



On Categories of L -Fuzzifying Approximation Spaces, L -Fuzzifying Pretopological Spaces and L -Fuzzifying Closure Spaces

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Abstract. This paper investigates the essential connections among several categories with a weaker structure than that of L -fuzzifying topology, namely category of L -fuzzifying approximation spaces based on reflexive L -fuzzy relations, category of L -fuzzifying pretopological spaces and category of L -fuzzifying interior (closure) spaces. The interrelations among these structures are established in categorical setup.

Keywords: L -fuzzifying approximation space · L -fuzzifying pretopological space · Čech L -fuzzifying interior (closure) spaces · Galois connection

1 Introduction

Since the introduction of the rough set by Pawlak [11], this powerful theory drawn the attention of many researchers due to its importance in the study of intelligent systems with insufficient and incomplete information. Several generalizations of rough sets have been made by replacing the equivalence relation by an arbitrary relation. Dubois and Prade [3] generalized this theory and introduced the concept of fuzzy rough set. Various types of fuzzy rough approximation operators have been introduced and studied (c.f. [9, 17–21]) in the context of fuzzy rough set theory. The most well known introduced fuzzy rough set is obtained by replacing the crisp relations with fuzzy relations and the crisp subset of the universe by fuzzy sets. Further, a rough fuzzy set was introduced in [23] by considering the fuzzy approximated subsets and crisp relations. In [25] Yao, introduced another kind of fuzzy rough set which is based on fuzzy relations and crisp approximated subsets, and is further studied by Pang [10] through the constructive and axiomatic approach. Several interesting studies have been carried on relating the theory of fuzzy rough sets with fuzzy topologies (cf., [2, 6, 13, 16, 19, 22]). Further, Ying [26] introduced a logical approach to study the fuzzy topology and proposed the notion of fuzzifying topology. In brief, a

fuzzifying topology on a set X assigns to every crisp subset of X a certain degree of being open. A number of articles were published based on this new approach (cf., [4, 5, 8, 24, 29, 30]). Fang [4, 5] showed the one to one correspondence between fuzzifying topologies and fuzzy preorders and Shi [24] discussed the relationship of fuzzifying topology and specialization preorder in the sense of Lai and Zhang [7]. In 1999, Zhang [28] studied the fuzzy pretopology through the categorical point of view and Perfilieva et al. in [12, 14] discussed its relationship with F-transform. Further following the approach of Ying [26], Lowen and Xu [8], Zhang [30] discussed the categorical study of fuzzifying pretopology.

Recently, Pang [10] followed the approach of Ying [26] and studied *L*-fuzzifying approximation operators through the constructive and axiomatic approaches. So far, the relationship among *L*-fuzzifying pretopological spaces, Čech *L*-fuzzifying interior (closure) spaces and *L*-fuzzifying approximation spaces has not been studied yet. In this paper, we will discuss such relationship in more details. It is worth to mention that our motivation is different from Qiao and Hu [15], in which such connection is established in the sense of Zhang [28] rather than *L*-fuzzifying pretopological setting. Specifically, we established the Galois connection between *L*-fuzzifying reflexive approximation space and *L*-fuzzifying pretopological spaces. Finally, we investigate the categorical relationship between Čech *L*-fuzzifying interior spaces and *L*-fuzzy relational structure.

2 Preliminaries

Throughout this paper, L denotes a De Morgan algebra $(L, \vee, \wedge, ', 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a complete lattice with the least element 0 and greatest element 1 and an order reversing involution “ $'$ ”. For any $a \subseteq L$, $\bigvee a$ and $\bigwedge a$ are respectively the least upper bound and the greatest lower bound of a . In particular, we have $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$.

Let X be a nonempty set. The set of all subsets of X will be denoted by $\mathcal{P}(X)$ and called powerset of X . For $\lambda \in \mathcal{P}(X)$, λ^c is the complement of λ and characteristic function of λ is 1_λ . Let X, Y be two nonempty sets and $f : X \rightarrow Y$ be a mapping, then it can be extended to the powerset operator $f^\rightarrow : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ and $f^\leftarrow : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ such that for each $C \in \mathcal{P}(X)$, $f^\rightarrow(C) = \{f(x) : x \in C\}$ and for each $D \in \mathcal{P}(Y)$, $f^\leftarrow(D) = f^{-1}(D) = \{x : f(x) \in D\}$.

A map $f : X \rightarrow Y$ can be extended to the powerset operators $f^\rightarrow : L^X \rightarrow L^Y$ and $f^\leftarrow : L^Y \rightarrow L^X$ such that $\lambda \in L^X, \mu \in L^Y, y \in Y$,

$$f^\rightarrow(\lambda)(y) = \bigvee_{x, f(x)=y} \lambda(x), \quad f^\leftarrow(\mu) = \mu \circ f.$$

For a nonempty set X , L^X denotes the collection of all *L*-fuzzy subsets of X , i.e. a mapping $\lambda : X \rightarrow L$. Also, for all $a \in L$, $\mathbf{a}(x) = a$ is a constant *L*-fuzzy set on X . The greatest and least element of L^X is denoted by 1_X and 0_X respectively. For the sake of terminological economy, we will use the notation λ for both crisp

set and L -fuzzy set. Further, an L -fuzzy set $1_y \in L^X$ is called a *singleton*, if it has the following form

$$1_y(x) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Let X be a nonempty set. Then for $\lambda, \mu \in L^X$, we can define new L -fuzzy sets as follows:

$$\begin{aligned} \lambda = \mu &\iff \lambda(x) = \mu(x), \lambda \leq \mu \iff \lambda(x) \leq \mu(x), \\ (\lambda \wedge \mu)(x) &= \lambda(x) \wedge \mu(x), (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x), \\ (\lambda)'(x) &= (\lambda(x))', \forall x \in X. \end{aligned}$$

Let I be a set of indices, $\lambda_i \in L^X$, $i \in I$. The *meet* and *join* of elements from $\{\lambda_i \mid i \in I\}$ are defined as follows:

$$\left(\bigwedge_{i \in I} \lambda_i\right)(x) = \bigwedge_{i \in I} \lambda_i(x), \left(\bigvee_{i \in I} \lambda_i\right)(x) = \bigvee_{i \in I} \lambda_i(x).$$

Throughout this paper, all the considered categories are concrete. A *concrete category* (or construct) [1] is defined over **Set**. Specifically, it is a pair (\mathbf{C}, \mathbb{U}) , with \mathbf{C} as a category and $\mathbb{U} : \mathbf{C} \rightarrow \mathbf{Set}$ is a faithful (forgetful) functor. We say $\mathbb{U}(X)$ the underlying set for each \mathbf{C} -object X . We write simply \mathbf{C} for the pair (\mathbf{C}, \mathbb{U}) , since \mathbb{U} is clear from the context.

A *concrete functor* between concrete categories (\mathbf{C}, \mathbb{U}) and (\mathbf{D}, \mathbb{V}) is a functor $\mathbb{F} : \mathbf{C} \rightarrow \mathbf{D}$ with $\mathbb{U} = \mathbb{V} \circ \mathbb{F}$. It means, \mathbb{F} only changes structures on the underlying sets. For more on category we refer to [1].

Now we recall the following definition of L -fuzzy relation from [27].

Definition 1 [27]. *Let X be a nonempty set. An L -fuzzy relation θ on X is an L -fuzzy subset of $X \times X$. An L -fuzzy relation θ is called reflexive if $\theta(x, x) = 1$, $\forall x \in X$.*

A set X equipped with an L -fuzzy relation θ is denoted by (X, θ) and is called a *L -fuzzy relational structure*.

Below we define the category **FRS** of L -fuzzy relational structures.

1. The pairs (X, θ) with reflexive L -fuzzy relation θ on X are the *objects*, and
2. for the pairs (X, θ) and (Y, ρ) a *morphism* $f : (X, \theta) \rightarrow (Y, \rho)$ is a map $f : X \rightarrow Y$ such that $\forall x, y \in X, \theta(x, y) \leq \rho(f(x), f(y))$.

3 L -Fuzzifying Approximation Operators

In this section, we recall the notion of L -fuzzifying approximation operators and its properties presented in [10]. We also define the category of L -fuzzifying approximation space and show that this category is isomorphic to the category of fuzzy relational structures.

Definition 2 [10]. Let θ be an L -fuzzy relation on X . Then upper (lower) L -fuzzifying approximation of λ is a map $\bar{\theta}, \underline{\theta} : \mathcal{P}(X) \rightarrow L^X$ defined by;

$$\begin{aligned} (\forall \lambda \in \mathcal{P}(X), x \in X), \quad \bar{\theta}(\lambda)(x) &= \bigvee_{y \in \lambda} \theta(x, y), \\ (\forall \lambda \in \mathcal{P}(X), x \in X), \quad \underline{\theta}(\lambda)(x) &= \bigwedge_{y \notin \lambda} \theta(x, y)'. \end{aligned}$$

We call $\underline{\theta}, \bar{\theta}$ the lower L -fuzzifying approximation operator and the upper L -fuzzifying approximation operator respectively. Further, the pair $(\bar{\theta}, \underline{\theta})$ is called L -fuzzifying rough set and (X, θ) is called an L -fuzzifying approximation space based on L -fuzzy relation θ .

- (i) It is important to note that, if $\lambda = \{y\} \in \mathcal{P}(X)$ for some $y \in X$, then we have the upper L -fuzzifying approximation $\bar{\theta}(\{y\})(x) = \theta(x, y)$ for each $x \in X$. If $\lambda = X - \{y\} \in \mathcal{P}(X)$ for some $y \in X$, then we have the lower L -fuzzifying approximation $\underline{\theta}(X - \{y\})(x) = \theta(x, y)'$ for each $x \in X$.
- (ii) Let X be a nonempty set and θ be reflexive L -fuzzy relation on X . We call the pair (X, θ) , an L -fuzzifying reflexive approximation space.

Now, we give some useful properties of L -fuzzifying upper (lower) approximation operators from [10]. These properties will be used in the further text.

Proposition 1 [10]. Let (X, θ) be an L -fuzzifying reflexive approximation space. Then for $\lambda \in \mathcal{P}(X)$ and $\{\lambda_i \mid i \in I\} \subseteq \mathcal{P}(X)$, the following holds.

- (i) $\bar{\theta}(\phi) = 0_X, \underline{\theta}(X) = 1_X,$
- (ii) $\bar{\theta}(\lambda) = \underline{\theta}(\lambda^c)', \underline{\theta}(\lambda) = \bar{\theta}(\lambda^c)',$
- (iii) $\bar{\theta}(\lambda) \geq 1_\lambda, \underline{\theta}(\lambda) \leq 1_\lambda,$
- (iv) $\bar{\theta}(\bigcup_{i \in I} \lambda_i) = \bigvee_{i \in I} \bar{\theta}(\lambda_i), \underline{\theta}(\bigcap_{i \in I} \lambda_i) = \bigwedge_{i \in I} \underline{\theta}(\lambda_i).$

Below we give the notion of morphism between two L -fuzzifying reflexive approximation spaces.

Definition 3. The morphism $f : (X, \theta) \rightarrow (Y, \rho)$ between two L -fuzzifying reflexive approximation spaces (X, θ) and (Y, ρ) is given by

$$f^{\leftarrow}(\underline{\rho}(\lambda)) \leq \underline{\theta}(f^{\leftarrow}(\lambda)) \quad \forall \lambda \in \mathcal{P}(Y).$$

It is easy to verify that all L -fuzzifying reflexive approximation spaces as objects and morphism defined above form a category. We denote this category by **L-FYAPP**.

Theorem 1. The category **L-FYAPP** is isomorphic to the category **FRS**.

Proof. The proof is divided into two parts. On one hand we can see that both the categories have the identical objects. It only remains to show that both the

categories have the identical morphisms. Let $f : (X, \theta) \rightarrow (Y, \rho)$ be a morphism in the category **FRS**, then for any $\lambda \in \mathcal{P}(Y)$ and $x \in X$ we have

$$\begin{aligned} \underline{\theta}(f^{\leftarrow}(\lambda))(x) &= \bigwedge_{y \notin f^{\leftarrow}(\lambda)} \theta(x, y)' \geq \bigwedge_{f(y) \notin \lambda} \rho(f(x), f(y))' \\ &\geq \bigwedge_{t \notin \lambda} \rho(f(x), t)' = f^{\leftarrow}(\underline{\rho}(\lambda))(x). \end{aligned}$$

On the other hand, let f be a morphism in the category **L-FYAPP**. Then for all $x, y \in X$ we have

$$\begin{aligned} \rho(f(x), f(y))' &= \bigwedge_{t \notin (Y - \{f(y)\})} \rho(f(x), t)' = \underline{\rho}(Y - \{f(y)\})(f(x)) \\ &= f^{\leftarrow}(\underline{\rho}(Y - \{f(y)\}))(x) \leq \underline{\theta}(f^{\leftarrow}(Y - \{f(y)\}))(x) \\ &= \bigwedge_{y \notin f^{\leftarrow}(Y - \{f(y)\})} \theta(x, y)' \\ &= \underline{\theta}(X - \{y\})(x) = \theta(x, y)'. \end{aligned}$$

Hence we get $\rho(f(x), f(y))' \leq \theta(x, y)'$. Since “ \prime ” is order reversing, hence $\theta(x, y) \leq \rho(f(x), f(y))$ holds and f is a morphism in the category **FRS**. We denote this isomorphism by \mathbb{N} .

4 L -Fuzzifying Approximation Space and L -Fuzzifying Pretopological Space

This section is towards the categorical relationship among L -fuzzifying pretopological space, Čech (L -fuzzifying) interior space and L -fuzzifying approximation space. We discuss how to generate an L -fuzzifying pretopology by an reflexive L -fuzzy relation and our approach is based on the L -fuzzifying approximation operator studied in L -fuzzifying rough set theory.

Below, we present the definition of L -fuzzifying pretopological space which is similar (but not identical) to that in [8].

Definition 4. A set of functions $\tau_X = \{p_x : \mathcal{P}(X) \rightarrow L \mid x \in X\}$ is called an L -fuzzifying pretopology on X if for each $\lambda, \mu \in \mathcal{P}(X)$, and $x \in X$, it satisfies,

- (i) $p_x(X) = 1$,
- (ii) $p_x(\lambda) \leq 1_\lambda(x)$,
- (iii) $p_x(\lambda \cap \mu) = p_x(\lambda) \wedge p_x(\mu)$.

For an L -fuzzifying pretopology τ_X , the pair (X, τ_X) is called an L -fuzzifying pretopological space.

An L -fuzzifying pretopological space (X, τ_X) is called Alexandroff, if

- (iv) $p_x(\bigcap_{i \in I} \lambda_i) = \bigwedge_{i \in I} p_x(\lambda_i)$.

With every L -fuzzifying pretopological space $\tau_X = \{p_x : \mathcal{P}(X) \rightarrow L \mid x \in X\}$ and each $\lambda \in \mathcal{P}(X)$, we can associate another L -fuzzy set $\phi_\lambda \in L^X$ such that for all $x \in X$, $\phi_\lambda(x) = p_x(\lambda)$. Obviously, $\phi : \lambda \mapsto \phi_\lambda$ is an operator on X .

A mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ between two L -fuzzifying pretopological spaces is called *continuous* if for all $x \in X$ and for each $\lambda \in \mathcal{P}(Y)$, $q_{f(x)}(\lambda) \leq p_x(f^\leftarrow(\lambda))$, where $\tau_X = \{p_x : \mathcal{P}(X) \rightarrow L \mid x \in X\}$, $\tau_Y = \{q_{f(x)} : \mathcal{P}(Y) \rightarrow L \mid f(x) \in Y\}$ and $f^\leftarrow(\lambda) = \{x : f(x) \in \lambda\}$. It can be verified that all L -fuzzifying pretopological spaces as objects and their continuous maps as morphisms form a category, denoted by **L-FYPT**.

Now, we define the concepts of Čech L -fuzzifying interior (closure) operators by considering the domain as crisp power set $\mathcal{P}(X)$ rather than L -fuzzy set L^X .

Definition 5. A mapping $\hat{i} : \mathcal{P}(X) \rightarrow L^X$ is called a Čech (L -fuzzifying) interior operator on X if for each $\lambda, \mu \in \mathcal{P}(X)$, and $x \in X$, it satisfies

- (i) $\hat{i}(X) = 1_X$,
- (ii) $\hat{i}(\lambda) \leq 1_\lambda$,
- (iii) $\hat{i}(\lambda \cap \mu) = \hat{i}(\lambda) \wedge \hat{i}(\mu)$.

The pair (X, \hat{i}) is called a Čech (L -fuzzifying) interior space.

A Čech (L -fuzzifying) interior operator (X, \hat{i}) is called Alexandroff, if

- (iv) $\hat{i}(\bigcap_{i \in I} \lambda_i) = \bigwedge_{i \in I} \hat{i}(\lambda_i)$.

The map $f : (X, \hat{i}) \rightarrow (Y, \hat{j})$ between two Čech L -fuzzifying interior spaces is called *continuous* if for each $x \in X$ and $\lambda \in \mathcal{P}(Y)$, $\hat{j}(\lambda)(f(x)) \leq \hat{i}(f^\leftarrow(\lambda))(x)$. It is trivial to verify that all Čech (L -fuzzifying) interior spaces as objects and continuous maps as morphisms form a category. We denote this category by **L-FYIC**. Moreover, we denote the subcategory (full) **L-AFYIC** of **L-FYIC** with Čech Alexandroff L -fuzzifying interior operators as objects.

The notion of Čech (L -fuzzifying) closure operator can be defined using the duality of L .

Definition 6. A mapping $c_X : \mathcal{P}(X) \rightarrow L^X$ is called a Čech (L -fuzzifying) closure operator on X if for each $\lambda, \mu \in \mathcal{P}(X)$, and $x \in X$, it satisfies

- (i) $c_X(\phi) = 0_X$,
- (ii) $c_X(\lambda) \geq 1_\lambda$,
- (iii) $c_X(\lambda \cup \mu) = c_X(\lambda) \vee c_X(\mu)$.

The pair (X, c_X) is called a Čech L -fuzzifying closure space.

A Čech (L -fuzzifying) closure space (X, c_X) is called Alexandroff, if

- (iv) $c_X(\bigcup_{i \in I} \lambda_i) = \bigvee_{i \in I} c_X(\lambda_i)$.

A mapping $f : (X, c_X) \rightarrow (Y, c_Y)$ between two Čech (L -fuzzifying) closure spaces is called *continuous* if for all $x \in X$ and $\lambda \in \mathcal{P}(X)$, $f^\rightarrow(c_X(\lambda)) \leq c_Y(f^\rightarrow(\lambda))$.

Remark 1. For a De Morgan algebra L , the L -fuzzifying pretopologies, Čech L -fuzzifying interior operators and Čech L -fuzzifying closure operators are generally considered as equivalent and can be defined using the immanent duality of L in the following manner.

$$c_X(\lambda) = (i_X(\lambda^c))^c, \quad \forall \lambda \in \mathcal{P}(X)$$

From now on, we will only study the relationship between L -fuzzifying approximation spaces, L -fuzzifying pretopological spaces and Čech L -fuzzifying interior spaces. Since the similar results can be obtained for Čech L -fuzzifying closure spaces.

The following Proposition is an easy consequence of Definitions 4 and 5.

Proposition 2. *The set of functions $\tau_X = \{p_x : \mathcal{P}(X) \rightarrow L \mid x \in X\}$ is an L -fuzzifying pretopology on X iff the map $\hat{i}_{\tau_X} : \mathcal{P}(X) \rightarrow L^X$ such that for all $x \in X$,*

$$\hat{i}_{\tau_X}(\lambda)(x) = p_x(\lambda), \quad (1)$$

is a Čech L -fuzzifying interior operator. Moreover, if L -fuzzifying pretopology τ_X is Alexandroff, then the map \hat{i}_{τ_X} is a Čech-Alexandroff L -fuzzifying interior operator.

Theorem 2. *The category **L-FYPT** and **L-FYIS** are isomorphic.*

Proof. Let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is a morphism (continuous map) in **L-FYPT**. We define the functor \mathbb{G} as follows

$$\mathbb{G} : \begin{cases} \mathbf{L-FYPT} & \longrightarrow \mathbf{L-FYIS} \\ (X, \tau_X) & \longmapsto (X, \hat{i}_{\tau_X}) \\ f & \longmapsto f, \end{cases}$$

and for all $\lambda \in \mathcal{P}(X)$, $x \in X$, $\hat{i}_{\tau_X}(\lambda)(x) = p_x(\lambda)$. Since (X, \hat{i}_{τ_X}) is the object of category **L-FYIS**, then $f : (X, \hat{i}_{\tau_X}) \rightarrow (Y, \hat{j}_{\tau_Y})$ is a continuous map, i.e. $\forall \lambda \in \mathcal{P}(Y)$, $\hat{j}_{\tau_Y}(\lambda)(x) = q_{f(x)}(\lambda) \leq p_x(f^{\leftarrow}(\lambda)) = \hat{i}_{\tau_X}(f^{\leftarrow}(\lambda))(x)$.

Conversely, let $f : (X, \hat{i}_{\tau_X}) \rightarrow (Y, \hat{j}_{\tau_Y})$ is a continuous map in the category **L-FYIS**. We define the inverse functor \mathbb{G}^{-1} as follows

$$\mathbb{G} : \begin{cases} \mathbf{L-FYIS} & \longrightarrow \mathbf{L-FYPT} \\ (X, \hat{i}_{\tau_X}) & \longmapsto (X, \tau_X) \\ f & \longmapsto g, \end{cases}$$

and for all $\lambda \in \mathcal{P}(X)$, $p_x(\lambda) = \hat{i}_{\tau_X}(\lambda)(x)$. Then clearly \mathbb{G}^{-1} is an inverse functor with the inverse \mathbb{G} .

In the next proposition, we show that an L -fuzzifying pretopology on X can be represented by an L -fuzzifying lower approximations of sets on X with respect to a reflexive L -fuzzy relation.

Proposition 3. *Suppose that (X, θ) be an *L*-fuzzifying reflexive approximation space. Let for all $\lambda \in \mathcal{P}(X)$, $x \in X$, we denote*

$$p_x^\theta(\lambda) = \bigwedge_{y \notin \lambda} \theta(x, y)'. \quad (2)$$

*Then $\tau_\theta = \{p_x^\theta : \mathcal{P}(X) \rightarrow L | x \in X\}$, is an *L*-fuzzifying pretopology on X .*

Proof. For all $x \in X$ and $\lambda \in \mathcal{P}(X)$, from Proposition 1, it can be easily verified that τ_θ as defined in Eq. 2 satisfies the properties (i)–(iii) of lower *L*-fuzzifying approximation operator.

Proposition 4. *Let (X, τ_X) be an *L*-fuzzifying pretopological space. Then for any $x \in X$, we define*

$$\Theta_{\tau_X}(x, y) = p_x(X - \{y\})'.$$

*Then, Θ_{τ_X} is a reflexive *L*-fuzzy relation and (X, Θ_{τ_X}) is an *L*-fuzzifying reflexive approximation space.*

Proof. For all $x \in X$ and from the Definition 4 we have, $\Theta_{\tau_X}(x, x) = p_x(X - \{x\})' \leq 1_{X - \{x\}}(x)' = 0' = 1$. Which shows that Θ_{τ_X} is a reflexive *L*-fuzzy relation and hence (X, Θ_{τ_X}) is an *L*-fuzzifying reflexive approximation space.

Proposition 5. *If $f : (X, \theta) \rightarrow (Y, \rho)$ is a morphism between two *L*-fuzzifying reflexive approximation spaces. Then f is continuous function between two *L*-fuzzifying pretopological spaces (X, τ_θ) and (Y, τ_ρ) .*

Proof. The proof directly follows from Proposition 3.

Thus from the Propositions 3 and 5 we obtain a concrete functor τ as follows:

$$\tau : \begin{cases} \mathbf{L-FYAPP} & \longrightarrow \mathbf{L-FYPT} \\ (X, \theta) & \longmapsto (X, \tau_\theta) \\ f & \longmapsto f. \end{cases}$$

Next, we prove a result, which gives a concrete functor $\Theta : \mathbf{FYPT} \rightarrow \mathbf{FYAPP}$.

Proposition 6. *If f is a continuous function between two *L*-fuzzifying pretopological spaces (X, τ_X) and (Y, τ_Y) . Then $f : (X, \Theta_{\tau_X}) \rightarrow (Y, \Theta_{\tau_Y})$ is a morphism between two *L*-fuzzifying reflexive approximation spaces.*

Proof. Let $\lambda \in \mathcal{P}(Y)$ and $x \in X$, we have

$$\begin{aligned}
 f^-(\underline{\Theta}_{\tau_Y}(\lambda))(x) &= \underline{\Theta}_{\tau_Y}(\lambda)(f(x)) = \bigwedge_{t \notin \lambda} \Theta_{\tau_Y}(f(x), t)' \\
 &= \bigwedge_{t \notin \lambda} q_{f(x)}(Y - \{t\}) = \bigwedge_{f(y) \notin \lambda} q_{f(x)}(Y - \{f(y)\}) \\
 &\leq \bigwedge_{y \notin f^-(\lambda)} p_x(f^-(Y - \{f(y)\})) \\
 &\leq \bigwedge_{y \notin f^-(\lambda)} p_x(X - \{y\}) = \bigwedge_{y \notin f^-(\lambda)} \Theta_{\tau_X}(x, y)' \\
 &= \underline{\Theta}_{\tau_X}(f^-(\lambda))(x).
 \end{aligned}$$

Hence, we have $f : (X, \Theta_{\tau_X}) \rightarrow (Y, \Theta_{\tau_Y})$ is a morphism between two L -fuzzifying reflexive approximation spaces (X, Θ_{τ_X}) and (Y, Θ_{τ_Y}) . In particular, we obtain a concrete functor Θ as follows:

$$\Theta : \begin{cases} \mathbf{L-FYPT} & \longrightarrow \mathbf{L-FYAPP} \\ (X, \tau_X) & \longmapsto (X, \Theta_{\tau_X}) \\ f & \longmapsto f. \end{cases}$$

In the next theorem we prove the adjointness between the categories **L-FYAPP** and **L-FYPT**. Now we have the following.

Theorem 3. *Let (X, θ) be an L -fuzzifying reflexive approximation space. Then $\tau : \mathbf{L-FYAPP} \rightarrow \mathbf{L-FYPT}$ is a left adjoint of $\Theta : \mathbf{L-FYPT} \rightarrow \mathbf{L-FYAPP}$. Moreover $\Theta \circ \tau(X, \theta) = (X, \theta)$ i.e., Θ is a left inverse of τ .*

Proof: The proof is divided into two parts. At first, we show that for any L -fuzzifying reflexive approximation space (X, θ) , $\mathbb{I}_X : (X, \theta) \rightarrow (X, \Theta_{\tau_\theta})$ is a morphism between L -fuzzifying reflexive approximation spaces.

For any $\lambda \in \mathcal{P}(X)$ and $x \in X$, we have

$$\begin{aligned}
 \underline{\Theta}_{\tau_\theta}(\lambda)(x) &= \bigwedge_{y \notin \lambda} \Theta_{\tau_\theta}(x, y)' \\
 &= \bigwedge_{y \notin \lambda} (p_x^\theta(X - \{y\}))' \quad (\text{from Proposition 4}) \\
 &= \bigwedge_{y \notin \lambda} p_x^\theta(X - \{y\}) \quad (\text{by involution of “ ’ ”}) \\
 &= \bigwedge_{y \notin \lambda} \underline{\theta}(X - \{y\}) = \bigwedge_{y \notin \lambda} \theta(x, y)' = \underline{\theta}(\lambda)(x).
 \end{aligned}$$

Hence, $\mathbb{I}_X : (X, \theta) \rightarrow (X, \Theta_{\tau_\theta})$ is a morphism between L -fuzzifying reflexive approximation spaces.

On the other hand, for any $\lambda \in \mathcal{P}(X)$, $x \in X$, we have

$$\begin{aligned} p_x^{\Theta_{\tau_X}}(\lambda) &= \underline{\Theta_{\tau_X}}(\lambda)(x) = \bigwedge_{y \notin \lambda} \Theta_{\tau_X}(x, y)' \\ &= \bigwedge_{y \notin \lambda} (p_x(X - \{y\}))' \\ &= \bigwedge_{y \notin \lambda} p_x(X - \{y\}) \quad (\text{by involution of “ ’ ”}) \\ &\geq p_x \bigcap_{y \notin \lambda} (X - \{y\}) = p_x(\lambda). \end{aligned}$$

Hence, we show that $\mathbb{I}_X : (X, \tau_{\Theta_{\tau_X}}) \rightarrow (X, \tau_X)$ is continuous.

Therefore, $\tau : \mathbf{L-FYAPP} \rightarrow \mathbf{L-FYPT}$ is a left adjoint of $\Theta : \mathbf{L-FYPT} \rightarrow \mathbf{L-FYAPP}$ (Fig. 1).

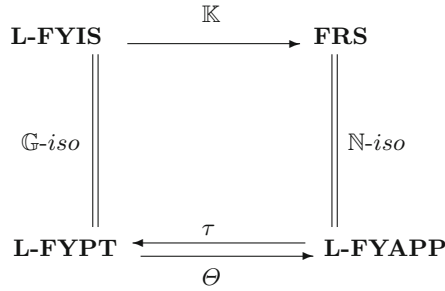


Fig. 1. Commutative diagram of Theorems 1, 2 and 3.

5 L -Fuzzy Relational Structures and Čech L -Fuzzifying Interior Space

In this section, we establish the categorical relationship between the category **FRS** of L -fuzzy relational structures and the category **L-FYIS** of Čech L -fuzzifying interior spaces. Now we have the following.

Proposition 7. *Let $f : (X, \theta) \rightarrow (Y, \rho)$ be a morphism in the category **FRS**, then $f : (X, \hat{i}_\theta) \rightarrow (Y, \hat{j}_\rho)$ is a continuous function (morphism) in the category **L-FYIS**.*

Proof: Given that $f : (X, \theta) \rightarrow (Y, \rho)$ be a morphism in the category **FRS**. We define a functor \mathbb{F} as follows;

$$\mathbb{F} : \begin{cases} \mathbf{FRS} & \longrightarrow \mathbf{L-FYIS} \\ (X, \theta) & \longmapsto (X, \hat{i}_\theta) \\ f & \longmapsto f, \end{cases}$$

and $\forall \lambda \in \mathcal{P}(X), x \in X, \hat{i}_\theta(\lambda)(x) = \bigwedge_{y \notin \lambda} \theta(x, y)'$. As (X, \hat{i}_θ) is the object of category **L-AFYIS**, we need to show that $f : (X, \hat{i}_\theta) \rightarrow (Y, \hat{j}_\rho)$ is a continuous function (morphism) in the category **L-AFYIS**. For all $\lambda \in \mathcal{P}(Y), x \in X$ we have

$$\begin{aligned} \hat{j}_\rho(\lambda)(f(x)) &= \bigwedge_{z \notin \lambda} \rho(f(x), z)' \leq \bigwedge_{f(y) \notin \lambda} \rho(f(x), f(y))' \\ &\leq \bigwedge_{y \notin f^{-1}(\lambda)} \theta(x, y)' = \hat{i}_\theta(f^{-1}(\lambda))(x). \end{aligned}$$

Hence \mathbb{F} is a functor.

Proposition 8. *Let $f : (X, \hat{i}_\theta) \rightarrow (Y, \hat{j}_\rho)$ be a continuous function (morphism) in the category **L-FYIS**, then $f : (X, \theta) \rightarrow (Y, \rho)$ is a morphism in the category **FRS**.*

Proof: Let $f : (X, \hat{i}_\theta) \rightarrow (Y, \hat{j}_\rho)$ is a continuous function. Define a functor \mathbb{K} as follows;

$$\mathbb{K} : \begin{cases} \mathbf{L-FYIS} & \longrightarrow \mathbf{FRS} \\ (X, \hat{i}_\theta) & \longmapsto (X, \theta), \\ f & \longmapsto f, \end{cases}$$

and $\theta(x, y) = \hat{i}_\theta(X - \{y\})'(x)$. Clearly θ is reflexive. Since \hat{i}_θ is anti-extensive, hence $\theta(x, x) = \hat{i}_\theta(X - \{x\})'(x) \leq (X - \{x\})'(x) = 0' = 1$. It remains to show that $f : (X, \theta) \rightarrow (Y, \rho)$ is a morphism in the category **FRS**, i.e., $\theta(x, y) \leq \rho(f(x), f(y))$, or

$$\begin{aligned} \hat{i}_\theta(X - \{y\})'(x) &\leq \hat{j}_\rho(Y - \{f(y)\})'(f(x)), \\ \text{or, } \hat{i}_\theta(X - \{y\})(x) &\geq \hat{j}_\rho(Y - \{f(y)\})(f(x)). \end{aligned} \quad (3)$$

Since $f : (X, \hat{i}_\theta) \rightarrow (Y, \hat{j}_\rho)$ is a continuous function, we have $\hat{j}_\rho(\lambda)(f(x)) \leq \hat{i}_\theta(f^{-1}(\lambda))(x)$. Therefore for $\lambda = (Y - \{f(y)\})$, we get

$$\begin{aligned} \hat{j}_\rho(Y - \{f(y)\})(f(x)) &\leq \hat{i}_\theta(f^{-1}(Y - \{f(y)\}))(x) \\ &\leq \hat{i}_\theta(X - \{y\})(x). \end{aligned}$$

Hence (3) holds and $f : (X, \theta) \rightarrow (Y, \rho)$ is a morphism in the category **FRS**.

Proposition 9. *Let (X, \hat{i}_θ) be a Čech Alexandroff L -fuzzifying interior space and $\mathbb{F} : \mathbf{FRS} \rightarrow \mathbf{L-AFYIS}$, $\mathbb{K} : \mathbf{L-AFYIS} \rightarrow \mathbf{FRS}$ be the concrete functors. Then $\mathbb{FK}(X, \hat{i}_\theta) = (X, \hat{i}_\theta)$ (Fig. 2).*

Proof: Let $\mathbb{FK}(X, \hat{i}_\theta) = (X, \hat{j}_\rho)$, where,

$$\forall \lambda \in \mathcal{P}(X), \hat{j}_\rho(\lambda)(x) = \bigwedge_{y \notin \lambda} \hat{i}_\theta(X - \{y\})(x) = \bigwedge_{y \notin \lambda} \theta(x, y)'.$$

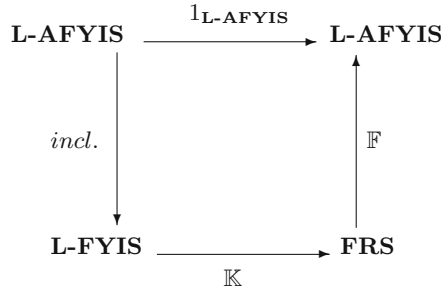


Fig. 2. Commutative diagram of Proposition 9.

As we know that, any arbitrary set $\lambda \in \mathcal{P}(X)$ can be decomposed as

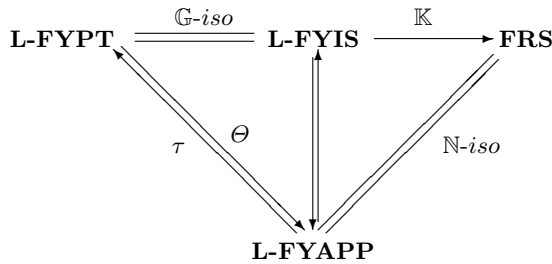
$$\lambda = \bigcap_{y \notin \lambda} (X - \{y\}).$$

Therefore for Čech Alexandroff L -fuzzifying interior operator \hat{i}_θ , we have

$$\begin{aligned} \hat{i}_\theta(\lambda)(x) &= \hat{i}_\theta \left(\bigcap_{y \notin \lambda} (X - \{y\})(x) \right) = \bigwedge_{y \notin \lambda} \hat{i}_\theta(X - \{y\})(x) \\ &= \bigwedge_{y \notin \lambda} \theta(x, y)' = \hat{j}_\rho(\lambda)(x). \end{aligned}$$

Hence we have $\mathbb{FK}(X, \hat{i}_\theta) = (X, \hat{i}_\theta)$.

In the end, we give a graph to collect the relationships among the discussed categories.



6 Conclusion

This paper contributes to the theory of L -fuzzifying topology, which originates from [26]. We have considered various categories that are weaker than the category of L -fuzzifying topology, e.g., category of L -fuzzifying reflexive approximation spaces, category of L -fuzzifying pretopological spaces and the category of Čech L -fuzzifying interior (closure) spaces. At first, we have shown how an L -fuzzifying pretopology can be generated by a reflexive L -fuzzy relation. Further, we have shown interconnections among these categories and some of their subcategories using the commutative diagram. Finally, we have established the relationship between L -fuzzy relational structure and Čech L -fuzzifying interior space by means of Galois connection.

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