Transition Property for α-Power Free Languages with $\alpha \geq 2$ and $k \geq 3$ Letters

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Abstract. In 1985, Restivo and Salemi presented a list of five problems concerning power free languages. Problem 4 states: Given α-power-free words u and v, decide whether there is a transition from u to v. Problem 5 states: Given α-power-free words u and v, find a transition word w, if it exists.

Let Σ_k denote an alphabet with k letters. Let $L_{k,\alpha}$ denote the α-power free language over the alphabet Σ_k, where α is a rational number or a rational "number with +". If α is a "number with +" then suppose $k \geq 3$ and $\alpha \geq 2$. If α is "only" a number then suppose $k = 3$ and $\alpha > 2$ or $k > 3$ and $\alpha \geq 2$. We show that: If $u \in L_{k,\alpha}$ is a right extendable word in $L_{k,\alpha}$ and $v \in L_{k,\alpha}$ is a left extendable word in $L_{k,\alpha}$ then there is a (transition) word w such that $uwv \in L_{k,\alpha}$. We also show a construction of the word w.

Keywords: Power free languages · Transition property · Dejean's conjecture

1 Introduction

The power free words are one of the major themes in the area of combinatorics on words. An *α-power* of a word r is the word $r^\alpha = rr\dots rt$ such that $\frac{|r^\alpha|}{|r|} = \alpha$ and t is a prefix of r, where $\alpha \geq 1$ is a rational number. For example $(1234)^3 = 123412341234$ and $(1234)^{\frac{7}{4}} = 1234123$. We say that a finite or infinite word w is *α-power free* if w has no factors that are β-powers for $\beta \geq \alpha$ and we say that a finite or infinite word w is α^+-power free if w has no factors that are β-powers for $\beta > \alpha$, where $\alpha, \beta \geq 1$ are rational numbers. In the following, when we write "α-power free" then α denotes a number or a "number with +". The power free words, also called repetitions free words, include well known square free (2-power free), overlap free (2^+-power free), and cube free words (3-power free). Two surveys on the topic of power free words can be found in [8] and [13].

One of the questions being researched is the construction of infinite power free words. We define the *repetition threshold* RT(k) to be the infimum of all rational numbers α such that there exists an infinite α-power-free word over an alphabet with k letters. Dejean's conjecture states that RT(2) = 2, RT(3) = $\frac{7}{4}$,

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$\mathrm{RT}(4) = \frac{7}{5}$, and $\mathrm{RT}(k) = \frac{k}{k-1}$ for each $k > 4$ [3]. Dejean's conjecture has been proved with the aid of several articles [1–3,5,6,9].

It is easy to see that α-power free words form a factorial language [13]; it means that all factors of a α-power free word are also α-power free words. Then Dejean's conjecture implies that there are infinitely many finite α-power free words over Σ_k, where $\alpha > \mathrm{RT}(k)$.

In [10], Restivo and Salemi presented a list of five problems that deal with the question of extendability of power free words. In the current paper we investigate Problem 4 and Problem 5:

– Problem 4: Given α-power-free words u and v, decide whether there is a transition word w, such that uwu is α-power free.
– Problem 5: Given α-power-free words u and v, find a transition word w, if it exists.

A recent survey on the progress of solving all the five problems can be found in [7]; in particular, the problems 4 and 5 are solved for some overlap free (2^+-power free) binary words. In addition, in [7] the authors prove that: For every pair (u, v) of cube free words (3-power free) over an alphabet with k letters, if u can be infinitely extended to the right and v can be infinitely extended to the left respecting the cube-freeness property, then there exists a "transition" word w over the same alphabet such that uwv is cube free.

In 2009, a conjecture related to Problems 4 and Problem 5 of Restivo and Salemi appeared in [12]:

Conjecture 1. [12, Conjecture 1] Let L be a power-free language and let $e(L) \subseteq L$ be the set of words of L that can be extended to a bi-infinite word respecting the given power-freeness. If $u, v \in e(L)$ then $uwv \in e(L)$ for some word w.

In 2018, Conjecture 1 was presented also in [11] in a slightly different form.

Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{Q}$ denote the set of rational numbers.

Definition 1. *Let*

$$\begin{aligned} \Upsilon = &\{(k,\alpha) \mid k \in \mathbb{N} \textit{ and } \alpha \in \mathbb{Q} \textit{ and } k = 3 \textit{ and } \alpha > 2\} \\ &\cup\{(k,\alpha) \mid k \in \mathbb{N} \textit{ and } \alpha \in \mathbb{Q} \textit{ and } k > 3 \textit{ and } \alpha \geq 2\} \\ &\cup\{(k,\alpha^+) \mid k \in \mathbb{N} \textit{ and } \alpha \in \mathbb{Q} \textit{ and } k \geq 3 \textit{ and } \alpha \geq 2\}. \end{aligned}$$

Remark 1. The definition of Υ says that: If $(k, \alpha) \in \Upsilon$ and α is a "number with +" then $k \geq 3$ and $\alpha \geq 2$. If $(k, \alpha) \in \Upsilon$ and α is "just" a number then $k = 3$ and $\alpha > 2$ or $k > 3$ and $\alpha \geq 2$.

Let L be a language. A finite word $w \in \mathrm{L}$ is called *left extendable* (resp., *right extendable*) in L if for every $n \in \mathbb{N}$ there is a word $u \in \mathrm{L}$ with $|u| = n$ such that $uw \in \mathrm{L}$ (resp., $wu \in \mathrm{L}$).

In the current article we improve the results addressing Problems 4 and Problem 5 of Restivo and Salemi from [7] as follows. Let Σ_k denote an alphabet with k letters. Let $\mathrm{L}_{k,\alpha}$ denote the α-power free language over the alphabet Σ_k. We show that if $(k,\alpha) \in \Upsilon$, $u \in \mathrm{L}_{k,\alpha}$ is a right extendable word in $\mathrm{L}_{k,\alpha}$, and $v \in \mathrm{L}_{k,\alpha}$ is a left extendable word in $\mathrm{L}_{k,\alpha}$ then there is a word w such that $uwv \in \mathrm{L}_{k,\alpha}$. We also show a construction of the word w.

We sketch briefly our construction of a "transition" word. Let u be a right extendable α-power free word and let v be a left extendable α-power free word over Σ_k with $k > 2$ letters. Let $\bar{u}$ be a right infinite α-power free word having u as a prefix and let $\bar{v}$ be a left infinite α-power free word having v as a suffix. Let x be a letter that is recurrent in both $\bar{u}$ and $\bar{v}$. We show that we may suppose that $\bar{u}$ and $\bar{v}$ have a common recurrent letter. Let t be a right infinite α-power free word over $\Sigma_k \setminus \{x\}$. Let $\bar{t}$ be a left infinite α-power free word such that the set of factors of $\bar{t}$ is a subset of the set of recurrent factors of t. We show that such $\bar{t}$ exists. We identify a prefix $\tilde{u}xg$ of $\bar{u}$ such that g is a prefix of t and $\tilde{u}xt$ is a right infinite α-power free word. Analogously we identify a suffix $\bar{g}x\tilde{v}$ of $\bar{v}$ such that $\bar{g}$ is a suffix of $\bar{t}$ and $\bar{t}x\tilde{v}$ is a left infinite α-power free word. Moreover our construction guarantees that u is a prefix of $\tilde{u}xt$ and v is a suffix of $\bar{t}x\tilde{v}$. Then we find a prefix hp of t such that $px\tilde{v}$ is a suffix of $\bar{t}x\tilde{v}$ and such that both h and p are "sufficiently long". Then we show that $\tilde{u}xhpx\tilde{v}$ is an α-power free word having u as a prefix and v as a suffix.

The very basic idea of our proof is that if u, v are α-power free words and x is a letter such that x is not a factor of both u and v, then clearly uxv is α-power free on condition that $\alpha \geq 2$. Just note that there cannot be a factor in uxv which is an α-power and contains x, because x has only one occurrence in uxv. Our constructed words $\tilde{u}xt$, $\bar{t}x\tilde{v}$, and $\tilde{u}xhpx\tilde{v}$ have "long" factors which does not contain a letter x. This will allow us to apply a similar approach to show that the constructed words do not contain square factor rr such that r contains the letter x.

Another key observation is that if $k \geq 3$ and $\alpha > \mathrm{RT}(k-1)$ then there is an infinite α-power free word $\bar{w}$ over $\Sigma_k \setminus \{x\}$, where $x \in \Sigma_k$. This is an implication of Dejean's conjecture. Less formally said, if u, v are α-power free words over an alphabet with k letters, then we construct a "transition" word w over an alphabet with $k-1$ letters such that uwv is α-power free.

Dejean's conjecture imposes also the limit to possible improvement of our construction. The construction cannot be used for $\mathrm{RT}(k) \leq \alpha < \mathrm{RT}(k-1)$, where $k \geq 3$, because every infinite (or "sufficiently long") word w over an alphabet with $k-1$ letters contains a factor which is an α-power. Also for $k = 2$ and $\alpha \geq 1$ our technique fails. On the other hand, based on our research, it seems that our technique, with some adjustments, could be applied also for $\mathrm{RT}(k-1) \leq \alpha \leq 2$ and $k \geq 3$. Moreover it seems to be possible to generalize our technique to bi-infinite words and consequently to prove Conjecture 1 for $k \geq 3$ and $\alpha \geq \mathrm{RT}(k-1)$.

2 Preliminaries

Recall that Σ_k denotes an alphabet with k letters. Let ϵ denote the empty word. Let Σ_k^* denote the set of all finite words over Σ_k including the empty word ϵ, let

$\Sigma_k^{\mathbb{N},R}$ denote the set of all right infinite words over Σ_k, and let $\Sigma_k^{\mathbb{N},L}$ denote the set of all left infinite words over Σ_k. Let $\Sigma_k^{\mathbb{N}} = \Sigma_k^{\mathbb{N},L} \cup \Sigma_k^{\mathbb{N},R}$. We call $w \in \Sigma_k^{\mathbb{N}}$ an infinite word.

Let $\operatorname{occur}(w,t)$ denote the number of occurrences of the nonempty factor $t \in \Sigma_k^* \setminus \{\epsilon\}$ in the word $w \in \Sigma_k^* \cup \Sigma_k^{\mathbb{N}}$. If $w \in \Sigma_k^{\mathbb{N}}$ and $\operatorname{occur}(w,t) = \infty$, then we call t a *recurrent* factor in w.

Let $\mathrm{F}(w)$ denote the set of all finite factors of a finite or infinite word $w \in \Sigma_k^* \cup \Sigma_k^{\mathbb{N}}$. The set $\mathrm{F}(w)$ contains the empty word and if w is finite then also $w \in \mathrm{F}(w)$. Let $\mathrm{F}_r(w) \subseteq \mathrm{F}(w)$ denote the set of all recurrent nonempty factors of $w \in \Sigma_k^{\mathbb{N}}$.

Let $\operatorname{Prf}(w) \subseteq \mathrm{F}(w)$ denote the set of all prefixes of $w \in \Sigma_k^* \cup \Sigma_k^{\mathbb{N},R}$ and let $\operatorname{Suf}(w) \subseteq \mathrm{F}(w)$ denote the set of all suffixes of $w \in \Sigma_k^* \cup \Sigma_k^{\mathbb{N},L}$. We define that $\epsilon \in \operatorname{Prf}(w) \cap \operatorname{Suf}(w)$ and if w is finite then also $w \in \operatorname{Prf}(w) \cap \operatorname{Suf}(w)$.

We have that $\mathrm{L}_{k,\alpha} \subseteq \Sigma_k^*$. Let $\mathrm{L}_{k,\alpha}^{\mathbb{N}} \subseteq \Sigma_k^{\mathbb{N}}$ denote the set of all infinite α-power free words over Σ_k. Obviously $\mathrm{L}_{k,\alpha}^{\mathbb{N}} = \{w \in \Sigma_k^{\mathbb{N}} \mid \mathrm{F}(w) \subseteq \mathrm{L}_{k,\alpha}\}$. In addition we define $\mathrm{L}_{k,\alpha}^{\mathbb{N},R} = \mathrm{L}_{k,\alpha}^{\mathbb{N}} \cap \Sigma_k^{\mathbb{N},R}$ and $\mathrm{L}_{k,\alpha}^{\mathbb{N},L} = \mathrm{L}_{k,\alpha}^{\mathbb{N}} \cap \Sigma_k^{\mathbb{N},L}$; it means the sets of right infinite and left infinite α-power free words.

3 Power Free Languages

Let $(k,\alpha) \in \Upsilon$ and let u, v be α-power free words. The first lemma says that uv is α-power free if there are no word r and no nonempty prefix $\bar{v}$ of v such that rr is a suffix of $u\bar{v}$ and rr is longer than $\bar{v}$.

Lemma 1. *Suppose* $(k,\alpha) \in \Upsilon$, $u \in \mathrm{L}_{k,\alpha}$, *and* $v \in \mathrm{L}_{k,\alpha} \cup \mathrm{L}_{k,\alpha}^{\mathbb{N},R}$. *Let*

$$\Pi = \{(r,\bar{v}) \mid r \in \Sigma_k^* \setminus \{\epsilon\} \text{ and } \bar{v} \in \operatorname{Prf}(v) \setminus \{\epsilon\} \text{ and } rr \in \operatorname{Suf}(u\bar{v}) \text{ and } |rr| > |\bar{v}|\}.$$

If $\Pi = \emptyset$ *then* $uv \in \mathrm{L}_{k,\alpha} \cup \mathrm{L}_{k,\alpha}^{\mathbb{N},R}$.

Proof. Suppose that uv is not α-power free. Since u is α-power free, then there are $t \in \Sigma_k^*$ and $x \in \Sigma_k$ such that $tx \in \operatorname{Prf}(v)$, $ut \in \mathrm{L}_{k,\alpha}$ and $utx \notin \mathrm{L}_{k,\alpha}$. It means that there is $r \in \operatorname{Suf}(utx)$ such that $r^\beta \in \operatorname{Suf}(utx)$ for some $\beta \geq \alpha$ or $\beta > \alpha$ if α is a "number with +"; recall Definition 1 of Υ. Because $\alpha \geq 2$, this implies that $rr \in \operatorname{Suf}(r^\beta)$. If follows that $(tx, r) \in \Pi$. We proved that $uv \notin \mathrm{L}_{k,\alpha} \cup \mathrm{L}_{k,\alpha}^{\mathbb{N},R}$ implies that $\Pi \neq \emptyset$. The lemma follows. $\square$

The following technical set $\Gamma(k,\alpha)$ of 5-tuples (w_1, w_2, x, g, t) will simplify our propositions.

Definition 2. *Given* $(k,\alpha) \in \Upsilon$, *we define that* $(w_1, w_2, x, g, t) \in \Gamma(k,\alpha)$ *if*

1. $w_1, w_2, g \in \Sigma_k^*$,
2. $x \in \Sigma_k$,
3. $w_1 w_2 x g \in \mathrm{L}_{k,\alpha}$,

4. $t \in \mathrm{L}^{\mathbb{N},R}_{k,\alpha}$*,*
5. $\mathrm{occur}(t, x) = 0$*,*
6. $g \in \mathrm{Prf}(t)$*,*
7. $\mathrm{occur}(w_2xgy, xgy) = 1$*, where* $y \in \Sigma_k$ *is such that* $gy \in \mathrm{Prf}(t)$*, and*
8. $\mathrm{occur}(w_2, x) \geq \mathrm{occur}(w_1, x)$*.*

Remark 2. Less formally said, the 5-tuple (w_1, w_2, x, g, t) is in $\Gamma(k, \alpha)$ if w_1w_2xg is α-power free word over Σ_k, t is a right infinite α-power free word over Σ_k, t has no occurrence of x (thus t is a word over $\Sigma_k \setminus \{x\}$), g is a prefix of t, xgy has only one occurrence in w_2xgy, where y is a letter such that gy is a prefix of t, and the number of occurrences of x in w_2 is bigger than the number of occurrences of x in w_1, where w_1, w_2, g are finite words and x is a letter.

The next proposition shows that if (w_1, w_2, x, g, t) is from the set $\Gamma(k, \alpha)$ then w_1w_2xt is a right infinite α-power free word, where (k, α) is from the set Υ.

Proposition 1. *If* $(k, \alpha) \in \Upsilon$ *and* $(w_1, w_2, x, g, t) \in \Gamma(k, \alpha)$ *then* $w_1w_2xt \in \mathrm{L}^{\mathbb{N},R}_{k,\alpha}$*.*

Proof. Lemma 1 implies that it suffices to show that there are no $u \in \mathrm{Prf}(t)$ with $|u| > |g|$ and no $r \in \Sigma_k^* \setminus \{\epsilon\}$ such that $rr \in \mathrm{Suf}(w_1w_2xu)$ and $|rr| > |u|$. Recall that w_1w_2xg is an α-power free word, hence we consider $|u| > |g|$. To get a contradiction, suppose that such r, u exist. We distinguish the following distinct cases.

- If $|r| \leq |u|$ then: Since $u \in \mathrm{Prf}(t) \subseteq \mathrm{L}_{k,\alpha}$ it follows that $xu \in \mathrm{Suf}(r^2)$ and hence $x \in \mathrm{F}(r^2)$. It is clear that $\mathrm{occur}(r^2, x) \geq 1$ if and only if $\mathrm{occur}(r, x) \geq 1$. Since $x \notin \mathrm{F}(u)$ and thus $x \notin \mathrm{F}(r)$, this is a contradiction.
- If $|r| > |u|$ and $rr \in \mathrm{Suf}(w_2xu)$ then: Let $y \in \Sigma_k$ be such that $gy \in \mathrm{Prf}(t)$. Since $|u| > |g|$ we have that $gy \in \mathrm{Prf}(u)$ and $xgy \in \mathrm{Prf}(xu)$. Since $|r| > |u|$ we have that $xgy \in \mathrm{F}(r)$. In consequence $\mathrm{occur}(rr, xgy) \geq 2$. But Property 7 of Definition 2 states that $\mathrm{occur}(w_2xgy, xgy) = 1$. Since $rr \in \mathrm{Suf}(w_2xu)$, this is a contradiction.
- If $|r| > |u|$ and $rr \notin \mathrm{Suf}(w_2xu)$ and $r \in \mathrm{Suf}(w_2xu)$ then:
  Let $w_{11}, w_{12}, w_{13}, w_{21}, w_{22} \in \Sigma_k^*$ be such that $w_1 = w_{11}w_{12}w_{13}$, $w_2 = w_{21}w_{22}$, $w_{12}w_{13}w_{21} = r$, $w_{12}w_{13}w_2xu = rr$, and $w_{13}w_{21} = xu$; see Figure below.

<table>
<tr><td></td><td></td><td colspan="2">xu</td><td></td><td></td><td></td></tr>
<tr><td>w_{11}</td><td>w_{12}</td><td>w_{13}</td><td>w_{21}</td><td>w_{22}</td><td>x</td><td>u</td></tr>
<tr><td></td><td colspan="3">r</td><td colspan="3">r</td></tr>
</table>

  It follows that $w_{22}xu = r$ and $w_{22} = w_{12}$. It is easy to see that $w_{13}w_{21} = xu$. From $\mathrm{occur}(u, x) = 0$ we have that $\mathrm{occur}(w_2, x) = \mathrm{occur}(w_{22}, x)$ and $\mathrm{occur}(w_{13}, x) = 1$. From $w_{22} = w_{12}$ it follows that $\mathrm{occur}(w_1, x) > \mathrm{occur}(w_2, x)$. This is a contradiction to Property 8 of Definition 2.

– If $|r| > |u|$ and $rr \notin \mathrm{Suf}(w_2xu)$ and $r \notin \mathrm{Suf}(w_2xu)$ then: Let $w_{11}, w_{12}, w_{13} \in \Sigma_k^*$ be such that $w_1 = w_{11}w_{12}w_{13}$, $w_{12} = r$ and $w_{13}w_2xu = r$; see Figure below.

w_{11}	w_{12}	w_{13}	w_2	x	u
	r	r			

It follows that

$$\mathrm{occur}(w_{12}, x) = \mathrm{occur}(w_{13}, x) + \mathrm{occur}(w_2, x) + \mathrm{occur}(xu, x).$$

This is a contradiction to Property 8 of Definition 2.

We proved that the assumption of existence of r, u leads to a contradiction. Thus we proved that for each prefix $u \in \mathrm{Prf}(t)$ we have that $w_1w_2xu \in \mathrm{L}_{k,\alpha}$. The proposition follows. □

We prove that if $(k, \alpha) \in \Upsilon$ then there is a right infinite α-power free word over Σ_{k-1}. In the introduction we showed that this observation could be deduced from Dejean's conjecture. Here additionally, to be able to address Problem 5 from the list of Restivo and Salemi, we present in the proof also examples of such words.

Lemma 2. *If* $(k, \alpha) \in \Upsilon$ *then the set* $\mathrm{L}_{k-1,\alpha}^{\mathbb{N},R}$ *is not empty.*

Proof. If $k = 3$ then $|\Sigma_{k-1}| = 2$. It is well known that the Thue Morse word is a right infinite 2^+-power free word over an alphabet with 2 letters [11]. It follows that the Thue Morse word is α-power free for each $\alpha > 2$.

If $k > 3$ then $|\Sigma_{k-1}| \geq 3$. It is well known that there are infinite 2-power free words over an alphabet with 3 letters [11]. Suppose $0, 1, 2 \in \Sigma_k$. An example is the fixed point of the morphism θ defined by $\theta(0) = 012$, $\theta(1) = 02$, and $\theta(2) = 1$ [11]. If an infinite word t is 2-power free then obviously t is α-power free and α^+-power free for each $\alpha \geq 2$.

This completes the proof. □

We define the sets of extendable words.

Definition 3. *Let* $\mathrm{L} \subseteq \Sigma_k^*$. *We define*

$$\mathrm{lext}(\mathrm{L}) = \{w \in \mathrm{L} \mid w \textit{ is left extendable in } \mathrm{L}\}$$

and

$$\mathrm{rext}(\mathrm{L}) = \{w \in \mathrm{L} \mid w \textit{ is right extendable in } \mathrm{L}\}.$$

If $u \in \mathrm{lext}(\mathrm{L})$ *then let* $\mathrm{lext}(u, \mathrm{L})$ *be the set of all left infinite words* $\bar{u}$ *such that* $\mathrm{Suf}(\bar{u}) \subseteq \mathrm{L}$ *and* $u \in \mathrm{Suf}(\bar{u})$. *Analogously if* $u \in \mathrm{rext}(\mathrm{L})$ *then let* $\mathrm{rext}(u, \mathrm{L})$ *be the set of all right infinite words* $\bar{u}$ *such that* $\mathrm{Prf}(\bar{u}) \subseteq \mathrm{L}$ *and* $u \in \mathrm{Prf}(\bar{u})$.

We show the sets $\mathrm{lext}(u, \mathrm{L})$ and $\mathrm{rext}(v, \mathrm{L})$ are nonempty for left extendable and right extendable words.

Lemma 3. *If* $\mathrm{L} \subseteq \Sigma_k^*$ *and* $u \in \mathrm{lext}(\mathrm{L})$ *(resp.,* $v \in \mathrm{rext}(\mathrm{L})$*) then* $\mathrm{lext}(u, \mathrm{L}) \neq \emptyset$ *(resp.,* $\mathrm{rext}(v, \mathrm{L}) \neq \emptyset$*).*

Proof. Realize that $u \in \mathrm{lext}(\mathrm{L})$ (resp., $v \in \mathrm{rext}(\mathrm{L})$) implies that there are infinitely many finite words in L having u as a suffix (resp., v as a prefix). Then the lemma follows from König's Infinity Lemma [4,8]. □

The next proposition proves that if $(k, \alpha) \in \Upsilon$, w is a right extendable α-power free word, $\bar{w}$ is a right infinite α-power free word having the letter x as a recurrent factor and having w as a prefix, and t is a right infinite α-power free word over $\Sigma_k \setminus \{x\}$, then there are finite words w_1, w_2, g such that the 5-tuple (w_1, w_2, x, g, t) is in the set $\Gamma(k, \alpha)$ and w is a prefix of $w_1 w_2 x g$.

Proposition 2. *If* $(k, \alpha) \in \Upsilon$*,* $w \in \mathrm{rext}(\mathrm{L}_{k,\alpha})$*,* $\bar{w} \in \mathrm{rext}(w, \mathrm{L}_{k,\alpha})$*,* $x \in \mathrm{F}_r(\bar{w}) \cap \Sigma_k$*,* $t \in \mathrm{L}_{k,\alpha}^{\mathbb{N},R}$*, and* $\mathrm{occur}(t, x) = 0$ *then there are finite words* w_1, w_2, g *such that* $(w_1, w_2, x, g, t) \in \Gamma(k, \alpha)$ *and* $w \in \mathrm{Prf}(w_1 w_2 x g)$*.*

Proof. Let $\omega = \mathrm{F}(\bar{w}) \cap \mathrm{Prf}(xt)$ be the set of factors of $\bar{w}$ that are also prefixes of the word xt. Based on the size of the set ω we construct the words w_1, w_2, g and we show that $(w_1, w_2, x, g, t) \in \Gamma(k, \alpha)$ and $w_1 w_2 x g \in \mathrm{Prf}(\bar{w}) \subseteq \mathrm{L}_{k,\alpha}$. The Properties 1, 2, 3, 4, 5, and 6 of Definition 2 are easy to verify. Hence we explicitly prove only properties 7 and 8 and that $w \in \mathrm{Prf}(w_1 w_2 x g)$.

– If ω is an infinite set. It follows that $\mathrm{Prf}(xt) = \omega$. Let $g \in \mathrm{Prf}(t)$ be such that $|g| = |w|$; recall that t is infinite and hence such g exists. Let $w_2 \in \mathrm{Prf}(\bar{w})$ be such that $w_2 x g \in \mathrm{Prf}(\bar{w})$ and $\mathrm{occur}(w_2 x g, x g) = 1$. Let $w_1 = \epsilon$.
  Property 7 of Definition 2 follows from $\mathrm{occur}(w_2 x g, x g) = 1$. Property 8 of Definition 2 is obvious, because w_1 is the empty word. Since $|g| = |w|$ and $w \in \mathrm{Prf}(\bar{w})$ we have that $w \in \mathrm{Prf}(w_1 w_2 x g)$.
– If ω is a finite set. Let $\bar{\omega} = \omega \cap \mathrm{F}_r(\bar{w})$ be the set of prefixes of xt that are recurrent in $\bar{w}$. Since x is recurrent in $\bar{w}$ we have that $x \in \bar{\omega}$ and thus $\bar{\omega}$ is not empty. Let $g \in \mathrm{Prf}(t)$ be such that xg is the longest element in $\bar{\omega}$. Let $w_1 \in \mathrm{Prf}(w)$ be the shortest prefix of $\bar{w}$ such that if $u \in \omega \setminus \bar{\omega}$ is a non-recurrent prefix of xt in $\bar{w}$ then $\mathrm{occur}(w_1, u) = \mathrm{occur}(\bar{w}, u)$. Such w_1 obviously exists, because ω is a finite set and non-recurrent factors have only a finite number of occurrences. Let w_2 be the shortest factor of $\bar{w}$ such that $w_1 w_2 x g \in \mathrm{Prf}(\bar{w})$, $\mathrm{occur}(w_1, x) < \mathrm{occur}(w_2, x)$, and $w \in \mathrm{Prf}(w_1 w_2 x g)$. Since xg is recurrent in $\bar{w}$ and $w \in \mathrm{Prf}(\bar{w})$ it is clear such w_2 exists.
  We show that Property 7 of Definition 2 holds. Let $y \in \Sigma_k$ be such that $gy \in \mathrm{Prf}(t)$. Suppose that $\mathrm{occur}(w_2 x g, x g y) > 0$. It would imply that xgy is recurrent in $\bar{w}$, since all occurrences of non-recurrent words from ω are in w_1. But we defined xg to be the longest recurrent word ω. Hence it is contradiction to our assumption that $\mathrm{occur}(w_2 x g, x g y) > 0$.
  Property 8 of Definition 2 and $w \in \mathrm{Prf}(w_1 w_2 x g)$ are obvious from the construction of w_2.

This completes the proof. $\square$

We define the *reversal* w^R of a finite or infinite word $w = \Sigma_k^* \cup \Sigma_k^{\mathbb{N}}$ as follows: If $w \in \Sigma_k^*$ and $w = w_1w_2\dots w_m$, where $w_i \in \Sigma_k$ and $1 \leq i \leq m$, then $w^R = w_mw_{m-1}\dots w_2w_1$. If $w \in \Sigma_k^{\mathbb{N},L}$ and $w = \dots w_2w_1$, where $w_i \in \Sigma_k$ and $i \in \mathbb{N}$, then $w^R = w_1w_2\dots \in \Sigma_k^{\mathbb{N},R}$. Analogously if $w \in \Sigma_k^{\mathbb{N},R}$ and $w = w_1w_2\dots$, where $w_i \in \Sigma_k$ and $i \in \mathbb{N}$, then $w^R = \dots w_2w_1 \in \Sigma_k^{\mathbb{N},L}$.

Proposition 1 allows one to construct a right infinite α-power free word with a given prefix. The next simple corollary shows that in the same way we can construct a left infinite α-power free word with a given suffix.

Corollary 1. *If $(k,\alpha) \in \Upsilon$, $w \in \mathrm{lext}(\mathrm{L}_{k,\alpha})$, $\bar{w} \in \mathrm{lext}(w, \mathrm{L}_{k,\alpha})$, $x \in \mathrm{F}_r(\bar{w}) \cap \Sigma_k$, $t \in \mathrm{L}_{k,\alpha}^{\mathbb{N},L}$, and $\mathrm{occur}(t,x) = 0$ then there are finite words w_1, w_2, g such that $(w_1^R, w_2^R, x, g^R, t^R) \in \Gamma(k,\alpha)$, $w \in \mathrm{Suf}(gxw_2w_1)$, and $txw_2w_1 \in \mathrm{L}_{k,\alpha}^{\mathbb{N},L}$.*

Proof. Let $u \in \Sigma_k^* \cup \Sigma_k^{\mathbb{N}}$. Realize that $u \in \mathrm{L}_{k,\alpha} \cup \mathrm{L}_{k,\alpha}^{\mathbb{N}}$ if and only if $u^R \in \mathrm{L}_{k,\alpha} \cup \mathrm{L}_{k,\alpha}^{\mathbb{N}}$. Then the corollary follows from Proposition 1 and Proposition 2. $\square$

Given $k \in \mathbb{N}$ and a right infinite word $t \in \Sigma_k^{\mathbb{N},R}$, let $\Phi(t)$ be the set of all left infinite words $\bar{t} \in \Sigma_k^{\mathbb{N},L}$ such that $\mathrm{F}(\bar{t}) \subseteq \mathrm{F}_r(t)$. It means that all factors of $\bar{t} \in \Phi(t)$ are recurrent factors of t. We show that the set $\Phi(t)$ is not empty.

Lemma 4. *If $k \in \mathbb{N}$ and $t \in \Sigma_k^{\mathbb{N},R}$ then $\Phi(t) \neq \emptyset$.*

Proof. Since t is an infinite word, the set of recurrent factors of t is not empty. Let g be a recurrent nonempty factor of t; g may be a letter. Obviously there is $x \in \Sigma_k$ such that xg is also recurrent in t. This implies that the set $\{h \mid hg \in \mathrm{F}_r(t)\}$ is infinite. The lemma follows from König's Infinity Lemma [4,8]. $\square$

The next lemma shows that if u is a right extendable α-power free word then for each letter x there is a right infinite α-power free word $\bar{u}$ such that x is recurrent in $\bar{u}$ and u is a prefix of $\bar{u}$.

Lemma 5. *If $(k,\alpha) \in \Upsilon$, $u \in \mathrm{rext}(\mathrm{L}_{k,\alpha})$, and $x \in \Sigma_k$ then there is $\bar{u} \in \mathrm{rext}(u, \mathrm{L}_{k,\alpha})$ such that $x \in \mathrm{F}_r(\bar{u})$.*

Proof. Let $w \in \mathrm{rext}(u, \mathrm{L}_{k,\alpha})$; Lemma 3 implies that $\mathrm{rext}(u, \mathrm{L}_{k,\alpha})$ is not empty. If $x \in \mathrm{F}_r(w)$ then we are done. Suppose that $x \notin \mathrm{F}_r(w)$. Let $y \in \mathrm{F}_r(w) \cap \Sigma_k$. Clearly $x \neq y$. Proposition 2 implies that there is $(w_1, w_2, y, g, t) \in \Gamma(k,\alpha)$ such that $u \in \mathrm{Prf}(w_1w_2yg)$. The proof of Lemma 2 implies that we can choose t in such a way that x is recurrent in t. Then $w_1w_2yt \in \mathrm{rext}(u, \mathrm{L}_{k,\alpha})$ and $x \in \mathrm{F}_r(w_1w_2yt)$. This completes the proof. $\square$

The next proposition shows that if u is left extendable and v is right extendable then there are finite words $\tilde{u}, \tilde{v}$, a letter x, a right infinite word t, and a left infinite word $\bar{t}$ such that $\tilde{u}xt, \bar{t}x\tilde{v}$ are infinite α-power free words, t has no occurrence of x, every factor of $\bar{t}$ is a recurrent factor in t, u is a prefix of $\tilde{u}xt$, and v is a suffix of $\bar{t}x\tilde{v}$.

Proposition 3. *If* $(k,\alpha)\in\Upsilon$, $u\in\operatorname{rext}(\mathrm{L}_{k,\alpha})$, *and* $v\in\operatorname{lext}(\mathrm{L}_{k,\alpha})$ *then there are* $\tilde{u},\tilde{v}\in\Sigma_k^*$, $x\in\Sigma_k$, $t\in\Sigma_k^{\mathbb{N},R}$, *and* $\bar{t}\in\Sigma_k^{\mathbb{N},L}$ *such that* $\tilde{u}xt\in\mathrm{L}_{k,\alpha}^{\mathbb{N},R}$, $\bar{t}x\tilde{v}\in\mathrm{L}_{k,\alpha}^{\mathbb{N},L}$, $\operatorname{occur}(t,x)=0$, $\mathrm{F}(\bar{t})\subseteq\mathrm{F}_r(t)$, $u\in\operatorname{Prf}(\tilde{u}xt)$, *and* $v\in\operatorname{Suf}(\bar{t}x\tilde{v})$.

Proof. Let $\bar{u}\in\operatorname{rext}(u,\mathrm{L}_{k,\alpha})$ and $\bar{v}\in\operatorname{lext}(v,\mathrm{L}_{k,\alpha})$ be such that $\mathrm{F}_r(\bar{u})\cap\mathrm{F}_r(\bar{v})\cap\Sigma_k\neq\emptyset$. Lemma 5 implies that such $\bar{u},\bar{v}$ exist. Let $x\in\mathrm{F}_r(\bar{u})\cap\mathrm{F}_r(\bar{v})\cap\Sigma_k$. It means that the letter x is recurrent in both $\bar{u}$ and $\bar{v}$.

Let t be a right infinite α-power free word over $\Sigma_k\setminus\{x\}$. Lemma 2 asserts that such t exists. Let $\bar{t}\in\Phi(t)$; Lemma 4 shows that $\Phi(t)\neq\emptyset$. It is easy to see that $\bar{t}\in\mathrm{L}_{k,\alpha}^{\mathbb{N},L}$, because $\mathrm{F}(\bar{t})\subseteq\mathrm{F}_r(t)$ and $t\in\mathrm{L}_{k,\alpha}^{\mathbb{N},R}$.

Proposition 2 and Corollary 1 imply that there are $u_1,u_2,g,v_1,v_2,\bar{g}\in\mathrm{L}_{k,\alpha}$ such that

- $(u_1,u_2,x,g,t)\in\Gamma(k,\alpha)$,
- $(v_1^R,v_2^R,x,\bar{g}^R,\bar{t}^R)\in\Gamma(k,\alpha)$,
- $u\in\operatorname{Prf}(u_1u_2xg)$, and
- $v^R\in\operatorname{Prf}(v_1^Rv_2^Rx\bar{g}^R)$; it follows that $v\in\operatorname{Suf}(\bar{g}xv_2v_1)$.

Proposition 1 implies that $u_1u_2xt, v_1^Rv_2^Rx\bar{t}^R\in\mathrm{L}_{k,\alpha}^{\mathbb{N},R}$. It follows that $\bar{t}xv_2v_1\in\mathrm{L}_{k,\alpha}^{\mathbb{N},L}$. Let $\tilde{u}=u_1u_2$ and $\tilde{v}=v_2v_1$. This completes the proof. □

The main theorem of the article shows that if u is a right extendable α-power free word and v is a left extendable α-power free word then there is a word w such that uwv is α-power free. The proof of the theorem shows also a construction of the word w.

Theorem 1. *If* $(k,\alpha)\in\Upsilon$, $u\in\operatorname{rext}(\mathrm{L}_{k,\alpha})$, *and* $v\in\operatorname{lext}(\mathrm{L}_{k,\alpha})$ *then there is* $w\in\mathrm{L}_{k,\alpha}$ *such that* $uwv\in\mathrm{L}_{k,\alpha}$.

Proof. Let $\tilde{u},\tilde{v},x,t,\bar{t}$ be as in Proposition 3. Let $p\in\operatorname{Suf}(\bar{t})$ be the shortest suffix such that $|p|>\max\{|\tilde{u}x|,|x\tilde{v}|,|u|,|v|\}$. Let $h\in\operatorname{Prf}(t)$ be the shortest prefix such that $hp\in\operatorname{Prf}(t)$ and $|h|>|p|$; such h exists, because p is a recurrent factor of t; see Proposition 3. We show that $\tilde{u}xhpx\tilde{v}\in\mathrm{L}_{k,\alpha}$.

We have that $\tilde{u}xhp\in\mathrm{L}_{k,\alpha}$, since $hp\in\operatorname{Prf}(t)$ and Proposition 3 states that $\tilde{u}xt\in\mathrm{L}_{k,\alpha}^{\mathbb{N},R}$. Lemma 1 implies that it suffices to show that there are no $g\in\operatorname{Prf}(\tilde{v})$ and no $r\in\Sigma_k^*\setminus\{\epsilon\}$ such that $rr\in\operatorname{Suf}(\tilde{u}xhpxg)$ and $|rr|>|xg|$. To get a contradiction, suppose there are such r,g. We distinguish the following cases.

- If $|r|\leq|xg|$ then $rr\in\operatorname{Suf}(pxg)$, because $|p|>|x\tilde{v}|$ and $xg\in\operatorname{Prf}(x\tilde{v})$. This is a contradiction, since $px\tilde{v}\in\operatorname{Suf}(\bar{t}x\tilde{v})$ and $\bar{t}x\tilde{v}\in\mathrm{L}_{k,\alpha}^{\mathbb{N},L}$; see Proposition 3.
- If $|r|>|xg|$ then $|r|\leq\frac{1}{2}|\tilde{u}xhpxg|$, otherwise rr cannot be a suffix of $\tilde{u}xhpxg$. Because $|h|>|p|>\max\{|\tilde{u}x|,|x\tilde{v}|\}$ we have that $r\in\operatorname{Suf}(hpxg)$. Since $\operatorname{occur}(hp,x)=0$, $|h|>|p|>|x\tilde{v}|$, and $xg\in\operatorname{Suf}(r)$ it follows that there are words h_1,h_2 such that $\tilde{u}xhpxg=\tilde{u}xh_1h_2pxg$, $r=h_2pxg$ and $r\in\operatorname{Suf}(\tilde{u}xh_1)$. It follows that $xg\in\operatorname{Suf}(\tilde{u}xh_1)$ and because $\operatorname{occur}(h_1,x)=0$ we have that $|h_1|\leq|g|$. Since $|p|>|\tilde{u}x|$ we get that $|h_2pxg|>|\tilde{u}xg|\geq|\tilde{u}xh_1|$; hence $|r|>|\tilde{u}xh_1|$. This is a contradiction.

We conclude that there is no word r and no prefix $g \in \mathrm{Prf}(\tilde{v})$ such that $rr \in \mathrm{Suf}(\tilde{u}xhpxg)$. Hence $\tilde{u}xhpx\tilde{v} \in \mathrm{L}_{k,\alpha}$. Due to the construction of p and h we have that $u \in \mathrm{Prf}(\tilde{u}xhpx\tilde{v})$ and $v \in \mathrm{Suf}(\tilde{u}xhpx\tilde{v})$. This completes the proof. $\square$

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