

Fractal Dimensions of Networks

Eric Rosenberg

Fractal Dimensions of Networks

 Springer

Eric Rosenberg
AT&T Labs
Middletown, NJ, USA

ISBN 978-3-030-43168-6 ISBN 978-3-030-43169-3 (eBook)
<https://doi.org/10.1007/978-3-030-43169-3>

© Springer Nature Switzerland AG 2020

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To David and Alexa

Acknowledgements

I gratefully acknowledge the comments and suggestions of Lazaros Gallos, Robert Kelley, Robert Murray, Erin Pearse, Curtis Provost, and Brendan Ruskey. Special thanks to Ron Levine for slogging through an early draft of this book and providing enormously helpful feedback. I am of course responsible for any remaining mistakes in this book, and I welcome corrections from readers: send email to FractalDimensionsOfNetworks@gmail.com. Many thanks to the artist Jug Cerovic for providing permission to reproduce their artwork, a beautiful map of the New York City subway system, in the cover of this book. Finally, many thanks to Paul Drougas, Senior Editor at Springer, for his encouragement and support.

About the Author

Eric Rosenberg received a B.A. in Mathematics from Oberlin College and a Ph.D. in Operations Research from Stanford University. He recently retired from AT&T Labs in Middletown, New Jersey, and has joined the faculty of Georgian Court University in New Jersey. Dr. Rosenberg has taught undergraduate and graduate courses in modeling and optimization at Princeton University, New Jersey Institute of Technology, and Rutgers University. He has authored or co-authored 19 patents and has published in the areas of convex analysis and nonlinearly constrained optimization, computer-aided design of integrated circuits and printed wire boards, telecommunications network design and routing, and fractal dimensions of networks. He is the author of *A Primer of Multicast Routing* (Springer, 2012) and *A Survey of Fractal Dimensions of Networks* (Springer, 2018).

Contents

1	Introduction	1
1.1	Intended Readership	8
1.2	Notation and Terminology	9
2	Networks: Introductory Material	17
2.1	Basic Network Definitions	17
2.2	Small-World Networks	25
2.3	Random Networks	32
2.4	Scale-Free Networks	33
2.5	Skeleton of a Network	40
3	Fractals: Introductory Material	43
3.1	Introduction to Fractals	43
3.2	Fractals and Power Laws	49
3.3	Iterated Function Systems	51
3.4	Convergence of an IFS	54
4	Topological and Box Counting Dimensions	61
4.1	Topological Dimension	61
4.2	Box Counting Dimension	67
4.3	A Deeper Dive into Box Counting	73
4.4	Fuzzy Fractal Dimension	77
4.5	Minkowski Dimension	78
5	Hausdorff, Similarity, and Packing Dimensions	83
5.1	Hausdorff Dimension	84
5.2	Similarity Dimension	89
5.3	Fractal Trees and Tilings from an IFS	96
5.4	Packing Dimension	99
5.5	When Is an Object Fractal?	103

6	Computing the Box Counting Dimension	107
6.1	Lower and Upper Box Sizes	107
6.2	Computing the Minimal Box Size	111
6.3	Coarse and Fine Grids	115
6.4	Box Counting Dimension of a Spiral	119
6.5	Some Applications of the Box Counting Dimension	122
7	Network Box Counting Dimension	131
7.1	Node Coverings and Arc Coverings	132
7.2	Diameter-Based and Radius-Based Boxes	135
7.3	A Closer Look at Network Coverings	140
7.4	The Origin of Fractal Networks	143
8	Network Box Counting Heuristics	145
8.1	Node Coloring Formulation	145
8.2	Node Coloring for Weighted Networks	148
8.3	Random Sequential Node Burning	150
8.4	Set Covering Formulation and a Greedy Method	154
8.5	Box Burning	157
8.6	Box Counting for Scale-Free Networks	161
8.7	Other Box Counting Methods	163
8.8	Lower Bounds on Box Counting	165
9	Correlation Dimension	177
9.1	Dynamical Systems	177
9.2	Pointwise Mass Dimension	182
9.3	The Correlation Sum	183
9.4	The Correlation Dimension	186
9.5	Comparison with the Box Counting Dimension	191
10	Computing the Correlation Dimension	195
10.1	How Many Points are Needed?	195
10.2	Analysis of Time Series	201
10.3	Constant Memory Implementation	204
10.4	Applications of the Correlation Dimension	205
10.5	The Sandbox Dimension	217
11	Network Correlation Dimension	221
11.1	Basic Definitions	222
11.2	Variants of the Correlation Dimension	226
11.3	Cluster Growing and Random Walks	230
11.4	Overall Slope Correlation Dimension	234
11.5	Comparing the Box Counting and Correlation Dimensions	243
11.6	Generalized Correlation Dimension	245

12	Dimensions of Infinite Networks	247
12.1	Mass Dimension	249
12.2	Transfractal Networks	253
12.3	Volume and Surface Dimensions	256
12.4	Conical Graphs with Prescribed Dimension	262
12.5	Dimension of a Percolating Network	265
13	Similarity Dimension of Infinite Networks	267
13.1	Generating Networks from an IFS	270
13.2	Analysis of the Infinite Sequence	272
14	Information Dimension	279
14.1	Shannon Entropy and Hartley Entropy	280
14.2	Dimension of a Probability Distribution	286
14.3	Information Dimension Bounds the Correlation Dimension . .	290
14.4	Fixed Mass Approach	295
14.5	Rényi and Tsallis Entropy	297
14.6	Applications of the Information Dimension	300
15	Network Information Dimension	305
15.1	Entropy of a Node and a Network	305
15.2	Defining the Network Information Dimension	311
15.3	Comparison with the Box Counting Dimension	317
15.4	Maximal Entropy Minimal Coverings	318
15.5	Binning	324
16	Generalized Dimensions and Multifractals	325
16.1	Multifractal Cantor Set	327
16.2	Generalized Dimensions	329
16.3	Spectrum of the Multifractal Cantor Set	333
16.4	Spectrum of a Multifractal	338
16.5	Computing the Multifractal Spectrum and Generalized Dimensions	346
16.6	The Sandbox Method for Geometric Multifractals	353
16.7	Particle Counting Dimension	359
16.8	Applications of Multifractals	361
16.9	Generalized Hausdorff Dimensions	363
17	Multifractal Networks	365
17.1	Lexico Minimal Coverings	365
17.2	Non-monotonicity of the Generalized Dimensions	373
17.3	Multifractal Network Generator	383
17.4	Computational Results for Multifractal Networks	386
17.5	Sandbox Method for Multifractal Networks	388

18	Generalized Hausdorff Dimensions of Networks	391
18.1	Defining the Hausdorff Dimension of a Network	391
18.2	Computing the Hausdorff Dimension of a Network	396
18.3	Generalized Hausdorff Dimensions of a Network	407
19	Lacunarity	413
19.1	Gliding Box Method	415
19.2	Lacunarity Based on an Optimal Covering	420
19.3	Applications of Lacunarity	421
20	Other Dimensions	425
20.1	Spectral Dimension	425
20.2	Lyapunov Dimension	427
20.3	Three More Dimensions	429
20.4	Uses and Abuses of Fractal Dimensions	431
21	Coarse Graining and Renormalization	437
21.1	Renormalization Basics	438
21.2	Renormalizing a Small-World Network	442
21.3	Renormalizing Unweighted, Undirected Networks	445
21.4	Renormalization of Scale-Free Networks	448
21.5	Universality Classes for Networks	452
22	Other Network Dimensions	455
22.1	Dimension of a Tree	455
22.2	Area-Length Dimension	457
22.3	Metric Dimension	458
22.4	Embedding Dimension	459
22.5	Zeta Dimension	461
22.6	Dimensions Relating to Graph Curvature	465
22.7	Spectral Dimension of Networks	467
22.8	Other Avenues	468
23	Supplemental Material	471
23.1	Rectifiable Arcs and Arc Length	471
23.2	The Gamma Function and Hypersphere Volumes	473
23.3	Laplace's Method of Steepest Descent	474
23.4	Stirling's Approximation	475
23.5	Convex Functions and the Legendre Transform	476
23.6	Graph Spectra	480
	Bibliography	485
	Index	517

List of Figures

1.1	Four generations of the Sierpiński triangle evolution	3
1.2	Convex and concave functions	11
2.1	Complete graphs K_4 and K_5	20
2.2	Arc betweenness	22
2.3	Network motifs	23
2.4	Closing in on the target	27
2.5	Regular lattice with $N = 16$ and $k = 4$	28
2.6	Commellas and Sampels first construction for $N = 6$ and $k = 4$	31
2.7	Preferential attachment	35
2.8	IP alias resolution	38
2.9	ATM cloud	38
3.1	Length of the coast of Britain	44
3.2	Three generations of the Cantor set	46
3.3	IFS for the Sierpiński triangle	52
3.4	Another IFS for the Sierpiński triangle	53
3.5	The L_1 , L_2 , and L_∞ metrics	55
3.6	Illustration of Hausdorff distance	56
4.1	A set that is open, and a set that is not open	63
4.2	Covering of a square by six open sets	64
4.3	Line segment has $d_T = 1$	65
4.4	Topological dimension of a square	65
4.5	Covering a square by 4 sets	66
4.6	Box counting for a line and for a square	67
4.7	3^E boxes cover the set	69
4.8	Starfish box counting	71
4.9	Six iterations of the Hilbert curve	73
4.10	Two generations of the Sierpiński carpet	74
4.11	Randomized Sierpiński carpet	75
4.12	Cantor dust	76

4.13	Minkowski sausage in two dimensions	79
4.14	astrocytes	82
5.1	Self-similarity with $K = 8$ and $\alpha = 1/4$	90
5.2	Three generations of a self-similar fractal	90
5.3	Sierpiński triangle	95
5.4	The Open Set Condition for the Sierpiński gasket	95
5.5	Sinuosity generator $\mathcal{I}_U$ based on the Koch fractal	96
5.6	Topology generator $\mathcal{I}_T$	97
5.7	$\mathcal{I} \equiv \mathcal{I}_U \mathcal{I}_T$ applied to a straight line initiator	97
5.8	IFS for goose down	98
5.9	Packing the diamond region with four balls	100
5.10	Packing a square with long skinny rectangles	100
5.11	Constructing the Lévy curve	104
5.12	The Lévy curve after 12 iterations	105
5.13	Pythagoras tree after 15 iterations	105
6.1	Maximal box size for the Sierpiński carpet	110
6.2	s_{max} is reached when $B(s_{max}) = 1$	110
6.3	Python code for binary search	114
6.4	% error in s_{min} for $N = 1000$ and $\epsilon = 0.001$, for $1 \leq d_B \leq 2$. .	114
6.5	s_{min} for $N = 1000$ and for three values of ϵ , for $1 \leq d_B \leq 2$. .	115
6.6	Coarse-grained and fine-grained images of clover	116
6.7	Spiral (left) and spiral and line (right)	119
6.8	Spiral radius for $J = 4$	120
6.9	Clothoid double spiral	122
7.1	A node 3-covering (i) and an arc 3-covering (ii)	133
7.2	Arc s -coverings for $s = 2, 3, 4$	134
7.3	A node covering which is not an arc covering	134
7.4	An arc covering which is not a node covering	134
7.5	Comparing $B_N(2)$ and $B_A(2)$	135
7.6	Optimal $B_D(3)$ box (left), and optimal $B_D(5)$ and $B_D(7)$ boxes (right)	136
7.7	Diameter-based versus radius-based boxes	137
7.8	Frequent associations among 62 dolphins	138
7.9	Node degrees for the <i>dolphins</i> network	139
7.10	Diameter-based box counting results for the <i>dolphins</i> network .	139
7.11	Chain network	140
7.12	$\log \tilde{B}(s)$ versus $\log s$ using wide boxes	140
7.13	Radius-based box	141
7.14	Covering a 6×6 grid	142
7.15	Fractal versus non-fractal scaling	143
8.1	Example network for node coloring	146
8.2	(i) Auxiliary graph with $s=3$, and (ii) its coloring	146

8.3	Minimal covering for $s = 3$	147
8.4	(i) Auxiliary graph with $s=4$, and (ii) minimal covering for $s=4$	147
8.5	A weighted network	148
8.6	Auxiliary graph for box size $s_1 = 0.85$	149
8.7	Two boxes are required for box size s_1	149
8.8	<i>USAir97</i> network	150
8.9	<i>Random Sequential Node Burning</i>	151
8.10	Covering using <i>Random Sequential Node Burning</i>	152
8.11	<i>Random Sequential Node Burning</i> applied to a hub and spoke network	153
8.12	Double ring hub and spoke network	153
8.13	Network with 7 nodes and 8 arcs	154
8.14	<i>Maximum Excluded Mass Burning</i>	156
8.15	<i>Box Burning</i>	157
8.16	Network for illustrating box burning	158
8.17	Box burning 2-covering and a minimal 2-covering	158
8.18	Disconnected boxes in a minimal covering	159
8.19	<i>Compact Box Burning</i>	160
8.20	Adjacent nodes only connected through a hub	162
8.21	Overlapping boxes	163
8.22	Example network with 7 nodes and 8 arcs	166
8.23	Increasing u_i to 1 blocks other dual variable increases	169
8.24	Results of applying <i>Dual Ascent</i> to <i>Zachary's Karate Club</i> network	169
8.25	Closing nodes in <i>Zachary's Karate Club</i> network	172
8.26	Linear program results for <i>Lada Adamic's social network</i>	174
8.27	BCLP is infeasible for <i>C. elegans</i>	174
10.1	Embedding dimension	202
10.2	Packing of powder particles	207
10.3	Radius of gyration	216
11.1	1-dimensional grid with 8 nodes	224
11.2	1-dimensional grid with $K = 100$	225
11.3	Three examples of a Voronoi tessellation	229
11.4	$\mathbb{N}_k^0(n)$ for $k = 0, 1, 2$	231
11.5	A hub and spoke network	233
11.6	Non-truncated and truncated boxes in a 2-dimensional grid	235
11.7	Neighbors in a 6×6 grid	236
11.8	Correlation dimension of $K \times K$ grid versus K	239
11.9	Correlation dimension of $K \times K \times K$ grid versus $\log_{10} K$	242
11.10	Triangulated hexagon	243
12.1	Constructing $\mathbb{G}_{t+1}$ from $\mathbb{G}_t$	249
12.2	Three generations with $p = 1$	250

12.3	Three generations with $p = 0$	250
12.4	$\mathbb{G}_0$ and $\mathbb{G}_1$ for $x = 2$ and $m = 3$	252
12.5	$\mathbb{G}_0$ and $\mathbb{G}_1$ and $\mathbb{G}_2$ for $\alpha = 0$ and $x = 2$ and $m = 2$	253
12.6	Three generations of a (1,3)-flower	254
12.7	Construction of a conical graph for which $d_V(n^*) = 5/3$	263
13.1	A network (i) and two ways to interconnect K copies (ii)	268
13.2	$\mathbb{G}_0$ has four border nodes	268
13.3	Constructing $\mathbb{G}_2$ from $\mathbb{G}_1$	269
13.4	Generating a network from the Maltese cross fractal	270
13.5	A network satisfying Assumption 13.2 but not Assumption 13.3	272
13.6	$\mathbb{G}_1$ and $\mathbb{G}_2$ for a chain network	273
13.7	Another example of $\mathbb{G}_0$ and $\mathbb{G}_1$	274
13.8	Ω_0 and $\mathbb{G}_0$ for the Maltese cross	274
13.9	Two invalid choices of Ω_0	275
13.10	Example for which $2R_0 + 1 \neq 1/\alpha$	275
13.11	$\mathbb{G}_0$ has no center node	277
13.12	Network generator $\mathbb{G}_0$ (left) and $\mathbb{G}_2$ (right)	278
14.1	Shannon entropy for the distribution $(p, 1 - p)$	282
14.2	Farmer's shades of grey example	290
14.3	H_q versus q for four values of p	298
15.1	Networks (i) and (ii)	308
15.2	Networks (iii) and (iv)	308
15.3	<i>Hub and spoke with a tail</i> network	314
15.4	Coverings for $K = 6$, for $s = 3$ and $s = 5$	314
15.5	Results for the <i>hub and spoke with a tail</i> network	316
15.6	Three hubs connected to a superhub	316
15.7	$\log B(s)$ and $H(s)$ for the <i>hub and spoke with a tail</i> network	317
15.8	Network for which $d_I > d_B$: minimal coverings for $s = 2$ and $s = 3$	318
15.9	Maximal entropy minimal 3-covering	319
15.10	<i>Erdős-Rényi</i> network	320
15.11	Results for the <i>Erdős-Rényi</i> network	321
15.12	Results for the <i>jazz</i> network	322
15.13	Results for <i>Lada Adamic's social network</i>	322
15.14	<i>Communications</i> network	323
15.15	Results for the <i>communications</i> network	323
16.1	Two generations in a multifractal construction	326
16.2	Generation 1 of the multifractal Cantor set	327
16.3	Four generations of the multifractal Cantor set for $p_1 = 0.2$	328
16.4	Logarithm of generation 4 probabilities for $p_1 = 0.2$	329
16.5	Generalized dimensions of the multifractal Cantor set	332
16.6	f_q versus p_1 for the multifractal Cantor set	336

16.7	$f(\alpha)$ spectrum for the multifractal Cantor set	337
16.8	Geometry of the Legendre transform of $f(\alpha)$	344
16.9	The six points in the unit square	349
16.10	Specifying to which box the boundaries belong	350
16.11	Nine points covered by two boxes	354
16.12	Two generations of the Tél-Viscek multifractal	359
17.1	Two minimal 3-coverings and a minimal 2-covering for the chair network	367
17.2	D_∞ for three networks	372
17.3	Two plots of the generalized dimensions of the <i>chair</i> network .	374
17.4	Box counting and $\log Z(x(s), q)$ versus $\log(s/\Delta)$ for the dolphins network	375
17.5	Secant estimate of D_q for the dolphins network for different L and U	376
17.6	$\log R(s)$ versus $\log s$ for the <i>dolphins</i> network	381
17.7	<i>jazz</i> box counting (left) and D_q versus q for various L, U (right)	381
17.8	Local minimum and maximum of D_q for the <i>jazz</i> network and $L = 4, U = 5$	382
17.9	$\log R(s)$ versus $\log s$ for the <i>jazz</i> network	382
17.10	<i>C. elegans</i> box counting (left) and $D_q(L, U)$ versus q for various L, U (right)	382
17.11	Generation-1 (left) and generation-2 (right) rectangles and their probabilities	384
17.12	The second generation probabilities $P_{20}^{(T)}$	385
18.1	Two minimal 3-coverings of a 7 node chain	394
18.2	<i>chair</i> network (left) and d_H calculation (right)	397
18.3	Two 7 node networks with the same d_B but different d_H	397
18.4	Box counting results for $s = 2$ (left) and $s = 3$ (right)	398
18.5	Box of radius $r = 1$ in a 6×6 rectilinear grid	399
18.6	$\log B(s)$ versus $\log s$ for the <i>dolphins</i> network	400
18.7	$B(s)[s/(\Delta + 1)]^d$ versus d for $s \in \mathcal{S}$ (left) and $m(\mathbb{G}, s, d)$ versus d (right) for <i>dolphins</i>	400
18.8	$\sigma(\mathbb{G}, \mathcal{S}, d)$ versus d for <i>dolphins</i> for $d \in [0, 3]$ (left) and $d \in [2.26, 2.40]$ (right)	401
18.9	$\log B(s)$ versus $\log s$ for the <i>USAir97</i> network	402
18.10	$\log B(s)$ versus $\log s$ (left) and $m(\mathbb{G}, s, d)$ versus d (right) for the <i>jazz</i> network	402
18.11	$\log B(s)$ versus $\log s$ (left) and $m(\mathbb{G}, s, d)$ versus d (right) for the <i>C. elegans</i> network	403
18.12	$\sigma(\mathbb{G}, \mathcal{S}, d)$ versus d for <i>C. elegans</i> for $d \in [0, 5]$ (left) and $d \in [2.26, 2.40]$ (right)	404
18.13	$\log B(s)$ versus $\log s$ for $s \in [4, 14]$ for the <i>protein</i> network . .	404
18.14	Summary of d_B and d_H computations	405

18.15	Perturbations of the <i>jazz</i> network	406
18.16	d_B and d_H of a network growing with preferential attachment .	406
18.17	D_q^H versus q for the network $\mathbb{G}_1$ of Fig. 18.3	409
18.18	D_q^H versus q for the <i>dolphins</i> network	409
18.19	D_q^H versus q for the <i>jazz</i> network	410
18.20	D_q^H versus q for the <i>C. elegans</i> network	410
19.1	Cantor dust lacunarity	414
19.2	Lacunarity for a Cantor set	414
19.3	Gliding box method for the top row	416
19.4	Gliding box example continued	416
19.5	Three degrees of lacunarity	417
19.6	Lacunarity with three hierarchy levels	418
19.7	A covering of Ω by balls	420
19.8	Ω is partitioned into 16 identical smaller pieces	424
20.1	Lyapunov numbers	427
20.2	F maps a square to a parallelogram	428
20.3	Cell structure	432
21.1	Kadanoff's block renormalization	441
21.2	Renormalizing the small-world model for $k = 1$	443
21.3	Two-site scaling	446
21.4	Two different minimal 3-coverings of the same network	447
21.5	Procedure <i>Network Renormalization</i>	447
21.6	Network renormalization with diameter-based boxes	448
22.1	Horton's ordering of streams	455
22.2	Metric dimension example	458
22.3	Tetrahedron and cube	460
22.4	Wheel network and complete bicolored graph	460
22.5	Computing S_r^E by summing over cross sections	463
22.6	Two-dimensional triangular lattice $\mathcal{L}$	465
22.7	A finite graph $\mathbb{G}$ embedded in $\mathcal{L}$	466
22.8	1-dimensional (left) and 2-dimensional (right) graphs	466
22.9	A tree whose nodes have degree 1 or 3	469
23.1	Rectifiable Arc	472
23.2	Computing the length of a rectifiable arc	472
23.3	A differentiable convex function lies above each tangent line . .	477
23.4	Two supporting hyperplanes of a convex function	477
23.5	The Legendre transform G of F	478
23.6	The conjugate F^* of F	479
23.7	Small network to illustrate Hall's method	481
23.8	Small hub and spoke network	483