f_i(v), f_i(t) \} \mid 1 \le i \le 2n - 1 \} \cup \{ \{ f_i(t), f_i(s) \} \mid 1 \le i \le 2n - 1 \}$$
$$\cup \{ \{ f_i(s), f_i(w) \} \mid n + 1 \le i \le 2n - 1 \} \cup \{ \{ f_i(u), f_i(s), f_i(w) \} \mid 1 \le i \le n \}.$$

▶ Claim 14. If φ is an H-model in G, then $\varphi(H) \supseteq Q$ for some $Q \in \mathcal{Q}$.

Proof. Let φ be an arbitrary H-model in G. We know from Proposition 9 that $G[\varphi(L)]$ contains a biconnected subgraph on at least |L| vertices. Let B be such a biconnected subgraph in G. Subgraph B must be fully contained in a biconnected component of G. Such a biconnected component must contain least $|B| \geq |L|$ vertices. We do a case distinction over all biconnected components in G with size at least |L|, and prove that if B is contained in them, then $\varphi(H) \supseteq Q$ for some $Q \in Q$.

- $L_{2,i}$ for any $1 \le i \le 2n-1$: We know $|L_{2,i}| = |L|$ so B = L, hence $f_i(t), f_i(s) \in \varphi(L)$.
- $L_{3,i}$ for any $1 \le i \le 2n-1$: We know $|L_{3,i}| = |L|$ so B = L, hence $f_i(v), f_i(t) \in \varphi(L)$.
- $L_{1,i}$ for any $n+1 \le i \le 2n-1$: We know $|L_{1,i}| = |L|$ so B=L, hence $f_i(s), f_i(w) \in \varphi(L)$.
- $L_{1,i}$ for any $1 \leq i \leq n$: We know $|L_{1,i}| = |L|$ so B = L, hence $f_i(s), f_i(w) \in \varphi(L)$. If $f_i(t) \in \varphi(H)$ or $f_i(u) \in \varphi(H)$ then clearly $Q \ni \{f_i(t), f_i(s)\} \subseteq \varphi(H)$ or $Q \ni \{f_i(u), f_i(s), f_i(w)\} \subseteq \varphi(H)$. If $f_{n+i}(t) \in \varphi(H)$ then $Q \ni \{f_{n+i}(v), f_{n+i}(t)\} \subseteq \varphi(H)$,

since $f_i(w) = f_{n+i}(v)$. Suppose $\varphi(H)$ does not contain $f_i(t)$, $f_i(u)$, or $f_{n+i}(t)$, then φ must be an H-model in the graph $G' := (H_{1,i} - f_i(u)) \cup (L_{2,i} - f_i(t)) \cup (L_{3,n+i} - f_i(t))$, so $H \leq G'$. By Proposition 11 we know that |L|-prune $(H) \leq |L|$ -prune(G'). Clearly |L|-prune(H) = H. The graph G' contains at least two leaf blocks that are smaller than |L|, namely $L_{2,i} - f_i(t)$ and $L_{3,n+i} - f_i(t)$, so |L|-prune(G') is a subgraph of $H_{1,i} - f_i(u)$. But then |V(|L|-prune(G'))| < |V(H)| so |L|-prune(G') cannot contain an H-model. Contradiction.

- There can be biconnected components of size at least |L| in $R_{2,i}$ for any $1 \le i \le 2n-1$. Suppose $f_i(t) \notin \varphi(H)$, then φ must be an H-model in the graph $R_{2,i} f_i(t)$. Clearly this is not possible since $|V(R_{2,i} f_i(t))| < |V(H)|$, so $f_i(t) \in \varphi(H)$. If $f_i(v) \in \varphi(H)$ or $f_i(s) \in \varphi(H)$ then $Q \ni \{f_i(v), f_i(t)\} \subseteq \varphi(H)$ or $Q \ni \{f_i(t), f_i(s)\} \subseteq \varphi(H)$. Suppose $\varphi(H)$ does not contain $f_i(v)$ or $f_i(s)$, then φ must be an H-model in the graph $G' := R_{2,i} \cup (L_{3,i} f_i(v)) \cup (L_{2,i} f_i(s))$, so $H \preceq G'$. By Proposition 11 we know that |L|-prune $(H) \preceq |L|$ -prune(G'), so $H \preceq |L|$ -prune $(G') = R_{2,i}$. This is a contradiction since $R_{2,i}$ cannot contain H as a minor.
- There can be biconnected components of size at least |L| in $R_{1,i}$ for any $n+1 \leq i \leq 2n-1$. Suppose $f_i(s) \notin \varphi(H)$, then φ must be an H-model in the graph $R_{1,i} - f_i(s)$. Clearly this is not possible since $|V(R_{1,i} - f_i(s))| < |V(H)|$, so $f_i(s) \in \varphi(H)$. If $f_i(t) \in \varphi(H)$ or $f_i(w) \in \varphi(H)$ then $Q \ni \{f_i(t), f_i(s)\} \subseteq \varphi(H)$ or $Q \ni \{f_i(s), f_i(w)\} \subseteq \varphi(H)$. Suppose $\varphi(H)$ does not contain $f_i(t)$ or $f_i(w)$, then φ must be an H-model in the graph $G' := R_{1,i} \cup (L_{2,i} - f_i(t)) \cup (L_{1,i} - f_i(w))$, so $H \preceq G'$. By Proposition 11 we know that |L|-prune $(H) \preceq |L|$ -prune(G'), so $H \preceq |L|$ -prune $(G') = R_{1,i}$. This is a contradiction since $R_{1,i}$ cannot contain H as a minor.
- There can be biconnected components of size at least |L| in $R_{1,i}$ for any $1 \leq i \leq n$. Suppose $f_{n+i-1}(s) \in \varphi(H)$ then $f_i(u) = f_{n+i-1}(w) \in \varphi(H)$ since any path in G connecting $f_{n+i-1}(s)$ to any vertex in $R_{1,i}$ includes $f_i(u)$. So $Q \ni \{f_{n+i-1}(s), f_{n+i-1}(w)\} \subseteq \varphi(H)$. Similarly if $f_i(t) \in \varphi(H)$ then $Q \ni \{f_i(t), f_i(s)\} \subseteq \varphi(H)$ and if $f_{n+i}(t) \in \varphi(G)$ then $Q \ni \{f_{n+i}(v), f_{n+i}(t)\} \subseteq \varphi(H)$. Suppose $\varphi(H)$ does not contain $f_{n+i-1}(s), f_i(t)$ or $f_{n+i}(t)$, then φ must be an H-model in $G' := H_{1,i} \cup (L_{1,n+i-1} f_{n+i-1}(s)) \cup (L_{2,i} f_i(t)) \cup (L_{3,n+i} f_{n+i}(t))$. If $Q \ni \{f_i(u), f_i(s), f_i(w)\} \subseteq \varphi(H)$, then the claim holds, so suppose $\{f_i(u), f_i(s), f_i(w)\} \not\subseteq \varphi(H)$, then for some $p \in \{f_i(u), f_i(s), f_i(w)\}$ we have that φ is an H-model in G' p. Therefore $H \preceq G' p$ and by Proposition 11 we know |L|-prune $(H) \preceq |L|$ -prune(G' p), so $H \preceq |L|$ -prune(G' p) = |L|-prune $(H_{1,i} p)$. However |L|-prune $(H_{1,i} p)$ has at most $|V(H_{1,i} p)| = |V(H)| 1$ vertices, so it cannot contain an H-model. Contradiction.

This concludes the proof of Claim 14.

We now proceed to prove condition 4 of the lemma statement. Let $f_j(v) \in S$ be an arbitrary vertex in S, implying $1 \leq j \leq n$, and take

$$X = \left(\bigcup_{1 \le i < j} \{ f_i(t), f_i(w), f_{i+n}(s) \} \right) \cup \{ f_j(v), f_j(s) \} \cup \left(\bigcup_{j < i \le n} \{ f_i(t), f_i(u), f_{i+n-1}(t) \} \right).$$

In Fig. 4 the vertices in Z are shown in graph G as a red cross. Observe that |X| = 3n - 1 and $f_j(v) \in X$. Furthermore X contains at least one element from each set in Q, hence G - X so is H-minor-free by Claim 14. This shows condition 4a and 4b of the lemma statement holds. We proceed to show conditions 4c and 4d.

(4c) Consider the graph $G' := \lambda(H)$ -prune(G - X). Figure 4 shows a supergraph of G' in gray for the case that $a \in V(R)$. Every connected component in G' must contain

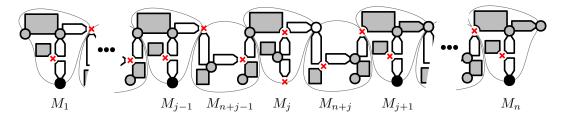


Figure 4 The graph G-X. Vertices in X that are removed from the graph are marked by a cross. Vertices in S are marked black. A supergraph of $\lambda(H)$ -prune(G-X) is shown in gray. Note that when |V(R)| = |V(L)|, not all subgraphs and vertices marked gray are necessarily part of $\lambda(H)$ -prune(G-X).

a biconnected component with at least $\lambda(H) = |L|$ vertices. Consider all biconnected components in G-X containing at least |L| vertices, these can only be contained in the following subgraphs of G: $R_{2,i}$ for any $1 \le i \le 2n-1$, $H_{1,i}$ for any $1 \le i \le n$, and $R_{1,i}$ for any $n+1 \le i \le 2n-1$. Note that any path from a vertex of one of these subgraphs to a vertex of another contains at least one vertex in X, hence any connected component in G contains vertices of at most one of these subgraphs. Since all other biconnected components in G-X have size less than |L| we know that each connected component in |L|-prune(G-X) is a subgraph of $R_{1,i}$, $R_{2,i}$ or $H_{1,i}$ for some i, hence |L|-prune $(G-X) \lesssim H$.

(4d) Finally we show that all connected components in G-X that contain a vertex of S have size less than |L|. Since we have $f_j(v) \in X$, there is no connected component in G-X containing $f_j(v)$. For all $i \neq j$ we have $f_i(t) \in X$ so the connected components in G-X containing a vertex from S are $L_{3,i} - f_i(t)$ for all $1 \leq i < j$ or $j < i \leq n$. These all have size |L| - 1 and contain exactly one vertex of S.

Using the clause gadget described in Lemma 13 we give a polynomial-parameter transformation for the case where \mathcal{F} contains a single, connected graph H.

- ▶ Lemma 15. For any connected graph H with at least three vertices there exists a polynomial-time algorithm that, given a CNF-formula Φ with k variables, outputs a graph G and an integer ℓ such that all of the following are true:
- 1. there is a set $S \subseteq V(G)$ of at most 2k vertices such that $tw(G S) \le tw(H)$,
- **2.** G contains ℓ vertex-disjoint H-subgraphs,
- 3. if Φ is not satisfiable then there does not exist a set $X \subseteq V(G)$ of size at most ℓ such that G X is H-subgraph-free,
- **4.** if Φ is satisfiable then there exists a set $X \subseteq V(G)$ of size at most ℓ such that G X is H-minor-free, $\lambda(H)$ -prune $(G X) \preceq H$, and $\operatorname{tw}(G X) \leq \operatorname{tw}(H)$.

Proof. Let x_1, \ldots, x_k denote the variables of Φ , let C_1, \ldots, C_m denote the sets of literals in each clause of Φ , and let n denote the total number of occurrences of literals in Φ , i.e. $n = \sum_{1 \leq j \leq m} |C_j|$. Let H_1, \ldots, H_k be copies of H. In each copy H_i we arbitrarily label one vertex v_{x_i} and another $v_{\neg x_i}$. Let G_{var} be the graph obtained from the disjoint union of H_1, \ldots, H_k . For each clause C_j of Φ we create a graph called W_j and vertex set $S_j \subseteq V(W_j)$ by invoking Lemma 13 with H and $|C_j|$. Let G be the graph obtained from the disjoint union of W_1, \ldots, W_m and G_{var} where we identify the vertices in S_j with the appropriate v_{x_i} or $v_{\neg x_i}$ as follows: For each clause C_j let $s_1, \ldots, s_{|C_j|}$ be the vertices in S_j in some arbitrary order, and let $c_1, \ldots, c_{|C_j|}$ be the literals in C_j , then we identify s_i and v_{c_i} for each $1 \leq i \leq |C_j|$.

Finally let $\ell = k + 3n - 2m$ and $S = \bigcup_{1 \le i \le k} \{v_{x_i}, v_{\neg x_i}\}$. Note that $S_j \subseteq S$ for all $1 \le j \le m$. This concludes the description of G, ℓ , and S.

It is easy to see they can be constructed in polynomial time. We proceed to show that all conditions in the lemma statement are met.

- (1) Clearly |S| = 2k and since every connected components in G S is a subgraph of H_1, \ldots, H_k or W_1, \ldots, W_m , we can easily see that $\operatorname{tw}(G - S) \leq \operatorname{tw}(H)$.
- (2) For all $1 \leq j \leq m$ we know from Lemma 13 that $W_j S$ contains a packing of $3|C_i|-2$ H-subgraphs. Since W_j-S and W_i-S are vertex-disjoint for $j\neq i$ we can combine these packings to obtain a packing in G-S of $\sum_{1 \le i \le m} 3|C_i|-2=3n-2m$ vertex-disjoint H-subgraphs. Note that this packing does not contain vertices from H_1, \ldots, H_k , so we can add these to the packing and obtain a packing of $k+3n-2m=\ell$ vertex disjoint H-subgraphs in G.
- (3) We now show that if Φ is not satisfiable, then there does not exist a set $X \subseteq V(G)$ of size at most ℓ such that G-X is H-subgraph-free. Suppose there exists such a set X. Since there is a packing of ℓ vertex-disjoint H-subgraphs in G, we know that X contains exactly one vertex from each H-subgraph in the packing. Since v_{x_i} and $v_{\neg x_i}$ belong to the same subgraph, they cannot both be contained in X. Consider the variable assignment where x_i is assigned true if $v_{x_i} \in X$ or false otherwise. Since we assumed Φ is not satisfiable, there is at least one clause in Φ that evaluates to false with this variable assignment. Let C_i denote such a clause. Since C_i evaluates to false, all of its literals must be false, so for all variables x_i that are not negated in C_j we have $x_i = false$ and therefore $v_{x_i} \notin X$. For all negated variables x_i in C_j we know $x_i = true$ meaning $v_{x_i} \in X$, so $v_{\neg x_i} \notin X$. This means that $\emptyset = X \cap S_j = X \cap V(W_j) \cap V(G_{var})$, but since G_{var} contains k vertex-disjoint H-subgraphs we have $|X \cap V(G_{var})| \geq k$, so then $|X \cap (V(G_{var}) \setminus V(W_j))| \geq k$. For all i there is a packing of $3|C_i|-2$ vertex-disjoint H-subgraphs in $W_i-S=W_i-V(G_{var})$, so in the graph $G - V(W_j)$ there are $k + \sum_{i \neq j} (3|C_i| - 2)$ vertex-disjoint H-subgraphs. This means that $|X \cap (V(G) \setminus V(W_j))| \ge k + \sum_{i \ne j} (3|C_i| - 2)$, and since $|X| = k + \sum_{1 \le i \le m} (3|C_i| - 2)$ we know that $|X \cap V(W_j)| \leq 3|C_j| - 2$. However W_j contains $3|C_j| - 1$ vertex-disjoint H-subgraphs, so G - X cannot be H-subgraph-free. Contradiction.
- (4) Finally we show that if Φ is satisfiable then there exists a set $X \subseteq V(G)$ of size at most ℓ such that G-X is H-minor-free, $\lambda(H)$ -prune $(G-X) \preceq H$, and $\operatorname{tw}(G-X) \leq \operatorname{tw}(H)$. Since Φ is satisfiable there exists a variable assignment such that each clause contains at least one literal that is true. Consider the set X' consisting of all vertices v_{x_i} when x_i is true and $v_{\neg x_i}$ when x_i is false. Since every clause contains one literal that is true, we know for each $1 \leq j \leq m$ that W_j contains at least one vertex from X'. So for each $1 \leq j \leq m$ we have $X' \cap S_j \neq \emptyset$. Take and arbitrary vertex $v_j \in X' \cap S_j$ and let $X_i \subseteq V(W_i)$ be the vertex set containing v_i obtained from condition 4 of Lemma 13. Let $X = X' \cup \bigcup_{1 \le j \le m} X_j$. For all $1 \le j \le m$ we know $|X' \cap X_j| \ge 1$ since $v_j \in X' \cap X_j$. So $|X| \le |X'| + \sum_{1 \le j \le m} (3|C_j| - 2) = k + 3n - 2m = \ell.$

By condition 4b we have that $W_j - X_j$ is H-minor-free for all $1 \leq j \leq m$, so clearly $W_i - X$ is also H-minor-free. Consider an arbitrary connected component G' of G - X. If G' is also a connected component of $W_j - X$ for some $1 \leq j \leq m$, then we have that G' is H-minor-free, $\lambda(H)$ -prune $(G') \leq H$ (by condition 4c), and $\operatorname{tw}(G') \leq \operatorname{tw}(W_i) \leq \operatorname{tw}(H)$. If G' is not a connected component of $W_j - X$ for any $1 \leq j \leq m$, then it contains a connected component of $H_i - X$ as a subgraph, for some $1 \le i \le k$. When G' does not contain any vertices of S we know that G' must be a subgraph of H_i , so G' is H-minor-free, $\lambda(H)$ -prune $(G') \leq G' \leq H$, and $\operatorname{tw}(G') \leq \operatorname{tw}(H_i) = \operatorname{tw}(H)$.

Suppose on the other hand G' does contain a vertex $v \in S$. No connected component

of $W_i - X_i$ contains more than one vertex from S and each connected component of G_{var} contains exactly two vertices of S, one of which is in X. So v is the only vertex in G' that is contained in S. Moreover, since S is the only overlap between the graphs G_{var} and W_j for all $1 \le j \le m$, we have that v is a cut vertex in G', such that for some $1 \le i \le k$, each biconnected component of G' is a subgraph of $H_i - X$ or $W_j - X$ for any $1 \le j \le m$. So each of these biconnected components of G' has treewidth at most tw(H), hence $tw(G') \le tw(H)$. Also, each biconnected component in G' that is a subgraph of $W_i - X = W_i - X_i$ for some $1 \leq j \leq m$ contains a vertex from S and therefore has size at most $\lambda(H) - 1$ by condition 4d on the choice of X_i . So we have that $\lambda(H)$ -prune(G') is a subgraph of H_i , hence $\lambda(H)$ -prune $(G') \leq G' \leq H$. Additionally since H_i contains at least one vertex that is not contained in G' we have $H \not\preceq \lambda(H)$ -prune(G'). Because $H = \lambda(H)$ -prune(H) we can conclude by Proposition 12 that G' is H-minor-free. Since H is connected, and all connected components of G-X are H-minor-free, G-X must also be H-minor-free. We also know for all connected components G' of G-X that $\lambda(H)$ -prune $(G') \leq H$, so $\lambda(H)$ -prune $(G-X) \lesssim H$. Finally since $tw(G') \leq tw(H)$ for each connected component G' of G - X we have that $tw(G - X) \le tw(H)$.

The construction from Lemma 15 can directly be used to give a polynomial-parameter transformation from CNF-SAT parameterized by the number of variables. Observe that if G-X is \mathcal{F} -minor-free, then G-X is also \mathcal{F} -subgraph-free. Similarly, if G-X contains an H-subgraph for all $X\subseteq V(G)$ with $|X|\leq \ell$, then G-X also contains an H-minor. Therefore, for any $type\in\{\text{minor}, \text{subgraph}\}$ and \mathcal{F} consisting of one connected graph on at least three vertices, Lemma 15 gives a polynomial-parameter transformation from CNF-SAT parameterized by the number of variables to \mathcal{F} -type-FREE DELETION parameterized by deletion distance to min tw(\mathcal{F}).

When \mathcal{F} contains multiple graphs, each containing a connected component of at least three vertices, it is possible to select a connected component H of one of the graphs in \mathcal{F} such that the construction described in Lemma 15 forms the main ingredient for a polynomial-parameter transformation. We formally argue this in the following lemma.

- ▶ Lemma 16. For a set \mathcal{F} of graphs, all with a connected component of at least 3 vertices, and a CNF-formula Φ with k variables, we can create in polynomial time a graph G and integer ℓ such that all of the following are true:
- 1. there exists a set $S \subseteq V(G)$ of at most $k^{\mathcal{O}(1)}$ vertices such that $\operatorname{tw}(G-S) \leq \min \operatorname{tw}(\mathcal{F})$,
- **2.** if Φ is not satisfiable then there does not exist a set $X \subseteq V(G)$ of size at most ℓ such that G X is \mathcal{F} -subgraph-free, and
- 3. if Φ is satisfiable then there exists a set $X \subseteq V(G)$ of size at most ℓ such that G X is \mathcal{F} -minor-free.

Proof. Let x_1, \ldots, x_k denote the variables of Φ , let C_1, \ldots, C_m denote the sets of literals in each clause of Φ and let n denote the total number of literals in Φ , i.e. $n = \sum_{1 \le i \le m} |C_i|$.

Note that as a consequence of Observation 4, there is a graph $F \in \mathcal{F}$ that is \preceq -minimal with $\operatorname{tw}(F) = \min \operatorname{tw}(\mathcal{F})$. Let $\mathcal{F}_{\downarrow} \subseteq \mathcal{F}$ denote the set of all \preceq -minimal graphs in \mathcal{F} that have treewidth $\min \operatorname{tw}(\mathcal{F})$. Let H_{\uparrow} denote a \preceq -maximal component of a graph in \mathcal{F}_{\downarrow} such that no other \preceq -maximal component of a graph in \mathcal{F}_{\downarrow} has a leaf-block smaller than $\lambda(H_{\uparrow})$. Let $H \in \mathcal{F}$ denote the graph that contains H_{\uparrow} as \preceq -maximal component, so H is \preceq -minimal in \mathcal{F} and $\operatorname{tw}(H) = \min \operatorname{tw}(\mathcal{F})$. Let $c \geq 1$ denote the number of connected components in H isomorphic to H_{\uparrow} and let Y denote the set of vertices contained in these connected components, so $G[Y] = c \cdot H_{\uparrow}$.

Note that H_{\uparrow} contains at least 3 vertices since otherwise H_{\uparrow} would be a minor of at least one connected component of H containing at least 3 vertices, which contradicts H_{\uparrow} being a \preceq -maximal component of H. We use Lemma 15 to construct a graph G' and integer ℓ' satisfying conditions 1–4. Let $S' \subseteq V(G')$ be the vertex set obtained from condition 1.

Let $G_1 := (2c-1) \cdot G'$, and let the set S be the union of all 2c-1 corresponding copies of S'. Take $\ell = (2c-1) \cdot \ell'$ and let $G_2 := (\ell+1) \cdot (H-Y)$. We make the following claim about G_2 :

▶ Claim 17. G_2 has the following properties: (1) $G_2 \lesssim H$, (2) $\operatorname{tw}(G_2) \leq \operatorname{tw}(H)$, (3) G_2 is H-minor-free, and (4) G_2 is \mathcal{F} -minor-free.

Proof. Property (1) follows directly from the construction and Property (2) follows directly from Property (1). To show Property (3), we show that G_2 is H_{\uparrow} -minor-free. Suppose for contradiction that G_2 contains H_{\uparrow} as minor then, since H_{\uparrow} is connected, there is a connected component H' of G_2 that contains H_{\uparrow} as minor. H' is also a connected component of H. Since H_{\uparrow} is a \leq -maximal component of H and $H_{\uparrow} \leq H'$ we know $H' \leq H_{\uparrow}$, and it follows from Observation 3 that H' is isomorphic to H_{\uparrow} . This is a contradiction since G_2 contains only connected components of H that are not isomorphic to H_{\uparrow} .

Having shown that G_2 is H_{\uparrow} -minor-free, Property (4) is easily shown by contradiction. Suppose G_2 is not \mathcal{F} -minor-free, then there exists a graph $B \in \mathcal{F}$ such that $B \leq G_2$. It follows from $G_2 \lesssim H$ that $B \lesssim H$ and since H is \lesssim -minimal in \mathcal{F} we have that $H \lesssim B \leq G_2$, but then $H_{\uparrow} \leq G_2$. This is a contradiction since G_2 is H_{\uparrow} -minor-free.

Consider the graph $G = G_2 \cup G_1$. It follows from condition 1 of Lemma 15 that $\operatorname{tw}(G_1 - S) \leq \operatorname{tw}(H_{\uparrow}) \leq \operatorname{tw}(H)$ and since $\operatorname{tw}(G_2) \leq \operatorname{tw}(H)$ by Claim 17, we obtain $\operatorname{tw}(G - S) \leq \operatorname{tw}(H) = \min \operatorname{tw}(\mathcal{F})$. Note that $|S| = (2c - 1) \cdot 2k \leq \mathcal{O}(k)$, hence condition 1 holds. We proceed to show the other conditions of the lemma statement.

(2) Suppose Φ is not satisfiable, and take an arbitrary $X \subseteq V(G)$ of size at most ℓ . We prove G-X is not \mathcal{F} -subgraph-free by showing that G-X contains an H-subgraph. First note that $G_2 - X$ contains at least one copy of $H - Y = H - c \cdot H_{\uparrow}$, so it remains to show that $G_1 - X$ contains c vertex-disjoint H_{\uparrow} -subgraphs. Recall that G_1 is the disjoint union of 2c-1 copies of G'. Consider the subgraph \hat{G}_1 of G_1 consisting of the G'-subgraphs in G_1 that contain at most ℓ' vertices of X. Since Φ is not satisfiable, G' leaves at least one H_{\uparrow} -subgraph when ℓ' or fewer vertices are removed, so each G'-subgraph in \hat{G}_1 leaves at least one H_{\uparrow} -subgraph in $G_1 - X$. When \hat{G}_1 contains at least c vertex-disjoint G'-subgraphs, we know that there are at least c vertex-disjoint H_{\uparrow} -subgraphs in $G_1 - X$, concluding the proof. Suppose instead that \hat{G}_1 contains less than c vertex-disjoint G'-subgraphs. Let xbe the number of G'-subgraphs in $G_1 - V(\hat{G}_1)$. Since G_1 contains 2c-1 vertex-disjoint G'-subgraphs we have $x \geq c$. Each of the G'-subgraphs in $G_1 - V(\hat{G}_1)$ contain at least $\ell' + 1$ vertices of X, so \hat{G}_1 contains at most $\ell - x(\ell' + 1)$ vertices of X. We also know \hat{G}_1 contains $\ell'((2c-1)-x)$ vertex-disjoint H_{\uparrow} -subgraphs since G' contains ℓ' vertex-disjoint H_{\uparrow} -subgraphs and there are (2c-1)-x vertex-disjoint G'-subgraphs in \hat{G}_1 . We conclude that the number of vertex-disjoint H_{\uparrow} -subgraphs in $\hat{G}_1 - X$, and therefore also in $G_1 - X$, is at least

$$\ell'((2c-1)-x) - (\ell - x(\ell'+1)) = \ell'((2c-1)-x) - ((2c-1)\cdot\ell' - x\ell' - x)$$
$$= \ell'((2c-1)-x) - \ell'((2c-1)-x) + x$$
$$= x > c.$$

This concludes the proof of condition 2.

(3) When Φ is satisfiable we know that there exists a set $X' \subseteq V(G')$ of size at most ℓ' such that G' - X' is H_{\uparrow} -minor-free and $\lambda(H_{\uparrow})$ -prune $(G' - X') \lesssim H_{\uparrow}$. So then there exists a set $X \subseteq V(G_1)$ of size at most $(2c-1) \cdot \ell' = \ell$ such that $G_1 - X$ is H_{\uparrow} -minor-free and $\lambda(H_{\uparrow})$ -prune $(G_1 - X) \lesssim H_{\uparrow}$. Since G_2 is also H_{\uparrow} -minor-free we know that G - X is H_{\uparrow} -minor-free and therefore also H-minor-free. We now show that G - X is also \mathcal{F} -minor-free.

First observe the following:

$$\lambda(H_{\uparrow})$$
-prune $(G_2 - X) \leq G_2 - X \leq G_2 \lesssim H$, and (1)

$$\lambda(H_{\uparrow})$$
-prune $(G_1 - X) \preceq H_{\uparrow} \preceq H$. (2)

We now deduce

$$\lambda(H_{\uparrow})$$
-prune $(G - X) = \lambda(H_{\uparrow})$ -prune $(G_2 - X) \cup (G_1 - X)$)
= $\lambda(H_{\uparrow})$ -prune $(G_2 - X) \cup \lambda(H_{\uparrow})$ -prune $(G_1 - X)$)
 $\lesssim H$ (by Equation 1 and 2)

Suppose G-X is not \mathcal{F} -minor-free, then for some $H' \in \mathcal{F}$ we have $H' \preceq G-X$. There must exist a graph $B \in \mathcal{F}$ such that B is \preceq -minimal in \mathcal{F} and $B \preceq G-X$ since if H' is \preceq -minimal in \mathcal{F} then H' forms such a graph B, and if on the other hand H' is not \preceq -minimal in \mathcal{F} then there exists a graph $H'' \in \mathcal{F}$ such that $H'' \preceq H'$ and H'' is \preceq -minimal in \mathcal{F} , meaning H'' forms such a graph B.

Since $B \lesssim G - X$ we know by Observation 4 that $\operatorname{tw}(B) \leq \operatorname{tw}(G - X)$. Recall that $\operatorname{tw}(G - X) \leq \min \operatorname{tw}(\mathcal{F})$ so then $B \in \mathcal{F}_{\downarrow}$. Because of how we chose H_{\uparrow} , we know for all \leq -maximal components B_{\uparrow} of B that $\lambda(B_{\uparrow}) \geq \lambda(H_{\uparrow})$. Therefore

$$B \lesssim \lambda(H_{\uparrow})$$
-prune (B) since $B = \lambda(H_{\uparrow})$ -prune (B) $\lesssim \lambda(H_{\uparrow})$ -prune $(G - X)$ by Proposition 12 since $B \lesssim G - X$ $\lesssim H$.

Since H is \preceq -minimal in \mathcal{F} , it follows that $H \preceq B$. By definition of \preceq we have $H_{\uparrow} \preceq B \preceq G - X$. Since H_{\uparrow} is connected we conclude $H_{\uparrow} \preceq G - X$. This is a contradiction since G - X is H_{\uparrow} -minor-free.

We conclude that a polynomial-parameter transformation exists for all $type \in \{\text{minor}, \text{subgraph}\}$ and \mathcal{F} containing only graphs with a connected component on at least three vertices. Together with the fact that CNF-SAT is MK[2]-hard and does not admit a polynomial kernel unless $NP \subseteq coNP/poly$ (cf. [22, Lemma 9]), this proves the following generalization of Theorem 1.

▶ Theorem 18. For type \in {minor, subgraph} and a set \mathcal{F} of graphs, all with a connected component of at least three vertices, \mathcal{F} -type-FREE DELETION parameterized by vertex-deletion distance to a graph of treewidth min tw(\mathcal{F}) is MK[2]-hard and does not admit a polynomial kernel unless NP \subseteq coNP/poly.

4 A polynomial Turing kernelization

In this section we consider the case where \mathcal{F} contains a graph with no connected component of more than two vertices; or in short \mathcal{F} contains a P_3 -subgraph-free graph. This graph consists of isolated vertices and disjoint edges. Let $\mathrm{isol}(G)$ denote the set of isolated vertices in a graph G, i.e. $\mathrm{isol}(G) = \{v \in V(G) \mid \deg(v) = 0\}$. We first show that the removal of all isolated vertices from all graphs in \mathcal{F} only changes the answer to \mathcal{F} -MINOR-FREE DELETION and \mathcal{F} -Subgraph-Free Deletion when the input is of constant size.

▶ **Lemma 19.** For $type \in \{minor, subgraph\}$ and any family of graphs \mathcal{F} containing a P_3 -subgraph-free graph, let $\mathcal{F}' = \{F - isol(F) \mid F \in \mathcal{F}\}$. For any graph G, if G is \mathcal{F} -type-free but not \mathcal{F}' -type-free, then $|V(G)| < \max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3)$.

Proof. We first prove the lemma for type = subgraph. Suppose G is \mathcal{F} -subgraph-free but not \mathcal{F}' -subgraph-free. Now G contains an H'-subgraph for some graph $H' \in \mathcal{F}'$. This subgraph consists of |V(H')| vertices. Let $H \in \mathcal{F}$ be the graph for which H' = H - isol(H). G cannot contain |isol(H)| vertices in addition to the vertices in the H'-subgraph because otherwise G trivially contains an \mathcal{F} -subgraph. Hence $|V(G)| < |V(H')| + |\text{isol}(H)| = |V(H)| \le \max_{F \in \mathcal{F}} |V(F)|$.

Next, we show the lemma holds for type = minor. If some graph G is \mathcal{F} -minor-free but not \mathcal{F}' -minor-free then for some graph $H \in \mathcal{F}$ we have $H' \preceq G$ but not $H \preceq G$ where H' = H - isol(H). Let φ be a minimal H'-model in G. The graph G has less than |V(isol(H))| vertices that are not in any branch set of φ , since otherwise an H-model could be constructed in G by taking the branch sets of φ and adding |V(isol(H))| branch sets consisting of a single vertex.

The number of vertices in G that are contained in a branch set of φ can also be limited. For an arbitrary vertex $v \in V(H')$ consider a spanning tree T of $G[\varphi(v)]$. If $\varphi(v)$ contains multiple vertices then for each leaf p of T, there must be a vertex $u \in N_{H'}(v)$ and $q \in N_G(p) \cap \varphi(u)$, such that p is the only vertex from $\varphi(v)$ that is adjacent to $\varphi(u)$; otherwise, removing leaf p from the branch set $\varphi(v)$ would yield a smaller H'-model in G. Hence there can only be $\max\{1, \deg_{H'}(v)\}$ leaves in T.

To give a bound on the size of each branch set consider a smallest graph $D \in \mathcal{F}'$ that is P_3 -subgraph-free. Take $\ell = |V(D)|$ and note that $D \preceq P_\ell$. Since we know that G is \mathcal{F}' -minor-free, G must also be P_ℓ -subgraph-free, therefore T is also P_ℓ -subgraph-free. Consider an arbitrary vertex r in T. Since T is a tree, there is exactly one path from r to each leaf of T and every vertex of T lies on at least one path from r to a leaf of T. Since there are no more than $\max\{1,\deg_{H'}(v)\}$ leaves in T there are at most $\max\{1,\deg_{H'}(v)\}$ such paths, and all these paths contain less than ℓ vertices since T is P_ℓ -subgraph-free, hence in total T contains less than $\deg_{H'}(v) \cdot \ell$ vertices. We can now give a bound on the total number of vertices in G as follows:

$$\begin{split} |V(G)| &< |\operatorname{isol}(H)| + \sum_{v \in H - \operatorname{isol}(H)} |\varphi(v)| \\ &\leq |\operatorname{isol}(H)| + \sum_{v \in H - \operatorname{isol}(H)} (\deg_{H'}(v) \cdot \ell) \\ &\leq |\operatorname{isol}(H)| + 2 \cdot |E(H)| \cdot \ell \\ &\leq |V(H)| + 2 \cdot |V(H)|^2 \cdot |V(D)| \\ &\leq \max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3) \end{split}$$

This concludes the proof.

After the removal of isolated vertices in \mathcal{F} to obtain \mathcal{F}' , we know that \mathcal{F}' contains a graph consisting entirely of disjoint edges, i.e. this graph is isomorphic to $c \cdot P_2$ for some integer $c \geq 0$. If c = 0 then \mathcal{F} -type-free graphs have constant size and the problem is polynomial-time solvable. We proceed assuming $c \geq 1$. Let the matching number of a graph G, denoted as $\nu(G)$, be the size of a maximum matching in G. We make the following observation.

▶ **Observation 20.** For all $c \ge 1$, a graph G is $c \cdot P_2$ -subgraph-free if and only if $\nu(G) \le c - 1$.

We give a characterization of graphs with bounded matching number, based on an adaptation of the Tutte-Berge formula [4]. We use odd(G) to denote the number of connected components in G that consist of an odd number of vertices.

- ▶ **Lemma 21.** For any graph G and integer m we have $\nu(G) \leq m$ if and only if V(G) can be partitioned into three disjoint sets U, R, S such that all of the following are true:
- \blacksquare all connected components in G[R] have an odd size of at least 3,
- = G[S] is independent,
- $N_G(S) \subseteq U$, and
- $|U| + \frac{1}{2}(|R| \text{odd}(G[R])) \le m.$

Proof. Consider the Tutte-Berge formula [4] (cf. [36, Chapter 24]):

$$\nu(G) = \frac{1}{2} \min_{U \subset V(G)} (|V(G)| - \text{odd}(G - U) + |U|).$$

Suppose $\nu(G) \leq m$. It follows from the Tutte-Berge formula that there exists a $U_1 \subseteq V(G)$ such that $\frac{1}{2}(|V(G)| - \operatorname{odd}(G - U_1) + |U_1|) = \nu(G) \leq m$. From each connected component in $G - U_1$ on an even number of vertices, select a single non-cut vertex and add the selected vertices to a set U_2 . Now take $U = U_1 \cup U_2$. Note that G - U contains only odd sized connected components and $\operatorname{odd}(G - U) = \operatorname{odd}(G - U_1) + |U_2|$. Let S be the set of isolated vertices in G - U and let $R = V(G) \setminus (U \cup S)$. Observe that U, R, and S satisfy the first three conditions in the lemma statement: G[R] contains only connected components with an odd size of at least S, S is independent, and S satisfy the first odd S contains only connected components with an odd size of at least S contains only connected components with odd S contains only connected components with S contains of S contains of S contains of S contains only connected S contains

$$|U| + \frac{1}{2}(|R| - \operatorname{odd}(G[R])) = \frac{1}{2}(2|U| + |S| - |S| + |R| - \operatorname{odd}(G[R])$$

$$= \frac{1}{2}((|U| + |S| + |R|) - (\operatorname{odd}(G[R]) + |S|) + |U|)$$

$$= \frac{1}{2}(|V(G)| - \operatorname{odd}(G - U) + |U|)$$

$$= \frac{1}{2}(|V(G)| - (\operatorname{odd}(G - U_1) + |U_2|) + |U_1| + |U_2|)$$

$$= \frac{1}{2}(|V(G)| - \operatorname{odd}(G - U_1) + |U_1|)$$

$$= \nu(G) < m$$

For the reverse direction of the proof, suppose V(G) can be partitioned into disjoint sets U, R, S as described in the lemma statement. A maximum matching in G[R] has size at most $\frac{1}{2}(|R| - \operatorname{odd}(G[R]))$ since at least one vertex in each odd component remains unmatched and every matching edge covers two vertices. Since $N_G(S) \subseteq U$ we know that S is isolated in G - U, so $\nu(G - U) = \nu(G[R]) \leq \frac{1}{2}(|R| - \operatorname{odd}(G[R]))$. Since a matching in G is at most |U| edges larger than a matching in G - U we conclude $\nu(G) \leq |U| + \frac{1}{2}(|R| - \operatorname{odd}(G[R])) \leq m$.

Let us showcase how Lemma 21 can be used to attack \mathcal{F} -MINOR-FREE DELETION when \mathcal{F} consists of a single graph $c \cdot P_2$, so that the problem is to find a set $X \subseteq G$ of size at most ℓ such that G - X has matching number less than c.

▶ Theorem 22. For any constant c, the $\{c \cdot P_2\}$ -MINOR-FREE DELETION problem parameterized by the size k of a feedback vertex set, can be solved in polynomial time using an oracle that answers VERTEX COVER instances with $\mathcal{O}(k^3)$ vertices.

Proof. If an instance (G, ℓ) admits a solution X, then Lemma 21 guarantees that V(G - X) can be partitioned into U, R, S satisfying the four conditions for m = c - 1. We try all relevant options for the sets U and R in the partition, of which there are only polynomially many since $|U| + \frac{1}{3}|R| \le m \in \mathcal{O}(1)$.

For given sets $U, R \subseteq V(G)$, we can decide whether there is a solution X of size at most ℓ for which U, R, and $S := V(G) \setminus (U \cup R \cup X)$ form the partition witnessing that G - Xhas matching number at most m, as follows. If some component of G[R] has even size, or less than three vertices, we reject outright. Similarly, if $|U| + \frac{1}{2}(|R| - \text{odd}(G[R])) > m$, we reject. Now, if U and R were guessed correctly, then Lemma 21 guarantees that the only neighbors of R in the graph G-X belong to U. Hence we infer that all vertices of $X' := N_G(R) \setminus U$ must belong to the solution X. Note that since S is an independent set in G-X, the solution X forms a vertex cover of $G-(U\cup R)$, so that $X'':=X\setminus X'$ is a vertex cover of $G' := G - (U \cup R \cup X')$. On the other hand, for every vertex cover X''of G', the graph $G - (X' \cup X'')$ will have matching number at most m, as witnessed by the partition. Hence the problem of finding a minimum solution X whose corresponding graph G-X has U and R as two of the classes in its witness partition, reduces to finding a minimum vertex cover of the graph G'. In terms of the decision problem, this means G has a solution of size at most ℓ with U and R as witness partite sets, if and only if G' has a vertex cover of size at most $\ell - |X'|$. Since $FVS(G') \leq FVS(G)$, we can apply the known [25] kernel for Vertex Cover parameterized by the feedback vertex number to reduce $(G', \ell - |X'|)$ to an equivalent instance with $\mathcal{O}(\text{FVS}(G)^3)$ vertices, which is queried to the oracle. If the oracle answers positively to any query, then (G,ℓ) has answer YES; otherwise the answer is NO.

We remark that by using the polynomial-time reduction guaranteed by NP-completeness, the queries to the oracle can be posed as instances of the original \mathcal{F} -MINOR-FREE DELETION problem, rather than VERTEX COVER. We now present our general (non-adaptive) Turing kernelization for the minor-free and subgraph-free deletion problems for all families \mathcal{F} containing a P_3 -subgraph-free graph, combining three ingredients. Lemma 19 allows us to focus on families whose graphs have no isolated vertices. The guessing strategy of Theorem 22 is the second ingredient. The final ingredient is required to deal with the fact that a solution subgraph G-X that is $c \cdot P_2$ -minor-free for some $c \cdot P_2 \in \mathcal{F}$, may still have one of the other graphs in \mathcal{F} as a forbidden minor. To cope with this issue, we show in Lemma 24 that if G-X has no matching of size c (i.e., G-X has a vertex cover of size at most 2c), but does contain a minor model of some graph in \mathcal{F} , then there is such a minor model of constant size. By employing a more expensive (but still polynomially bounded) guessing step, this allows us to complete the Turing kernelization. In the following lemmas VC(G) will denote the vertex cover number and $\Delta(G)$ will denote the maximum degree of G.

- ▶ Proposition 23 ([13, Proposition 1]). If G contains H as a minor, then there is a subgraph G^* of G containing an H-minor such that $\Delta(G^*) \leq \Delta(H)$ and $|V(G^*)| \leq |V(H)| + \text{VC}(G^*) \cdot (\Delta(H) + 1)$.
- ▶ **Lemma 24.** For any type \in {minor, subgraph}, let \mathcal{F} be a family of graphs, let G be a graph with vertex cover C, and let S = V(G C). If G contains an \mathcal{F} -type, then there exists $S' \subseteq S$ such that $G[C \cup S']$ contains an \mathcal{F} -type and $|S'| \le \max_{H \in \mathcal{F}} |V(H)| + |C| \cdot (\Delta(H) + 1)$.
- **Proof.** Suppose type = minor, then by Proposition 23 we know that if G contains $H \in \mathcal{F}$ as a minor, then there is a subgraph G^* of G containing an H-minor such that $|V(G^*)| \leq |V(H)| + \text{VC}(G^*) \cdot (\Delta(H) + 1)$. Take $S' = V(G^*) \cap S$, then $G[C \cup S'] = G[C \cup V(G^*)]$ contains

an \mathcal{F} -minor and

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\begin{split} |S'| &\leq |V(G^*)| \\ &\leq |V(H)| + \operatorname{VC}(G^*) \cdot (\Delta(H) + 1) \\ &\leq |V(H)| + |C| \cdot (\Delta(H) + 1) \\ &\leq \max_{H \in \mathcal{F}} |V(H)| + |C| \cdot (\Delta(H) + 1). \end{split}
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On the other hand, when type = subgraph then G contains an H-subgraph for some $H \in \mathcal{F}$, and trivially there exists a set $X \subseteq V(G)$ of |V(H)| vertices such that G[X] contains an H-subgraph. Take S' = X - C and clearly $G[C \cup S']$ contains an H-minor.

Armed with Lemma 24 we now present the proof of the general Turing kernelization.

▶ **Theorem 2.** Let \mathcal{F} be a finite set of graphs, such that some $H \in \mathcal{F}$ has no connected component of three or more vertices. Then \mathcal{F} -MINOR-FREE DELETION and \mathcal{F} -SUBGRAPH-FREE DELETION admit polynomial Turing kernels when parameterized by the vertex-deletion distance to a graph of treewidth min tw(\mathcal{F}).

Proof. Fix some $type \in \{ \min, \operatorname{subgraph} \}$. First, consider input instances (G, ℓ) for which $|V(G)| - \ell \leq \max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3)$. For these instances there exists a vertex set X of size at most ℓ such that G - X is \mathcal{F} -type-free if and only if there exists a vertex set Y of size at most $|V(G)| - \ell \leq \max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3)$ such that G[Y] is \mathcal{F} -type-free. Since there are only polynomially many such vertex sets Y, and for each Y we can check in polynomial time whether G[Y] contains an \mathcal{F} -type [34], we can apply brute force to solve the instance in polynomial time.

So from now on we only consider instances (G, ℓ) for which $|V(G)| - \ell > \max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3)$. This means that for any vertex set X of size at most ℓ , the graph G - X contains more than $\max_{F \in \mathcal{F}} (|V(F)| + 2|V(F)|^3)$ vertices. Take $\mathcal{F}' = \{F - \mathrm{isol}(F) \mid F \in \mathcal{F}\}$ and we obtain from Lemma 19 that if G - X is \mathcal{F} -type-free, it is also \mathcal{F}' -type-free, and clearly if G - X contains an \mathcal{F} -type it is also contains an \mathcal{F}' -type. Hence the \mathcal{F} -type-FREE DELETION instance (G, ℓ) is equivalent to the \mathcal{F}' -type-FREE DELETION instance (G, ℓ) . Note that if \mathcal{F}' contains an empty graph, the instance is trivially false since every graph contains the empty graph as a subgraph. In the rest of the algorithm we assume each graph in \mathcal{F}' contains at least one edge.

Since every graph in \mathcal{F} contains an edge and at least one graph in \mathcal{F} has no component of three vertices or more, we have $\min \operatorname{tw}(\mathcal{F}) = 1$. Therefore the parameter, the deletion distance to treewidth $\min \operatorname{tw}(\mathcal{F})$, is equal to $\operatorname{FVS}(G)$.

To complete the Turing kernelization for \mathcal{F} -type-Free Deletion, it suffices to give a polynomial-time algorithm solving \mathcal{F}' -type-Free Deletion using an oracle that can solve \mathcal{F} -Minor-Free Deletion instances (G',ℓ') for which $|V(G')| \leq \text{FVS}(G)^{\mathcal{O}(1)}$ and $\text{FVS}(G') \leq \text{FVS}(G)^{\mathcal{O}(1)}$. Note that the latter condition on (G',ℓ') is redundant since FVS(G') < |V(G')| for any graph G'.

We will use the \mathcal{F} -type-Free Deletion oracle to solve Vertex Cover instances on induced subgraphs G_0 of G, for which $\text{FVS}(G_0) \leq \text{FVS}(G)$. This is done by a subroutine called VCoracle. A call to VCoracle (G_0, ℓ_0) decides whether G_0 has a vertex cover of size at most ℓ_0 , by the following process.

1. Compute a 2-approximate feedback vertex set S_0 on G_0 in polynomial time, for example using the algorithm by Bafna et al. [2].

- 2. Apply the kernelization by Jansen and Bodlaender [25] for VERTEX COVER parameterized by feedback vertex set to the instance (G_0, ℓ_0) and the approximate feedback vertex set S_0 . This takes polynomial time, and results in an instance (G_1, ℓ_1) of VERTEX COVER on $\mathcal{O}(|S_0|^3) \leq \mathcal{O}(\text{FVS}(G_0)^3)$ vertices that is equivalent to (G_0, ℓ_0) .
- 3. Since every graph in \mathcal{F} contains at least one edge, the \mathcal{F} -type-FREE DELETION problem is NP-complete [30]. The VERTEX COVER problem is also known to be NP-complete, hence there exists a polynomial-time algorithm that transforms the VERTEX COVER instance (G_1, ℓ_1) into an equivalent \mathcal{F} -type-FREE DELETION instance (G_2, ℓ_2) . Since this algorithm runs in polynomial time and the size of its input is $FVS(G_0)^{\mathcal{O}(1)}$, the number of vertices in G_2 is upper-bounded by $FVS(G_0)^{\mathcal{O}(1)}$.
- **4.** Query the instance (G_2, ℓ_2) of size $FVS(G_0)^{\mathcal{O}(1)}$ to the \mathcal{F} -type-Free Deletion oracle, and output the answer as the result of the call $VCoracle(G_0, \ell_0)$.

Whenever G_0 is an induced subgraph of G, we have $FVS(G_0) \leq FVS(G)$ so that the procedure above allows the Turing kernelization to implement an oracle for answering vertex cover queries on G_0 , using queries to the \mathcal{F} -type-Free Deletion oracle of size bounded polynomially in the parameter.

Using this subroutine, the Turing kernelization algorithm is given in Algorithm 1. The high-level idea is as follows. Let M be the smallest graph in \mathcal{F}' that has no component of three or more vertices, or equivalently, which is P_3 -subgraph-free; then M consists of isolated edges. The Turing kernelization first guesses the sets U and R as per Lemma 21 witnessing that the graph G-X obtained after removing the unknown solution X, does not have a matching of |E(M)| edges (i.e. that G-X does not contain $M \in \mathcal{F}'$ as both a minor and a subgraph). Since Lemma 21 guarantees that in the graph G-X we have $N_{G-X}(R)\subseteq U$, it follows that $N_G(R) \setminus U$ must belong to the unknown solution X if this guess was correct. The algorithm then considers the remaining vertices $Q := V(G) \setminus (U \cup R \cup N_G(R))$ and classifies them into $2^{|U|}$ types based on their adjacency to U. An additional guessing step attempts to guess up to α vertices of each type in G-X, which will be part of the set S in the partition of Lemma 21, by taking them into the range of the function f. The algorithm tests whether the graph $G[f(2^U) \cup U \cup R]$ is \mathcal{F}' -type-free. If not, then the guess was incorrect. If so, then for each type of which fewer than α vertices were guessed to remain behind in G-X, the algorithm collects the remaining vertices of that type in a set Q' to be added to the solution X, and a VERTEX COVER instance is formulated on the remaining vertices of Q. For types of which α vertices remained behind, no vertices have to be added to Q'or the solution X in this step, because using Lemma 24 it can be guaranteed that having more vertices of that type will not lead to an \mathcal{F}' -type. The algorithm returns true if the formulated instance of Vertex Cover has a solution that yields a set of size at most ℓ when combined with the vertices of $N_G(R) \setminus U$ and Q'.

Correctness When the algorithm returns true, then consider the values of U, R, Q, f, and Q' at the time that true is returned. There exists a vertex cover X' of size at most $\ell - |(N_G(R) \setminus U) \cup Q'|$ in G[Q] - Q'. Let $X = X' \cup (N_G(R) \setminus U) \cup Q'$, which has size at most ℓ . The set X is a vertex cover in $G - (U \cup R)$, since $G - (U \cup R) - X = (G[Q] - Q') - X'$. Hence $S := V(G - (U \cup R)) \setminus X$ is an independent set, even in G. The sets G, G, G, G form a partition of G. See Fig. 5a for a visual representation of these sets. We will show that G is a solution to G-type-Free Deletion on G.

Consider an arbitrary vertex $v \in S$. Note that since $N_G(R) \subseteq U \cup X$ we have $S = V(G) \setminus (U \cup R \cup X) \subseteq V(G) \setminus (U \cup R \cup N_G(R)) = Q$, so $v \in Q$. By definition of X we know $Q' \subseteq X$ so $v \notin Q'$. Then by definition of Q' on line 15 we observe the following:

Algorithm 1: Solving \mathcal{F}' -type-Free Deletion instances using VCoracle with \mathcal{F}' containing a P_3 -subgraph-free graph and no empty graphs or graphs with isolated vertices.

```
input: A graph G and an integer \ell
   output: true if there exists a set X of size at most \ell such that G-X is \mathcal{F}'-type-free,
               or false otherwise.
 1 Let m = |E(M)| - 1 where M is a smallest P_3-subgraph-free graph in \mathcal{F}'
 2 Let \alpha = \max_{H \in \mathcal{F}'} |V(H)| + 3m(\Delta(H) + 1)
 3 forall U \subseteq V(G) with |U| \leq m do
        forall R \subseteq V(G-U) such that
            all connected components in G[R] have an odd size of at least 3
 5
            and |U| + \frac{1}{2}(|R| - \text{odd}(G[R])) \le m
 6
            Q := V(G) \setminus (U \cup R \cup N_G(R))
            for all functions f: 2^U \to 2^Q such that
 9

ightharpoonup Recall that f(2^U) = \bigcup_{Y \subset U} f(Y)
                 G[f(2^U)] is independent and
10
                 G[f(2^U) \cup U \cup R] is \mathcal{F}'-type-free and
11
                 \forall_{Y \subset U} |f(Y)| \leq \alpha and
12
                 \forall_{Y \subseteq U} \forall_{v \in f(Y)} N_G(v) \cap U = Y
13
                 Q' := \{ v \in Q \setminus f(2^U) \mid |f(N_G(v) \cap U)| < \alpha \}
15
                 if VCoracle(G[Q] - Q', \ell - |(N_G(R) \setminus U) \cup Q'|) then
16
                     return true
17
18 return false
```

▶ **Observation 25.** For all $v \in S$ we have $v \in f(2^U)$ or $|f(N_G(v) \cap U)| \ge \alpha$.

Assume for a contradiction that $G-X=G[U\cup R\cup S]$ contains an \mathcal{F}' -type. Since G[S] is independent, $U\cup R$ is a vertex cover in G-X, and by Lemma 24 there exists a set $S'\subseteq S$ with $|S'|\leq \max_{H\in\mathcal{F}'}|V(H)|+|U\cup R|\cdot (\Delta(H)+1)$ such that $G[U\cup R\cup S']$ contains an \mathcal{F}' -type. Note that $|R|-3\operatorname{odd}(G[R])\geq 0$ since every connected component in G[R] contains at least 3 vertices, so then

```
\begin{split} |S'| & \leq \max_{H \in \mathcal{F}'} |V(H)| + |U \cup R| \cdot (\Delta(H) + 1) \\ & \leq \max_{H \in \mathcal{F}'} |V(H)| + (|U| + |R| + \frac{1}{2}(|R| - 3\operatorname{odd}(G[R]))) \cdot (\Delta(H) + 1) \\ & \leq \max_{H \in \mathcal{F}'} |V(H)| + 3(|U| + \frac{1}{2}(|R| - \operatorname{odd}(G[R]))) \cdot (\Delta(H) + 1) \\ & \leq \max_{H \in \mathcal{F}'} |V(H)| + 3m(\Delta(H) + 1) \\ & = \alpha. \end{split}
```

▶ Claim 26. The graph $G[U \cup R \cup S']$ is isomorphic to a subgraph of $G[U \cup R \cup f(2^U)]$.

Proof. Observe that S contains no neighbors of R, and since G[S] is independent, we know for all $v \in S$ that $N_G(v) \subseteq U \cup X$ and therefore $N_{G-X}(v) = N_G(v) \cap U$. From Observation 25 it follows for all $v \in S'$ that $v \in f(2^U)$ or $|f(N_G(v) \cap U)| \ge \alpha$. In the latter case v is a

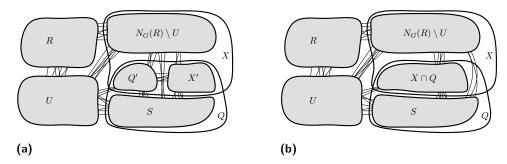


Figure 5 We show two partitions of G. Figure 5a shows a partition of G given that Algorithm 1 returns true, while Fig. 5b shows a partition of G given that G - X is \mathcal{F} -type-free. Note that in both cases there can be no edges between R and Q.

false twin of any vertex $u \in f(N_G(v) \cap U)$ in G - X since by definition of f we have $N_G(u) \cap U = N_G(v) \cap U$ for all vertices $u \in f(N_G(v) \cap U)$. We have $|S'| \leq |f(N_G(v) \cap U)|$ for all $v \in S'$, so there exists a bijection that maps all vertices $v \in S'$ to a vertex in $u \in f(2^U)$ that is a false twin of v in G - X. Any two false twins in G - X are interchangeable in G - X, hence $G[U \cup R \cup S']$ is isomorphic to a subgraph of $G[U \cup R \cup f(2^U)]$.

Since f is chosen such that $G[U \cup R \cup f(2^U)]$ is \mathcal{F}' -type-free on line 11, Claim 26 leads to a contradiction with the fact that $G[U \cup R \cup S']$ contains an \mathcal{F}' -type. We conclude that if the algorithm returns true a set X of size ℓ exists such that G - X is \mathcal{F}' -type-free.

Next, we consider the reverse direction. We show that the algorithm returns true when there exists a set X of size at most ℓ such that G-X is \mathcal{F}' -type-free. Let m=|E(M)|-1 where M is the smallest P_3 -subgraph-free graph in \mathcal{F}' , i.e. M is isomorphic to $(m+1)\cdot P_2$ since no graph in \mathcal{F}' contains isolated vertices. The graph G-X is \mathcal{F}' -type-free so it is also $(m+1)\cdot P_2$ -subgraph-free, and by Observation 20 we know $\nu(G-X)\leq m$. Therefore by Lemma 21 there exists a partition U',R',S of V(G-X) such that all of the following are true:

- \blacksquare all connected components in (G-X)[R']=G[R'] have an odd size of at least 3,
- (G-X)[S] = G[S] is independent,
- $N_{G-X}(S) \subseteq U$ or equivalently $N_G(S) \subseteq U \cup X$, and
- $|U'| + \frac{1}{2}(|R'| \text{odd}(G[R'])) \le m.$

Clearly U' and R' are such that there is an iteration in the algorithm where U=U' and R=R'. Let Q be the set as defined on line 8 in this iteration, see Fig. 5b. Let $g\colon 2^U\to 2^S$ be defined as $g(Y)=\{v\in S\mid Y=N_G(v)\cap U\}$ for all $Y\subseteq U$. We define a function $f'\colon 2^U\to 2^S$ that maps any $Y\subseteq U$ to an arbitrary subset of g(Y) of size $\min\{|g(Y)|,\alpha\}$. We make the following observations:

- Since $N_G(S) \subseteq U \cup X$ we have $N_G(R) \cap S = \emptyset$ so $S = S \setminus N_G(R) = V(G) \setminus (U \cup R \cup X \cup N_G(R)) \subseteq V(G) \setminus (U \cup R \cup N_G(R)) = Q$, so $f' : 2^U \to 2^Q$.
- \blacksquare G[S] is independent, so $G[f'(2^U)]$ is also independent because $f'(2^U) \subseteq S$.
- $G[U \cup R \cup f'(2^U)]$ is a subgraph of $G[U \cup R \cup S] = G X$, and since G X is \mathcal{F}' -type-free, $G[U \cup R \cup f'(2^U)]$ is also \mathcal{F}' -type-free.
- \blacksquare Clearly $\forall_{Y \subset U} |f'(Y)| \leq \alpha$, and
- $\forall Y \in U \forall v \in f'(Y) N_G(v) \cap U = Y.$

Hence f' satisfies all conditions stated in line 9 of the algorithm, so there is an iteration of the algorithm where f = f'. Let Q' be the set as defined on line 15 in this iteration. We now show that there exists a vertex cover of size at most $\ell - |(N_G(R) \setminus U) \cup Q'|$ in G[Q] - Q'.

Since G[S] is independent, X is a vertex cover in $G[X \cup S] = G - (U \cup R)$. Then clearly $X \setminus (N_G(R) \setminus U)$ is a vertex cover in $G - (U \cup R \cup (N_G(R) \setminus U))$ and since $N_G(R) \setminus U \subseteq X$ we have $|X \setminus (N_G(R) \setminus U)| \le \ell - |N_G(R) \setminus U|$. Similarly consider the set $A = (N_G(R) \setminus U) \cup Q'$. Clearly $X \setminus A$ is a vertex cover in $G - (U \cup R \cup A)$ and $|X \setminus A| \le \ell - |A|$ if $A \subseteq X$. We will show that $A \subseteq X$. We know $N_G(R) \setminus U \subseteq X$ so it remains to be shown that $Q' \subseteq X$. Consider an arbitrary $v \in Q'$ and suppose $v \notin X$. Since $Q' \subseteq Q$ we obtain from the definition of Q that $v \notin U$ and $v \notin R$, so then $v \in S$. We also note from the definition of Q' that $|f(N_G(v) \cap U)| < \alpha$. Since f = f' we have $|f'(N_G(v) \cap U)| < \alpha$, and from the definition of Q' we know that if $|f'(Y)| < \alpha$ for some $Y \subseteq U$, then |f'(Y)| = |g(Y)|. By definition of Q' we have $|f(N_G(v) \cap U)| = |f(2^U)|$ by definition of Q'.

Now we have shown that $X \setminus A$ is a vertex cover of size at most $\ell - |A| = \ell - |(N_G(R) \setminus U) \cup Q'|$ in $G - (U \cup R \cup A) = G[Q] - Q'$, hence the **VCoracle** should report that a vertex cover exists on line 8.

Running time and query size Since \mathcal{F} is fixed m and α are constants, the sets U and R are of constant size so there are only polynomially many possibilities for U and R. The function f maps subsets of U to sets of a maximum size of α , so there are only polynomially many possible functions over which to iterate. For each possibility we run VCoracle which takes polynomial time.

The VCoracle subroutine is invoked on induced subgraphs of G which therefore have a feedback vertex number of at most FVS(G). Hence after computing the vertex cover kernel we invoke the oracle for vertex cover instances with $\mathcal{O}(FVS(G)^3)$ vertices.

5 Conclusion

Earlier work [8, 23, 25, 37] has shown that several \mathcal{F} -MINOR-FREE DELETION problems admit polynomial kernelizations when parameterized by the feedback vertex number. In this paper we showed that when \mathcal{F} contains a forest and each graph in \mathcal{F} has a connected component of at least three vertices, the \mathcal{F} -MINOR-FREE DELETION problem does *not* admit such a polynomial kernel unless $NP \subseteq coNP/poly$. This lower bound generalizes to any \mathcal{F} where each graph has a connected component of at least three vertices, when we consider the vertex-deletion distance to treewidth min $tw(\mathcal{F})$ as parameter.

For all other choices of \mathcal{F} we showed that a polynomial Turing kernelization exists for \mathcal{F} -MINOR-FREE DELETION parameterized by the feedback vertex number. The size of the VERTEX COVER queries generated by the Turing kernelization does not depend on \mathcal{F} : the Turing kernelization can be shown to be *uniformly polynomial* (cf. [20]). However, it remains unknown whether the *running time* can be made uniformly polynomial, and whether the Turing kernelization can be improved to a traditional kernelization.

Our results leave open the possibility that all \mathcal{F} -MINOR-FREE DELETION problems admit a polynomial kernel when parameterized by the vertex-deletion distance to a *linear forest*, i.e. a collection of paths. Resolving this question may be an interesting direction for future work.

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