On the arithmetic of Knuth's powers and some computational results about their density

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Abstract. The object of the paper are the so-called "unimaginable numbers". In particular, we deal with some arithmetic and computational aspects of the Knuth's powers notation and move some first steps into the investigation of their density. Many authors adopt the convention that unimaginable numbers start immediately after 1 *googol* which is equal to 10^{100}, and G.R. Blakley and I. Borosh have calculated that there are exactly 58 integers between 1 and 1 googol having a nontrivial *"kratic representation"*, i.e., are expressible nontrivially as Knuth's powers. In this paper we extend their computations obtaining, for example, that there are exactly 2 893 numbers smaller than $10^{10\,000}$ with a nontrivial kratic representation, and we, moreover, investigate the behavior of some functions, called *krata*, obtained by fixing at most two arguments in the Knuth's power $a\uparrow^b c$.

Keywords: Unimaginable numbers · Knuth up-arrow notation · Algebraic recurrences · Computational number theory

1 Introduction: The unimaginable numbers

An unimaginable number, intuitively and suggestively, is a number that go beyond the human imagination. There is not a completely accepted standard formal definition of unimaginable numbers, but one of the most used is the following: a number is called *unimaginable* if it is greater than 1 googol, where *1 googol* is equal to 10^{100}. To better understand the size of numbers like these, consider that it is estimated that in the observable universe there are at most 10^{82} atoms; this justifies the term *unimaginable*. The first appearance of the unimaginable numbers was, to our knowledge, in Magna Graecia in the work of Archimedes of Syracuse. Archimedes in his work called "*Arenarius*" in Latin, or the "*Sand Reckoner*" in English, describes, using words of the natural language, an extremely large number that, in exponential notation, is equal to

$$10^{8\cdot 10^{16}} \quad = \quad 10^{80\,000\,000\,000\,000\,000}. \tag{1}$$

Obviously, writing this number without any kind of modern mathematical notation, as Archimedes did, is very very difficult. Let us jump to modern times and introduce the most used notation that allows to write numbers so large that are definitely beyond the common experience of a human being.

Definition 1 (Knuth's up-arrow notation). *For all non-negative integers a, b, n, we set*

$$a\uparrow^n b \ := \begin{cases} a \cdot b & \textit{if} \ \ n = 0; \\ 1 & \textit{if} \ \ n \geq 1 \ \ \textit{and} \ \ b = 0; \\ a\uparrow^{n-1}(a\uparrow^n(b-1)) & \textit{if} \ \ n \geq 1 \ \ \textit{and} \ \ b \geq 1. \end{cases} \tag{2}$$

For $n = 1$ we obtain the ordinary exponentiation, e.g., $3\uparrow 4 = 3^4$; for $n = 2$ we obtain *tetration*, for $n = 3$ *pentation*, and so on. Hence (2) represents the so called *n-hyperoperation.* Now, using Knuth's notation, the Archimedes' number (1) can be easily written as follows

$$\big((10\uparrow 8)\uparrow^2 2\big)\uparrow(10\uparrow 8).$$

In this paper we use an alternative notation to that introduced in Definition 1. Denoting by $\mathbb{N}$ the set of natural numbers (i.e., non-negative integers) we define the *Knuth's function k* as follows

$$\begin{aligned} k \,:\, \mathbb{N}\times\mathbb{N}\times\mathbb{N} \;&\longrightarrow\; \mathbb{N} \\ (B, d, T) \;&\mapsto\; k(B, d, T) := B\uparrow^d T \end{aligned} \tag{3}$$

and we call the first argument of k (i.e., B) the *base*, the second (i.e., d) the *depth* and the third (i.e., T) the *tag* (see [3]).

The paper is organized as follows: in Section 2 we introduce some general computational problems, while in Section 3, which is the core of this work, we deal with density and representational problems related to Knuth's powers. In particular, Proposition 2 and Corollary 1 give some simple results which characterize the difference of digits in base 10 between two "consecutive" Knuth's powers of the simplest "non-trivial" type, i.e., $a\uparrow^2 2$. Proposition 3 extends, instead, a computation by Blakley and Borosh (see [3, Proposition 1.1]): they found that there are exactly 58 numbers smaller than 1 *googol* $(= 10^{100})$ nontrivially expressible through the Knuth's function k. We obtained that such number increases to 2893 if we consider integers lesser than $10^{10\,000}$. Among these 2893 numbers, 2888 are expressible through the aforementioned form $a\uparrow^2 2$.

We conclude the introductory section by giving the reader some brief information on some useful references to deepen the issues related to unimaginable numbers. In addition to article [3] which, for our purposes, represents the main reference, the same authors investigate the modular arithmetic of Knuth's powers in [4]. Knuth himself had instead introduced the notation (2) a few years earlier in [16] (1976), but these ideas actually date from the beginning of the century (see [1, 2, 13, 19]). More recent works that start from "extremely large" or "infinite" numbers are [7, 8, 12, 14, 15, 17, 18, 20, 21, 23]. There are also the online resources [5, 6, 22]. While [10] provides the reader with a brief general introduction with some further reference.

2 Representational and computational problems

It is obvious that almost all the numbers are unimaginable, hence a first natural question is: can we write every unimaginable number using Knuth's up-arrow notation? The answer is trivial: we cannot. Actually there are just very few numbers that are expressible using this notation. More precisely let $\mathcal{K}_0$ denote the image of the map k, i.e., the set of those natural numbers that are expressible via Knuth's notation. As customary one can consider the ratio

$$\rho_0(x) = \frac{\#\big(\mathcal{K}_0 \cap \{m \in \mathbb{N} : m < x\}\big)}{x}. \tag{4}$$

The computed values of $\rho_0(x)$ are very close to zero and $\rho_0(x)$ appears to be quickly converging to zero as $x \to +\infty$. In the next section we compute some values of a ratio related to this.

In recent years dozens of systems and notations have been developed to write unimaginable numbers (for example see [1], [6], [7], [14], [15]), most of them can reach bigger numbers in a more compact notation than Knuth's notation can, but the difference between two consecutive numbers with a compact representation in a specific notation often increases quicker than in Knuth's notation. Hence, almost all unimaginable numbers remain inaccessible to write (and to think about?) and the problem of writing an unimaginable number in a convenient way is open.

A strictly related open problem is to find a good way to represent an unimaginable number on a computer. It is not possible to represent with usual decimal notation numbers like $3 \uparrow^3 3$ on a computer at the present time. Hence, to do explicit computations involving these numbers is quite hard. Therefore finding a way, compatible with classical operations, to represent these kind of numbers on a computer not only would make many computations faster but it would also help to deeper develop the mathematical properties and the applications related to unimaginable numbers.

We recall, for the convenience of the reader, some basic properties of Knuth's up-arrow notation that will be used in the next section.

Proposition 1. *For all positive integers a, b, n, with $b > 1$, we have:*

(i) $a \uparrow^n b < (a+1) \uparrow^n b$;
(ii) $a \uparrow^n b < a \uparrow^n (b+1)$;
(iii) $a \uparrow^n b \leq a \uparrow^{n+1} b$, *where the equality holds if and only if* $a = 1$ *or* $a = b = 2$.

Proof. See [3, Theorem 1.1].

3 About the density of numbers with kratic representation

We follow the nomenclature used in [3] and we say that a positive integer x has a *non-trivial kratic representation* if there are integers a, b, n all greater than 1 such that $x = a \uparrow^n b$. Note that a kratic representation should not be confused with a kratos: a *kratos* (pl. *krata*)[4] is a function h that comes from the Knuth's function k by fixing at most two arguments (see [3]). It is then a natural question to ask "how many" numbers have a non-trivial kratic representation.

Example 1. The least positive integer with non-trivial kratic representation is 4, in fact $2 \uparrow^n 2 = 4$ for all positive integers n.

It is easy to see that numbers with kratic representation of the form $a \uparrow^2 2 = a^a$ are more frequent than those with other types of kratic representation. The following proposition states how often they appear with respect to the number of digits, i.e., it calculates the increment of the number of digits between two "consecutive" numbers with kratic representation of that form. We need a further piece of notation: as usual Log denotes the logarithm with base 10, $\lfloor \alpha \rfloor$ the floor of a real number α and $\nu(a)$ the number of digits of a positive integer a (in base 10). Using these notation we have

$$\nu(a) = \lfloor \mathrm{Log}\ a \rfloor + 1 \tag{5}$$

for all positive integers a.

Proposition 2. *For every integer $a \geq 1$ we have*

$$\nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) = \left\lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}\left(1 + \frac{1}{a}\right) \right\rfloor \tag{6}$$

or

$$\nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) = \left\lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}\left(1 + \frac{1}{a}\right) \right\rfloor + 1. \tag{7}$$

Proof. The proposition states that the difference between the number of digits of $(a+1)^{a+1}$ and a^a is given by Formula (6) or (7). For any integer $a \geq 1$ we have

$$\begin{aligned} \nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) &= \left\lfloor \mathrm{Log}(a+1)^{a+1} \right\rfloor - \left\lfloor \mathrm{Log}\ a^a \right\rfloor \\ &= \lfloor (1+a)\mathrm{Log}(a+1) \rfloor - \lfloor a\,\mathrm{Log}\ a \rfloor \\ &= \lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}(a+1) - a\,\mathrm{Log}\ a \\ &\quad + a\,\mathrm{Log}\ a \rfloor - \lfloor a\,\mathrm{Log}\ a \rfloor \\ &= \left\lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}\left(1 + \frac{1}{a}\right) + a\,\mathrm{Log}\ a \right\rfloor \qquad (8) \\ &\quad - \lfloor a\,\mathrm{Log}\ a \rfloor . \end{aligned}$$

[4] Kratos, written in ancient Greek $\kappa\rho\acute{\alpha}\tau o\varsigma$, indicated the "personification of power".

Since for all real numbers α and β the following inequalities hold

$$\lfloor \alpha \rfloor + \lfloor \beta \rfloor \leq \lfloor \alpha + \beta \rfloor \leq \lfloor \alpha \rfloor + \lfloor \beta \rfloor + 1, \tag{9}$$

then, combining (9) with (8) we obtain

$$\begin{aligned} \left\lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}\left(1+\frac{1}{a}\right) \right\rfloor \; &\leq \; \nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) \\ &\leq \; \left\lfloor \mathrm{Log}(a+1) + a\,\mathrm{Log}\left(1+\frac{1}{a}\right) \right\rfloor + 1, \end{aligned}$$

proving the proposition.

Corollary 1. *For every integer $a \geq 1$ the following inequalities hold*

$$\lfloor \mathrm{Log}(a+1) \rfloor \; \leq \; \nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) \; \leq \; \lfloor \mathrm{Log}(a+1) \rfloor + 2. \tag{10}$$

Proof. The first inequality in (10) is an immediate consequence of (6). For the second one note that, using the previous proposition, the second inequality in (9) and the well-known bound

$$\left(1+\frac{1}{a}\right)^a \; < \; e, \qquad \text{for all } a \geq 1, \tag{11}$$

we obtain

$$\begin{aligned} \nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) \; &\leq \; \left\lfloor \mathrm{Log}(a+1) + \mathrm{Log}\left(1+\frac{1}{a}\right)^a \right\rfloor + 1 \\ &\leq \; \lfloor \mathrm{Log}(a+1) \rfloor + \left\lfloor \mathrm{Log}\left(1+\frac{1}{a}\right)^a \right\rfloor + 2 \\ &= \; \lfloor \mathrm{Log}(a+1) \rfloor + 2. \end{aligned}$$

The two possibilities given by (6) and (7) in Proposition 2 and the three given by Corollary 1 $\left(\text{that is, } \nu\left((a+1)^{a+1}\right) - \nu\left(a^a\right) - \lfloor \mathrm{Log}(a+1) \rfloor = 0,\ 1,\ 2\right)$ are all effectively realized: it is sufficient to look at the values $a = 1, 2, 7$ in Table 1.

Table 1. The first 10 values of a^a.

a	1	2	3	4	5	6	7	8	9	10
a^a	1	4	27	256	3 125	46 656	823 543	16 777 216	387 420 489	10 000 000 000

Remark 1. Note that using (11) and the lower bound $2 \leq (1+1/a)^a$, for $a \geq 1$, we obtain

$$2(a+1) \; \leq \; \frac{(a+1){\uparrow}^2 2}{a{\uparrow}^2 2} \; < \; e(a+1) \tag{12}$$

for all integers $a \geq 1$. It is also interesting that the ratio of two consecutive numbers of that form can be approximated by a linear function in the base a.

The previous remark implies that given a number with kratic representation of the form $a\!\uparrow^2 2$, the subsequent one, $(a+1)\!\uparrow^2 2$, is rather close to it. Instead, numbers with kratic representation of other forms are much more sporadic: the following proposition gives a more precise idea of this phenomenon.

Proposition 3. *There are exactly* 2893 *numbers smaller than* $10^{10\,000}$ *that admit a non-trivial kratic representation. Among them,* 2888 *have a representation of the form* $a\!\uparrow^2 2$*, and only* 5 *do not have such a representation.*

Proof. By [3, Proposition 1.1] there are exactly 58 numbers with less than 10^2 digits in decimal notation that have a non-trivial kratic representation; we collect them in the following set

$$E_2 \;=\; \{a\!\uparrow^2 2 : 2 \le a \le 56\} \sqcup \{2\!\uparrow^2 3,\; 3\!\uparrow^2 3,\; 2\!\uparrow^2 4\}.$$

Note also that some of them have more than one representation:

$$2\!\uparrow^2 2 = 4 = 2\!\uparrow^d 2 \;\; \forall d \ge 2, \quad 3\!\uparrow^2 3 = 3^{27} = 3\!\uparrow^3 2, \quad 2\!\uparrow^2 4 = 2^{16} = 2\!\uparrow^3 3.$$

We look for the numbers we need to add to E_2 to obtain the desired set

$$E := \left\{n \in \mathbb{N} : n < 10^{10\,000} \text{ and } n \text{ has a non-trivial kratic representation}\right\}.$$

We consider different cases depending on the depth d.

(i) “$d = 2$”. Since

$$\mathrm{Log}(2889\!\uparrow^2 2) \approx 9998.1 \quad \text{and} \quad \mathrm{Log}(2890\!\uparrow^2 2) \approx 10001.99,$$

we have to add to E_2 the numbers from $57\!\uparrow^2 2$ to $2889\!\uparrow^2 2$. Then, since

$$\mathrm{Log}(5\!\uparrow^2 3) \approx 2184.28 \quad \text{and} \quad \mathrm{Log}(6\!\uparrow^2 3) \approx 36305.4,$$

the numbers $4\!\uparrow^2 3$ and $5\!\uparrow^2 3$ belong to E as well. Instead,

$$\mathrm{Log}(3\!\uparrow^2 4) \approx 3638334640024.1 \quad \text{and} \quad \mathrm{Log}(2\!\uparrow^2 5) \approx 19728.3 \tag{13}$$

guarantee, by using Proposition 1, that there are no other elements with $d = 2$ in E.

(ii) “$d = 3$”. Note that $4\!\uparrow^3 2 = 4\!\uparrow^2 4 > 3\!\uparrow^2 4$, and $3\!\uparrow^3 3 = 3\!\uparrow^2 3\!\uparrow^2 3 > 3\!\uparrow^2 4$, and $2\!\uparrow^3 4 = 2\!\uparrow^2 2^{16}$, hence, by using (13), we have that E does not contain any new element with $d = 3$.

(iii) “$d = 4$”. Since $3\uparrow^4 2 = 3\uparrow^3 3$ and $2\uparrow^4 3 = 2\uparrow^3 4$, part (ii) yields that they do not belong to E. Therefore, it has no new elements with $d = 4$.

Now, only the (trivial) case “$d \ge 5$” remains, but since $3\uparrow^d 2 > 3\uparrow^4 2$, (iii) yields that there are no new elements with $d \ge 5$ in E. In conclusion, we have proved that

$$E = E_2 \sqcup \{a\!\uparrow^2 2 : 57 \le a \le 2889\} \sqcup \{4\!\uparrow^2 3,\; 5\!\uparrow^2 3\}$$

and its cardinality is 2893. From the proof, it is also clear that the only elements of E having no representation of the type $a\uparrow^2 2$ are $2\uparrow^2 3,\; 3\uparrow^2 3,\; 2\uparrow^2 4,\; 4\uparrow^2 3$ and $5\!\uparrow^2 3$.

We conclude the paper with some observations about the frequency of numbers with non-trivial kratic representation. Let $\mathcal{K}$ denote the set of integers that admit a non-trivial kratic representation. Define the *kratic representation ratio* $\rho(x)$ as:

$$\rho(x) := \frac{\#\big(\mathcal{K} \cap \{m \in \mathbb{N} : m < x\}\big)}{x}.$$

(Note the differences with respect to (4).) We find the following values:

$$\begin{aligned}
\rho(10) &= \frac{1}{10} = 0.1 \cdot 10^{-1}, \\
\rho\left(10^2\right) &= \frac{3}{10^2} = 0.3 \cdot 10^{-1}, \\
\rho\left(10^4\right) &= \frac{5}{10^4} = 0.5 \cdot 10^{-3}, \\
\rho\left(10^{10}\right) &= \frac{9}{10^{10}} = 0.9 \cdot 10^{-9}, \\
\rho\left(10^{10^2}\right) &= \frac{58}{10^{10^2}} = 0.58 \cdot 10^{-98}, \\
\rho\left(10^{10^4}\right) &= \frac{2893}{10^{10^4}} = 0.2893 \cdot 10^{-9996}.
\end{aligned}$$

These data seems to indicate that $\rho(x)$ tends rapidly to zero for $x \to +\infty$. However, it is not known, to our knowledge, an explicit formula for $\rho(x)$ or, equivalently, for the cardinality of the set $\mathcal{K}(x) = \mathcal{K} \cap \{m \in \mathbb{N} : m < x\}$ itself.

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