



On Semantic Properties of Fuzzy Quantifiers over Fuzzy Universes: Restriction and Living on

Antonín Dvořák and Michal Holčápek^(✉)

CE IT4I - IRAFM, University of Ostrava, 30. dubna 22,
701 03 Ostrava 1, Czech Republic
{antonin.dvorak,michal.holcapek}@osu.cz
<http://ifm.osu.cz>

Abstract. The article investigates important semantic properties of fuzzy quantifiers, namely restriction and living on a (fuzzy) set. These properties are introduced in the novel frame of fuzzy quantifiers over fuzzy universes.

Keywords: Generalized quantifiers · Fuzzy quantifiers · Semantic properties

1 Introduction

In this paper, we continue our investigation of fuzzy quantifiers and their semantic properties [3–5, 8]. We are working with the general concept of generalized quantifiers originating from works of Mostowski [12], Lindström [11] and many others. For details, we refer to monograph [13]. For example, a generalized quantifier Q with one argument (so-called type $\langle 1 \rangle^1$) over a set universe M can be understood as a mapping from the powerset of M to the set of truth values $\{0, 1\}$ (i.e., false and true, respectively).

At first [3, 8], we investigated a straightforward generalization of these generalized quantifiers, where arguments of fuzzy quantifiers were fuzzy sets and the set of truth values $\{0, 1\}$ has been replaced by some more general structure. Namely, we used a residuated lattice L and defined fuzzy quantifiers on M as mappings from the power set of M to L . However, these generalized quantifiers were still defined over a *crisp* universe M . Gradually, we started to be aware of severe limitations of this approach. For example, it was not possible to define the important operation of *relativization* in a satisfactory way (see [5]). The reason

¹ This notation originated in [11], where quantifiers are understood as classes of relational structures of a certain type (representing a number of arguments and variable binding). It is widely used in the literature on generalized quantifiers [13].

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is that in the definition of relativization, the first argument of a quantifier (i.e., a fuzzy set) becomes a new universe for the relativized quantifier. But, only crisp sets have been permitted as universes for fuzzy quantification. To overcome this limitation, in [5] we defined the so-called *C*-fuzzy quantifiers, where a fuzzy set *C* served as a universe of quantification. We showed there that relativization can be satisfactorily defined in this frame. However, also the approach of *C*-fuzzy quantifiers has its limitations. In this contribution, we present initial observations on a more general approach, in which pairs (M, C) , where *M* is a crisp set and *C* is a fuzzy subset of *M*, serve as universes for fuzzy quantification. As case studies, important semantic notions of *restriction* and of *living on a fuzzy set* in the setting of fuzzy quantifiers over fuzzy universes are investigated.

This paper is structured as follows: In Sect. 2, we summarize necessary notions on algebras of truth values and on fuzzy sets. Section 3 contains basic definitions of generalized quantifiers, restricted quantifiers and quantifiers living on a set. These notions are then generalized in Sect. 4. Finally, Sect. 5 contains conclusions and directions of further research.

2 Preliminaries

2.1 Algebraic Structures of Truth Values

In this article we assume that the algebraic structure of truth values is a *complete residuated lattice*, i.e., an algebraic structure $L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice, where 0 is the least element and 1 is the greatest element of *L*, $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e., \otimes is associative, commutative and the identity $a \otimes 1 = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \quad (1)$$

holds for each $a, b, c \in L$, where \leq denotes the corresponding lattice ordering, i.e., $a \leq b$ if $a \wedge b = a$ for $a, b \in L$. A residuated lattice *L* is said to be *divisible* if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$. The operation of *negation* on *L* is defined as $\neg a = a \rightarrow 0$ for $a \in L$. A residuated lattice *L* satisfies the *law of double negation* if $\neg \neg a = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. A residuated lattice is said to be *linearly ordered* if the corresponding lattice ordering is linear, i.e., $a \leq b$ or $b \leq a$ holds for any $a, b \in L$.

Obviously, the two elements residuated lattice, i.e., $L = \{0, 1\}$, is a Boolean algebra. We put $\mathbf{2} = L = \{0, 1\}$. Other examples of complete residuated lattices can be determined from left-continuous *t*-norms on the unit interval:

Example 1. The algebraic structure

$$L_T = \langle [0, 1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where *T* is a left continuous *t*-norm on $[0, 1]$ and $a \rightarrow_T b = \bigvee \{c \in [0, 1] \mid T(a, c) \leq b\}$, defines the residuum, is a complete residuated lattice (see, e.g., [2, 7, 10]).

2.2 Fuzzy Sets

Let L be a complete residuated lattice, and let M be a non-empty universe of discourse. A function $A : M \rightarrow L$ is called a *fuzzy set* (*L-fuzzy set*) on M . A value $A(m)$ is called a *membership degree of m in the fuzzy set A* . The set of all fuzzy sets on M is denoted by $\mathcal{F}(M)$. A fuzzy set A is called *crisp* if there is a subset $Z \subseteq M$ such that $A = 1_Z$, where 1_Z denotes the characteristic function of Z . Obviously, a crisp fuzzy set can be uniquely identified with a subset of M . The symbol \emptyset denotes the empty fuzzy set on M , i.e., $\emptyset(m) = 0$ for any $m \in M$. The set of all crisp fuzzy sets on M (i.e., the power set of M) is denoted by $\mathcal{P}(M)$. The set $\text{Supp}(A) = \{m \in M \mid A(m) > 0\}$ is called the *support* of a fuzzy set A .

Let $A, B \in \mathcal{F}(M)$. We say that A is *less than or equal to* B and denoted it as $A \subseteq B$ if $A(m) \leq B(m)$ for any $m \in M$. Moreover, A is *equal to* B if $A \subseteq B$ and $B \subseteq A$.

Let $\{A_i \mid i \in I\}$ be a non-empty family of fuzzy sets on M . Then the *union* and *intersection* of A_i are defined as

$$\left(\bigcup_{i \in I} A_i\right)(m) := \bigvee_{i \in I} A_i(m) \quad \text{and} \quad \left(\bigcap_{i \in I} A_i\right)(m) := \bigwedge_{i \in I} A_i(m), \quad (2)$$

for any $m \in M$, respectively. Further, extensions of the operations \otimes and \rightarrow on L to the operations on $\mathcal{F}(M)$ are given by

$$(A \otimes B)(m) := A(m) \otimes B(m) \quad \text{and} \quad (A \rightarrow B)(m) := A(m) \rightarrow B(m), \quad (3)$$

respectively, for any $A, B \in \mathcal{F}(M)$ and $m \in M$. Finally, we introduce the difference of fuzzy sets A and B on M as follows:

$$(A \setminus B)(m) = A(m) \otimes \neg B(m) \quad (4)$$

for any $m \in M$.

3 NL-Quantifiers and Generalized Quantifiers

By *NL-quantifiers*, we in this paper understand natural language expressions such as “for all”, “many”, “several”, etc. For our purposes it is not necessary to delineate the class of NL-quantifiers formally. In fact, we are interested in *general mathematical models* of these NL-quantifiers. For the sake of comprehensibility, we in the following informal explanation consider NL-quantifiers with two arguments, such as “some” in the sentence “Some people are clever.”

3.1 Generalized Quantifiers

Generally (see [13]), a model of the NL-quantifier “some” takes the form of a functional (the so-called *global* quantifier) **some** that to any universe of discourse

M assigns a *local* quantifier some_M . This local quantifier is a mapping that to any two subsets A and B of M assigns a truth value $\text{some}_M(A, B)$. In the following, if we speak about a quantifier Q , we have in mind some *global* quantifier, that is, the functional as above. If we consider only (classical) sets A and B and the truth value of $\text{some}_M(A, B)$ can be only either *true* or *false*, we say that this *some* is a *generalized quantifier*. If A and B are fuzzy sets and the truth value of $\text{some}_M(A, B)$ is taken from some many-valued structure of truth degrees, we say that this *some* is a *fuzzy quantifier*.

Definition 1 (Local generalized quantifier). *Let M is a universe of discourse. A local generalized quantifier Q_M of type $\langle 1^n, 1 \rangle$ over M is a function $\mathcal{P}(M)^n \times \mathcal{P}(M) \rightarrow \mathbf{2}$ that to any sets A_1, \dots, A_n and B from $\mathcal{P}(M)$ assigns a truth value $Q_M(A_1, \dots, A_n, B)$ from $\mathbf{2}$.*

Definition 2 (Global generalized quantifier). *A global generalized quantifier Q of type $\langle 1^n, 1 \rangle$ is a functional that to any universe M assigns a local generalized quantifier $Q_M: \mathcal{P}(M)^n \times \mathcal{P}(M) \rightarrow \mathbf{2}$ of type $\langle 1^n, 1 \rangle$.*

Among the most important examples of generalized quantifiers of type $\langle 1 \rangle$ belong the classical quantifiers \forall and \exists . Their definitions are as follows: $\forall_M(B) = 1$ if and only if $B = M$ and $\exists_M(B) = 1$ if and only if $B \neq \emptyset$ for any $B \in \mathcal{P}(M)$. Formally, these definitions can be also expressed as $\forall_M(B) := B = M$ and $\exists_M(B) := B \neq \emptyset$. The important examples of type $\langle 1, 1 \rangle$ generalized quantifiers are *all* and *some*, defined as $\text{all}_M(A, B) = 1$ if and only if $A \subseteq B$ and $\text{some}_M(A, B) = 1$ if and only if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{P}(M)$. Note that the universe M does not appear on the right side of definitions of *all* and *some*, which is a difference from the quantifier \forall_M , therefore, the truth values of these quantifiers are not directly influenced by their universes. The quantifiers, which possess this essential (semantic) property, are in the generalized quantifier theory said to satisfy the *extension*. Among further essential properties of generalized quantifiers belong the *permutation* and *isomorphism invariance* or the *conservativity*. More about these properties can be found in [9, 13].

Let us recall the definition of the relativization of generalized quantifiers [13] mentioned in Sect. 1.

Definition 3. *Let Q be a global generalized quantifier of type $\langle 1^n, 1 \rangle$. The relativization of Q is a global generalized quantifier Q^{rel} of type $\langle 1^{n+1}, 1 \rangle$ defined as*

$$(Q^{\text{rel}})_M(A, A_1, \dots, A_n, B) := Q_A(A \cap A_1, \dots, A \cap A_n, A \cap B) \quad (5)$$

for all $A, A_1, \dots, A_n, B \in \mathcal{P}(M)$. For the most common case of relativization from type $\langle 1 \rangle$ to type $\langle 1, 1 \rangle$,

$$(Q^{\text{rel}})_M(A, B) = Q_A(A \cap B). \quad (6)$$

It is well known that $\forall^{\text{rel}} = \text{all}$ and $\exists^{\text{rel}} = \text{some}$. In [13], the authors argue that all models of NL-quantifiers of the most common type $\langle 1, 1 \rangle$ should be conservative and satisfy the property of extension. It is very interesting that each type $\langle 1, 1 \rangle$ generalized quantifier, which possesses the above mentioned properties, is the relativization of a type $\langle 1 \rangle$ generalized quantifier.

3.2 Restriction

In the theory of generalized quantifiers, we can distinguish two interesting notions, namely, global generalized quantifiers restricted to a set and local generalized quantifiers living on a set, that play an undoubtedly important rôle in the characterization of generalized quantifiers, but they have not been taken into account for fuzzy quantifiers yet. Note that these notions are considered in [13] for quantifiers of type $\langle 1 \rangle$. In this part, we recall (for type $\langle 1 \rangle$) and extend (for type $\langle 1^n, 1 \rangle$) the concept of the generalized quantifier restricted to a set.

Definition 4. Let Q be a type $\langle 1 \rangle$ global generalized quantifier, and let A be a set. The quantifier Q is restricted to A if for any M and $B \subseteq M$ we have

$$Q_M(B) = Q_A(A \cap B). \quad (7)$$

The set of all type $\langle 1 \rangle$ global generalized quantifiers restricted to A is denoted by $RST_{\langle 1 \rangle}(A)$.

A natural generalization of the concept of generalized quantifiers restricted to a set for quantifiers of type $\langle 1^n, 1 \rangle$ can be provided as follows:

Definition 5. Let Q be a type $\langle 1^n, 1 \rangle$ global generalized quantifier, and let A be a set. The quantifier Q is restricted to A if for any M and $A_1, \dots, A_n, B \subseteq M$ we have

$$Q_M(A_1, \dots, A_n, B) = Q_A(A \cap A_1, \dots, A \cap A_n, A \cap B). \quad (8)$$

The set of all type $\langle 1^n, 1 \rangle$ global generalized quantifiers restricted to A is denoted by $RST_{\langle 1^n, 1 \rangle}(A)$.

One can see that if a global generalized quantifier Q is restricted to a set A , then its evaluations over all considered universes are determined from the local generalized quantifier Q_A .

Lemma 1. If Q is a type $\langle 1^n, 1 \rangle$ global generalized quantifier restricted to a set A , then Q is restricted to any set A' such that $A \subseteq A'$, i.e., $RST_{\langle 1^n, 1 \rangle}(A) \subseteq RST_{\langle 1^n, 1 \rangle}(A')$.

Now we show that global generalized quantifiers restricted to sets can be used to introduce the relativization of a global generalized quantifier Q . More precisely, let Q be a type $\langle 1^n, 1 \rangle$ generalized quantifier. Then, for each set A , we can introduce the type $\langle 1^n, 1 \rangle$ global generalized quantifier $Q^{[A]}$ as

$$(Q^{[A]})_M(A_1, \dots, A_n, B) = Q_A(A \cap A_1, \dots, A \cap A_n, A \cap B)$$

for any M and $A_1, \dots, A_n, B \subseteq M$. It is easy to show that $Q^{[A]} \in RST_{\langle 1^n, 1 \rangle}(A)$. It should be noted that Q and $Q^{[A]}$ are different quantifiers. They become identical if Q is already restricted to A . Now, if we define the global generalized quantifier Q' of type $\langle 1^{n+1}, 1 \rangle$ as

$$Q'_M(A, A_1, \dots, A_n, B) = (Q^{[A]})_M(A_1, \dots, A_n, B) \quad (9)$$

for all M and $A_1, \dots, A_n, B \subseteq M$, then it is easy to show that $Q' = Q^{\text{rel}}$, where Q^{rel} has been defined in Definition 3.

3.3 Generalized Quantifiers Living on a Set

A concept related to that of a global generalized quantifier restricted to a set is the concept of a local generalized quantifier living on a set.

Definition 6. *Let M and A be sets. A local generalized quantifier Q_M of type $\langle 1 \rangle$ lives on A if, for any $B \subseteq M$, we have*

$$Q_M(B) = Q_M(A \cap B). \quad (10)$$

One can see that each local generalized quantifier Q_M , where $Q \in \text{RST}_{\langle 1 \rangle}(A)$ (that is, Q is a global generalized quantifier restricted to A and Q_M is the corresponding local quantifier over M), lives on A . Indeed, by (7), we have

$$Q_M(B) = Q_A(A \cap B) = Q_A(A \cap (A \cap B)) = Q_M(A \cap B).$$

Note that A need not be the smallest set on which Q_M lives ($Q \in \text{RST}_{\langle 1 \rangle}(A)$). Moreover, the concept of *conservativity* can be introduced in terms of local generalized quantifiers living on sets. Let Q be a global conservative quantifier of type $\langle 1, 1 \rangle$.² For any M and $A \subseteq M$, define a type $\langle 1 \rangle$ local quantifier $(Q[A])_M$ as follows

$$(Q[A])_M(B) = Q_M(A, B) \quad (11)$$

for all $B \subseteq M$. One can see that, as a simple consequence of the conservativity of Q , we obtain that $(Q[A])_M$ lives on A . Vice versa, if each local generalized quantifier $(Q[A])_M$ lives on A for any M , then Q is conservative. Note that Barwise and Cooper expressed the concept of conservativity as we described above using the live-on property [1].

A natural generalization of the concept of a local generalized quantifier living on a set for quantifiers of type $\langle 1^n, 1 \rangle$ can be defined as follows:

Definition 7. *Let M and A be sets. The local generalized quantifier Q_M of type $\langle 1^n, 1 \rangle$ lives on A if, for any $A_1, \dots, A_n, B \subseteq M$, we have*

$$Q_M(A_1, \dots, A_n, B) = Q_M(A \cap A_1, \dots, A \cap A_n, A \cap B). \quad (12)$$

Also in this case, a conservativity of global generalized quantifiers of type $\langle 1^n, 1 \rangle$ can be expressed in terms of the live-on property. Indeed, let Q be a global quantifier of type $\langle 1^n, 1 \rangle$ for $n \geq 1$. For any M and $A_1, \dots, A_n \subseteq M$, define local quantifier $(Q[A_1, \dots, A_n])_M$ of type $\langle 1 \rangle$ as

$$(Q[A_1, \dots, A_n])_M(B) = Q_M(A_1, \dots, A_n, B). \quad (13)$$

² Recall that a type $\langle 1, 1 \rangle$ global generalized quantifier is *conservative* if $Q_M(A, B) = Q_M(A, B')$ holds for any $A, B, B' \in \mathcal{P}(M)$ such that $A \cap B = A \cap B'$. The conservativity for a type $\langle 1^n, 1 \rangle$ generalized quantifier is defined analogously such that $Q_M(A_1, \dots, A_n, B) = Q_M(A_1, \dots, A_n, B')$ holds for any $A_1, \dots, A_n, B, B' \in \mathcal{P}(M)$ such that $A_i \cap B = A_i \cap B'$ for $i = 1, \dots, n$.

It is easy to see that Q is conservative if and only if each local generalized quantifier $(Q[A_1, \dots, A_n])_M$ lives on $A = \bigcup_{i=1}^n A_i$.

The following lemma contains useful facts about generalized quantifiers living on sets (cf. [13, Section 3.2.2]).

Lemma 2. *Let Q be a type $\langle 1^n, 1 \rangle$ global generalized quantifier, and let M be a set. Then*

- (i) Q_M lives on M .
- (ii) Q_M lives on \emptyset if and only if Q_M is trivial (i.e., $Q_M(A_1, \dots, A_n, B) = 0$ for any $A_1, \dots, A_n, B \subseteq M$ or $Q_M(A_1, \dots, A_n, B) = 1$ for any $A_1, \dots, A_n, B \subseteq M$).
- (iii) If Q_M lives on C_1 and C_2 , then it lives on $C_1 \cap C_2$. Hence, if M is finite, there is always a smallest set on which Q_M lives. This fails, however, when M is infinite.
- (iv) $(Q^{[A]})_M$ lives on A and its supersets. If $(Q^{[A]})_M$ is non-trivial, A need not be the smallest set on which $(Q^{[A]})_M$ lives.

4 Fuzzy Quantifiers over Fuzzy Universes

The aim of this section is to introduce the concept of fuzzy quantifiers defined over fuzzy universes. We start with the introduction of (local and global) fuzzy quantifiers over crisp universes, where we demonstrate the limitation of their definitions. This motivates us to introduce the concept of fuzzy universe and define (local and global) fuzzy quantifiers over such universes.

4.1 Fuzzy Quantifiers over Crisp Universes

An immediate generalization of Definitions 1 and 2 consists of replacing classical sets A_1, \dots, A_n and B by fuzzy subsets of M and of using a residuated lattice L instead of the Boolean algebra $\mathbf{2}$ (see [6, 8]).

Definition 8 (Local fuzzy quantifier). *Let M be a universe of discourse. A local fuzzy quantifier Q_M of type $\langle 1^n, 1 \rangle$ over M is a function $Q_M : \mathcal{F}(M)^n \times \mathcal{F}(M) \rightarrow L$ that to any fuzzy sets A_1, \dots, A_n and B from $\mathcal{F}(M)$ assigns a truth value $Q_M(A_1, \dots, A_n, B)$ from L .*

Definition 9 (Global fuzzy quantifier). *A global fuzzy quantifier Q of type $\langle 1^n, 1 \rangle$ is a functional that to any universe M assigns a local fuzzy quantifier $Q : \mathcal{F}(M)^n \times \mathcal{F}(M) \rightarrow L$ of type $\langle 1^n, 1 \rangle$. The set of all global fuzzy quantifiers of type $\langle 1^n, 1 \rangle$ will be denoted by $\text{QUANT}_{\langle 1^n, 1 \rangle}$.*

Among the important examples of global fuzzy quantifiers of type $\langle 1 \rangle$ are, again, \forall and \exists . They are standardly defined as

$$\forall_M(B) := \bigwedge_{m \in M} B(m) \quad (14)$$

and

$$\exists_M(B) := \bigvee_{m \in M} B(m). \quad (15)$$

Important examples of global fuzzy quantifiers of type $\langle 1, 1 \rangle$ are **all** and **some**, defined as

$$\text{all}_M(A, B) := \bigwedge_{m \in M} (A \rightarrow B)(m) \quad (16)$$

and

$$\text{some}_M(A, B) := \bigvee_{m \in M} (A \cap B)(m). \quad (17)$$

As we have mentioned in Sect. 1, there is a principal problem to introduce the relativization of a fuzzy quantifier, because one argument of a fuzzy quantifier becomes a universe and a fuzzy set as the universe is not permitted. Therefore, in [8], the relativization of a global fuzzy quantifier Q of type $\langle 1 \rangle$ has been proposed:

$$Q_M^{\text{rel}}(A, B) := Q_{\text{Supp}(A)}(A \cap B), \quad (18)$$

where $\text{Supp}(A)$ is used as the universe of the fuzzy quantifier Q instead of A , which is the fuzzy set in the first argument of fuzzy quantifier Q_M^{rel} . Unfortunately, this solution is generally not satisfactory. For example, one can simply derive that

$$(\forall^{\text{rel}})_M(A, B) := \bigwedge_{m \in \text{Supp}(A)} (A \cap B)(m) \neq \text{all}_M(A, B),$$

which is counter-intuitive. Hence, the definition of relativization does not seem to be well established. Another concept called weak relativization was also provided in [8], but again it generally fails. In [5], it was proved that there is no satisfactory definition of relativization of fuzzy quantifiers of type $\langle 1 \rangle$. However, an introduction of fuzzy sets as universes for fuzzy quantifiers is not motivated only by relativization of fuzzy quantifiers. The second example can be the concept of restriction of a fuzzy quantifier to a fuzzy set (see Definition 5). Thus, the absence of fuzzy sets as universes for fuzzy quantifiers brings significant limitations in the development of the fuzzy quantifier theory that should possibly cover a wide part of topics studied in the field of generalized quantifiers.

4.2 Fuzzy Universes

Let A be a fuzzy set on N , and let M be a set. Define $A_M : M \rightarrow L$ as

$$A_M(m) = \begin{cases} A(m), & \text{if } m \in M \cap N, \\ 0, & \text{otherwise.} \end{cases} \quad (19)$$

The fuzzy set A_M represents A (or its part) on the universe M . Obviously, if $A \in \mathcal{F}(M)$, then $A_M = A$. It is easy to see that $\text{Supp}(A_M) = \text{Supp}(A) \cap M$. Moreover, if A, B are fuzzy sets on N and M is a set, then we have $(A \cap B)_M = A_M \cap B_M$ and $(A \cup B)_M = A_M \cup B_M$.

A pair (M, A) , where M is a set and A is a fuzzy set on M , is called a *fuzzy universe*. Obviously, if $M = \emptyset$ in (M, A) , then $A = \emptyset$ is the empty function. A fuzzy universe (M, A) is said to be *crisp* if A is crisp and $A = 1_M$. Let (M, A) and (N, B) be fuzzy universes. The basic (fuzzy) set operations for fuzzy universes are defined as follows:

- $(M, A) \cap (N, B) = (K, A_K \cap B_K)$, where $K = M \cap N$;³
- $(M, A) \cup (N, B) = (K, A_K \cup B_K)$, where $K = M \cup N$;
- $(M, A) \setminus (N, B) = (M, A_M \setminus B_M)$.

For the sake of simplicity, we write simply $(M, A) \cap (N, B) = (M \cap N, A \cap B)$ and assume that $A \cap B$ is well introduced on the universe $M \cap N$ according to the definition above. A similar notation can be also used for the union and the difference of fuzzy universes.

A non-empty class \mathcal{U} of fuzzy universes is said to be *well defined* if

- C1) $(M, A) \in \mathcal{U}$ implies $(M, B) \in \mathcal{U}$ for any $B \in \mathcal{F}(M)$;
 C2) \mathcal{U} is closed under the intersection, union and difference.

In what follows, we assume that each class of fuzzy universes is well defined.

Fundamental binary relations for fuzzy universes in a class \mathcal{U} can be introduced as follows. We say that (M, A) is *equal to* (N, B) , and denote it by $(M, A) = (N, B)$, if $M = N$ and $A = B$. Moreover, we say that (M, A) is *equal to* (N, B) *up to negligible elements*, and denote it by $(M, A) \sim (N, B)$, if $\text{Supp}(A) = \text{Supp}(B)$ and $A_{\text{Supp}(A)} = B_{\text{Supp}(A)}$. Obviously, $(M, A) \sim (N, B)$ if and only if $(\text{Supp}(A), A_{\text{Supp}(A)}) = (\text{Supp}(B), B_{\text{Supp}(B)})$.⁴ Note that if $\text{Supp}(A) = \emptyset = \text{Supp}(B)$, then $A = \emptyset$ on M and $B = \emptyset$ on N . It is easy to see that for any (M, A) and a set $N \supseteq M$, there exists exactly one fuzzy set A' on N such that $(M, A) \sim (N, A')$. This fuzzy set A' is called the *extension of A from M to N* . We say that (M, A) is a *subset of* (N, B) , and denote it by $(M, A) \subseteq (N, B)$, if $(M, A) \cap (N, B) \sim (M, A)$ (or, equivalently, $(M, A) \cup (N, B) \sim (N, B)$).

The following two statements show properties of the equality relation up to negligible elements.

Lemma 3. *If $(M, A) \sim (M', A')$, then $(N, A_N) = (N, A'_N)$ for any set N such that $(N, A_N) \in \mathcal{U}$.*

Theorem 1. *The binary relation \sim on \mathcal{U} is a congruence with respect to the intersection, union and difference of fuzzy universes.*

³ If $K = M \cap N = \emptyset$, then $A_K \cap B_K$ is the empty mapping.

⁴ Note that $(\text{Supp}(A), A_{\text{Supp}(A)}) \notin \mathcal{U}$ in general, but it is not a problem to extend the class \mathcal{U} by such fuzzy universes. Then we can use this equality for the verification that $(M, A) \sim (N, B)$.

4.3 Fuzzy Quantifiers over Fuzzy Universes

In Sect. 4.1, we demonstrated the limitations of the definition of fuzzy quantifiers over set universes. In this subsection we introduce the concept of fuzzy quantifiers defined over fuzzy universes (see Sect. 4.2) in such a way that it overcomes these limitations.

Definition 10 (Local fuzzy quantifier over a fuzzy universe). *Let (M, C) be a fuzzy universe. A local fuzzy quantifier $Q_{(M,C)}$ of type $\langle 1^n, 1 \rangle$ on (M, C) is a function $Q_{(M,C)} : \mathcal{F}(M)^n \times \mathcal{F}(M) \rightarrow L$ that to any fuzzy sets A_1, \dots, A_n and B from $\mathcal{F}(M)$ assigns a truth value $Q_{(M,C)}(A_1, \dots, A_n, B)$ from L and*

$$Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(M,C)}(A'_1, \dots, A'_n, B') \quad (20)$$

holds for any $A_1, \dots, A_n, B, A'_1, \dots, A'_n, B' \in \mathcal{F}(M)$ such that $A_i \cap C = A'_i \cap C$ for any $i = 1, \dots, n$ and $B \cap C = B' \cap C$.

One can see that the local fuzzy quantifier $Q_{(M,C)}$, which is defined over a fuzzy universe (M, C) , is, in fact, a fuzzy quantifier defined on M that lives on a fuzzy set C (cf. Definition 14). Hence, an analysis of properties of fuzzy quantifiers over fuzzy universes can be practically restricted to fuzzy subsets of the fuzzy set C as it was proposed in [5].

Definition 11 (Global fuzzy quantifier over fuzzy universes). *Let \mathcal{U} be a class of fuzzy universes. A global fuzzy quantifier Q of type $\langle 1^n, 1 \rangle$ is a functional assigning to any fuzzy universe $(M, C) \in \mathcal{U}$ a local fuzzy quantifier $Q_{(M,C)}$ of type $\langle 1^n, 1 \rangle$ such that for any $(M, C), (M', C') \in \mathcal{U}$ with $(M, C) \sim (M', C')$, it holds that*

$$Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(M',C')}(A'_1, \dots, A'_n, B') \quad (21)$$

for any $A_1, \dots, A_n, B \in \mathcal{F}(M)$ and $A'_1, \dots, A'_n, B' \in \mathcal{F}(M')$ such that $(M, A_i) \sim (M', A'_i)$ for any $i = 1, \dots, n$ and $(M, B) \sim (M', B')$.

We should note that condition (21) ensures that if two fuzzy universes are equal up to negligible elements, then the fuzzy quantifiers defined over them are practically identical. More precisely, their evaluations coincide for fuzzy sets that together with their universes are equal up to negligible elements.

Before we provide an example of fuzzy quantifiers defined over fuzzy universes, let us define a binary fuzzy relation of fuzzy equivalence for fuzzy sets on a fuzzy universe. Let $(M, C) \in \mathcal{U}$ be a fuzzy universe. A mapping $\cong_{(M,C)} : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow L$ defined as

$$(A \cong_{(M,C)} B)(m) = \bigwedge_{m \in M} ((A \cap C)(m) \leftrightarrow (B \cap C)(m)) \quad (22)$$

is called a *fuzzy equivalence* on (M, C) .

Example 2. Let \mathcal{U} be a family of fuzzy universes. A global fuzzy quantifier over fuzzy universes \forall of type $\langle 1 \rangle$ assigns to any $(M, C) \in \mathcal{U}$ a local fuzzy quantifier over fuzzy universes $\forall_{(M,C)}: \mathcal{F}(M) \rightarrow L$ defined for any $B \in \mathcal{F}(M)$ as

$$\forall_{(M,C)}(B) := B \cong_{(M,C)} C,$$

where $\cong_{(M,C)}$ is the fuzzy equivalence (22) on (M, C) .⁵

Due to the definition of the fuzzy equivalence on (M, C) , we can write

$$\begin{aligned} \forall_{(M,C)}(B) = B \cong_{(M,C)} C &= \bigwedge_{m \in M} ((B \cap C)(m) \leftrightarrow (C \cap C)(m)) = \\ &= \bigwedge_{m \in M} ((B \cap C)(m) \leftrightarrow C(m)) = \bigwedge_{m \in M} ((B(m) \wedge C(m)) \leftrightarrow C(m)) = \\ &= \bigwedge_{m \in M} (C(m) \rightarrow B(m)) = \bigwedge_{m \in M} (C \rightarrow B)(m), \end{aligned} \quad (23)$$

where we used the equality $(a \wedge b) \leftrightarrow b = b \rightarrow a$ holding for any $a, b \in L$ in every residuated lattice L . If (M, C) is crisp, then

$$\forall_{(M,C)}(B) = \bigwedge_{m \in M} (C(m) \rightarrow B(m)) = \bigwedge_{m \in M} (1 \rightarrow B(m)) = \bigwedge_{m \in M} B(m),$$

that is, it coincides with the standard definition of the fuzzy quantifier \forall_M provided in (14).

Finally, we define *relativization* for fuzzy quantifiers defined over fuzzy universes as follows (cf. Definition 3).

Definition 12 (Relativization of fuzzy quantifiers over fuzzy universes). Let Q be a global fuzzy quantifier of type $\langle 1^n, 1 \rangle$ over fuzzy universes. The relativization of Q is a global fuzzy quantifier Q^{rel} of type $\langle 1^{n+1}, 1 \rangle$ over fuzzy universes defined as

$$(Q^{\text{rel}})_{(M,C)}(A, A_1, \dots, A_n, B) := Q_{(M, C \cap A)}(A \cap A_1, \dots, A \cap A_n, A \cap B) \quad (24)$$

for all $A, A_1, \dots, A_n, B \in \mathcal{F}(M)$. For the most common case of relativization from type $\langle 1 \rangle$ to type $\langle 1, 1 \rangle$,

$$(Q^{\text{rel}})_{(M,C)}(A, B) = Q_{(M, C \cap A)}(A \cap B). \quad (25)$$

4.4 Restriction

As we mentioned in Subsect. 4.1, we are unable to extend the concept of restriction for fuzzy quantifiers defined over crisp universes. In this part, we show that if we employ fuzzy universes for fuzzy quantification, we can introduce this concept

⁵ Note the structural similarity of this definition with the definition of the generalized quantifier $\forall_M(B) := B = M$ given below Definition 2.

in an elegant way following the standard definition. In the rest of the paper, by a fuzzy quantifier we mean a fuzzy quantifier over fuzzy universes unless stated otherwise.

The concept of fuzzy quantifiers of type $\langle 1^n, 1 \rangle$ restricted to a fuzzy set can be introduced as follows.

Definition 13. Let Q be a type $\langle 1^n, 1 \rangle$ global fuzzy quantifier, and let A be a fuzzy set on N . The fuzzy quantifier Q is restricted to A if for any $(M, C) \in \mathcal{U}$ and $A_1, \dots, A_n, B \in \mathcal{F}(M)$ we have

$$Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(N,A)}(A \cap (C \cap A_1)_N, \dots, A \cap (C \cap A_n)_N, A \cap (C \cap B)_N). \quad (26)$$

The set of all type $\langle 1^n, 1 \rangle$ global fuzzy quantifiers restricted to a fuzzy set A on a universe N is denoted by $FRST_{\langle 1^n, 1 \rangle}(N, A)$.

Similarly to the classical case of the restriction to a set, a global fuzzy quantifier, which is restricted to a fuzzy set A on a universe N , is determined from the local fuzzy quantifier $Q_{(N,A)}$. One can simply verify that the previous definition of the restriction to a fuzzy set is correct in the sense of Definition 11. Note that such verification is useless for the global generalized quantifiers, because their definition has no requirement on the functionals defining them. Obviously, an equivalent definition could be as follow. A fuzzy quantifier Q is restricted to a fuzzy set A on a universe N if for any $(M, C) \in \mathcal{U}$ and $A_1, \dots, A_n, B \in \mathcal{F}(M)$ we have

$$Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(N,A \cap C_N)}(A \cap A_{1,N}, \dots, A \cap A_{n,N}, A \cap B_N). \quad (27)$$

The following lemma is a generalization of Lemma 1 for fuzzy quantifiers defined over fuzzy universes.

Lemma 4. If Q is a type $\langle 1^n, 1 \rangle$ global generalized quantifier restricted to a fuzzy set A on N , then Q is restricted to any fuzzy set A' on an arbitrary universe N' such that $(N, A) \subseteq (N', A')$, i.e., $FRST_{\langle 1^n, 1 \rangle}(N, A) \subseteq FRST_{\langle 1^n, 1 \rangle}(N', A')$.

Let us show that the global fuzzy quantifiers restricted to fuzzy sets can be used to introduce the relativization of global fuzzy quantifiers. Let Q be a type $\langle 1^n, 1 \rangle$ fuzzy quantifier. Then, for each fuzzy set A on a universe N , we can introduce the type $\langle 1^n, 1 \rangle$ global fuzzy quantifier $Q^{[(N,A)]}$ as

$$(Q^{[(N,A)]})_{(M,C)}(A_1, \dots, A_n, B) = Q_{(N,A)}(A \cap (C \cap A_1)_N, \dots, A \cap (C \cap A_n)_N, A \cap (C \cap B)_N)$$

for any $(M, C) \in \mathcal{U}$ and $A_1, \dots, A_n, B \in \mathcal{F}(M)$. It is easy to see that $Q^{[(N,A)]} \in FRST_{\langle 1^n, 1 \rangle}(N, A)$. It should be noted that Q and $Q^{[(N,A)]}$ are different quantifiers. They become identical if Q is already restricted to the fuzzy set A on the universe N . Now, if we define the global fuzzy quantifier Q' of type $\langle 1^{n+1}, 1 \rangle$ as

$$Q'_{(M,C)}(A, A_1, \dots, A_n, B) = (Q^{[(M,A)]})_{(M,C)}(A_1, \dots, A_n, B) \quad (28)$$

for all $(M, C) \in \mathcal{U}$ and $A_1, \dots, A_n, B \in \mathcal{F}(M)$, then using (27) it is easy to show that $Q' = Q^{\text{rel}}$, where Q^{rel} has been defined in Definition 12.

4.5 Fuzzy Quantifiers Living on a Fuzzy Set

A concept related to that of a global fuzzy quantifier restricted to a fuzzy set is the concept of a local fuzzy quantifier living on a fuzzy set.

Definition 14. Let $(M, C) \in \mathcal{U}$ and A be a fuzzy set on a universe N . The local fuzzy quantifier $Q_{(M,C)}$ of type $\langle 1^n, 1 \rangle$ lives on A if, for any $A_1, \dots, A_n, B \in \mathcal{F}(M)$, we have

$$Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(M,C)}(A_M \cap A_1, \dots, A_M \cap A_n, A_M \cap B). \quad (29)$$

If the distributivity of \wedge over \vee is satisfied in a residuated lattice L (e.g., L is an MV-algebra), the conservativity of global fuzzy quantifiers of type $\langle 1^n, 1 \rangle$ can be expressed in terms of the live-on property.⁶ Indeed, let Q be a global fuzzy quantifier of type $\langle 1^n, 1 \rangle$ for $n \geq 1$. For any fuzzy universe (M, C) and $A_1, \dots, A_n \in \mathcal{F}(M)$, define local quantifier $(Q[A_1, \dots, A_n])_{(M,C)}$ of type $\langle 1 \rangle$ as

$$(Q[A_1, \dots, A_n])_{(M,C)}(B) = Q_{(M,C)}(A_1, \dots, A_n, B). \quad (30)$$

One can show that Q is conservative if and only if each local generalized quantifier $(Q[A_1, \dots, A_n])_M$ lives on $A = \bigcup_{i=1}^n A_i$. Note that the distributivity of \wedge over \vee ensures the crucial equality $(\bigcup_{i=1}^n A_i) \cap B = \bigcup_{i=1}^n (A_i \cap B)$ important in the proof of a characterization of the conservativity of type $\langle 1^n, 1 \rangle$ global fuzzy quantifiers.

The following lemma specifies basic facts about fuzzy quantifiers living on fuzzy sets (cf. Lemma 2).

Lemma 5. Let Q be a type $\langle 1^n, 1 \rangle$ global fuzzy quantifier, let (M, C) be a fuzzy universe from \mathcal{U} , and let $C_1 \in \mathcal{F}(N)$ and $C_2 \in \mathcal{F}(N')$ be fuzzy sets. Then

- (i) $Q_{(M,C)}$ lives on C .
- (ii) $Q_{(M,C)}$ lives on \emptyset if and only if $Q_{(M,C)}$ is trivial ($Q_{(M,C)}(A_1, \dots, A_n, B) = a$ for any $A_1, \dots, A_n, B \in \mathcal{F}(M)$ with $a \in L$).
- (iii) If $Q_{(M,C)}$ lives on C_1 and C_2 , then it lives on $C_{1,N \cap N'} \cap C_{2,N \cap N'}$.
- (iv) $(Q^{[(N,A)]})_{(M,C)}$ lives on A .

5 Conclusion

In this article, we proposed a novel framework for fuzzy quantifiers which are defined over fuzzy universes. We introduced the concept of a fuzzy universe and define a class of fuzzy universes closed under the operations of intersection, union

⁶ A global fuzzy quantifier Q of type $\langle 1^n, 1 \rangle$ over fuzzy universes is *conservative*, if for any $(M, C) \in \mathcal{U}$ and $A_1, \dots, A_n, B, B' \in \mathcal{F}(M)$ it holds that if $A_i \cap B = A_i \cap B'$ for $i = 1, \dots, n$, then $Q_{(M,C)}(A_1, \dots, A_n, B) = Q_{(M,C)}(A_1, \dots, A_n, B')$.

and difference. Moreover, we established a binary relation called the equivalence up to negligible elements, which is essential in the definition of global fuzzy quantifiers over fuzzy universes. Further, we generalized the fuzzy quantifiers defined over crisp universes to fuzzy quantifiers defined over fuzzy universes. The novel conception of fuzzy quantifiers naturally allows us to introduce, for example, the notion of relativization of fuzzy quantifiers, which principally could not be defined in the case of fuzzy quantifiers if only crisp universes are permitted. For an illustration, we investigated the important semantic notions of restriction to a fuzzy set and living on a fuzzy set in our novel framework for fuzzy quantifiers defined over fuzzy universes. It should be noted that the notion of restriction to a fuzzy set also could not be introduced for fuzzy quantifiers defined over crisp universes. Although the presented work is only an initial study, it shows that the fuzzy quantifiers over fuzzy universes enable us to develop the fuzzy quantifier theory in the same fashion as in the theory of generalized quantifiers. In our future research, we will concentrate on investigation of further semantic properties of fuzzy quantifiers over fuzzy universes, e.g., the property of extension and isomorphism invariance.

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