Optimal Control Under Fuzzy Conditions for Dynamical Systems Associated with the Second Order Linear Differential Equations

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Abstract. This paper is devoted to an optimal trajectory planning problem with uncertainty in location conditions considered as a problem of constrained optimal control for dynamical systems. Fuzzy numbers are used to incorporate uncertainty of constraints into the classical setting of the problem under consideration. The proposed approach applied to dynamical systems associated with the second order linear differential equations allows to find an optimal control law at each α-level using spline-based methods developed in the framework of the theory of splines in convex sets. The solution technique is illustrated by numerical examples.

Keywords: Dynamical system · Fuzzy constraints · Optimal control

1 Introduction

Optimal control is the process of determining control and state trajectories for a dynamical system over a period of time to minimize an objective function. In this paper we analyse the special case of the following control theory problem:

$$x'(t) = Mx(t) + \beta u(t), y(t) = \gamma^\top x(t), \quad t \in [a, b], \tag{1}$$

considered with the initial condition

$$x(a) = c. \tag{2}$$

Here x is a vector-valued absolutely continuous function defined on $[a, b]$, M is a given quadratic constant matrix and β, γ are given constant vectors of compatible dimensions. We consider system (1) as the curve $z = y(t)$ generator. The goal is

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to find a control law $u \in L_2[a,b]$ which drives the scalar output trajectory close to a sequence of set points at fixed times

$$\{(t_i, z_i) : i = 1, 2, \ldots, n\}, \quad \text{where} \ \ a < t_1 < t_2 < \ldots < t_n \leq b, \tag{3}$$

by minimization of the objective functional

$$\int_a^b (u(t))^2 dt. \tag{4}$$

In some applications of such type of control problems, for example, doing trajectory planning in traffic control, we need to be able to generate curves that pass through predefined states at given times since we need to be able to specify the position in which the system will be in at a sequence of times (see, e.g., [1]). In this case we refer to the classical setting of the problem under consideration:

$$\int_a^b (u(t))^2 dt \rightarrow \min_{u \in L_2[a,b]:\ \ x(a)=c,\ y(t_i)=z_i,\ i=1,\ldots,n}, \tag{5}$$

where x and y depend on u by means of (1). It is shown in [1] and the references therein that a number of interpolation and path planning problems can be incorporated into control problem and studied using control theory and optimization techniques on Hilbert spaces with efficient numerical spline-based schemes. Control splines give a richer class of smoothing curves relative to polynomial curves. They have been proved to be useful for trajectory planning in [2], mobile robots in [3], contour modelling of images in [4], probability distribution estimation in [5] and so on.

However, in many situations, it is not really crucial that we pass a trajectory through these points exactly, but rather that we go reasonably close to them, while minimizing the objective functional. Such approach is closely related to the idea of smoothing under fuzzy interpolation conditions. We propose to use fuzzy numbers $Z_i,\ i = 1, \ldots, n$, in (5) instead of crisp $z_i,\ i = 1, \ldots, n$, to incorporate uncertainty of location conditions (3) into the model. According to this idea, we rewrite optimisation problem (5) in the following way:

$$\int_a^b (u(t))^2 dt \rightarrow \min_{u \in L_2[a,b]:\ \ x(a)=c,\ y(t_i) \text{is}\ Z_i,\ i=1,\ldots,n}, \tag{6}$$

where x and y depend on u by means of (1).

In this paper, the main attention is paid to the special case of problem (1):

$$M = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

For this case problem (6) can be rewritten as

$$\int_a^b (g''(t) + pg'(t) + qg(t))^2 dt \longrightarrow \min_{\substack{g \in L_2^2[a,b]:\ \ g(a)=c_1,\ g'(a)=c_2, \\ y(t_i) \text{is}\ Z_i,\ i=1,\ldots,n}}, \tag{7}$$

where

$$y(t) = \gamma_1 g(t) + \gamma_2 g'(t),$$
$$u(t) = g''(t) + pg'(t) + qg(t),$$

and g is used to denote x_1.

2 Control Problem at α-Levels

In this paper we suggest a method for construction of solutions of (7) and finding corresponding control laws at each α-level with respect to fuzzy numbers used in the model by applying results from the theory of splines in convex sets.

To rewrite (7) for α-levels we introduce notations to deal with fuzzy numbers $Z_i,\ i = 1, \ldots, n$. Fuzzy real number Z_i is a normal fuzzy subset of $I\!R$ that satisfies the condition: all α-cuts of Z_i are closed bounded intervals.

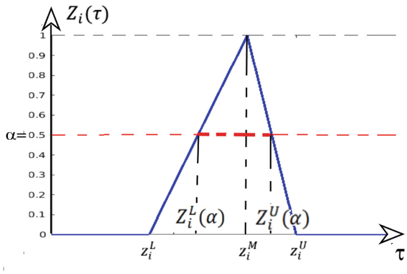

Fig. 1. Triangular fuzzy number

The α-cut ($\alpha \in (0,\ 1]$) of fuzzy number Z_i is the crisp set $(Z_i)_\alpha$ defined as

$$(Z_i)_\alpha = \{\tau \in I\!R \mid Z_i(\tau) \geq \alpha\}.$$

If $\alpha = 0$, then α-cut $(Z_i)_0$ can be defined as the support of function Z_i. The constraints "$y(t_i)$ is $Z_i,\ i = 1, \ldots, n$," can be written at α-levels using α-cuts:

$$y(t_i) \in (Z_i)_\alpha,\ i = 1, \ldots, n.$$

For each α-level the α-cut of Z_i is the closed interval

$$(Z_i)_\alpha = [Z_i^L(\alpha), Z_i^U(\alpha)].$$

Therefore problem (7) at α-level can be written in the following form:

$$\int_a^b (g''(t) + pg'(t) + qg(t))^2 dt \longrightarrow \min_{\substack{g \in L_2^2[a,b]:\ \ g(a)=c_1,\ g'(a)=c_2, \\ Z_i^L(\alpha) \leq y(t_i) \leq Z_i^U(\alpha), i=1,\ldots,n}}, \tag{8}$$

where
$$y(t) = \gamma_1 g(t) + \gamma_2 g'(t),\ u(t) = g''(t) + pg'(t) + qg(t).$$

We apply triangular fuzzy numbers Z_i (see Fig. 1) given by triples (z_i^L, z_i^M, z_i^U):

$$Z_i(\tau) = \begin{cases} (\tau - z_i^L)(z_i^M - z_i^L)^{-1} & \text{if } \tau \in [z_i^L, z_i^M], \\ (z_i^U - \tau)(z_i^U - z_i^M)^{-1} & \text{if } \tau \in (z_i^M, z_i^U], \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$Z_i^L(\alpha) = z_i^L + \alpha(z_i^M - z_i^L),\ Z_i^U(\alpha) = z_i^U - \alpha(z_i^U - z_i^M) \text{ for all } \alpha \in [0,\ 1].$$

3 Spline-Based Approach

We consider problem (8) as the special case of the following more general conditional minimization problem:

$$||Tg||_{L_2[a,b]} \longrightarrow \min_{\substack{g \in L_2^r[a,b]:\ (Ag)_0 = c_1,\ (Ag)_{n+1} = c_2, \\ Z_i^L(\alpha) \le (Ag)_i \le Z_i^U(\alpha),\ i=1,\dots,n}}, \tag{9}$$

where linear operators $T :\ L_2^r[a,b] \ \rightarrow\ L_2[a,b]$, and $A : L_2^r[a,b] \rightarrow I\!R^{n+2}$ are continuous (here $L_2^r[a,b]$ is the Sobolev space), vector $c \in I\!R^2$ is given and $Z_i^L(\alpha), Z_i^U(\alpha), i = 1,\dots,n$, are known. We assume that $A(L_2^r[a,b]) = I\!R^{n+2}$. In the case under consideration $r = 2$ and

$$Tg = g'' + pg' + qg = u, (Ag)_i = \gamma_1 g(t_i) + \gamma_2 g'(t_i), i = 1, ..., n,$$
$$(Ag)_0 = g(a), (Ag)_{n+1} = g'(a). \tag{10}$$

The solution of problem (9) will be considered for different α-levels. Value $\alpha = 1$ corresponds to the case when we pass the output trajectory through points (3) exactly (the case $z_i = z_i^M$). In this case problem (9) turns into the interpolating problem. For $\alpha < 1$ problem (9) will be considered applying smoothing splines.

3.1 Interpolating Splines

Problem (9) in the case $\alpha = 1$ corresponds to the following interpolating problem:

$$||Tg||_{L_2[a,b]} \longrightarrow \min_{\substack{g \in L_2^r[a,b]:\ (Ag)_0 = c_1,\ (Ag)_{n+1} = c_2, \\ (Ag)_i = z_i^M,\ i=1,\dots,n}}. \tag{11}$$

The conditions of existence and uniqueness of solution of (11) and its characterization follow from the well known theorems (see, e.g., Theorems 4.4.2. and 4.5.9. in [6]).

Proposition 1. *Under the assumption, that $kerT \cap ker(A) = \{0\}$ and $kerT$ is finite-dimensional, the unique solution of problem (11) exists. An element $s \in L_2^r[a,b]$, such as $(As)_0 = c_1$, $(As)_i = z_i^M$, $i = 1, \ldots, n$, and $(As)_{n+1} = c_2$, is a solution of (11) if and only if there exists vector $\lambda \in \mathbb{R}^{n+2}$ such that*

$$T^*Ts = A^*\lambda. \tag{12}$$

This result implies that a solution of problem (11) is a spline from the space

$$S(T,A) = \{s \in L_2^r[a,b] \mid \forall x \in kerA \quad \langle Ts, Tx\rangle = 0\}.$$

Here and in the sequel the corresponding inner product is denoted by $\langle\cdot,\cdot\rangle$, and $kerA$ is the kernel of operator A.

The view of splines from the space $S(T,A)$ in depending on parameters p and q for the considered case of operators (10) is obtained in [7] using the general theorem (see Theorem 1 in [8]) and applying functional analysis tools. For example, if $p = q = 0$ then elements of $S(T,A)$ are polynomial cubic splines from $C^1[a,b]$, i.e., they are cubic polynomials on each interval $[t_{i-1}, t_i]$, $i = 1, \ldots, n+1$, where $t_0 = a$ and $t_{n+1} = b$.

3.2 Splines in Convex Sets

Problem (9) in the case $\alpha < 1$ corresponds to the following smoothing problem (problem on splines in a convex set):

$$||Tg||_{L_2[a,b]} \longrightarrow \min_{\substack{g \in L_2^r[a,b]:\quad (Ag)_0 = c_1,\ (Ag)_{n+1} = c_2, \\ Z_i^L(\alpha) \le (Ag)_i \le Z_i^U(\alpha),\ i=1,\ldots,n}} \tag{13}$$

considered under assumption $Z_i^L(\alpha) < Z_i^U(\alpha)$.

The conditions of existence and uniqueness of solution of (13) follow from the known theorem (see Theorem 7 in [8]).

Proposition 2. *Under the assumption that $kerT$ is finite-dimensional a solution of problem (13) exists. An element $s \in L_2^r[a,b]$, such as $(As)_0 = c_1$, $(As)_{n+1} = c_2$, $Z_i^L(\alpha) \le (As)_i \le Z_i^U(\alpha)$, $i = 1, \ldots, n$, is a solution of (13) if and only if there exists vector $\lambda \in \mathbb{R}^{n+2}$ such that*

$$T^*Ts = A^*\lambda \tag{14}$$

and components λ_i, $i = 1, \ldots, n$, satisfy the conditions

$$\begin{array}{lll} \lambda_i = 0, & \text{if} & Z_i^L(\alpha) < (As)_i < Z_i^U(\alpha), \\ \lambda_i \ge 0, & \text{if} & (As)_i = Z_i^L(\alpha), \\ \lambda_i \le 0, & \text{if} & (As)_i = Z_i^U(\alpha). \end{array} \tag{15}$$

Under the additional assumption $kerT \cap ker(A) = \{0\}$ this solution is unique.

This result implies that a solution of problem (13) belongs to the space $S(T,A)$. To find it we can use the method of adding-removing interpolation knots which is considered in details, for example, in [9] or [10]. It is an iterative method. On the k-th step of it we need to solve the following interpolation problem: to construct a spline $s^k \in S(T,A)$ such that the initial conditions $(As^k)_0 = c_1,\ (As^k)_{n+1} = c_2$, and the interpolation conditions written in the form $(As^k)_i = d_i^k,\ i \in I^k$, are satisfied. The set of indices $I^k \subset \{1,\dots,n\}$ and numbers d_i^k are specified during the iterations. The knots t_i for $i \in I^k$ are considered as interpolation knots on the k-th step. We start with a solution s_1 obtained using only the initial conditions, i.e., $I^1 = \emptyset$. The iterative step from I^k to I^{k+1} is done by adding to I^k all indices $i \in \{1,\dots,n\}$ such that the restriction $Z_i^L \le (As^k)_i \le Z_i^U$ is not satisfied. For the added index i we take $d_i^{k+1} = Z_i^L(\alpha)$ if $Z_i^L(\alpha) > (As^k)_i$, and we take $d_i^{k+1} = Z_i^U(\alpha)$ if $(As^k)_i > Z_i^U(\alpha)$. On the other hand, we remove from I^k all indices $i \in I^k$ such that the rule (15) is not satisfied for the corresponding coefficient of s^k. To finish the k-th step we also denote $d_i^{k+1} = d_i^k$ for $i \in I^{k+1} \cap I^k$. If $I^{k+1} = I^k$ then the algorithm ends and the obtained s^k is a solution of (13).

4 Numerical Solutions

In this paper we consider problem (8) as (9) with operator T and A defined by (10). According to Proposition 1 and Proposition 2 in this case solutions of (8) at each α-level belong to the space $S(T,A)$. The view of splines from the corresponding $S(T,A)$ (i.e., the view of solutions of problem (8)) is obtained in [7]. This view in [7] is given depending on the roots r_1, r_2 of the equation $r^2 + pr + q = 0$:

- Class 1 (exponential splines with polynomial coefficients): $r_1 = r_2 \in I\!R \setminus \{0\}$.
- Class 2 (exponential splines): $r_1, r_2 \in I\!R,\ r_1 \neq r_2$.
- Class 3 (polynomial-exponential splines): $r_1, r_2 \in I\!R,\ r_1 \neq r_2, r_1 \neq 0, r_2 = 0$.
- Class 4 (polynomial splines): $r_1 = r_2 = 0$.
- Class 5 (trigonometric splines with polynomial coefficients): $r_{1,2} = \pm i\eta \neq 0$.
- Class 6 (trigonometric splines with exponential-polynomial coefficients): $r_{1,2} = \zeta \pm i\eta$ with $\eta \neq 0$ and $\zeta \neq 0$.

The simplest case with $p = q = 0$, i.e., $Tg = g''$, corresponds to the classical smoothing problem in the theory of splines according to which a solution of (8) without the initial conditions is a cubic spline. Taking into account the initial conditions we get the following form for solution s of problem (8) for this case:

$$s(t) = c_1 + c_2(t-a) + \frac{\lambda_0}{6}(t-a)_+^3 - \frac{\lambda_{n+1}}{2}(t-a)_+^2 + \sum_{i=1}^{n} \lambda_i\left(\frac{\gamma_1}{6}(t-t_i)_+^3 - \frac{\gamma_2}{2}(t-t_i)_+^2\right)$$

with the following conditions on coefficients

$$\lambda_0 + \sum_{i=1}^{n} \lambda_i(\gamma_1 + \gamma_2) = 0, \quad \lambda_0 a + \lambda_{n+1} + \sum_{i=1}^{n} \lambda_i(\gamma_1 t_i + \gamma_2) = 0.$$

Here and in the sequel the truncated power function is defined as

$$(t - t_j)_+^k = \begin{cases} (t - t_j)^k, & t \geq t_j, \\ 0, & t < t_j. \end{cases}$$

The corresponding control function u could be obtained as $u = s''$.

Two numerical examples corresponding to more complicated cases are considered below for illustration of the proposed technique. Numerical results are obtained by using Maple.

4.1 Example 1: Exponential Splines

We consider the numerical example for the case $p = -3$ and $q = 2$, $\gamma_1 = 1$ and $\gamma_2 = 0$, interval $[a, b] = [0, 0.5]$, the initial conditions are with $c_1 = 1$, $c_2 = 1$. At equally spaced points of interval $[0.1, 0.5]$ with step size 0.1 we take the following fuzzy numbers $Z_i, i = 1, \dots, 5$: $(4, 5, 6), (1, 2, 3), (5, 6, 7), (2, 3, 4), (6, 7, 8)$. This case corresponds to the case of two nonzero roots of characteristic equation $r_1 = 1, r_2 = 2$, i.e., to the case when solutions belong to the class of exponential splines.

As it is obtained in [7], the class of exponential splines for problem (8) consists of splines

$$s(t) = \mu_1 e^{r_1(t-a)} + \mu_2 e^{r_2(t-a)} + \frac{1}{2(r_1^2 - r_2^2)}(\frac{(\lambda_0 - \lambda_{n+1} r_1) e^{r_1(t-a)}}{r_1}$$
$$- \frac{(\lambda_0 - \lambda_{n+1} r_2) e^{r_2(t-a)}}{r_2} + \sum_{i=1}^{n} \lambda_i (\frac{\gamma_1}{r_1} e^{r_1|t-t_i|} + \gamma_2 (e^{r_1(t_i - t)_+} - e^{r_1(t-t_i)_+})$$
$$- \frac{\gamma_1}{r_2} e^{r_2|t-t_i|} - \gamma_2 (e^{r_2(t_i-t)_+} - e^{r_2(t-t_i)_+}))). \qquad (16)$$

For the solution of (8) the coefficients are expressed by using the following system:

$$(\gamma_1 + \gamma_2 r_1) \sum_{i=1}^{n} \lambda_i e^{r_1 t_i} + (\lambda_0 + \lambda_{n+1} r_1) e^{r_1 a} = 0,$$
$$(\gamma_1 + \gamma_2 r_2) \sum_{i=1}^{n} \lambda_i e^{r_2 t_i} + (\lambda_0 + \lambda_{n+1} r_2) e^{r_2 a} = 0, \qquad (17)$$

and the system of interpolating conditions for $g(t_i) = z_i^M, i = 1, \dots, n$, in case $\alpha = 1$. For $\alpha < 1$ the interpolating conditions are precised by iterations of the method of adding-removing knots.

The corresponding control function u is given by

$$u(t) = \sum_{i=1}^{n} \frac{\lambda_i (t_i - t)_+^0}{r_1 - r_2} (\gamma_1 (e^{r_1(t_i - t)} - e^{r_2(t_i - t)}) + \gamma_2 (r_1 e^{r_1(t_i - t)} - r_2 e^{r_2(t_i - t)})). \tag{18}$$

For the considered case the conditions of the uniqueness of solution are satisfied. This solution is constructed for four α-levels: $\alpha_1 = 0$, $\alpha_2 = 0.25$, $\alpha_3 = 0.5$ and $\alpha_4 = 1$. The solution of problem (8) in this case is the exponential spline (16) with coefficients from Table 1. The control law u is obtained by (18). The corresponding graphs are considered in Fig. 2 and Fig. 3. The values of the objective functional for these α-levels are compared (see Table 2).

Table 1. Coefficients of the solution at α-level for *Example 1*

	$\alpha_1 = 0$	$\alpha_2 = 0.25$	$\alpha_3 = 0.5$	Interpolation ($\alpha_4 = 1$)
μ_1	-5.3984×10^1	-4.0037×10^1	-4.4686×10^1	-5.3984×10^1
μ_2	5.4984×10^1	4.1037×10^1	4.5686×10^1	5.4984×10^1
λ_0	-1.9715×10^4	-1.3245×10^4	-1.5402×10^4	-1.9715×10^4
λ_1	5.5236×10^4	3.5234×10^4	4.1902×10^4	5.5236×10^4
λ_2	-7.4174×10^4	-4.4561×10^4	-5.4432×10^4	-7.4174×10^4
λ_3	7.2526×10^4	4.2014×10^4	5.2185×10^4	7.2526×10^4
λ_4	-4.6963×10^4	-2.6943×10^4	-3.3617×10^4	-4.6963×10^4
λ_5	1.3092×10^4	7.4793×10^3	9.3504×10^3	1.3092×10^4
λ_6	-1.5895×10^2	-1.1711×10^2	-1.3105×10^2	-1.5895×10^2

By comparison of the objective functional at these α-levels from Table 2 we see that the minimum of this functional is obtained for $\alpha_1 = 0$ and the interpolating spline gives the biggest value.

Table 2. Comparison of the values of the objective functional for *Example 1*

α	$\alpha_1 = 0$	$\alpha_2 = 0.25$	$\alpha_3 = 0.5$	Interpolation ($\alpha_4 = 1$)
$\|u\|$	2.8622×10^2	3.7282×10^2	4.5988×10^2	6.34657×10^2

4.2 Example 2: Trigonometric Splines with Polynomial Coefficients

We consider the second numerical example for the case $p = 0$ and $q = 1$, $\gamma_1 = 1$ and $\gamma_2 = 0$, interval $[a, b] = [0, 0.5]$, the initial conditions are given with $c_1 = 1$ and $c_2 = 1$. At equally spaced points in interval $[0.1, 0.5]$ with

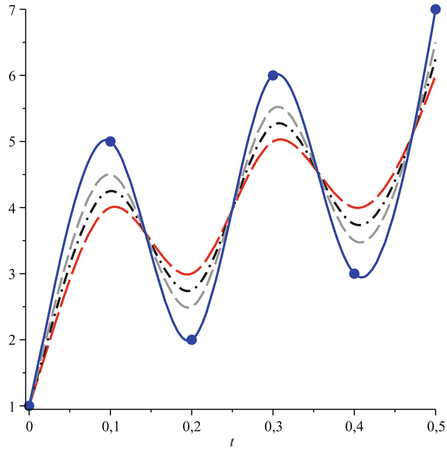

Fig. 2. State trajectories for *Example 1* (solid line for $\alpha = 1$, dash line for $\alpha = 0.25$, dash dot line for $\alpha = 0.5$, long dash line for $\alpha = 0$)

step size 0.1 we take the following values of fuzzy numbers $Z_i, i = 1, \ldots, 5$: $(2,3,4), (0,1,2), (4,5,6), (1,2,3), (3,4,5)$. This case corresponds to the case of complex roots of characteristic equation $r_{1,2} = \pm i$ (for this case $\eta = 1$), i.e., to the case when solutions belongs to the class of trigonometric splines with polynomial coefficients.

As it is obtained in [7], the class of trigonometric splines with polynomial coefficients for problem (8) in the considered case consists of splines

$$s(t) = c_1 cos\,(\eta(t-a)) + \frac{c_2}{\eta} sin\,(\eta(t-a)) + \frac{\lambda_0}{2\eta^3}(\eta(t-a)cos\,(\eta(t-a)) - sin\,(\eta(t-a)))$$
$$+\frac{\lambda_{n+1}}{2\eta}(t-a)sin\,(\eta(t-a)) + \frac{1}{2}\sum_{i=1}^{n}\lambda_i(\frac{\gamma_1}{\eta^3} sin\,(\eta(t-t_i)_+)$$
$$-\frac{(t-t_i)_+}{\eta^2}(\gamma_1 cos\,(\eta(t-t_i)) + \gamma_2 \eta sin\,(\eta(t-t_i))). \tag{19}$$

The coefficients fulfil the following conditions

$$\begin{aligned} &\sum_{i=1}^{n}\lambda_i(\gamma_1 sin\,(\eta t_i) + \gamma_2\eta cos\,(\eta t_i)) + \lambda_0 sin\,(\eta a) + \lambda_{n+1}\eta cos\,(\eta a) = 0, \\ &\sum_{i=1}^{n}\lambda_i(\gamma_1 cos\,(\eta t_i) - \gamma_2\eta sin\,(\eta t_i)) + \lambda_0 cos\,(\eta a) - \lambda_{n+1}\eta sin\,(\eta a) = 0, \end{aligned} \tag{20}$$

and the system of the interpolating conditions

$$\gamma_1 s(t_i) + \gamma_2 s'(t_i) = z_i^M, i = 1, \ldots, n,$$

in case $\alpha = 1$. For $\alpha < 1$ the interpolating conditions are specified by iterations of the method of adding-removing knots.

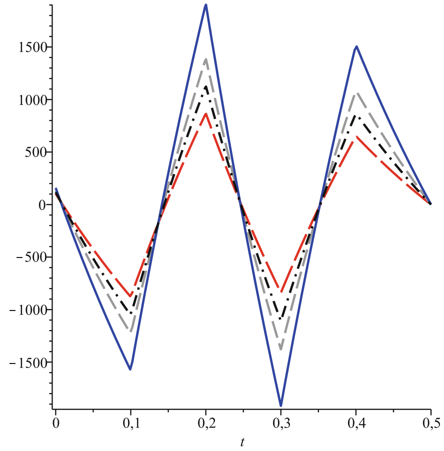

Fig. 3. Control law for *Example 1* (solid line for $\alpha = 1$, dash line for $\alpha = 0.25$, dash dot line for $\alpha = 0.5$, long dash line for $\alpha = 0$)

The corresponding control function u is given by

$$u(t) = \sum_{i=1}^{n} \lambda_i \left(\frac{\gamma_1}{\eta} sin\, \eta(t_i - t)_+ + \gamma_2 (t_i - t)_+^0 cos\, \eta(t_i - t)\right). \tag{21}$$

For the considered case the conditions on the uniqueness of solution are satisfied. This solution is constructed for four α-levels: $\alpha_1 = 0$, $\alpha_2 = 0.5, \alpha_3 = 0.75$ and $\alpha_4 = 1$. The solution of problem (8) in this case is the trigonometric splines with polynomial coefficients (19) with coefficients from Table 3. The control law u could be obtained by (21). The corresponding graphs are considered in Fig. 4 and Fig. 5. The values of the objective functional for considered α-levels are compared (see Table 4).

Table 3. Coefficients of the solutions at α-levels for *Example 2*

	$\alpha_1 = 0$	$\alpha_2 = 0.5$	$\alpha_3 = 0.75$	Interpolation ($\alpha_4 = 1$)
λ_0	9.0736×10^3	1.8084×10^4	2.2590×10^4	2.7095×10^4
λ_1	2.0746×10^4	4.1800×10^4	5.2327×10^4	6.2854×10^4
λ_2	-3.1238×10^4	-6.1490×10^4	-7.6616×10^4	-9.1742×10^4
λ_3	5.6274×10^4	1.1130×10^5	1.3881×10^5	1.6633×10^5
λ_4	-1.3993×10^5	-2.8177×10^5	-3.5268×10^5	-4.2360×10^5
λ_5	8.6636×10^5	1.7523×10^5	2.1953×10^5	2.63837×10^5
λ_6	4.6398×10^2	8.6472×10^2	1.0650×10^3	1.2654×10^3

Table 4. The values of the objective functional for *Example 2*

α	$\alpha_1 = 0$	$\alpha_2 = 0.5$	$\alpha_3 = 0.75$	Interpolation ($\alpha_4 = 1$)
$\|u\|$	2.8012×10^3	5.6865×10^3	7.1292×10^3	8.5718×10^3

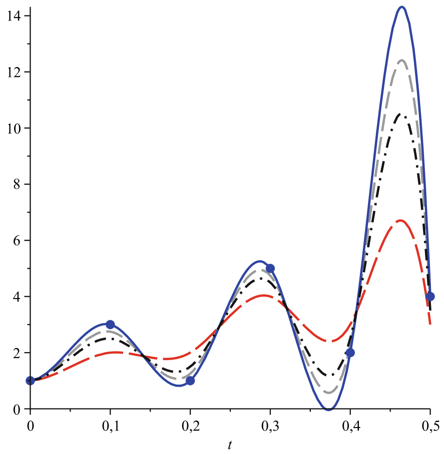

Fig. 4. State trajectories for *Example 2* (solid line for $\alpha = 1$, dash line for $\alpha = 0.5$, dash dot line for $\alpha = 0.75$, long dash line for $\alpha = 0$)

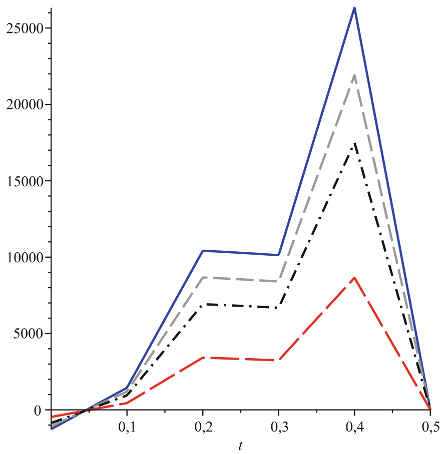

Fig. 5. Control law for *Example 2* (solid line for $\alpha = 1$, dash line for $\alpha = 0.5$, dash dot line for $\alpha = 0.75$, long dash line for $\alpha = 0$)

By comparison of the values of the objective functional from Table 4 we see that the minimum of this functional is obtained for $\alpha_1 = 0$ and the interpolating spline gives the biggest value.

5 Conclusion

The proposed method can be effectively used for dynamical systems associated with linear differential equations (the restriction on the order of equations is not essential) when uncertainty in location conditions is described by fuzzy numbers $Z_i,\ i = 1, \ldots, n$ (the restriction on the triangular type of fuzzy numbers is not essential). It seems natural to incorporate into the model also uncertainty of the sequence of times to be considered. For such purpose fuzzy numbers $T_i,\ i = 1, \ldots, n$, could be used instead of crisp $t_i,\ i = 1, \ldots, n$. In this case the constraints

$$y(t_i) \text{ is } Z_i,\ i = 1, \ldots, n,$$

could be rewritten using IF-THEN rules as

$$\text{IF } t \text{ is } T_i \text{ THEN } y(t) \text{ is } Z_i,\ i = 1, \ldots, n.$$

The future research could be devoted to development of the proposed approach for the following problem

$$\int_a^b (u(\tau))^2 d\tau \rightarrow \min_{\substack{u \in L_2[a,b]:\ \ x(a)=c, \\ \text{IF } t \text{ is } T_i \text{ THEN } y(t) \text{ is } Z_i,\ i=1,\ldots,n}}$$

considered in the context of (1)–(4).

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