Connectivity Keeping Trees in 2-Connected Graphs with Girth Conditions

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Abstract. Mader conjectured in 2010 that for any tree T of order m, every k-connected graph G with minimum degree at least $\lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is k-connected. This conjecture has been proved for $k = 1$; however, it remains open for general $k \geq 2$; for $k = 2$, partially affirmative answers have been shown, all of which restrict the class of trees to special subclasses such as trees of order at most 8, trees with diameter at most 4, trees with at most 5 internal vertices, and caterpillars. Instead of restricting the class of trees, we consider 2-connected graphs with girth conditions. We then show that Mader's conjecture is true for every 2-connected graph G with $g(G) \geq \delta(G) - 6$, where $g(G)$ and $\delta(G)$ denote the girth of G and the minimum degree of a vertex in G, respectively. Besides, we show that for every 2-connected graph G with $g(G) \geq \delta(G) - 3$, the lower bound of $m + 2$ on $\delta(G)$ in Mader's conjecture can be improved to $m + 1$ if $m \geq 6$. Moreover, the lower bound of $\delta(G) - 6$ (respectively, $\delta(G) - 3$) on $g(G)$ in these results can be improved to $\delta(G) - 7$ (respectively, $\delta(G) - 4$ with $m \geq 7$) if no six (respectively, four) cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in G. Mader's conjecture is interesting not only from a theoretical point of view but also from a practical point of view, since it may be applied to fault-tolerant problems in communication networks. Our proofs lead to $O(|V(G)|^4)$ time algorithms for finding a desired subtree in a given 2-connected graph G satisfying the assumptions.

Keywords: 2-connected graphs · Connectivity · Girth · Trees

1 Introduction

Throughout this paper, a graph $G = (V, E)$ means a simple undirected graph unless stated otherwise. The minimum degree of a vertex in G is denoted by $\delta(G)$. For a proper subset $S \subsetneq V(G)$, we denote by $G - S$ the graph obtained from G by deleting every vertex in S, where $G - \{v\}$ is abbreviated to $G - v$. For two sets A and B, we denote by $A \setminus B$ the set difference $\{x \mid x \in A, x \notin B\}$.

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For a nonempty subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $\langle S \rangle_G$, i.e., $\langle S \rangle_G = G - (V(G) \setminus S)$.

In 1972, Chartrand, Kaigars, and Lick proved the following result on the existence of a vertex whose removal does not influence k-connectedness of a graph.

Theorem 1 (Chartrand, Kaigars, and Lick [1]). *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor$ contains a vertex v such that $G - v$ is k-connected.*

After more than 30 years, Fujita and Kawarabayashi considered a similar problem for an edge of a graph and showed the following.

Theorem 2 (Fujita and Kawarabayashi [3]). *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + 2$ contains an edge uv such that $G - \{u, v\}$ is k-connected.*

In the same paper, they conjectured the next statement.

Conjecture 1. There is a function $f(m)$ such that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + f(m)$ contains a connected subgraph W of order m such that $G - V(W)$ is k-connected.

Note that the condition that W is connected is essential, since by iteratively applying Theorem 1, we can see that every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subgraph X of order m such that $G - V(X)$ is k-connected. In 2010, Mader [8] settled Conjecture 1 by showing the following result. Mader's result in fact improves the lower bound on $\delta(G)$ in Theorem 2 and generalizes Theorem 1.

Theorem 3 (Mader [8]). *Every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a path P of order m such that $G - V(P)$ is k-connected.*

Based on this result, Mader conjectured the following, i.e., a path in Theorem 3 can be generalized to any tree of the same order.

Conjecture 2 (Mader [8]). For any tree T of order m, every k-connected graph G with $\delta(G) \geq \lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is k-connected.

Mader's conjecture is a generalization not only from Theorem 1 but also from the next well-known result on the existence of a subtree isomorphic to any given tree.

Proposition 1. *For any tree T of order m, every graph G with $\delta(G) \geq m - 1$ contains a subtree $T' \cong T$.*

Apart from Mader's conjecture, Locke's conjecture concerning nonseparating trees in connected graphs is known. A k-cohesive graph is a non-trivial connected graph in which for any two distinct vertices u and v, the sum of the degrees of u and v and the distance between u and v is at least k.

Conjecture 3 (Locke [5]). For any tree T of order $m \geq 3$, every $2m$-cohesive graph G has a subtree $T' \cong T$ such that $G - V(T')$ is connected.

Motivated by Locke's conjecture, Diwan and Tholiya proved a theorem which is weaker than the conjecture, but it is the same as Mader's conjecture for $k = 1$ (Mader in fact mentioned their result in the paper [8]). Note that if G is connected and $\delta(G) \geq m$, then G is $2m$-cohesive.

Theorem 4 (Diwan and Tholiya [2]). *For any tree T of order m, every connected graph G with $\delta(G) \geq m$ contains a subtree $T' \cong T$ such that $G - V(T')$ is connected.*

For general $k \geq 2$, Mader's conjecture remains open; however for $k = 2$, partially affirmative answers have been shown. Tian et al. [10] first proved that Mader's conjecture for $k = 2$ is true when T is a star or a double-star, and they [11] further extended their results to a path-star or a path-double-star. Hasunuma and Ono [4] showed that for any tree T of order m, every 2-connected graph G with $\delta(G) \geq \max\{m + n(T) - 3, m + 2\}$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected, where $n(T)$ is the number of internal vertices of T. As a corollary, it follows that Mader's conjecture for $k = 2$ holds for any tree T with $n(T) \leq 5$ and for any tree of order $m \leq 8$. Lu and Zhang [6] also proved that Mader's conjecture for $k = 2$ is true for any tree with diameter at most 4. Very recently, it was reported that Lu and Ye [7] proved that Mader's conjecture for $k = 2$ holds for any caterpillars. Note that every known result which is a partially affirmative answer to Mader's conjecture for $k = 2$ restricts the class of trees to special subclasses. In this paper, we employ another approach to Mader's conjecture for $k = 2$. Namely, we add girth conditions to 2-connected graphs. The *girth* of a 2-connected graph G denoted by $g(G)$ is the length of a smallest cycle in G. We then show that Mader's conjecture is true for every 2-connected graph G with girth at least $\delta(G) - 6$. Note that for any given integers $r \geq 2$ and $g \geq 3$, there exists an r-regular graph with girth g, which has been shown in [12].

Theorem 5. *For any tree T of order m, every 2-connected graph G with $\delta(G) \geq m + 2$ and $g(G) \geq \delta(G) - 6$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

By increasing the lower bound of $\delta(G) - 6$ on $g(G)$, we can improve the lower bound of $m + 2$ on $\delta(G)$ to $m + 1$ if $m \geq 6$. Namely, a stronger statement holds in such a case.

Theorem 6. *For any tree T of order $m \geq 6$, every 2-connected graph G with $\delta(G) \geq m+1$ and $g(G) \geq \delta(G)-3$ contains a subtree $T' \cong T$ such that $G-V(T')$ is 2-connected.*

Moreover, by adding structural conditions, we can improve the girth conditions in Theorems 5 and 6.

Theorem 7. *For any tree T of order m, every 2-connected graph G with $\delta(G) \geq m+2$ and $g(G) \geq \delta(G)-7$ in which no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

Theorem 8. *For any tree T of order $m \geq 7$, every 2-connected graph G with $\delta(G) \geq m+1$ and $g(G) \geq \delta(G)-4$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G-V(T')$ is 2-connected.*

Mader's conjecture is interesting not only from a theoretical point of view but also from a practical point of view, since it may be applied to fault-tolerant problems in communication networks. That is, it is considered that Mader's conjecture guarantees the reliability of a communication network for a faulty subtree structure rather than a set of faulty vertices. Our proofs are constructive and lead to $O(|V(G)|^4)$ time algorithms for finding a desired subtree in a given 2-connected graph G in Theorems 5 and 6 (respectively, Theorems 7 and 8) if $g(G) \geq \delta(G) - 3$ (respectively, $g(G) \geq \delta(G) - 4$).

This paper is organized as follows. Section 2 presents notations, terminology, and known results used in this paper. Section 3 gives an outline of our proofs. Detailed proofs of Theorems 5 and 6 (respectively, Theorems 7 and 8) are given in Sect. 4 (respectively, Sect. 5). Section 6 concludes the paper with several remarks.

2 Preliminaries

For a nonempty subset $E' \subseteq E(G)$, we denote by $G - E'$ and $\langle E' \rangle$ the graph obtained from G by deleting every edge in E' and the edge-induced subgraph of G by E', respectively. For $v \in V(G)$, we denote by $N_G(v)$ the set of neighbors of v in G, i.e., vertices adjacent to v in G. The cardinality of $N_G(v)$ may be written by $\deg_G(v)$. Let $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$. For $S \subseteq V(G)$, $N_G(S)$ is defined as $(\cup_{v \in S} N_G(v)) \setminus S$. For $G' \subseteq G$, let $N_G(G') = N_G(V(G'))$.

A *component* of G is a maximal connected subgraph of G, while a *block* of G is a maximal connected subgraph of G without a cut vertex. A *cyclic block* is a block with order at least 3. For a tree T, the set of internal vertices, i.e., vertices with degree at least two, is denoted by $V_I(T)$, while the set of leaves, i.e., vertices with degree one, is denoted by $V_L(T)$. For a vertex v of a tree T, if v is adjacent to at least $\deg_T(v) - 1$ leaves, then v is called a *pseudo-leaf* of T. A *caterpillar* is a tree T such that $\langle V_I(T) \rangle_T$ is a path if $V_I(T) \neq \emptyset$.

We denote by $d_G(u, v)$ the distance between two vertices u and v in a connected graph G. The *eccentricity* $\mathrm{ecc}_G(v)$ of v in G is defined as $\max_{w \in V(G)} d_G(v, w)$. A *central vertex* of G is a vertex u with $\mathrm{ecc}_G(u) = \min_{v \in V(G)} \mathrm{ecc}_G(v)$, while a *peripheral vertex* is a vertex u with $\mathrm{ecc}_G(u) = \max_{v \in V(G)} \mathrm{ecc}_G(v)$. The *diameter* of a connected graph G denoted by $\mathrm{diam}(G)$ is the maximum distance for every pair of vertices in G, i.e., $\mathrm{diam}(G) = \max_{u,v \in V(G)} d_G(u, v)$. Let $\mathrm{diam}(G) = 0$ if $|V(G)| = 1$.

Proposition 1 can be stated in a more general form as follows.

Lemma 1 [4]**.** *Let T be a tree of order m and S a subtree obtained from T by deleting leaves adjacent to a vertex in $V_S \subseteq V_I(T)$. If a graph G contains a subtree $S' \cong S$ such that $\deg_G(u) \geq m-1$ for any $u \in \{\phi(v) \mid v \in V_S\}$ where ϕ is an isomorphism from $V(S)$ to $V(S')$, then G contains a subtree $T' \cong T$ such that $S' \subseteq T'$.*

Since any tree T of order m with $\mathrm{diam}(T) \geq m-2$ is a caterpillar and Mader's conjecture holds for any caterpillars [7], the following result is obtained.

Lemma 2. *For any tree T of order m with $\mathrm{diam}(T) \geq m-2$, Mader's conjecture for $k=2$ is true.*

An *orientation* D of a graph G is a directed graph obtained from G by replacing each edge by an arc (directed edge) with the same end-vertices. The *outdegree* $\deg_D^+(v)$ (respectively, *indegree* $\deg_D^-(v)$) of a vertex v in D is the number of arcs from (respectively, to) v in D. If for any $v \in V(G)$, $\deg_G(v)$ is even, then G is eulerian and has an orientation D in which for any $v \in V(D)$, $\deg_D^+(v) = \deg_D^-(v)$. If G has a vertex with odd degree, we can find a directed walk W connecting two vertices with odd degree, and by inductively applying a similar discussion for $G - E(W)$, we can see the following lemma holds. We here remark that Lemma 3 holds for multigraphs.

Lemma 3. *Every graph G has an orientation D such that $|\deg_D^+(v) - \deg_D^-(v)| \leq 1$ for any $v \in V(D)$.*

3 Outline of Proofs

In this section, we explain the outline of our constructive proofs and the time complexity for the algorithms based on the proofs.

Let T be a tree of order m. Let G be a 2-connected graph with $\delta(G) \geq m+2$. From Proposition 1, G contains a subtree $T' \cong T$. Let B be a maximum block in $G - V(T')$, i.e., a block with the maximum order among all the blocks in $G - V(T')$. Note that B is a cyclic block since $\delta(G - V(T')) \geq 2$. If $B = G - V(T')$, then T' is a desired subtree. Suppose that $B \neq G - V(T')$. Then there is a vertex in $G - V(T') \cup V(B)$. For any vertex $w \in V(G) \setminus (V(T') \cup V(B))$, $|N_G(w) \cap V(B)| \leq 1$. Now let $P = (v_1, v_2, \ldots, v_t)$, where $v_1, v_t \in V(B)$ and $v_i \notin V(B)$ for $1 < i < t$, be a shortest path among all the paths of G connecting two vertices in B such that every internal vertex is not in B. Since G is 2-connected, there are internally disjoint paths from a vertex in $G - V(B)$ to two vertices in B. Thus, P is well-defined. Suppose that $t \geq 4$. Then, we have that $N_G(v_2) \cap V(B) = \{v_1\}$ and $N_G(v_2) \cap V(P) = \{v_1, v_3\}$. Therefore, $|N_G(v_2) \setminus (V(P) \cup V(B))| \geq m+2-2 \geq m$, which implies that $V(G) \setminus (V(P) \cup V(B)) \neq \emptyset$. Let w be any vertex in $G - V(P) \cup V(B)$. By the definition of P, w can be adjacent to at most three vertices in $V(B) \cup V(P)$. Thus, $\delta(G - V(P) \cup V(B)) \geq m+2-3 = m-1$. Hence, by Proposition 1, $G - V(P) \cup V(B)$ contains a subtree $T'' \cong T$ such that $G - V(T'')$ has a block $B' \supseteq \langle V(B) \cup V(P) \rangle_G$. Thus, we can find a block with order at least $|V(B)| + 2$.

Suppose that $t = 3$. Then $v_2 \in V(T')$. If there exists a subtree T'' in $G - V(B) \cup \{v_2\}$ such that $T'' \cong T$, then $G - V(T'')$ has a block $B' \supseteq \langle V(B) \cup \{v_2\} \rangle_G$, i.e., we can find a block with order at least $|V(B)| + 1$. If we have a manipulation to find such a subtree T'', then by applying the manipulations for $t \geq 4$ or $t = 3$ iteratively, we finally obtain a desired subtree T'', i.e., $T'' \cong T$ such that $G - V(T'')$ is 2-connected. Therefore, if we can show the following statement, then it is concluded that Mader's conjecture for $k = 2$ is true.

Statement 1. Let T be a tree of order m and G a 2-connected graph with $\delta(G) \geq m + 2$. For any subtree $T' \cong T$ in G and a maximum block B in $G - V(T')$, if $B \neq G - V(T')$ and $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\} \neq \emptyset$, then there exist a vertex $v \in V_B(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$.

The above manipulations can be algorithmically described as follows.

1. Compute a subtree $T' \cong T$ in G.
2. Compute a maximum block B in $G - V(T')$.
3. If $B = G - V(T')$ then output T' as a desired subtree of G and stop.
4. If $B \neq G - V(T')$ then compute a shortest path P connecting vertices in B such that every internal vertex is not in B.
5. Compute a subtree T'' in $G - V(B) \cup V(P)$, let $T' = T''$, and return to Step 2.

We here check the complexity of the above algorithm under the assumption that there exists a constructive proof of Statement 1. A subtree $T' \cong T$ in G can be computed in $O(|E(G)|)$ time in Step 1, and a maximum block B can also be found in $O(|E(G)|)$ time in Step 2. In Step 4, a shortest path P can be found by computing all shortest paths for vertices of $V(B)$ in $G - E(B)$. Thus, it takes $O(|V(G)|^3)$ time. Since the number of iterations is $O(|V(G)|)$, if Statement 1 can be shown by a constructive proof from which a procedure within $O(|V(G)|^3)$ time is obtained, we have an $O(|V(G)|^4)$ time algorithm. These observations are summarized as follows.

Lemma 4. *If Statement 1 holds, then G contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected. Besides, if there is a procedure for Statement 1 within $O(|V(G)|^3)$ time, we have an $O(|V(G)|^4)$ time algorithm for finding a desired subtree.*

Next, we consider the case that a 2-connected graph G has no triangle, i.e., $g(G) \geq 4$. In such a case, we can show a similar lemma using the following statement. Note that the minimum degree condition $\delta(G) \geq m + 2$ is replaced with $\delta(G) \geq m + 1 \geq 3$.

Statement 2. Let T be a tree of order $m \geq 2$ and G a 2-connected graph with $\delta(G) \geq m+1$ and $g(G) \geq 4$. For any subtree $T' \cong T$ in G and a maximum block B in $G - V(T')$, if $B \neq G - V(T')$ and $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\} \neq \emptyset$, then there exist a vertex $v \in V_B(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$.

Lemma 5. *If Statement 2 holds, then G contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected. Besides, if there is a procedure for Statement 2 within $O(|V(G)|^3)$ time, we have an $O(|V(G)|^4)$ time algorithm for finding a desired subtree.*

Proof. We show that the algorithm for Lemma 4 also works well under the assumption that Statement 2 holds.

Let $T' \subset G$ such that $T' \cong T$. Let B be a maximum block in $G - V(T')$. Since $\delta(G - V(T')) \geq 1$, it may happen that B is not a cyclic block, i.e., B is a block with two vertices. Note that if B is not a cyclic block, then B is not 2-connected. Suppose that B is not a cyclic block. Assume that $B = G - V(T')$. Then, $|V(G)| = m + 2$. Since $\delta(G) \geq m + 1$, G must be a complete graph with at least four vertices, which contradicts the girth condition that $g(G) \geq 4$. Therefore, if B is not a cyclic block, then $B \neq G - V(T')$. Hence, in the case that $G - V(T')$ has no cyclic block, the algorithm does not incorrectly output a subtree in Step 3.

Let $P = (v_1, v_2, \ldots, v_t)$ be a shortest path between two vertices in B such that every internal vertex is not in B. Suppose that $t \geq 4$. By the definition of P and the girth condition $g(G) \geq 4$, any vertex w in $G - V(P) \cup V(B)$ can be adjacent to at most two vertices in $V(B) \cup V(P)$. Thus, $\delta(G - V(P) \cup V(B)) \geq m - 1$. Therefore, $G - V(P) \cup V(B)$ contains a subtree $T'' \cong T$. Hence, if $t \geq 4$, then we can find a subtree T'' in $G - V(P) \cup V(B)$ in Step 5. We here remark that the condition $m \geq 2$ is necessary to guarantee that $V(G) \setminus (V(P) \cup V(B)) \neq \emptyset$.

For the time complexity, similarly to Lemma 4, we have an $O(|V(G)|^4)$ time algorithm, if Statement 2 can be shown by a constructive proof which induces a procedure within $O(|V(G)|^3)$ time. $\square$

Note that in Statement 2, if B is not a cyclic block, then by the girth condition $g(G) \geq 4$, we have that $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\} = \emptyset$. Thus, in Statement 2, we may assume that a maximum block B is a cyclic block.

4 Proofs of Theorems 5 and 6

In order to show our main results, we prove the following lemma.

Lemma 6. *Let T be a tree of order m and G a 2-connected graph with $\delta(G) \geq m + 1$ and $g(G) \geq \mathrm{diam}(T) - 1$. For any subtree $T' \cong T$ in G and a maximum block B in $G - V(T')$, if $B \neq G - V(T')$ and $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\} \neq \emptyset$, then there exist a vertex $v \in V_B(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$ such that v and T'' can be found in $O(|E(G)|)$ time.*

Proof. Let $T' \subset G$ such that $T' \cong T$. Let B be a maximum block in $G - V(T')$ such that $B \neq G - V(T')$. Also, let $v \in V_B(T')$ and $H = G - V(T') \cup V(B)$. When $m \leq 2$, the lemma can be easily checked. Let $m \geq 3$. Suppose that v is a leaf of T' and for the neighbor v' of v in T', $v' \notin V_B(T')$, i.e., $|N_G(v') \cap V(B)| \leq 1$. Then, $|N_G(v') \cap V(H)| \geq 1$. For any $v'' \in N_G(v') \cap V(H)$, $T'' = \langle (E(T' - v) \cup \{v'v''\} \rangle \cong T$

such that $T'' \subset G - V(B) \cup \{v\}$. Thus, w.l.o.g., we may assume that v is not a leaf of T'. Let $S' = \langle V_I(T') \rangle_{T'}$. Then $v \in V(S')$. Since $\mathrm{diam}(S') = \mathrm{diam}(T') - 2$, $g(G) \geq \mathrm{diam}(S') + 1$. We regard S' as a rooted tree at v and denote by $C(u)$ the set of children of a vertex u in S'. Besides, we denote by $h(S')$ the height of S', i.e., $h(S') = \mathrm{ecc}_{S'}(v)$.

Since $\delta(G) \geq m + 1$, it holds that for any vertex $w \in V(H)$, $\deg_{G-V(B)\cup\{v\}}(w) \geq m - 1$. If there exists a subtree $W \subset \langle V(H) \cup V(T' - v) \rangle_G$ such that W is isomorphic to a subtree U obtained from T' by deleting leaves adjacent to a vertex in $V' \subseteq V(S')$ and $\phi(V') = \{\phi(u) \mid u \in V'\} \subseteq V(H)$ where ϕ is an isomorphism from $V(U)$ to $V(W)$, then by Lemma 1, there exists a subtree T'' in $G - V(B) \cup \{v\}$ such that $T'' \cong T$. In particular, if there exists a vertex w in H such that $C(v) \subseteq N_G(w)$, then we can employ the subtree $\langle E(T' - v) \cup \{wu \mid u \in C(v)\} \rangle$ as a desired subtree W where $V' = \{v\}$ and $\phi(V') = \{w\}$. Note that $C(v) = \emptyset$ when $\mathrm{diam}(S') = 0$, i.e., $|V(S')| = 1$. Suppose that v is a leaf of S'. Let $C(v) = \{v'\}$. If there is no vertex in H adjacent to v', i.e., $C(v) \not\subseteq N_G(w)$ for any $w \in V(H)$, then $\delta(H) \geq 1$ and $v' \in V_B(T')$. In such a case, we can employ $\langle \{xy\} \rangle$ as a desired subtree W for $xy \in E(H)$ where $V' = \{v, v'\}$ and $\phi(V') = \{x, y\}$ when $\mathrm{diam}(S') = 1$. From these observations, we may assume that there is no vertex w in H with $N_G(w) \supseteq C(v)$, v is not a leaf of S' (since we can employ v' instead of v if $v' \in V_B(T')$) and $\mathrm{diam}(S') \geq 2$.

Let $x \in V(H)$ and $C(v) \setminus N_G(x) = \{v_1, v_2, \ldots, v_p\}$. Since $|N_G(x) \cap V(B)| \leq 1$ and $|N_G(x) \cap V(T')| \leq m - p$, $|N_H(x)| \geq p$, i.e., there are at least p neighbors of x in H. Let $\{x_1, x_2, \ldots, x_p\} \subseteq N_H(x)$. If $h(S') = 1$, then we can employ $\langle E(T' - v) \cup \{xu \mid u \in C(v) \cap N_G(x)\} \cup \{xx_i \mid 1 \leq i \leq p\} \rangle$ as a desired subtree W where $V' = \{v, v_1, v_2, \ldots, v_p\}$ and $\phi(V') = \{x, x_1, x_2, \ldots, x_p\}$. Suppose that $h(S') \geq 2$. Let $|C(v_i) \setminus N_G(x_i)| = q_i$ for each i. Since $|C(v) \setminus N_G(x_i)| \geq 1$, there are at least $q_i + 1$ neighbors of x_i in H, which means that we can select q_i vertices $y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}$ as children of x_i in the subtree $\langle \{xx_i \mid 1 \leq i \leq p\} \rangle$ rooted at x. By letting these children correspond to the q_i children of v_i in $C(v_i) \setminus N_G(x_i)$ for each i with $q_i > 0$, we can obtain a desired subtree W if $h(S') = 2$. Note that when $\mathrm{diam}(S') = 3$, there is exactly one i such that $C(v_i) \neq \emptyset$, and if $q_i > 0$, then $\{x_1, x_2, \ldots, x_p\} \cap \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} = \emptyset$, since $g(G) \geq \mathrm{diam}(S') + 1 = 4$. When $\mathrm{diam}(S') = 4$, by the girth condition, we can see that $\{x_1, x_2, \ldots, x_p\} \cap \{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} = \emptyset$ for each i with $q_i > 0$ and $\{y_{i,1}, y_{i,2}, \ldots, y_{i,q_i}\} \cap \{y_{i',1}, y_{i',2}, \ldots, y_{i',q_{i'}}\} = \emptyset$ for any pair of i and i' with $q_i > 0$ and $q_{i'} > 0$. Thus, the subtree defined by $\langle E(T' - \{v, v_1, v_2, \ldots, v_p\}) \cup \{xu \mid u \in C(v) \cap N_G(x)\} \cup \{xx_i \mid 1 \leq i \leq p\} \cup (\cup_{1 \leq i \leq p}(\{x_i u \mid u \in C(v_i) \cap N_G(x_i)\} \cup \{x_i y_{i,j} \mid 1 \leq j \leq q_i\})) \rangle$ can be employed as a desired subtree W. If $h(S') \geq 3$, by inductively applying similar manipulations to descendants of x, we can finally obtain a desired subtree W. Note that in each extension step, disjointness of the sets of new children for descendants of x is guaranteed by the girth condition $g(G) \geq \mathrm{diam}(S') + 1$.

The assumption that v is neither a leaf of T' nor a leaf of S' can be realized by preferentially selecting a vertex in $V_B(T') \setminus (V_L(T') \cup V_L(S'))$ if $V_B(T') \setminus (V_L(T') \cup V_L(S')) \neq \emptyset$. For $v \in V_B(T') \setminus (V_L(T') \cup V_L(S'))$, we apply the manipulations for constructing W in a depth-first search order for S'. The selection process for new children of a descendant of x and the extension process from W to T'' can be done greedily. If $V_B(T') \setminus (V_L(T') \cup V_L(S')) = \emptyset$, then we can directly obtain either W or T''. Therefore, a desired subtree T'' can finally be found in $O(|E(G)|)$ time. $\square$

Lemma 6 is stronger than Statement 1 under the assumption that $g(G) \geq \mathrm{diam}(T) - 1$. Therefore, by Lemmas 4 and 6, we have the following.

Theorem 9. *For any tree T of order m, every 2-connected graph G with $\delta(G) \geq m + 2$ and $g(G) \geq \mathrm{diam}(T) - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

For any 2-connected graph G, it holds that $g(G) \geq 3$. Thus, the following result by Lu and Zhang [6] is obtained from Theorem 9.

Corollary 1 [6]**.** *For any tree T of order m with $\mathrm{diam}(T) \leq 4$, every 2-connected graph G with $\delta(G) \geq m + 2$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

Besides, by combining Lemmas 5 and 6, we have the following.

Theorem 10. *For any tree T of order $m \geq 2$, every 2-connected graph G with $\delta(G) \geq m + 1$ and $g(G) \geq \max\{\mathrm{diam}(T) - 1, 4\}$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

From Theorem 10, the following result for 2-connected graphs without a triangle is obtained.

Corollary 2. *For any tree T of order $m \geq 2$ with $\mathrm{diam}(T) \leq 5$, every 2-connected graph G with $\delta(G) \geq m + 1$ and $g(G) \geq 4$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

Now, we are ready to show our main two results stated in the introduction. Let T be a tree of order m. Suppose that G is a 2-connected graph with $\delta(G) \geq m + 2$ and $g(G) \geq \delta(G) - 6$. Then, $g(G) \geq m - 4$. From Lemma 2, it is sufficient to consider a tree T with $\mathrm{diam}(T) \leq m - 3$. That is, we have $g(G) \geq \mathrm{diam}(T) - 1$. Therefore, Theorem 5 follows from Theorem 9. Next, suppose that $m \geq 6$ and G is a 2-connected graph with $\delta(G) \geq m + 1$ and $g(G) \geq \delta(G) - 3$. Then, $g(G) \geq m - 2 \geq 4$, i.e., $g(G) \geq \max\{\mathrm{diam}(T) - 1, 4\}$. Hence, Theorem 6 follows from Theorem 10.

From Lemmas 4, 5 and 6, we can see that a desired subtree T' in Theorem 5 (respectively, Theorem 6) can be found in $O(|V(G)|^4)$ time if $g(G) \geq \delta(G) - 4$ (respectively, $g(G) \geq \delta(G) - 3$). Note that such a restriction on $g(G)$ for Theorem 5 follows from the fact that we use Lemma 2.

5 Proofs of Theorems 7 and 8

In this section, we try to improve the lower bounds on $g(G)$ in Theorems 5 and 6, and show that such improvements are possible if a 2-connected graph G satisfies a structural property on the smallest cycles. Note that for any two cycles C_1 and C_2 of length $g(G)$, it holds that $|E(C_1) \cap E(C_2)| \leq \left\lfloor \frac{g(G)}{2} \right\rfloor$.

Lemma 7. *Let T be a tree of order m. Let G be a 2-connected graph with $\delta(G) \geq m+2$ and $g(G) \geq \operatorname{diam}(T) - 2$ in which no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in G. For any subtree $T' \cong T$ in G and a maximum block B in $G - V(T')$, if $B \neq G - V(T')$ and $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \geq 2\} \neq \emptyset$, then there exist a vertex $v \in V_B(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$.*

Proof. We use the notations such as $T', B, v, H, S', C(u), W$, and x with the same meaning in the proof of Lemma 6. If $\operatorname{diam}(S') \leq 2$, then we can easily construct a desired subtree W without an additional structural property. Suppose that $\operatorname{diam}(S') \geq 3$. By the discussion in the proof of Lemma 6, we suppose that v is not a leaf of S' and there is no vertex w in H such that $C(v) \subseteq N_G(w)$. For $u \in C(v)$, we denote by S'_u the subtree rooted at u in S'. Let F be a component of H containing x. Note that $|N_G(F) \cap V(B)| \leq 1$. In the following discussion, w.l.o.g., we may assume that $N_G(F) \cap V(B) \neq \emptyset$. Then, let $N_G(F) \cap V(B) = \{v_B\}$ and $F' = \langle V(F) \cup \{v_B\}\rangle_G$.

Suppose that v is a pseudo-leaf of S' and v' is the non-leaf vertex adjacent to v in S'. If there exists a vertex y in H such that $v' \in N_G(y)$, then by letting the vertex y correspond to v, we can obtain a desired subtree W. If there is no vertex in H which is adjacent to v', then $v' \in V_B(T')$. Thus, we may assume that if $\operatorname{diam}(S') \geq 4$, v is not a pseudo-leaf of S', and if $\operatorname{diam}(S') = 3$, the central vertices v, v' are in $V_B(T')$ such that $\{v, v'\} \cap N_G(w) = \emptyset$ for any $w \in V(H)$. Suppose that $\operatorname{diam}(S') = 3$. Let $xy \in E(F)$. Then, $|N_{F'}(x)| \geq |C(v) \setminus N_G(x)| + 3$ and $|N_{F'}(y) \setminus \{x\}| \geq |C(v') \setminus N_G(y)| + 3$. The assumption on the smallest cycles implies that $|N_{F'}(x) \cap N_{F'}(y)| \leq 5$. Therefore, we can find $y \in C(x) \subset N_F(x)$ and $C(y) \subset N_F(y) \setminus \{x\}$ so that $C(x) \cap C(y) = \emptyset$, $|C(x)| = |C(v) \setminus N_G(x)|$ and $|C(y)| = |C(v') \setminus N_G(y)|$. Thus, a desired subtree W can be constructed. In what follows, we suppose that $\operatorname{diam}(S') \geq 4$.

It is sufficient to consider the case that $g(G) = \operatorname{diam}(T) - 2 = \operatorname{diam}(S')$. Let $P(S')$ and $Q(S')$ be the set of peripheral vertices in S' and the set of parents of a peripheral vertex in S', respectively. Let $S'' = S' - P(S')$. Note that $\operatorname{diam}(S'') = \operatorname{diam}(S') - 2$, and $v \notin P(S') \cup Q(S')$ since any vertex in $P(S')$ is a leaf of S' and any vertex in $Q(S')$ is a pseudo-leaf of S'. For the subtree S'', we apply the manipulations in the proof of Lemma 6. Let W' be the subtree obtained after such manipulations and let $W'_F = \langle V(W') \cap V(F)\rangle_{W'}$. Suppose that $\{z_1, z_2, \ldots, z_q\}$ is the set of vertices in W'_F which are corresponding to vertices in $Q(S')$. Note that q may be less than $|Q(S')|$. Let $\{u_1, u_2, \ldots, u_q\} \subseteq Q(S')$ such that u_i is corresponding to z_i for $1 \leq i \leq q$. For each $1 \leq i \leq q$, let $D(z_i) = N_F(z_i) \setminus \{p(z_i)\}$

where $p(z_i)$ is the parent of z_i in W'_F rooted at x. Also let $r_i = |C(u_i) \setminus N_G(z_i)|$ for each $1 \le i \le q$, where $C(u_i)$ is the set of children of u_i in S'. Since $g(G) = \mathrm{diam}(S')$, it may happen that $D(z_i) \cap D(z_j) \neq \emptyset$ for $1 \le i < j \le q$. It follows from $\delta(G) \ge m + 2$ and $|C(v) \setminus N_G(z_i)| \ge 1$ that $|D(z_i)| \ge r_i + 1$ for each i.

Suppose that $|D(z_k)| = r_k + 1$ for some $k \in \{1, 2, \ldots, q\}$. Then $|C(v) \setminus N_G(z_k)| = 1$ and z_k is adjacent to every vertex in T' except for ones in $(C(v) \cup C(u_k)) \setminus N_G(z_k)$. Thus, we may assume that $V_B(T') \subseteq N_{T'}(C(v) \setminus N_G(z_k)) \cup N_{T'}(C(u_k) \setminus N_G(z_k))$, since otherwise, there exists $v' \in V_B(T')$ such that $N_{T'}(v') \subseteq N_G(z_k)$. Let $C(v) \setminus N_G(z_k) = \{w_k\}$. Instead of x, we let z_k correspond to v and apply the manipulations in the proof of Lemma 6. Let W''_F be the resultant subtree in F. If w_k is a pseudo-leaf of S', then we can immediately obtain a desired subtree W in this setting. Otherwise, there is no pseudo-leaf adjacent to v in S' which corresponds to a vertex in the subtree W''_F. Thus, w.l.o.g., we may assume that $u_k \notin C(v)$. Consider the case that $u_k \in V_B(T')$. Since u_k is a pseudo-leaf of S', by the previous discussion, we may assume that $p(u_k) \in V_B(T')$ where $p(u_k)$ is the parent of u_k in S'. Since $p(u_k) \notin N_{T'}(C(u_k) \setminus N_G(z_k))$, $p(u_k) \in N_{T'}(w_k)$. This means that $w_k = p(p(u_k))$. Next consider the case that $u_k \notin V_B(T')$. In this case, we may assume that no descendant of u_k in T' is in $V_B(T')$. Hence, it is concluded that $V_B(T') \cap (\cup_{u \in C(v) \setminus \{w_k\}} V(S'_u)) = \emptyset$. Note that $w_k \notin V_B(T')$. Let $H' = \langle V(H) \cup (\cup_{u \in C(v) \setminus \{w_k\}} V(S'_u)) \rangle_G$. For every vertex $u' \in \cup_{u \in C(v) \setminus \{w_k\}} V(S'_u)$, $|N_G(u') \cap V(B)| \le 1$. Thus, it holds that $\delta(H') \ge 1 + \sum_{u \in C(v) \setminus \{w_k\}} |V(S'_u)|$. Let $w'_k \in N_G(w_k) \cap V(H)$. Then, there exists a subtree $U'_{H'}$ in H' which is isomorphic to $S' - V(S'_{w_k})$ such that w'_k corresponds to v in an isomorphism from $V(S') \setminus V(S'_{w_k})$ to $V(U'_{H'})$. Then, $\langle E(S'_{w_k}) \cup \{w_k w'_k\} \cup E(U'_{H'}) \rangle$ can be employed as a desired subtree W. Consequently, we may assume that any vertex z_i in $\{z_1, z_2, \ldots, z_q\}$ satisfies that $|D(z_i)| \ge r_i + 2$.

Let $D'(z_i) = N_{F'}(z_i) \setminus \{p(z_i)\}$ for $1 \le i \le q$. Then, $|D'(z_i)| \ge r_i + 3$ for each i. Note that either $D'(z_i) = D(z_i)$ or $D'(z_i) = D(z_i) \cup \{v_B\}$. Define I_G as the (multi)graph with vertex set $\{z_1, z_2, \ldots, z_q\}$ in which z_i and z_j are joined by $|D'(z_i) \cap D'(z_j)|$ edges. Note that I_G may be a multigraph only if $\mathrm{diam}(S') = 4$. The assumption that no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1 = \left\lceil \frac{\mathrm{diam}(S'')}{2} \right\rceil$ in G implies that $\Delta(I_G) \le 5$, i.e., each vertex in I_G is incident to at most five edges. Besides, the intersection of at least seven (respectively, three) sets in $\{D'(z_1), D'(z_2), \ldots, D'(z_q)\}$ is empty if $\mathrm{diam}(S'')$ is even (respectively, odd). Modify the graph I_G as follows, and let J_G be the resultant (multi)graph.

1. Delete every edge generated by a vertex in the intersection of at least three sets $D'(z_{i_1}), D'(z_{i_2})$, and $D'(z_{i_3})$.
2. Delete the edge generated by v_B if v_B is contained in exactly two sets $D'(z_{i_1})$ and $D'(z_{i_2})$.

Note that if v_B is contained in exactly one set $D'(z_i)$, then $|D(z_i)| \ge r_i + 2$ and $|D(z_j)| \ge r_j + 3$ for any $j \neq i$. By Lemma 3, J_G has an orientation D_G such that $|\deg^+_{D_G}(z) - \deg^-_{D_G}(z)| \le 1$ for any $z \in V(D_G)$ and if v_B is contained in exactly one set $D'(z_i)$ then $\deg^-_{D_G}(z_i) \le 2$. Note that if an orientation of J_G satisfying the first condition does not satisfy the second condition, the reverse orientation satisfies both the conditions since $\Delta(I_G) \le 5$. Based on D_G, we can disjointly select r_i vertices in $D(z_i)$ for $1 \le i \le q$ as follows. For each arc from z_{i_1} to z_{i_2} in D_G, we select the vertex in $D(z_{i_1}) \cap D(z_{i_2})$ corresponding to the edge $z_{i_1}z_{i_2}$ as a child of z_{i_1}. Note that we do not select the vertex v_B and any vertex in the intersection of at least three sets $D'(z_{i_1}), D'(z_{i_2})$, and $D'(z_{i_3})$. In this way, we can appropriately extend W'_F for a desired subtree W and finally obtain a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$. □

Next, we consider the case that $\delta(G) \ge m + 1$. In this case, we need to strengthen the structural condition on the smallest cycles in Lemma 7.

Lemma 8. *Let T be a tree of order m. Let G be a 2-connected graph with $\delta(G) \ge m + 1$ and $g(G) \ge \operatorname{diam}(T) - 2$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in G. For any subtree $T' \cong T$ in G and a maximum block B in $G - V(T')$, if $B \ne G - V(T')$ and $V_B(T') = \{u \in V(T') \mid |N_G(u) \cap V(B)| \ge 2\} \ne \emptyset$, then there exist a vertex $v \in V_B(T')$ and a subtree $T'' \cong T$ in $G - V(B) \cup \{v\}$.*

Proof. We use the notations in the proof of Lemma 7 with the same meaning. A desired subtree W can be constructed without an additional structural property if $\operatorname{diam}(S') \le 2$. Suppose that $\operatorname{diam}(S') = 3$. Applying a similar discussion in the proof of Lemma 7, we have that $|N_{F'}(x)| \ge |C(v) \setminus N_G(x)| + 2$ and $|N_{F'}(y) \setminus \{x\}| \ge |C(v') \setminus N_G(y)| + 2$. Since the condition on smallest cycles implies that $|N_{F'}(x) \cap N_{F'}(y)| \le 3$, we can find $y \in C(x) \subset N_F(x)$ and $C(y) \subset N_F(y) \setminus \{x\}$ so that $C(x) \cap C(y) = \emptyset$, $|C(x)| = |C(v) \setminus N_G(x)|$ and $|C(y)| = |C(v') \setminus N_G(y)|$. Suppose that $\operatorname{diam}(S') \ge 4$. From a similar discussion in the proof of Lemma 7, we may assume that every vertex z_i in $\{z_1, z_2, \ldots, z_q\}$ satisfies that $|D(z_i)| \ge r_i + 1$ and $|D'(z_i)| \ge r_i + 2$. Note that the degree condition $\delta(H') \ge \sum_{u \in C(v) \setminus \{w\}} |V(S'_u)|$ is sufficient to construct a subtree $U'_{H'}$ in H'. The assumption that no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ in G implies that $\Delta(I_G) \le 3$. By Lemma 3, J_G has an orientation D_G such that $|\deg^+_{D_G}(z) - \deg^-_{D_G}(z)| \le 1$ for any $z \in V(D_G)$ and if v_B is contained in exactly one set $D'(z_i)$ then $\deg^-_{D_G}(z_i) \le 1$. Based on D_G, we can disjointly select r_i vertices in $D(z_i)$ for $1 \le i \le q$. Hence, we can appropriately extend W'_F in order to obtain a desired subtree T''. □

From Lemmas 4, 5, 7, and 8, we have the following results.

Theorem 11. *For any tree T of order m, every 2-connected graph G with $\delta(G) \ge m + 2$ and $g(G) \ge \operatorname{diam}(T) - 2$ in which no six cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

Theorem 12. *For any tree T of order $m \geq 2$, every 2-connected graph G with $\delta(G) \geq m+1$ and $g(G) \geq \max\{\mathrm{diam}(T)-2, 4\}$ in which no four cycles of length $g(G)$ have a common path of length $\left\lceil \frac{g(G)}{2} \right\rceil - 1$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.*

Theorems 7 and 8 follow from Theorem 11 with Lemma 2 and Theorem 12, respectively. Manipulations in the proofs of Lemmas 7 and 8 can be done in $O(|E(G)|)$ time, although they are more complicated than those in the proof of Lemma 6. Therefore, we can find a desired subtree T' in Theorem 7 (respectively, Theorem 8) in $O(|V(G)|^4)$ time if $g(G) \geq \delta(G)-5$ (respectively, $g(G) \geq \delta(G)-4$).

6 Concluding Remarks

In this paper, we have shown that Mader's conjecture for $k = 2$ (with a weak degree condition $\delta(G) \geq m + 1$) holds for graphs with large girth. Mader's conjecture was posed in a purely mathematical interest; however, it has a potential application to fault-tolerant problems in communication networks. We then have shown that our constructive proofs lead to $O(|V(G)|^4)$ time algorithms.

Our lower bounds on the girth in Theorems 5 and 7 can be improved if the upper bound on the diameter of a tree for which Mader's conjecture for $k = 2$ holds is improved. Namely, the following result follows from Theorem 9.

Theorem 13. *If Mader's conjecture for $k = 2$ holds for any tree T with $\mathrm{diam}(T) \geq |V(T)| - \ell$, then Mader's conjecture for $k = 2$ holds for any 2-connected graph G with $g(G) \geq \delta(G) - \ell - 4$.*

In particular, by checking the proof in [7], we can see that Statement 2 holds for any caterpillars; thus, the lower bounds on $g(G)$ in Theorems 6 and 8 can be improved to $\delta(G) - 5$ and $\delta(G) - 6$, respectively. Besides, the restriction that $g(G) \geq \delta(G)-4$ (respectively, $g(G) \geq \delta(G)-5$) for an $O(|V(G)|^4)$ time algorithm can be removed for Theorem 5 (respectively, Theorem 7). On the other hand, in order to improve the lower bounds on the girth in Theorems 9, 10, 11, and 12 directly, we may need some other techniques.

Even though Mader's conjecture for $k = 2$ still remains open, from Lemma 5 and Corollary 2, we may conjecture the following.

Conjecture 4. For any tree T of order $m \geq 2$, every 2-connected graph G with $\delta(G) \geq m + 1$ and $g(G) \geq 4$ contains a subtree $T' \cong T$ such that $G - V(T')$ is 2-connected.

Although we consider Mader's conjecture only for $k = 2$, it would be interesting to approach Mader's conjecture for general $k \geq 2$ by considering girth conditions.

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