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Graphs with large total angular resolution ☆

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ABSTRACT

The total angular resolution of a straight-line drawing is the minimum angle between two edges of the drawing. It combines two properties contributing to the readability of a drawing: the angular resolution, which is the minimum angle between incident edges, and the crossing resolution, which is the minimum angle between crossing edges. We consider the total angular resolution of a graph, which is the maximum total angular resolution of a straight-line drawing of this graph.

We prove tight bounds for the number of edges for graphs for some values of the total angular resolution up to a finite number of well specified exceptions of constant size. In addition, we show that deciding whether a graph has total angular resolution at least 60° is NP-hard. Further we present some special graphs and their total angular resolution.

1. Introduction

We study angles between incident edges of straight-line drawings of graphs. In the following we mostly omit the word straight-line. The *total angular* resolution of a drawing D, or short TAR(D), is the smallest angle occurring in D, either between two edges incident to the same vertex or between two crossing edges. In other words, TAR(D) is the minimum of the angular resolution AR(D) and the crossing resolution CR(D) of the same drawing (where CR(D) $= 360°$ if D is plane). Furthermore, the total angular resolution of a graph G (or short TAR(G)) is defined as the maximum of TAR(D) over all drawings D of G. Similarly, the angular resolution and the crossing resolution of G are the maximum of AR(D) and CR(D), respectively, over all drawings D of G. Note that the total angular resolution of a graph can be smaller than the minimum of its crossing resolution and its angular resolution, see Fig. 1.

In 1993 Formann et al. [13] were the first to introduce the angular resolution of graphs. They showed that finding a drawing of a graph with angular resolution at least 90° is NP-hard for graphs with maximum degree 4. Four years later, Malitz and Papakostas [18] studied the angular resolution of planar graphs. For more results about the angular resolution see for example [12,11,17]. Another eleven years later, experiments by Huang et al. [14,16] showed that the crossing resolution

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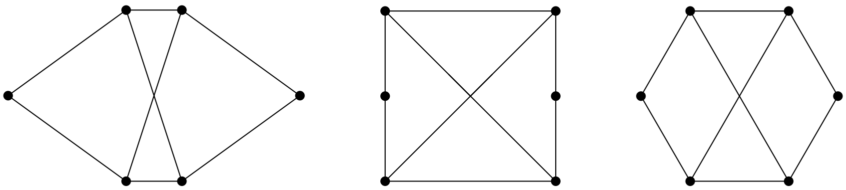


(a) Drawing with $\mathrm{AR}(D) = 67.5°$.
(b) Drawing with $\mathrm{CR}(D) = 90°$.
(c) Drawing with $\mathrm{TAR}(D) = 60°$.

Fig. 1. Three slightly different drawings D of a graph G with $\mathrm{TAR}(G) = 60°$.

plays a major role in the readability of drawings. Consequently, research in that direction was intensified. In particular, right angle crossing drawings (or short RAC drawings) were introduced by Didimo, Eades and Liotta [9]. Van Kreveld [20] showed among other results for RAC drawings, that the angular resolution in RAC drawings can be by an arbitrary factor larger than the angular resolution in plane drawings. The NP-hardness of deciding whether a given drawing admits an RAC drawing was proven by Argyriou, Bekos and Symvonis [3]. For a recent survey by Didimo on RAC drawings, see [8].

For αAC drawings (drawings with crossing resolution α), Dujmović et al. [10] showed an upper bound on the number of edges of $\frac{180°}{\alpha}(3n-6)$. For the two special classes of RAC drawings and αAC drawings with $60° < \alpha < 83°$ better upper bounds are known [1]. More precisely, Didimo et al. showed that RAC drawings have at most $4n-10$ edges [9] and that this bound is tight. For αAC drawings with $60° < \alpha < 83°$, Ackerman and Tardos [1] proved an upper bound of at most $6.5n-20$ edges. This bound is due to the fact that quasi-planar drawings (drawings without three pairwise crossing edges) have at most $6.5n-20$ edges and drawings that are not quasi-planar have a crossing angle of at most $60°$.

Argyriou, Bekos and Symvonis [4] were the first to study the total angular resolution, calling it just *total resolution*. They presented drawings of complete graphs and complete bipartite graphs with asymptotically optimal total angular resolution. Recently, Bekos et al. [5] presented a new heuristic for finding a drawing of a given graph with high total angular resolution which performed superior to earlier heuristics like [4,15] on the considered test cases. For a recent survey on angular resolution, crossing resolution and total angular resolution see Okamoto [19].

Outline

In this work we show that almost all graphs with $\mathrm{TAR}(G) > 60°$ have at most $2n-6$ edges, list the finitely many such graphs that have more than $2n-6$ edges and show that this bound is tight. Moreover, we show the following tight upper bounds on the number of edges for graphs with larger total angular resolution: $2n-2\sqrt{n}$ for $\mathrm{TAR}(G) \geq 90°$, $\frac{3}{2}n-\frac{5}{2}$ for $\mathrm{TAR}(G) > 90°$ (and $n \geq 3$), and n for $\mathrm{TAR}(G) > 120°$ (and $n \geq 7$). We also prove that it is NP-hard to determine whether $\mathrm{TAR}(G) \geq 60°$.

Further, we present an infinite family of graphs with $\mathrm{TAR}(G) = 60°$, for which every proper subgraph G' of G has $\mathrm{TAR}(G') > 60°$. We conclude this work with a bound on the number of edges that can be removed from the complete graph K_n without changing its total angular resolution.

2. Upper bounds on the number of edges

In this section we study the relation between the total angular resolution and the maximal number of edges. First we need some definitions. Every straight-line drawing D (of a graph G) partitions the plane into connected regions which are called *cells* of D. The *planarization* $P(D)$ of a drawing D is the drawing in which we replace every crossing by a vertex so that this new vertex splits both crossing edges into two edges. Furthermore, every edge in $P(D)$ has two *sides* and every side is incident to exactly one cell of $P(D)$. Note that both sides of an edge can be incident to the same cell. A connected drawing D is a drawing such that $P(D)$ corresponds to a drawing of a plane connected graph. We define the *size* of a cell of a connected drawing D as the number of sides in $P(D)$ incident to this cell.

2.1. Graphs with $\mathrm{TAR}(G) > 60°$

In this section we show that for almost all graphs with $\mathrm{TAR}(G) > 60°$ the number of edges is bounded by $2n-6$. We start by showing a bound for the number of edges in a connected drawing D depending on the size of the unbounded cell of D.

Lemma 1. *Let D be a connected drawing with $n \geq 1$ vertices, m edges and $\mathrm{TAR}(D) > 60°$. If the unbounded cell of D has size k, then $m \leq 2n-2-\lceil k/2 \rceil$.*

Proof. If at least three edges cross each other in a single point, then there exists an angle with at most $60°$ at this crossing point. Therefore, every crossing of the drawing D is incident to exactly two edges. We planarize D and get $n_P = n + \mathrm{cr}(D)$

and $m_P = m + 2\,\mathrm{cr}(D)$ where $\mathrm{cr}(D)$ is the number of crossings in D, n_P is the number of vertices of $P(D)$, and m_p is the number of edges of $P(D)$. Since $P(D)$ is a plane drawing, we can use Euler's formula to compute the number f_p of faces in $P(D)$ as

$$f_P = -n + m + \mathrm{cr}(D) + 2. \tag{1}$$

Moreover, every bounded cell of D has size at least 4, as otherwise $P(D)$ would contain a triangle, implying an angle of at most 60°. By definition, the unbounded cell of D has size k and we obtain the inequality

$$2m_P \geq 4(f_P - 1) + k. \tag{2}$$

Combining Equation (1) and Inequality (2) with $m_P = m + 2\,\mathrm{cr}(D)$ gives $m \leq 2n - 2 - \lceil k/2 \rceil$. □

From Lemma 1 it follows directly that a connected drawing D on $n \geq 3$ vertices and with $\mathrm{TAR}(D) > 60°$ fulfills $m \leq 2n-4$. Note that this bound is only two edges away from the optimal upper bound.

We proceed to prove the bound of $2n - 6$ for disconnected drawings.

Lemma 2. *Let D be a disconnected drawing on $n \geq 3$ vertices with $\mathrm{TAR}(D) > 60°$. Then, $m \leq 2n - 6$ or D consists of three vertices and one edge (Exception E0 in Fig. 2).*

Proof. Assume that D consists of $\ell \geq 2$ components C_i, $1 \leq i \leq \ell$, where C_i has $n_i \geq 1$ vertices and $m_i \geq 0$ edges. Furthermore, $\mathrm{TAR}(C_i) \geq \mathrm{TAR}(D) > 60°$ holds. By Lemma 1 we get $m_i \leq 2n_i - 2$ for every component. If $\ell \geq 3$, this directly implies

$$m = \sum_{i=1}^{\ell} m_i \leq \sum_{i=1}^{\ell} (2n_i - 2) = 2n - 2\ell \leq 2n - 6.$$

Now consider $\ell = 2$. If C_1 contains at least 2 edges, then the size of the unbounded cell of C_1 is at least 3. So we get $m_1 \leq 2n_1 - 4$ by Lemma 1. This gives

$$m = m_1 + m_2 \leq 2n_1 - 4 + 2n_2 - 2 = 2n - 6.$$

This implies that we only have to check drawings with exactly two connected components and each has at most one edge. If both C_1 and C_2 consist of two vertices and one edge, then we have $m = 2 = 2 \cdot 4 - 6$ edges. If D is a drawing on $n = 3$ vertices with $m = 1$ edges, then we have Exception E0. □

Now we prove the bound for connected drawings. To further improve the bound from Lemma 1, the following lemma is useful.

Lemma 3. *Let D be a plane connected drawing where the boundary of the unbounded cell is a simple polygon P with $p > 3$ vertices. Let the inner degree of a vertex v_i of P be the number $d'(v_i)$ of edges incident to v_i that lie in the interior of P. If $\mathrm{TAR}(D) > 60°$, then $\sum_{v_i \in V(P)} d'(v_i) \leq 2p - 7$ holds.*

Proof. Assume to the contrary that $\mathrm{TAR}(D) > 60°$ and $\sum_{v_i \in V(P)} d'(v_i) \geq 2p - 6$. The sum of all inner angles in any simple polygon P with p vertices is $180°(p-2)$. The number of all inner angles in D incident to vertices in P is $p + \sum_{v_i \in V(P)} d'(v_i) \geq 3p - 6$. Since every angle in D is larger than 60°, all inner angles incident to vertices of P sum up to strictly more than $180°(p-2)$. This contradicts that the sum of the inner angles is $180°(p-2)$. Therefore, we have $\sum_{v_i \in V(P)} d'(v_i) \leq 2p - 7$. □

Lemma 4. *Let D be a connected plane drawing on $n \geq 3$ vertices, where D is not a path on 3 vertices and not a 4-cycle. If $\mathrm{TAR}(D) > 60°$, then $m \leq 2n - 5$.*

Proof. The unbounded cell of D cannot have size 3, as in this case the convex hull of the drawing is a triangle and we have $\mathrm{TAR}(D) \leq 60°$. If the drawing D has an unbounded cell of size at least 5 and $\mathrm{TAR}(D) > 60°$, then $m \leq 2n - 5$ follows directly from Lemma 1. Otherwise, the unbounded cell of D has size 4, which, as D is not a path on 3 vertices, implies that the boundary of D is a 4-cycle F. By Lemma 3 and the fact that D is not a 4-cycle, D contains exactly one edge e with one vertex in the interior of F and the other vertex on the boundary of F. Let D' be the drawing we obtain by deleting all vertices and edges of F and also the edge e. The drawing D' is connected and has $n' \geq 1$ vertices and m' edges, where $n' = n - 4$ and $m' = m - 5$. By Lemma 1 we know that $m' \leq 2n' - 2$ and we derive $m = m' + 5 \leq 2n' - 2 + 5 = 2n - 5$. □

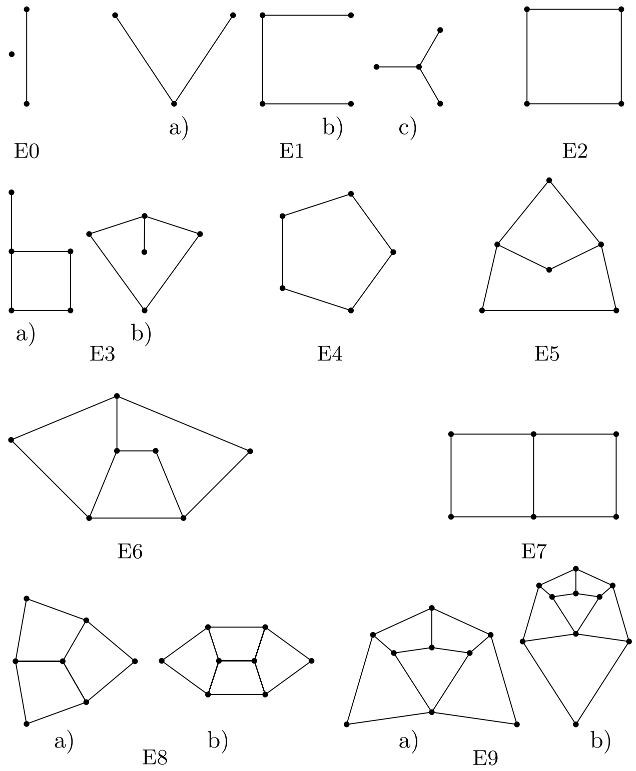


Fig. 2. All exceptions for Lemma 5 and Theorem 7.

Two drawings are *combinatorially equivalent* if all cells are bounded by the same edges, all crossing edge pairs are the same, and for each edge e the order of crossings along e is the same. We extend Lemma 4 in the following way.

Lemma 5. *Let D be a connected plane drawing on $n \geq 3$ vertices with m edges and $\mathrm{TAR}(D) > 60°$. Then, $m \leq 2n - 6$ unless D is combinatorially equivalent to one of the exceptions E1–E9 listed below and depicted in Fig. 2.*

- **E1** *A tree on at most 4 vertices.*
- **E2** *A plane 4-cycle.*
- **E3** *A plane 4-cycle with one additional vertex connected to one vertex of the 4-cycle. The vertex can be inside or outside the cycle.*
- **E4** *A plane 5-cycle.*
- **E5** *A plane 5-cycle with one vertex inside connected to two non-neighboring vertices of the 5-cycle.*
- **E6** *A plane 5-cycle with an edge inside, connected with 3 edges to the 5-cycle such that the interior of the 5-cycle is partitioned into two 4-faces and one 5-faces.*
- **E7** *A plane 6-cycle with an additional diagonal between opposite vertices.*
- **E8** *A plane 6-cycle with an additional vertex or edge inside, connected with 3 or 4, respectively, edges to the 6-cycle such that the interior of the 6-cycle is partitioned into 3 or 4, respectively, 4-faces.*
- **E9** *A plane 6-cycle with either a path on 3 vertices or a 4-cycle inside, connected with 5 edges to the 6-cycle such that the interior of the 6-cycle is partitioned into 4 or 5, respectively, 4-faces.*

Proof. Let D' be a subdrawing of D consisting of all vertices that are not on the unbounded cell and of all edges that are not incident to a vertex on the unbounded cell. Assume D' has n' vertices and m' edges. We distinguish four cases, depending on the size of the unbounded cell.

Case 1 The unbounded cell has size 4. If the drawing has only one cell, then it is Exception E1a. Otherwise, the boundary of the unbounded cell is a 4-cycle C and we have $n = n' + 4$. As by Lemma 3, there is at most one edge in D from a vertex of C to D', we have $m \leq m' + 5$.

If there is at most one vertex inside C, then we have Exception E2 or E3b. So assume that there are at least two vertices inside C. Since there is at most one edge from a vertex of C to the interior and D' is connected, D'

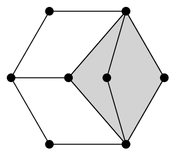


Fig. 3. Two separated vertices inside a 6-cycle.

thus has at least one edge. So the unbounded cell of D' has size at least 2. Hence, by Lemma 1 for D', it holds that $m' \leq 2n' - 3$ and we obtain

$$m \leq m' + 5 \leq 2n' - 3 + 5 = 2(n - 4) + 2 = 2n - 6.$$

Case 2 The unbounded cell has size 5. In this case, the outer boundary must be a 5-cycle: The only other possibility would be a triangle with an attached edge, but in that case we would have $\mathrm{TAR}(D) \leq 60°$. Hence, we have $n' = n - 5$. If there are at most two adjacent vertices inside the 5-cycle that are connected with edges to the 5-cycle, then we have one of the Exceptions E4, E5, or E6. So assume that there are at least 3 vertices in the interior. Due to Lemma 3, there are at most three edges connecting the interior to the 5-cycle and the 5-cycle itself has 5 edges, that is, $m \leq m' + 5 + 3 = m' + 8$. If D' is connected, then the size of the unbounded cell of D' is at least 3 and we have $m' \leq 2n' - 4$ by Lemma 1. Otherwise D' consists of two or three connected components. By Lemma 2 we have $m' \leq 2n' - 6$ unless D' consists of three vertices and an edge, which gives $m' \leq 2n' - 5$, or it contains fewer than three vertices. The only disconnected drawing with fewer than three vertices is a drawing consisting of two vertices, which gives $m' \leq 2n' - 4$. So, in all cases we get $m' \leq 2n' - 4$ and hence have

$$m \leq m' + 8 \leq 2n' - 4 + 8 = 2n - 6.$$

Case 3 The unbounded cell of the drawing D has size 6. If D has only one cell (i.e. only the unbounded cell), we have Exception E1b or E1c. Otherwise the boundary B of the unbounded cell of D either consists of two triangles sharing a vertex ($\mathrm{TAR}(D) \leq 60°$) or is a 4-cycle with an attached edge or a 6-cycle. So there are two cases we have to consider.

- If B is a 4-cycle with an attached edge, we use similar arguments as in Case 1. If there is no vertex inside the 4-cycle, then we have Exception E3a. If we have at least one point inside the 4-cycle, then by Lemma 1 we have $m' \leq 2n' - 2$. So we get

$$m = m' + 6 \leq 2n' - 2 + 6 = 2(n' + 5) - 6 = 2n - 6.$$

- If B is a 6-cycle, then by Lemma 3 we can have at most 5 edges connecting the interior to the 6-cycle. First we consider the case where D' is connected. If $\mathrm{TAR}(D) > 60°$ and $n' \geq 3$, then $\mathrm{TAR}(D') > 60°$ and D' fulfills $m' \leq 2n' - 5$ by Lemma 4 unless D' is a path on 3 vertices or a 4-cycle. Furthermore, we know that $n = n' + 6$ and $m \leq m' + 11$. If $m' \leq 2n' - 5$, then

$$m \leq m' + 11 \leq 2n' - 5 + 11 = 2n - 6.$$

  Consider now the case that $n' \leq 2$ or D' is a path on 3 vertices or a 4-cycle. These cases can be checked by hand. Therefore, we have Exceptions E7 and E8 if $n' \leq 2$, and Exceptions E9 if D' is a path on 3 vertices or a 4-cycle.
  If D' is not connected and $\mathrm{TAR}(D) > 60°$, then either D' fulfills $m' \leq 2n' - 6$, or D' consists of three vertices and an edge (by Lemma 2), or D' consists of two vertices. If D' fulfills $m' \leq 2n' - 6$ or consists of three vertices and an edge, then we have $m \leq 2n - 6$. So consider the case that D' consists of two non-adjacent vertices. Note that for $m > 2n - 6$, at least five edges must connect to D' in D. This means that one of the two inner vertices has degree at least 3 in the drawing D. If one vertex has degree 4, then there is a triangle in our drawing D which means that $\mathrm{TAR}(D) \leq 60°$. Otherwise, if one vertex has degree 3 and the other one has degree 2, then we have a drawing like in Fig. 3. The gray shaded 4-cycle has 2 edges in the interior. So due to Lemma 3 we have $\mathrm{TAR}(D) \leq 60°$.

Case 4 The unbounded cell has size at least 7. Then we have, by Lemma 1,

$$m \leq 2n - 2 - \left\lceil \frac{k}{2} \right\rceil = 2n - 2 - \left\lceil \frac{7}{2} \right\rceil \leq 2n - 6. \qquad \square$$

Note that Lemma 5 considers plane drawings. Next we consider drawings with at least one crossing, whose planarizations are in the exceptions of Lemma 5. If D has a crossing, then $P(D)$ has a vertex of degree at least 4. The only exceptions with such a vertex are the ones of E9; see again Fig. 2.

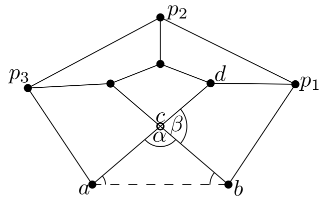


Fig. 4. Replacing the vertex of degree 4 of the drawing of E9a in Fig. 2 with a crossing.

Lemma 6. *If we replace the vertex of degree* 4 *in a drawing of E9 in Fig. 2 with a crossing, then the resulting drawings D have* $\mathrm{TAR}(D) \leq 60°$.

Proof. If we replace the vertex of degree 4 of Exception E9a in Fig. 2 with a crossing, then we get the drawing D_{cr} in Fig. 4, where the dashed edge is not part of the actual drawing. We want to show that $\mathrm{TAR}(D_{cr}) \leq 60°$. The crossing edge pair forms two angles. As indicated in Fig. 4, we denote $\angle acb$ as α and $\angle bcd$ as β, where c denotes the crossing and a, b and d are three of the endpoints of the crossing edges. Let p_1, p_2 and p_3 be the other three vertices on the unbounded cell. Since c is a crossing, c is inside the pentagon $abp_1p_2p_3$. The inner angles of a pentagon sum up to 540°. All eight inner angles of the drawing, that are incident to the pentagon $abp_1p_2p_3$, are larger than 60°. This implies that $\angle bac + \angle abc < 60°$. Furthermore we have $\alpha + \beta = 180° = \alpha + \angle bac + \angle abc$. This means we have $\beta = \angle bac + \angle abc < 60°$. However, β appears in D_{cr}, and so we have $\mathrm{TAR}(D_{cr}) \leq 60°$.

Now let D'_{cr} be the drawing we get if we replace the vertex of degree 4 in the drawing for E9b in Fig. 2 with a crossing. Then D_{cr} is a subdrawing of D'_{cr} and hence we get $\mathrm{TAR}(D'_{cr}) \leq \mathrm{TAR}(D_{cr}) \leq 60°$. □

So we have characterized all drawings D which have $\mathrm{TAR}(D) > 60°$ and $m > 2n - 6$ edges, such that $P(D)$ is in the exceptions of Lemma 5. This leads us to the following theorem.

Theorem 7. *Let G be a graph with $n \geq 3$ vertices, m edges and* $\mathrm{TAR}(G) > 60°$. *Then $m \leq 2n - 6$ unless either there exists a drawing of G that is an exception for Lemma 5 or G consists of exactly three vertices and one edge (Exception E0 in Fig. 2). Further, if G is a graph that forms an exception for Lemma 5, then every drawing D of G with* $\mathrm{TAR}(D) > 60°$ *is drawn plane and combinatorially equivalent to an exception of Lemma 5.*

Proof. Consider a graph G with $n \geq 3$ vertices, $m > 2n - 6$ edges and $\mathrm{TAR}(G) > 60°$. Then there exists a drawing D of G with $\mathrm{TAR}(D) > 60°$ and its planarization $P(D)$.

If G is disconnected, then by Lemma 2 it has either $m \leq 2n - 6$ edges or consists of three vertices and one edge. So for the rest of the proof we only consider connected graphs.

If three edges cross in a single point, then in $P(D)$ this point has degree 6, and therefore an angle with at most 60°. Hence every crossing involves exactly two edges and $P(D)$ has $m_P = m + 2\,\mathrm{cr}(D)$ edges and $n_P = n + \mathrm{cr}(D)$ vertices. By Lemma 5 we get that $m_P \leq 2n_P - 6$ or $P(D)$ is in the exceptions. If $m_P \leq 2n_P - 6$, then $m = m_P - 2\,\mathrm{cr}(D) \leq 2(n_P - \mathrm{cr}(D)) - 6 = 2n - 6$. If $P(D)$ is in the exceptions, then, as observed before, D is in the exceptions. □

The bound of Theorem 7 is the best possible in the sense that there are infinitely many graphs with $m = 2n - 6$ edges and $\mathrm{TAR}(G) > 60°$.

Proposition 8. *For every integer $n \geq 17$ there exists a graph G with n vertices and $m = 2n - 6$ edges such that* $\mathrm{TAR}(G) > 60°$.

Proof. Fig. 5 illustrates drawings D with $17 \leq n \leq 24$ vertices, $m = 2n - 6$ edges and $\mathrm{TAR}(D) > 60°$. We extend this family of drawings, such that for any number of vertices $n \geq 17$ we have a drawing with $m = 2n - 6$ edges and $\mathrm{TAR}(D) > 60°$, by adding layers of 8-cycles as illustrated in Fig. 6.

Let D be a drawing with n vertices, $m = 2n - 6$ edges, $\mathrm{TAR}(D) > 60°$ and whose boundary is a regular 8-cycle $C = p_1p_2p_3p_4p_5p_6p_7p_8$. We construct a bigger drawing D' in the following way. Let $C' = q_1q_2q_3q_4q_5q_6q_7q_8$ be a regular 8-cycle, which is concentric to C. Further, for any $1 \leq i \leq 8$ let p_i and q_i be on a ray from the common circumcenter of C and C'. We merge D with the 8-cycle C' and all edges p_iq_i with $1 \leq i \leq 8$ and call the resulting drawing D'.

First observe that since $\mathrm{TAR}(D) > 60°$ also $\mathrm{TAR}(D') > 60°$ holds, because the angles $\angle p_{i-1}q_{i-1}q_i = \angle q_{i-1}q_ip_i = 67.5°$ and $\angle q_ip_ip_{i-1} = \angle p_{i+1}p_iq_i = 112.5°$ with $q_0 = q_8$ and $p_0 = p_8$ for every $1 \leq i \leq 8$. The drawing D' contains $n' = n + 8$ vertices and $m' = m + 16$ edges (m edges of D, 8 edges on C' and 8 edges connecting C and C'). Since D has n vertices and $m = 2n - 6$ edges, also $m' = m + 16 = 2n - 6 + 16 = 2(n + 8) - 6 = 2n' - 6$. So by extending a drawing D to D' in this way we get eight more vertices. Since every drawing depicted in Fig. 5 has a regular 8-cycle as boundary, we are able to extend each of these drawings as described before. Doing this repeatedly, we are able to add $8k$ vertices to each of the drawings for every integer $k > 0$. The numbers of vertices of the drawings depicted in Fig. 5 cover all parities modulo eight. So there exists for any number $n \geq 17$ a graph G with $\mathrm{TAR}(G) > 60°$ and $m = 2n - 6$ edges. □

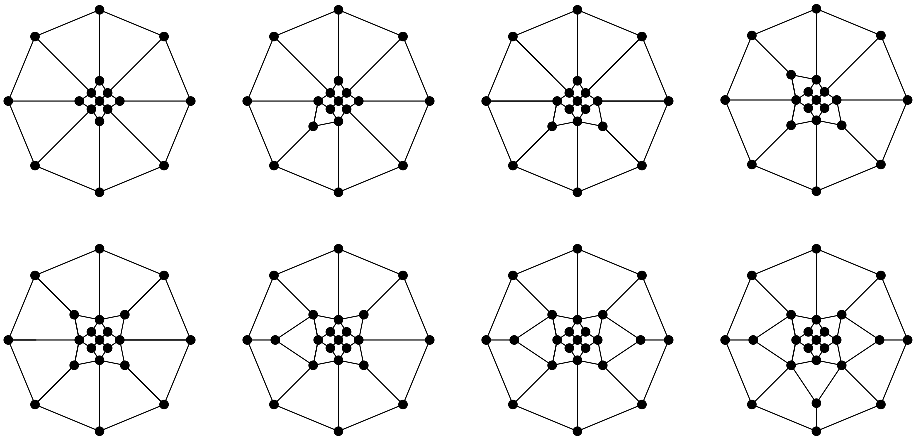


Fig. 5. Drawings of graphs G with $\mathrm{TAR}(G) > 60°$, n vertices and $m = 2n - 6$ edges for $17 \leq n \leq 24$.

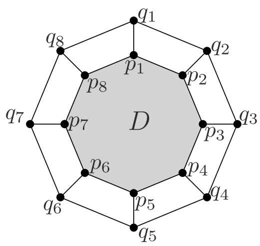


Fig. 6. Extending the drawings in Fig. 5.

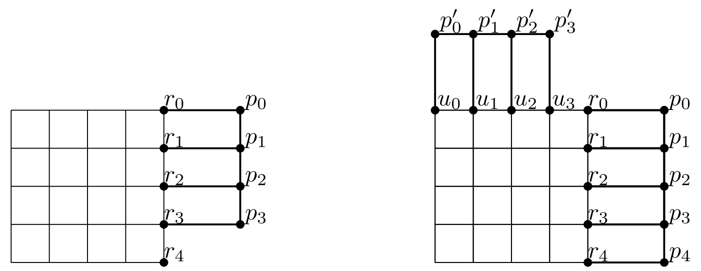
2.2. *Graphs with* $\mathrm{TAR}(G) \geq 90°$

Bodlaender and Tel [6] showed that if a graph can be embedded with angular resolution of at least 90°, then the graph can also be embedded such that all angles at vertices have one of the values 90°, 180°, 270°, and 360°. Note that in any such drawing, the angle between two crossing edges is exactly 90°. Hence, by [6], an angular resolution of at least 90° for a graph G implies $\mathrm{TAR}(G) \geq 90°$. In this section, we show that graphs with $\mathrm{TAR}(G) \geq 90°$ have at most $\lfloor 2n - 2\sqrt{n} \rfloor$ edges, which is tight.

Lemma 9. *For every $n \geq 1$, there exists a graph G with n vertices, $\lfloor 2n - 2\sqrt{n} \rfloor$ edges, and* $\mathrm{TAR}(G) = 90°$.

Proof. We will construct the graph G along with a drawing D for G that shows $\mathrm{TAR}(G) = 90°$. If we take a square grid with k vertices on each side, then we have in total $n = k^2$ vertices and $m = 2k^2 - 2k$ edges. So for a $k \times k$ grid we have $m = \lfloor 2n - 2\sqrt{n} \rfloor$, which proves the statement for $n = k^2$.

For $k^2 < n < (k+1)^2$ we extend the graph in the following way; see Fig. 7. We call the rightmost points of the grid $r_0, r_1, \ldots, r_{k-1}$ from top to bottom and the topmost points $u_0, \ldots, u_{k-1}$ from left to right. (Note that $r_0 = u_{k-1}$.)



(a) Grid drawing with $m = k^2 + r$ edges, $1 \leq r \leq k$.

(b) Grid drawing with $m = k^2 + r$ edges, $k + 1 \leq r < 2k + 1$.

Fig. 7. Different extensions of a grid drawing.

If $n = k^2 + r$ with $0 < r < k$, we place additional points $p_0, \ldots, p_{r-1}$, all on a common vertical line and each p_i is to the right of r_i for $0 \le i \le r-1$, as depicted in Fig. 7(a) where the added edges are drawn heavier than the old ones. Further, we add the edges $r_i p_i$, $0 \le i < r$, and $p_{j-1} p_j$, $1 \le j < r$. This gives us $r + (r-1) = 2r - 1$ new edges. So we have $m = 2k^2 - 2k + 2r - 1$ edges in total. On the other hand we get

$$\begin{aligned}\lfloor 2n - 2\sqrt{n} \rfloor &= \lfloor 2(k^2 + r) - 2\sqrt{k^2 + r} \rfloor \\ &= 2k^2 + 2r + \lfloor -2\sqrt{k^2 + r} \rfloor \\ &= 2k^2 - 2k + 2r - 1 = m\end{aligned}$$

because $k < \sqrt{k^2 + r} < k + \frac{1}{2}$ if $1 \le r \le k$. So $m = \lfloor 2n - 2\sqrt{n} \rfloor$ if $n = k^2 + r$ and $1 \le r \le k$.

For $n = k^2 + k + r$, with $1 \le r < k + 1$, we add k points as before and also add the same edges. The remaining r points $p'_0, \ldots, p'_{r-1}$ are placed on a horizontal line, such that p'_i is above u_i for $0 \le i \le r-1$. We further add the edges $p'_i u_i$, $0 \le i \le r-1$, and $p'_{j-1} p'_j$, $1 \le j \le r-1$, as depicted in Fig. 7(b) where the added edges are drawn heavier than the old ones. So we have $m = 2k^2 + 2r - 2$ edges, which is again $m = \lfloor 2n - 2\sqrt{n} \rfloor$ for $n = k^2 + k + r$ and $1 \le r \le k$. This means that for every n there exists a graph G with $\mathrm{TAR}(G) = 90°$, n vertices and $\lfloor 2n - 2\sqrt{n} \rfloor$ edges. □

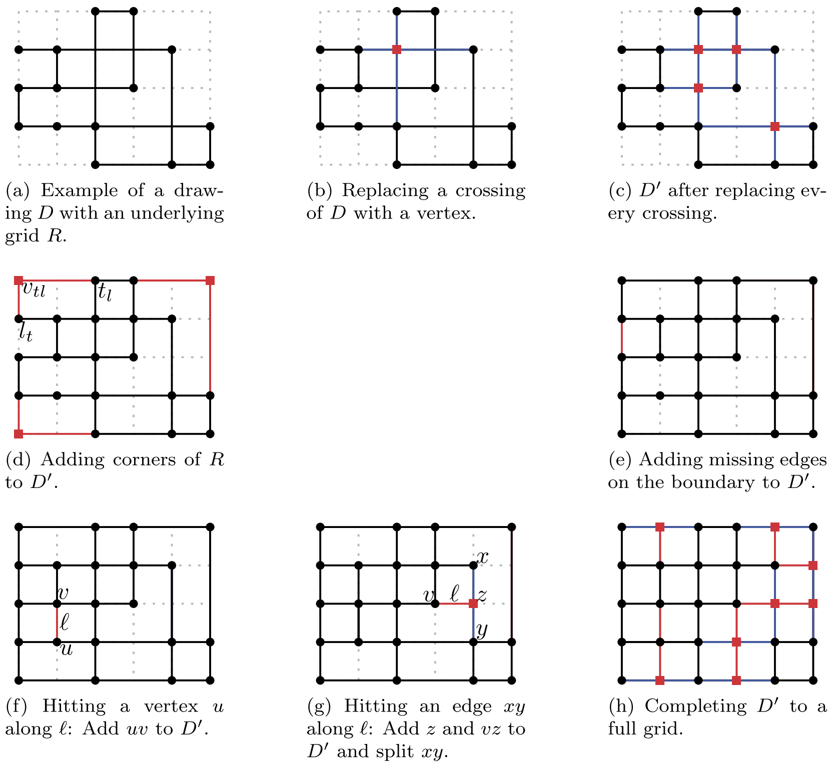
Lemma 10. *Every graph with* $\mathrm{TAR}(G) = 90°$ *has at most* $\lfloor 2n - 2\sqrt{n} \rfloor$ *edges.*

Proof. We prove the statement by contradiction. Let G be a graph with n vertices and m edges. By Bodlaender and Tel [6] we can embed our graph, such that every angle is $90°, 180°, 270°$ or $360°$. So we can embed our graph on some rectangular grid R with $a \times b$ points, such that in every column and every row there is at least one point and such that the edges are along the grid. We call this drawing D.

Now we add edges and vertices to D, so that this new drawing D' is the complete $a \times b$ grid.

At the beginning we set $D' = D$, see Fig. 8(a). First we add a vertex at every crossing of D' as depicted in Fig. 8(b) and Fig. 8(c). By doing this we get four edges instead of two and one new vertex. So we get two new edges and one new vertex for each crossing.

In the second step, we add corner vertices of R to D'. If all four corner vertices are already in the drawing, then we skip this step. Without loss of generality assume that the top left corner vertex of the grid is not a vertex in the drawing. We call



(a) Example of a drawing D with an underlying grid R.

(b) Replacing a crossing of D with a vertex.

(c) D' after replacing every crossing.

(d) Adding corners of R to D'.

(e) Adding missing edges on the boundary to D'.

(f) Hitting a vertex u along ℓ: Add uv to D'.

(g) Hitting an edge xy along ℓ: Add z and vz to D' and split xy.

(h) Completing D' to a full grid.

Fig. 8. Adding vertices and edges to get from an initial drawing D to a drawing of a grid. The underlying grid R is drawn in gray. The added vertices and edges are marked with red and all split edges are drawn in blue for each step. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

the topmost vertex in the leftmost column l_t and the leftmost vertex in the topmost row t_l as depicted in Fig. 8(d). Then we add the top left corner vertex v_{tl} of the grid together with the edges $v_{tl}l_t$ and $v_{tl}t_l$. Analogously, we add the remaining missing corner vertices. So for every added corner vertex we also added two edges to D'.

Next, we add edges between two points on the boundary of R as illustrated in Fig. 8(e), so that the outer face of D' is a rectangle with possibly some vertices on its sides. Since we already added the corner vertices in the second step, we only add edges in this step.

In the last step, we check if every vertex has full degree (degree 4 for inner vertices, degree 3 for vertices on the boundary, which are not corners, and degree 2 for corner vertices). Assume we have a vertex v without full degree. Then there is a line segment ℓ of R such that no edge of v is along ℓ. We draw a line segment e from v along ℓ until we hit a vertex or an edge of D'. If we hit a vertex u as depicted in Fig. 8(f), then we add the edge uv to D'. In this case, we do not add any vertex. If we hit an edge xy as depicted in Fig. 8(g), then we add a vertex z, where we hit the edge, to D'. Further we add the edge vz to D' and split xy into xz and yz in D'. In this case we have one additional vertex and increased the number of edges by two in D'.

If every vertex has full degree, then D' is equal to an $a \times b$ grid. Further, every time we added a vertex, we also increased the number of edges by two. So this new drawing D' has $n' = n + k$ vertices and m' edges where $m' \geq m + 2k$. On the other hand, an $a \times b$ rectangle grid has exactly $n' = ab$ vertices and $m' = 2ab - (a + b)$ edges. So we have

$$m' = 2ab - (a+b) \leq 2ab - 2\sqrt{ab} = 2(n+k) - 2\sqrt{n+k}, \text{ and hence}$$
$$m \leq m' - 2k \leq 2n - 2\sqrt{n+k} \leq 2n - 2\sqrt{n}.$$

This means that every graph G with $\mathrm{TAR}(G) = 90°$ and n vertices has at most $\lfloor 2n - 2\sqrt{n} \rfloor$ edges. □

2.3. *Graphs with* TAR(G) > 90°

If a graph has $\mathrm{TAR}(G) > 90°$, then this graph is planar, since a crossing would imply that at least one angle is at most 90°. Also note that the construction for a graph with $\mathrm{TAR}(G) = 90°$ and $\lfloor 2n - 2\sqrt{n} \rfloor$ edges heavily relied on 4-cycles. So we can improve the bound for graphs with $\mathrm{TAR}(G) > 90°$.

Theorem 11. *Every graph G with $n \geq 3$ vertices and $\mathrm{TAR}(G) > 90°$ has at most $\frac{3}{2}n - \frac{5}{2}$ edges. This bound is tight for infinitely many values of n.*

Proof. We observe that every vertex of a graph G with $\mathrm{TAR}(G) > 90°$ has degree at most 3. This already gives an upper bound of $\frac{3}{2}n$ edges for graphs with $\mathrm{TAR}(G) > 90°$. Let D be a drawing of G with $\mathrm{TAR}(D) > 90°$. Then every vertex on the boundary of the convex hull of D has degree at most 2. Further, consider the angles spanned by the convex hull edges of D. Assume that this angle is at most 90° for some convex hull vertex v. If v was incident to two edges of D, then these edges would span an angle of at most 90°. So v has degree at most 1 in this case.

If there are at least 5 vertices on the convex hull of D, then D has at most $(n-5)$ vertices of degree 3 and at least 5 vertices of degree at most 2. Therefore, D has at most $\frac{3}{2}(n-5) + \frac{2}{2} \cdot 5 = \frac{3}{2}n - \frac{5}{2}$ edges.

If there are exactly 4 vertices on the convex hull of D, then at least one of the inside angles of the boundary of the convex hull is at most 90°. Therefore, at least one vertex on the convex hull has degree 1. So D has at most $\frac{3}{2}n - \frac{5}{2}$ edges. Similarly, if there are exactly 3 vertices on the convex hull of D, then at least two vertices of those have degree 1. Again D has at most $\frac{3}{2}n - \frac{5}{2}$ edges. This means that every graph G with at least 3 vertices and $\mathrm{TAR}(G) > 90°$ has at most $\frac{3}{2}n - \frac{5}{2}$ edges.

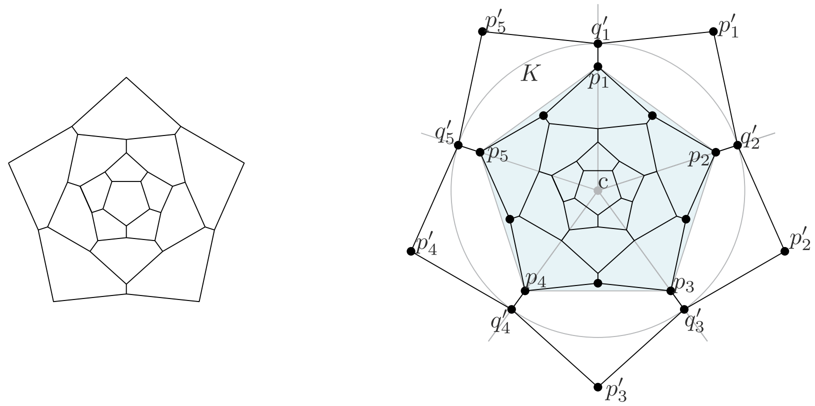
Let D be a drawing with $\mathrm{TAR}(D) \geq 96°$, n vertices and $\frac{3}{2}n - \frac{5}{2}$ edges such that the boundary B of the convex hull of D is a regular 5-gon $p_1p_2p_3p_4p_5$, also illustrated in Fig. 9(b). Note, that a regular 5-gon has these properties. Let c be the circumcenter of B and let K be a circle with center c such that D is inside K. We call the crossing of the ray cp_i with K q'_i, for every $1 \leq i \leq 5$. The rays cp_i, $1 \leq i \leq 5$, and the circle K are gray in Fig. 9(b). The vertex p'_i is the unique vertex with $\angle p_i q'_i p'_i = 96°$ and $\angle p'_i q'_{i+1} p_{i+1} = 96°$ for every $1 \leq i \leq 5$ where $q'_6 = q'_1$ and $p_6 = p_1$. Let D' be D together with the vertices q'_i and p'_i and the edges $p_iq'_i, q'_ip'_i$ and $p'_iq'_{i+1}$ for every $1 \leq i \leq 5$. By definition we have $\angle p_iq'_ip'_i = \angle p'_iq'_{i+1}p_{i+1} = 96°$. Looking at the quadrilateral $cq'_ip'_iq'_{i+1}$ we have $\angle q'_ip'_iq'_{i+1} = 96°$. Further the angles $\angle q'_{i+1}p_{i+1}p_i$ and $\angle p_{i+1}p_iq'_i$ are both inside an angle in D' and both have 126°. So we have $\mathrm{TAR}(D') = 96°$ with $n' = n + 10$ vertices and $\frac{3}{2}n' - \frac{5}{2}$ edges.

Starting with a regular 5-gon and doing this extension iteratively, there exists a drawing with $\mathrm{TAR}(D) = 96°$ with $n = 10k + 5$ vertices and $\frac{3}{2}n - \frac{5}{2}$ edges for every $k \geq 0$. For example, by doing this extension three times we get the drawing depicted in Fig. 9(a). □

2.4. *Graphs with* TAR(G) > 120°

For angles $\alpha > 120°$ we prove tight bounds on the number of edges of graphs G with $\mathrm{TAR}(G) > \alpha$.

Theorem 12. *Let $k > 6$. Every graph G with n vertices and $\mathrm{TAR}(G) \geq \frac{k-2}{k}180°$ has at most n edges for $n \geq k$, and at most $n - 1$ edges otherwise. These bounds are tight.*



(a) Drawing with $\mathrm{TAR}(D) = 96°$, $n = 35$ vertices and $\frac{3}{2}n - \frac{5}{2}$ edges.

(b) Extending Figure 9(a). The gray lines are for construction purposes and D is inside the cyan area.

Fig. 9. Drawings with $\mathrm{TAR}(D) > 90°$.

Proof. If a graph G has $\mathrm{TAR}(G) > 120°$ then every vertex has degree at most two. Any such graph is a collection of cycles, paths and isolated vertices. So this graph has at most n edges. If $n \geq k$, then a regular n-gon D has $\mathrm{TAR}(D) = \frac{n-2}{n}180° > 120°$ and exactly n edges. This means that a graph G with $\mathrm{TAR}(G) \geq \frac{k-2}{k}180°$ has at most n edges for $n \geq k$ and this bound is tight.

If $n < k$, then any cycle would prevent $\mathrm{TAR}(G) \geq \frac{k-2}{k}180°$. This means that a graph with $\mathrm{TAR}(G) \geq \frac{k-2}{k}180°$ is cycle-free and has at most $n-1$ edges. On the other hand, if G is a single path, then $\mathrm{TAR}(G) = 180°$. So a graph with $\mathrm{TAR}(G) \geq \frac{k-2}{k}180°$ and $n < k$ vertices is a path or a collection of disjoint paths. Hence it can have at most $n-1$ edges and this bound is tight. □

3. NP-hardness

Formann et al. [13] showed that the problem of determining whether a given graph G admits a drawing with angular resolution of 90° is NP-hard. Their proof, which is by reduction from 3SAT with exactly three different literals per clause, also implies the NP-hardness of deciding whether $\mathrm{TAR}(G) = 90°$. We adapt in the following their reduction to show the NP-hardness of the decision problem for $\mathrm{TAR}(G) \geq 60°$.

Note that every triangle of a drawing D must be equilateral if $\mathrm{TAR}(D) \geq 60°$ The idea of the construction is to build a rigid frame with triangles and add the clause gadgets such that they are also rigid; see Fig. 10 for depictions of the frame and the gadgets. Then, we add variable gadgets to the frame, such that they can only be oriented in two ways, which will correspond to the variable assignment.

Theorem 13. *It is* NP-*hard to decide whether a graph G has* $\mathrm{TAR}(G) \geq 60°$.

Proof. As input we are given a 3SAT formula with variables $x_1, x_2, \ldots, x_n$ and clauses $c_1, c_2, \ldots, c_m$, where every clause contains exactly three different literals. Cook [7] showed that the decision question for satisfiability of such a 3SAT formula is NP-complete.

We first construct a graph G for the formula. The basic building blocks of our construction consist of triangles, which, in order to obtain a total angular resolution of 60°, must all be equilateral. We use the following gadgets; see Fig. 10(a).

As clause gadget we use a sequence of four triangles that share a common vertex and in which consecutive triangles share an edge. The middle vertex with three incident edges, marked with C_j in the figure, will be used to connect the clause gadget to its literals. We refer to C_j as the *clause vertex*.

As variable gadget we use a triangle followed by a sequence of m hexagons and followed by another triangle. Each hexagon consists of six triangles sharing the center point. Each non-extreme hexagon of the sequence is incident to its neighboring hexagons via two "opposite" edges. The initial triangle is incident to the first hexagon via the edge opposite to the incidence with the second hexagon. The final triangle is incident to the last hexagon via the edge opposite to the incidence with the second to last hexagon. The vertices of the initial and the final triangle that are incident to none of the hexagons are denoted as $A_{i,1}$ and $A_{i,2}$, respectively.

For each variable x_i, we assign one side of the hexagonal path to the positive literal x_i and the other to the negative literal $\overline{x}_i$. The intermediate vertices of the jth hexagon of the path are denoted with $X_{i,j}$ and $\overline{X}_{i,j}$, respectively, and are called *literal vertices*. They will be used for connecting a literal to its clauses.

Additionally, we use a connector gadget. It consists of two triangles with a common edge. The two vertices that are incident to only one of the triangles are denoted by $A_{i,3}$ and $A_{i,4}$, respectively.

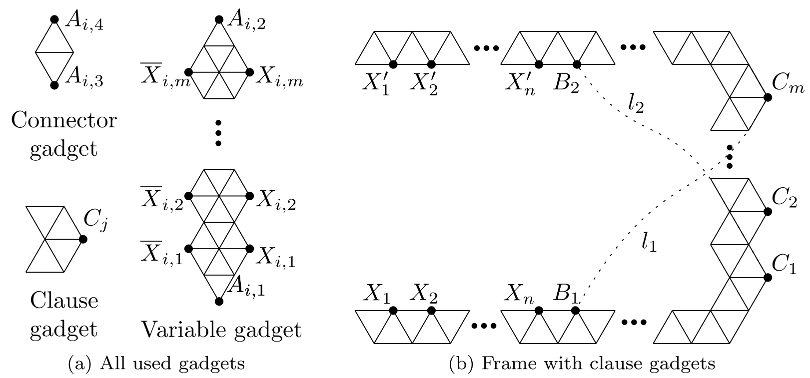


Fig. 10. Gadgets and frame of the NP-hardness proof.

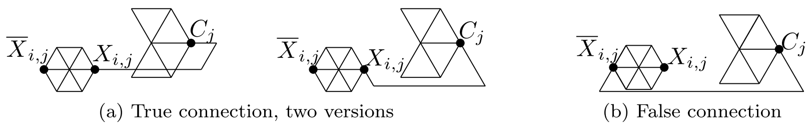


Fig. 11. Connections between clause and literal vertices in the NP-hardness proof.

Note that for all three gadgets, an embedding with total angular resolution 60° is unique up to rotation, scaling and reflection of the whole gadget. Especially, for each gadget, all triangles are congruent.

For connecting the gadgets, we first build a rigid 3-sided frame as depicted in Fig. 10(b). On the bottom, it consists of a straight path of $2n + 2m - 1$ triangles that alternatingly face up and down (the *bottom path*). On top of the rightmost triangle of this path, we add a sequence of m clause gadgets stacked on top of each other (one for each clause, with the clause vertices $C_1, \dots, C_m$ facing to the right). The top of the figure consists of a straight path of $2n + 2m - 1$ triangles that alternatingly face down and up (the *top path*). We denote the leftmost $n + 1$ vertices of degree four on the upper side of the bottom path with $X_1, \dots, X_n$, and B_1. The leftmost $n + 1$ vertices of degree four on the lower side of the top path are denoted $X'_1, \dots, X'_n$, and B_2. An embedding with total angular resolution 60° of this frame is again unique up to rotation, scaling, and reflection. We assume without loss of generality that it is embedded with $X_1, \dots, X_n$ being on a horizontal line, as depicted in Fig. 10(b). Then, for every $1 \le i \le n$, X'_i and X_i lie on a vertical line. Further, the line ℓ_1 spanned by B_1 and C_m has slope 60° and the line ℓ_2 through B_2 and C_1 has slope $-60°$.

We next add the variable gadgets in the following way. For each variable x_i, we identify the vertex $A_{i,1}$ of its gadget with X_i. Further, we connect the gadget to X'_i via a connector gadget by identifying $A_{i,2}$ with $A_{i,3}$ and $A_{i,4}$ with X'_i, respectively. Note that in any drawing with total angular resolution 60° of the construction so far, each variable gadget together with its connector gadget must be drawn vertically and between X_i and X'_i. Further, the variable gadgets can be scaled by adapting the height of the connector gadget. Independent of the scaling factor, the right side of each variable gadget is always to the left of the lines ℓ_1 and ℓ_2. Directionwise, variable gadgets can be drawn in two ways: either all $X_{i,j}$ are to the right of the $\overline{X}_{i,j}$ or the other way around.

To complete the construction, we add a path consisting of three consecutive edges between $X_{i,j}$ ($\overline{X}_{i,j}$) and C_j whenever x_i ($\overline{x}_i$) is a literal of clause c_j. An example of G with the 3SAT formula $(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$ is depicted in Fig. 12. To obtain a total angular resolution of 60° at every clause vertex C_i, all of these paths must start from C_i towards the right and one of them must start horizontally. We claim that the constructed graph G has a drawing D with $\mathrm{TAR}(D) \ge 60°$ if and only if the initial 3SAT formula is satisfiable.

Assume first that the 3SAT formula is satisfiable. Consider a truth assignment of the variables that satisfies the formula. We draw each variable gadget such that the side corresponding to its true literal is on the right. Further, we scale all the variable gadgets such that no two vertices of different variable gadgets or of a variable gadget and a clause gadget lie on a common horizontal line (except for the vertices X_i and X'_i). For every clause c_j, we choose a literal $v_i \in \{x_i, \overline{x}_i\}$ of c_i which is true. We draw the path between the corresponding clause vertex C_j and the matching literal vertex $V_{i,j} \in \{X_{i,j}, \overline{X}_{i,j}\}$ in the following way; see Fig. 11(a). We start with a horizontal edge from C_j to the right. Then, we continue with a $\pm 60°$ edge towards the left until we reach the height of $V_{i,j}$. We complete the path with a horizontal edge towards the left to $V_{i,j}$. For the other literals of c_j we draw a $\pm 60°$ edge from C_j to the right, followed by a horizontal edge to the left and a $\pm 60°$ edge to the left or right, depending on whether v_i is true or false; see Fig. 11. This way, all edges of the resulting drawing D are either horizontal or under an angle of $\pm 60°$ and no two edges overlap. Hence we have $\mathrm{TAR}(D) = 60°$ as desired.

For the other direction, assume that this graph G admits a drawing D with $\mathrm{TAR}(D) = 60°$. In D, consider a clause vertex C_j and the path $P = C_j M_1 M_2 V_{i,j}$ which starts horizontally towards the right at C_j. Then, the literal vertex $V_{i,j}$ must be on the right side of its variable gadget: If $V_{i,j}$ is a left vertex of a variable gadget, then P must enter $V_{i,j}$ from the left

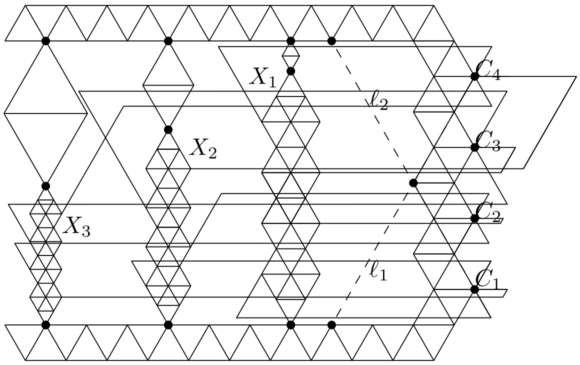


Fig. 12. Construction of the graph G for the 3SAT formula $(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$. For better readability, the variable gadgets are placed with a larger distance between them than in the description of the construction. The corresponding truth assignment is: x_1 is false, x_2 is true, x_3 is true.

under an angle of at most $\pm 60°$ with respect to the horizontal line. Hence, M_2 lies to the left of the lines ℓ_1 and ℓ_2. On the other hand, the second vertex M_1 of P lies horizontally to the right of C_j. However, to respect the $60°$ restriction at M_1, M_2 must lie to the right of the lines ℓ_1 and ℓ_2, a contradiction. Now consider the set of literal vertices that are an endpoint of a path starting horizontally at some clause vertex. As these literal vertices are on the right side of their corresponding variable gadgets, the set does not contain any pair $X_{i,j}, \overline{X}_{i,k}$. By setting all the corresponding literals to true, we obtain a non-contradicting (possibly partial) truth assignment of the variables which has at least one literal set true for every clause. Hence, any completion to a truth assignment of all variables satisfies the formula. □

4. TAR critical graph

In Section 2 we provided upper bounds on the number of edges a graph with a given total angular resolution can have, where the focus was on 60° and 90°. In the previous section, we saw that deciding if a graph can be drawn with a total angular resolution of a least 60° is NP-hard. So naturally it is of central interest to better understand the structure of graphs that do not allow for a certain total angular resolution. In the following we shed some light on graphs G with $\text{TAR}(G) = 60°$, such that removing any single edge from G increases its total angular resolution. In a certain sense these graphs have the minimal structure that forces $\text{TAR}(G)$ to be 60°. Thus, we call such graphs $\text{TAR}(G) - 60°$ critical graphs (see below for a proper definition). A better understanding of these graphs will help to see their structure and why some graphs need $\text{TAR}(G) = 60°$ while other, very similar, graphs can be drawn with a larger total angular resolution. We round up this picture in Subsection 4.2 by considering almost complete TAR critical graphs.

4.1. TAR-60° *critical graphs*

In this section we give a construction of a family of graphs that have a small number of edges and $\text{TAR}(G) \leq 60°$. Since we can construct connected graphs with a small number of edges and $\text{TAR}(G) \leq 60°$ by taking a triangle and adding a path to it, we only look at so called TAR-α *critical* graphs. These are connected graphs with $\text{TAR}(G) \leq \alpha$, but for all edges $e \in E$, $\text{TAR}(G \setminus \{e\}) > \alpha$ holds. In other words, $\text{TAR}(G) \leq \alpha$ and every proper subgraph H of G has $\text{TAR}(H) > \alpha$. We focus on TAR-60° critical graphs.

Theorem 14. *There exist* TAR-60° *critical graphs on n vertices with* $\frac{3}{2}n$ *edges for infinitely many values of n.*

This means there exist graphs with much fewer than $2n - 6$ edges, which have $\text{TAR}(G) \leq 60°$. We prove Theorem 14 by giving a construction of such graphs. Before we state our construction, we consider two 4-cycles that share an edge.

Lemma 15. *Two 4-cycles that share an edge (denoted by L) can be embedded with* $\text{TAR}(D) > 60°$ *only if L is drawn combinatorially equivalent to E7 in Fig. 2.*

Proof. First note that L has 6 vertices and 7 edges. Hence, L is an exception for Theorem 7. Therefore, L is drawn like $E7$ in Fig. 2. □

With the help of Lemma 15 we show the existence of a graph with $\frac{3}{2}n$ edges and $\text{TAR}(G) \leq 60°$. To this end, we first define a graph MS_k consisting of a sequence of 4-cycles glued together along opposite edges, which is essentially a circularly closed ladder graph on a Möbius strip.

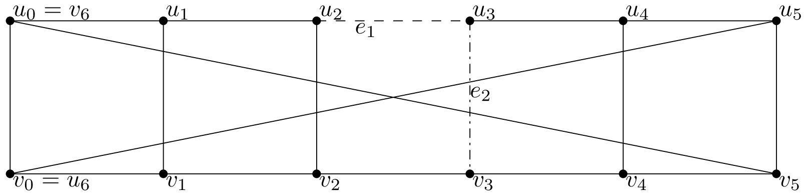


Fig. 13. Graph MS_6 with $\mathrm{TAR}(\mathrm{MS}_6) = 60°$ and $\frac{3}{2}n$ edges.

(a) Drawing of $\mathrm{MS}_6 \setminus \{e_1\}$ with $\mathrm{TAR}(\mathrm{MS}_6 \setminus \{e_1\}) > 60°$.

(b) Drawing of $\mathrm{MS}_6 \setminus \{e_2\}$ with $\mathrm{TAR}(\mathrm{MS}_6 \setminus \{e_2\}) > 60°$.

Fig. 14. Drawings of MS_6 without one edge.

Definition 16. We define the graph MS_k as follows. Let u_i, v_i, $0 \le i \le k-1$, be the vertices of MS_k. Further, we define $u_k = v_0$ and $v_k = u_0$. The edges of MS_k are

- (u_i, v_i) for $0 \le i \le k$,
- (u_i, u_{i+1}) and (v_i, v_{i+1}) for $0 \le i \le k-1$.

Fig. 13 depicts the graph MS_6, where the dashed and the dashed-dotted edge each highlight one instance of the two different edge types. Observe that MS_k has $n = 2k$ vertices and $\frac{3}{2}n = 3k$ edges.

Lemma 17. *Let $k \ge 6$ be an integer. Then MS_k is a TAR-60° critical graph.*

Proof. First we show that $\mathrm{TAR}(\mathrm{MS}_k) \le 60°$. We assume to the contrary that we can embed MS_k with $\mathrm{TAR}(\mathrm{MS}_k) > 60°$. Define C_i as the 4-cycle $u_i u_{i+1} v_{i+1} v_i$ for $0 \le i \le k-1$. If we can embed MS_k with $\mathrm{TAR}(\mathrm{MS}_k) > 60°$, then every C_i is a plane 4-cycle for every $0 \le i \le k-1$ and C_i and C_{i+1} are interior disjoint (with $C_k = C_0$) due to Lemma 15. First place a point c_i into every C_i. We draw a closed curve B through all c_i such that between c_i and c_{i+1} in C_i and C_{i+1}, B crosses only the edge $u_{i+1}v_{i+1}$ and this edge is crossed only once between c_i and c_{i+1}.

This is possible, because C_i and C_{i+1} are interior disjoint for any i. Since C_i and C_{i+1} are interior disjoint and every C_i is plane, all vertices u_i are on the same side and all the vertices v_i are on the other side of B if we walk along B. But then u_0 and v_k are on different sides which gives us the contradiction since $u_0 = v_k$. Therefore $\mathrm{TAR}(\mathrm{MS}_k) \le 60°$.

To show that MS_k is TAR-60° critical, we have to prove that for all edges e $\mathrm{TAR}(\mathrm{MS}_k \setminus \{e\}) > 60°$ holds. In Fig. 13 we see that the graph consists of edges like e_1, which are incident to one cycle of length 4 and edges like e_2, which are incident to two cycles of length 4. So we have two cases: Does $\mathrm{TAR}(\mathrm{MS}_6 \setminus \{e_1\}) > 60°$ hold and does $\mathrm{TAR}(\mathrm{MS}_6 \setminus \{e_2\}) > 60°$ hold?

Fig. 14(a) and Fig. 14(b) depict how $\mathrm{MS}_6 \setminus \{e_1\}$ and $\mathrm{MS}_6 \setminus \{e_2\}$, respectively, can be embedded with $\mathrm{TAR}(\mathrm{MS}_6 \setminus \{e_1\}) > 60°$. The dotted edge in both figures is the removed edge. For $k \ge 6$, the graph MS_k minus one edge can be embedded in a similar way (for example, by appropriately subdividing the two opposite edges of a crossing-free 4-cycle that are incident to only that 4-cycle and connecting the subdivision points). This completes the proof that MS_k is TAR-60° critical. □

Now we have all the results to prove Theorem 14.

Proof of Theorem 14. The graph MS_k with $k \ge 6$ is TAR-60° critical by Lemma 17. Furthermore MS_k has $\frac{3}{2}n$ edges since it is cubic. Therefore there exist TAR-60° critical graphs on n vertices with $\frac{3}{2}n$ edges for infinitely many values n. □

4.2. Almost complete graphs

Let K_n be the complete graph on n vertices. Argyriou, Bekos and Symvonis [4] showed that $\mathrm{TAR}(K_n) = \frac{180°}{n}$. In this section we show how the deletion of a few edges affects the total angular resolution. We start by showing that the removal of a small number of edges does not change the total angular resolution.

Theorem 18. *Every graph G with n vertices and at least $\binom{n}{2} - \frac{n-3}{3}$ edges has $\mathrm{TAR}(G) = \frac{180°}{n}$.*

Proof. Consider a drawing D of the complete graph K_n with $(n-k)$ vertices on the boundary B of the convex hull and k inner vertices.

A triangle T of D is called special if its vertices are on B and the three inner angles of T are split in total into at least n angles in D. Note, that the existence of a special triangle implies $\mathrm{TAR}(D) \leq \frac{180°}{n}$. If we delete a set E of at most $\frac{n-k-3}{2}$ edges of D, then there are three vertices on B which are not incident to any deleted edge. So these three vertices span a special triangle of $D \backslash E$ where $D \backslash E$ is the drawing D without the edges in E.

On the other hand, B is an $(n-k)$-cycle and its inner angles sum up to $(n-k-2)180°$. Since we have K_n, the inner angles of B are split into $(n-k)(n-2)$ angles. If the inner angles of B are split into at least $(n-k-2)n$ angles then the total angular resolution is at most $\frac{180°}{n}$. So we can delete up to $\frac{1}{2}((n-k)(n-2)-(n-k-2)n) = k$ edges and still have a drawing with $\mathrm{TAR}(D) \leq \frac{180°}{n}$.

Therefore, we want to minimize the maximum of k and $\frac{n-k-3}{2}$ over all possible values of k. This minimum is obtained for $k = \frac{n-3}{3}$. So any graph G with at least $\binom{n}{2} - \frac{n-3}{3}$ edges still has $\mathrm{TAR}(G) = \frac{180°}{n}$. □

Starting from the complete graph K_n, Theorem 18 implies that we have to delete more than $\frac{n-3}{3}$ edges to increase the total angular resolution. On the other hand, we can improve the total angular resolution by deleting $n-2$ edges, which are incident to the same vertex. This creates a graph G' that is essentially K_{n-1} with an additional vertex connected to the K_{n-1} by a single edge and thus $\mathrm{TAR}(G') = \frac{180°}{n-1} > \frac{180°}{n}$. We now show that the total angular resolution can be increased by removing even fewer edges.

Proposition 19. *For any $n \geq 12$ there exists a graph G with n vertices, at least $\binom{n}{2} - \frac{11n}{12} + 1$ edges and $\mathrm{TAR}(G) \geq \frac{180°}{n-1}$.*

Proof. We take a drawing D of K_{n-1} where the vertices $v_1, v_2, \ldots, v_{n-1}$ span a regular $(n-1)$-gon P. Note that $\mathrm{TAR}(D) = \frac{180°}{n-1}$. Let c be the circumcenter of P and C be the corresponding circumcircle. Let p be a point on the line spanned by c and v_1 such that $|cv_1| = |v_1 p|$ as in Fig. 15. Observe that the angle $\angle v_i p v_{i+1} < \angle p v_{i+1} v_i$ since $v_i v_{i+1}$ is the shortest edge of the triangle $p v_i v_{i+1}$.

Let the tangents of C through p touch C at the points t_1 and t_2. Let t_1 be the tangent point which lies on the arc between v_a and v_{a+1} such that v_a is closer to v_1 than v_{a+1}. Observe that $\angle v_{i-1} p v_i > \angle v_i p v_{i+1}$ holds for $1 < i < a$. Since $|cp| = 2 \cdot |ct_1|$ and $\angle c t_1 p = 90°$, the triangle $c t_1 p$ is half of an equilateral triangle. So $\angle p c t_1 = 60°$ and $\angle c p t_1 = 30°$. Further $\angle c v_1 v_2 < 90°$ holds and therefore $\angle v_1 p v_2 < \angle v_2 c v_1 = \frac{360°}{n-1}$. So $\angle v_i p v_{i+1} < \frac{360°}{n-1}$ holds for $1 \leq i < a$.

We place points b_j on C on the shorter arc between v_1 and v_a as depicted in Fig. 15 such that b_1 is v_1 and $\angle b_j p b_{j+1} = \frac{360°}{n-1}$ for $1 \leq j \leq \lfloor \frac{n-1}{12} \rfloor$. Note that the bound $\lfloor \frac{n-1}{12} \rfloor$ is implied by $\angle c p t_1 = 30°$. In Fig. 15 the points b_2 and b_3 are marked with a red cross. Since $\angle v_i p v_{i+1} < \frac{360°}{n-1}$ for $1 \leq i < a$, there is an index k with $1 < k < a$ such that v_k is on the arc between b_j and b_{j+1} including b_{j+1} for all $1 \leq j \leq \lfloor \frac{n-1}{12} \rfloor$.

Let S_1 be a set of vertices of P such that for any $1 \leq j \leq \lfloor \frac{\lfloor \frac{n-1}{12} \rfloor}{2} \rfloor$ there exists exactly one point v_k of S_1 such that v_k is on the arc from b_{2j} to b_{2j+1}. So S_1 contains $\lfloor \frac{\lfloor \frac{n-1}{12} \rfloor}{2} \rfloor$ points. Note that v_1 is not in S_1.

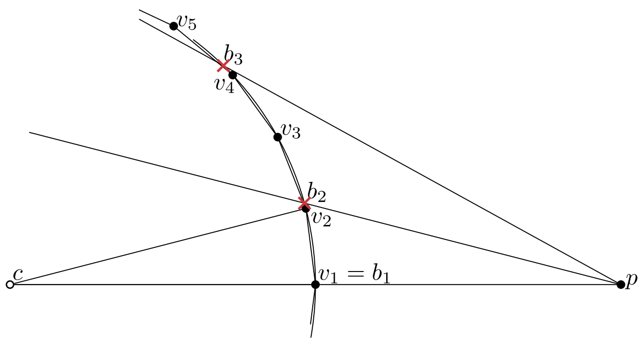


Fig. 15. Illustration of the proof of Proposition 19. Vertices of the graphs are marked with black disks.

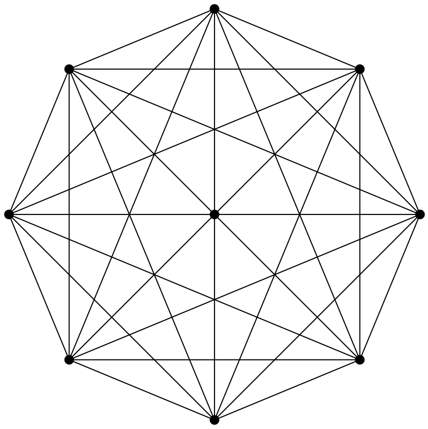


Fig. 16. A drawing with 32 edges and $\mathrm{TAR}(D) = 22.5°$.

Let S_2 be the set of vertices of P such that $v_{n+1-i} \in S_2$ if and only if $v_i \in S_1$. Let $S = S_1 \cup S_2 \cup \{v_1\}$. If $v_i, v_k \in S_1$ or $v_i, v_k \in S_2$, then $\angle v_i p v_k \geq \frac{360°}{n-1}$ holds by construction. If $v_i \in S_1$, then $\angle v_1 p v_i \geq \frac{360°}{n-1}$ holds since $\angle v_1 p v_i > \angle v_1 p b_2$. If $v_i \in S_2$, then $\angle v_1 p v_i \geq \frac{360°}{n-1}$ follows in a similar way. Therefore, for any two points $v_i, v_k \in S$, $\angle v_i p v_k \geq \frac{360°}{n-1}$ holds.

So the graph G with vertices $v_1, \dots, v_{n-1}$ and p, and edges $v_i v_j$ for any $1 \leq i, j \leq n-1$ and pv_i for $v_i \in S$ has $\mathrm{TAR}(G) \geq \frac{180°}{n-1}$ and $\binom{n-1}{2} + 2\lfloor \frac{\lfloor \frac{n-1}{12} \rfloor}{2} \rfloor + 1$ edges. Hence, G has at least $\binom{n}{2} - \frac{11n}{12}$ edges. □

Proposition 19 holds for all $n \geq 12$ but is most likely not tight. If the number of vertices is odd, then the following proposition gives us a better bound.

Proposition 20. *For any odd $n \geq 5$ there exists a graph G with n vertices, $\binom{n}{2} - \frac{n-1}{2}$ edges and* $\mathrm{TAR}(G) > \frac{180°}{n}$.

Proof. This is achieved by the following construction. We take a drawing D of K_{n-1} where the vertices $v_1, v_2, \dots, v_{n-1}$ form a regular $(n-1)$-gon. Next, we replace the common crossing of all main diagonals $(v_i, v_{i+(n-1)/2})$, $1 \leq i \leq (n-1)/2$, by a vertex v_n. We also replace every main diagonal $(v_i, v_{i+(n-1)/2})$ by the edges (v_i, v_n) and $(v_n, v_{i+(n-1)/2})$ for every $1 \leq i \leq \frac{n-1}{2}$. We denote the resulting drawing with D'. Fig. 16 depicts D' for $n = 9$ vertices.

Since we only replaced edges and do not have edges which are on top of each other, we have $\mathrm{TAR}(D') = \mathrm{TAR}(D) = \frac{180°}{n-1}$. Further, D' has $\frac{(n-1)^2}{2}$ edges. So D' has $\frac{n-1}{2}$ edges fewer than K_n. □

We have shown that in a complete graph K_n we can delete $\frac{11n}{12} - 1$ edges for arbitrary n and $\frac{n-1}{2}$ edges for odd n to increase the total angular resolution. For odd n, deleting any $\frac{n-3}{3}$ edges does not affect the total angular resolution but deleting $\frac{n-1}{2}$ edges can improve it. We conjecture that Proposition 20 is tight.

5. Conclusion and open problems

In this work we have shown that, up to a finite number of well specified exceptions of constant size, any graph G with $\mathrm{TAR}(G) > 60°$ has at most $2n - 6$ edges. For larger angles we were able to obtain similar bounds: For graphs with $\mathrm{TAR}(G) \geq 90°$ we have $m \leq 2n - 2\sqrt{n}$, for $\mathrm{TAR}(G) > 90°$ we have $m \leq \frac{3}{2}n - \frac{5}{2}$, and for $\mathrm{TAR}(G) > 120°$ we have $m \leq n$ for $n \geq 7$. These bounds are tight. We conjecture that almost all graphs with $\mathrm{TAR}(G) > \frac{k-2}{k}90°$ have at most $2n - 2 - \lfloor \frac{k}{2} \rfloor$ edges.

From a computational point of view, we have proven that deciding whether a given graph admits a drawing with total angular resolution at least 60° is in general NP-hard. The same was known before for at least 90° [13]. On the other hand, for large angles, the recognition problem eventually becomes easy (for example, G can be drawn with $\mathrm{TAR}(G) > 120°$ if and only if it is the union of cycles of at least 7 vertices and arbitrary paths). This yields the following open problem: At which angle(s) does the decision problem change from NP-hard to polynomial-time solvable?

We introduced TAR-α critical graphs and showed the existence of TAR-60° critical graphs with $\frac{3}{2}n$ edges. It remains open whether there are TAR-60° critical graphs with fewer than $\frac{3}{2}n$ edges. More generally, how many edges does the smallest TAR-α critical graph have for a fixed α? It is also open, for which values of α there exist TAR-α critical graphs with more than n vertices, where n is arbitrarily large. For the complete graph K_n we proved that we can delete any $\frac{n-1}{3}$ edges of K_n and still get $\mathrm{TAR}(G) = \frac{180°}{n}$. It is open whether this bound is tight. On the other hand we presented two families of drawings, which have $\mathrm{TAR}(G) > \frac{180°}{n}$ and many edges. As a related question, what is the smallest number of edges a graph with n vertices and $\mathrm{TAR}(G) = \frac{180°}{n}$ can have?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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