Fuzzy Neighborhood Semantics for Multi-agent Probabilistic Reasoning in Games

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Abstract. In this contribution we apply fuzzy neighborhood semantics to multiple agents' reasoning about each other's subjective probabilities, especially in game-theoretic situations. The semantic model enables representing various game-theoretic notions such as payoff matrices or Nash equilibria, as well as higher-order probabilistic beliefs of the players about each other's choice of strategy. In the proposed framework, belief-dependent concepts such as the strategy with the best expected value are formally derivable in higher-order fuzzy logic for any finite matrix game with rational payoffs.

Keywords: Probabilistic reasoning · Fuzzy logic · Modal logic · Neighborhood semantics · Matrix game

1 Introduction

In this paper, we propose a semantics for multi-agent reasoning about uncertain beliefs. Using a suitable fuzzy logic for its representation makes it possible to formalize doxastic reasoning under uncertainty in a rather parsimonious way, which is of particular importance, e.g., in software modeling of rational agents.

As a prominent measure of uncertainty, we apply a fuzzy probability measure to fuzzy doxastic propositions. Fuzzy-logical modeling of probability started with [13]. Common approaches include two-layered expansions of suitable fuzzy logics by a fuzzy modality *probably,* states of MV-algebras, and probabilistic fuzzy description logics [10,15,16]. Here we generalize the fuzzy modal approach of [13], overcoming some of its limitations given by its two-layered syntax. Generally, though, we do not want to restrict the framework to perfectly rational agents. Therefore, we introduce a more general semantics that admits also probabilistically incoherent assignments of certainty degrees. This paves the way not only for accommodating the reasoning of probabilistically irrational agents, but also for modeling the agents' reasoning about the other agent's (ir)rationality and its potential exploitability.

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As an illustration of the semantic framework, we apply it to probabilistic reasoning in game-theoretic situations. This application belongs to the broader research area of *logic in games* [4], which aims at a formal reconstruction of game-theoretical concepts by means of formal logic. As an interface between fuzzy logic and game theory, we employ the representation of (a broad class of) strategic games in fuzzy logic laid out in [2]. In the game-theoretic setting, the framework enables formalizing the player's beliefs about each other's choice of strategy, including higher-order beliefs (i.e., beliefs about the beliefs of others). In the game-theoretic setting, probabilistic beliefs are particularly important, due to the players' uncertainty about each others' choice of strategy, the possibility of using mixed (i.e., probabilistic) strategies, and is especially pronounced in games with incomplete information (such as most card games).

The paper is organized as follows. In Sect. 2, we gather prerequisites for developing fuzzy doxastic and probabilistic logic, including the logic ŁΠ and the notion of fuzzy probability measure. Fuzzy doxastic and probabilistic models for multi-agent reasoning are presented in Sect. 3. Next, in Sect. 4, we define fuzzy doxastic and probabilistic logic and discuss its relationships to various known logics. Section 5 provides an overview of game-theoretical notions formalized in fuzzy probabilistic or doxastic logic; subsequently, we apply these notions to represent probabilistic reasoning in a simple two-player game. Finally, the features of the introduced formalism and topics for future work are summarized in Sect. 6.

2 Preliminaries

For the formalization of probabilistic and doxastic reasoning in games, we will employ the expressively rich fuzzy logic ŁΠ. This choice is made for the sake of uniformity, even though many constructions described below can as well be carried out in some of its less expressive fragments such as $\text{Ł}_\triangle$ or $\text{PŁ}'_\triangle$. For details on the logic ŁΠ see [7,9]; here we just briefly recount the definition.

A salient feature of the logic ŁΠ is that it contains the connectives of many well-known fuzzy logics, including the three prominent t-norm based fuzzy logics (Gödel, Łukasiewicz, and product).

We use the symbols $\wedge, \vee, \neg, \otimes, \oplus, \sim, \Rightarrow, \odot, \oslash$, respectively, for the Gödel, Łukasiewicz, and product connectives $\&_\mathrm{G}, \vee_\mathrm{G}, \neg_\mathrm{G}, \&_\text{Ł}, \vee_\text{Ł}, \neg_\text{Ł}, \Rightarrow_\text{Ł}, \&_\Pi, \Rightarrow_\Pi$ of ŁΠ. Of these, $\Rightarrow$, $\oslash$, $\odot$, and the truth constant 0 can be taken as the only primitives; the others are definable.

The *standard semantics* of ŁΠ, or the *standard* ŁΠ-*algebra* $[0,1]_{\text{Ł}\Pi}$, interprets the connectives by the following truth functions on $[0,1]$:

$$
\begin{aligned}
x \wedge y &= \min(x,y) & x \otimes y &= \max(0, x+y-1)\\
x \vee y &= \max(x,y) & x \Rightarrow y &= \min(1, 1-x+y)\\
x \oplus y &= \min(1, x+y) & x \odot y &= x \cdot y\\
x \ominus y &= \max(0, x-y) & x \oslash y &= y/x \text{ if } x > y, \text{ otherwise } 1\\
\sim x &= 1-x & \neg x &= 1 - \text{sign } x\\
x \Leftrightarrow y &= 1 - |x-y| & \triangle x &= 1 - \text{sign}(1-x)
\end{aligned}
$$

An axiomatic system for ŁΠ consists of the axioms of Łukasiewicz and product logic respectively for Łukasiewicz and product connectives, the axioms $\triangle(\varphi \Rightarrow \psi) \Rightarrow (\varphi \oslash \psi)$, $\triangle(\varphi \oslash \psi) \Rightarrow (\varphi \Rightarrow \psi)$, $\varphi \odot (\psi \ominus \chi) \Leftrightarrow (\varphi \odot \psi) \ominus (\varphi \odot \chi)$, and the rules of modus ponens and $\triangle$-necessitation (from φ infer $\triangle\varphi$). The logic ŁΠ enjoys finite strong completeness of this axiomatic system w.r.t. its standard semantics on $[0, 1]$.

The general (linear) algebraic semantics of ŁΠ is given by the class of (linear) ŁΠ-*algebras* $\boldsymbol{L} = (L, \oplus, \sim, \oslash, \odot, 0, 1)$, where:

- $(L, \oplus, \sim, 0)$ is an MV-algebra,
- $(L, \vee, \wedge, \oslash, \odot, 0, 1)$ is a Π-algebra, and
- $x \odot (y \ominus z) = (x \odot y) \ominus (x \odot z)$ holds.

Like other fuzzy logics, ŁΠ also enjoys completeness w.r.t. the classes of linear and all ŁΠ-algebras. Except for the two-element ŁΠ-algebra $\{0, 1\}$, all non-trivial linear ŁΠ-algebras are isomorphic to the unit interval algebras of linearly ordered fields.

The first-order logic ŁΠ, denoted by ŁΠ∀, is defined as usual in fuzzy logic: in a first-order model $\boldsymbol{M} = (M, \boldsymbol{L}, I)$ over an ŁΠ-algebra $\boldsymbol{L}$, n-ary predicate symbols P are interpreted by $\boldsymbol{L}$-valued functions $I(P)\colon M^n \to \boldsymbol{L}$ and the quantifiers $\forall, \exists$ are evaluated as the infimum and supremum in $\boldsymbol{L}$. Safe ŁΠ∀-models (i.e., such that all required suprema and infima exist in $\boldsymbol{L}$) are axiomatized by the propositional axioms and rules of ŁΠ, generalization (from φ derive $(\forall x)\varphi$), and the axioms:

- $(\forall x)\varphi \Rightarrow \varphi(t)$, where t is free for x in φ, and
- $(\forall x)(\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow (\forall x)\varphi)$, where x is not free in χ.

First-order ŁΠ can be extended by the axioms for crisp identity, $x = x$ and $x = y \Rightarrow \triangle(\varphi(x) \Leftrightarrow \varphi(y))$, function symbols, and sorts of variables in a standard manner; see, e.g., [1].

The present paper will also make use of the logic ŁΠ of a higher order. Higher-order logic ŁΠ has been introduced in [1] and its Church-style notational variant in [20]. For the full description of higher-order ŁΠ we refer the reader to [1] or [3, Sect. A.3]; here we only highlight some of its features relevant to our purposes.

Of the full language of higher-order ŁΠ, in this paper we will only need its monadic fragment. Its syntax contains variables and constants for individuals $(x, y, \dots)$, first-order monadic predicates $(P, Q, \dots)$, second-order monadic predicates $(\mathcal{P}, \mathcal{Q}, \dots)$, etc. In a model over an ŁΠ-algebra $\boldsymbol{L}$, individual variables and constants are interpreted as elements of the model's domain X; first-order monadic predicates as fuzzy sets on X, i.e., elements of $\boldsymbol{L}^X$; second-order monadic predicates as fuzzy sets of fuzzy sets on X, i.e., elements of $\boldsymbol{L}^{\boldsymbol{L}^X}$; etc.

Besides the connectives of ŁΠ and the quantifiers of ŁΠ∀ (applicable to variables of any order), the language of monadic higher-order ŁΠ also contains *comprehension terms* $\{x^{(n)} \mid \varphi\}$, for all variables $x^{(n)}$ of any order n and all well-typed formulae φ. In $\boldsymbol{L}$-valued models, $\{x \mid \varphi(x)\}$ denotes the fuzzy set $A \in \boldsymbol{L}^X$

which assigns to each value of x the truth value of $\varphi(x)$. Analogously, $\{P \mid \varphi(P)\}$ denotes the second-order fuzzy set $\mathcal{A} \in \boldsymbol{L}^{\boldsymbol{L}^X}$ which assigns to each value of P the truth value of $\varphi(P)$, etc.

We will denote the higher-order logic ŁΠ by $Ł\Pi^\omega$. Its Henkin-style axiomatization in multi-sorted ŁΠ∀, consisting of the axioms of extensionality and comprehension for each type and complete w.r.t. Henkin models, can be found in [1,3].

In what follows we will need the following first-order fuzzy set operations (definable in $Ł\Pi^\omega$):

Definition 1. *Let X be a crisp set and $\boldsymbol{L}$ an ŁΠ-algebra. The fuzzy set operations $\Cap$, $\Cup$, $\smallsetminus$, and $\underline{0}$ are defined by setting for all $x \in X$ and $A, B \in \boldsymbol{L}^X$:*

$$
\begin{aligned}
(A \Cap B)(x) &= A(x) \otimes B(x) \\
(A \Cup B)(x) &= A(x) \oplus B(x) \\
(\smallsetminus A)(x) &= {\sim} A(x) \\
\underline{0}(x) &= 0
\end{aligned}
$$

The following sections also refer to fuzzy probability measures, which have been extensively studied in the literature; for an overview see [10]. Fuzzy probability measures are usually defined as valued in the real unit interval $[0, 1]$; here we use the definition generalized to any ŁΠ-algebra $\boldsymbol{L}$.

Definition 2. *Let $\boldsymbol{L}$ be an ŁΠ-algebra. A finitely additive $\boldsymbol{L}$-valued* fuzzy probability measure *on W is a function $\rho\colon \boldsymbol{L}^W \to \boldsymbol{L}$ such that the following conditions hold for all $A, B \in \boldsymbol{L}^W$:*

- $\rho(\underline{0}) = 0$
- $\rho(\smallsetminus A) = {\sim}\rho(A)$
- *If $\rho(A \Cap B) = 0$ then $\rho(A \Cup B) = \rho(A) \oplus \rho(B)$.*

Finally, let us briefly recall (multi-agent) *standard doxastic logic,* since fuzzy probabilistic and doxastic logics introduced below adapt its models to make them suitable for uncertain doxastic reasoning. For details on standard doxastic logic see, e.g., [18]. Standard multi-agent doxastic logic expands classical propositional logic by (freely nestable) unary modalities B_a for each agent a, where $\mathrm{B}_a\varphi$ is interpreted as "agent a believes that φ". Models for standard doxastic logic are given by Kripke-style (relational) semantics:

Definition 3. *A multi-agent* standard doxastic frame *is a tuple $\boldsymbol{F} = \big(W, A, \{R_a\}_{a\in A}\big)$, where:*

- *$W \neq \emptyset$ is a set of* possible worlds.
- *$A \neq \emptyset$ is a set of* agents.
- *$R_a \subseteq W^2$ is the* accessibility relation *for each agent a. In standard doxastic logic it is assumed that all R_a are serial, transitive, and Euclidean.*

A multi-agent standard doxastic model *over the frame* $\boldsymbol{F}$ *is a pair* $\boldsymbol{M} = \bigl(\boldsymbol{F}, e\bigr)$*, where* $e\colon Var \times W \to \{0,1\}$ *is an* evaluation *of (countably many) propositional variables* $p_i \in Var$ *in each world* $w \in W$*. The truth value, or the* extension $\|\varphi\|_w$ *of a formula* φ *in a world* w *of the model* $\boldsymbol{M}$ *is defined by the recursive Tarski conditions:*

$$\begin{aligned}
\|p\|_w = 1 &\quad \textit{iff} \quad e(p,w) = 1\\
\|\neg\varphi\|_w = 1 &\quad \textit{iff} \quad \|\varphi\|_w = 0\\
\|\varphi \Rightarrow \psi\|_w = 1 &\quad \textit{iff} \quad \|\varphi\|_w = 0 \textit{ or } \|\psi\|_w = 1\\
\|\mathrm{B}_a\varphi\|_w = 1 &\quad \textit{iff} \quad R_a ww' \textit{ implies } \|\varphi\|_{w'} = 1 \textit{ for all } w' \in W
\end{aligned}$$

The set $\|\varphi\| = \{w \in W\colon \|\varphi\|_w = 1\}$ *is called the* intension *of* φ *in* $\boldsymbol{M}$*. A formula is* valid *in* $\boldsymbol{M}$ *if* $\|\varphi\| = W$*. A formula is a doxastic* tautology *if it is valid in all doxastic models.*

Standard doxastic logic is axiomatized by adding the following axioms and rules to classical propositional logic, for all agents a:

(K)	$\mathrm{B}_a\varphi \wedge \mathrm{B}_a(\varphi \Rightarrow \psi) \Rightarrow \mathrm{B}_a\psi$	(logical rationality)
(D)	$\mathrm{B}_a\varphi \Rightarrow \neg\mathrm{B}_a\neg\varphi$	(consistency of beliefs)
(4)	$\mathrm{B}_a\varphi \Rightarrow \mathrm{B}_a\mathrm{B}_a\varphi$	(positive introspection)
(5)	$\neg\mathrm{B}_a\varphi \Rightarrow \mathrm{B}_a\neg\mathrm{B}_a\varphi$	(negative introspection)
(Nec)	from φ derive $\mathrm{B}_a\varphi$	(necessitation)

In standard doxastic frames, the intended role of accessibility relations R_a is such that the successor sets $wR_a = \{w' \mid R_a ww'\}$ comprise those worlds which the agent a in the world w does not rule out as being the actual world. The proposition "a believes that φ" is then considered true in w if φ is true in all worlds that a does not exclude in w, i.e., in all $w' \in wR_a$. The truth of $\mathrm{B}_a\varphi$ in w can thus be regarded as given by a (maxitive) two-valued measure $\beta_{a,w}$ on W:

$$\beta_{a,w}(A) = \begin{cases} 1 & \text{if } wR_a \subseteq A\\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

for all $A \subseteq W$. Then the Tarski condition for B_a can be written as $\|B_a\varphi\|_w = \beta_{a,w}(\|\varphi\|)$. In the next section, the maxitive two-valued measure $\beta_{a,w}$ implicit in standard doxastic frames will be generalized to a finitely additive fuzzy probability measure suitable for doxastic reasoning under uncertainty.

3 Fuzzy Doxastic Models

For the modeling of doxastic or probabilistic reasoning of agents, we will employ a suitable fuzzy variant of possible-world (intensional) semantics (cf. [3,6,8,21]). The multi-agent fuzzy doxastic frames introduced in the following definition are a variant of similar structures that have already been employed for the semantics of probabilistic reasoning in the literature [5,11,12,22]. They also generalize (an equivalent reformulation of) the Kripke frames of standard doxastic logic [18].

Definition 4. *A* multi-agent fuzzy doxastic frame *is a tuple* $\boldsymbol{F} = (W, \boldsymbol{L}, A, \nu)$, *where:*

- $W \neq \emptyset$ *is a crisp set of* possible worlds *(or world states, situations).*
- $\boldsymbol{L}$ *is an* ŁΠ*-algebra of degrees.*
- $A \neq \emptyset$ *is a set of agents.*
- $\nu = \{\nu_{a,w}\}_{a \in A, w \in W}$, *where* $\nu_{a,w} \colon \boldsymbol{L}^W \to \boldsymbol{L}$ *for each* $a \in A$ *and* $w \in W$.

Fuzzy subsets of the set W of possible worlds are called *(fuzzy) propositions* or, synonymously, *(fuzzy) events.* In a multi-agent fuzzy doxastic frame $\boldsymbol{F}$, the functions $\nu_{a,w} \colon \boldsymbol{L}^W \to \boldsymbol{L}$ thus assign degrees to events. The value $\nu_{a,w}(E)$ is understood as the degree of the agent a's certainty in the world w about the event E. Each $\nu_{a,w}$ can also be regarded as a fuzzy set of events, $\nu_{a,w} \in \boldsymbol{L}^{\boldsymbol{L}^W}$.

The system $\nu = \{\nu_{a,w}\}_{a \in A, w \in W}$ can equivalently be viewed as assigning to each agent $a \in A$ a *fuzzy neighborhood function* $\nu_a \colon W \times \boldsymbol{L}^W \to \boldsymbol{L}$. These are known from the fuzzy neighborhood semantics of fuzzy modal logics [8,21], which is a fuzzy generalization of the well known Scott–Montague neighborhood semantics of modal logics [19,23]. The applicability of fuzzy neighborhood semantics to probabilistic and doxastic reasoning has been made explicit in [22].

A fuzzy neighborhood function ν_a assigns to each world w a fuzzy set of fuzzy "neighborhoods". The fuzzy neighborhoods of w will be understood as events that the agent a in the world w considers probable (to the degree assigned by $\nu_{a,w}$). We will interpret $\nu_{a,w}$ as measuring the subjective probability of events, as assessed by agent a in world w. In the general setting of multi-agent fuzzy doxastic frames we impose no restriction on $\nu_{a,w}$. In the following definition we specify additional conditions suitable for the probabilistic interpretation of $\nu_{a,w}$.

Definition 5. *Let* $\boldsymbol{F} = (W, \boldsymbol{L}, A, \nu)$ *be a multi-agent fuzzy doxastic frame. We say that* $\boldsymbol{F}$ *is a* (multi-agent) fuzzy probabilistic frame *if each* $\nu_{a,w}$ *is a finitely additive fuzzy probability measure.*

Thus, in fuzzy probabilistic frames, the subjective probability measures $\nu_{a,w}$ of all agents a and in all world states w are supposed to satisfy the axioms of fuzzy probability from Definition 2. This corresponds to the assumption of probabilistic rationality of all agents. In fuzzy doxastic frames, this condition is relaxed, which makes it possible to model agents with incomplete or incoherent assignments of probabilities.

In the probabilistic setting, the most common choice of $\boldsymbol{L}$ will be that of the standard ŁΠ-algebra, $\boldsymbol{L} = [0,1]_{\text{ŁΠ}}$; nevertheless, the definition also admits other ŁΠ-algebras of certainty degrees that may be suitable for probabilistic or doxastic reasoning, including the two-valued, rational-valued, or hyperreal-valued ones.

We will use fuzzy doxastic and probabilistic frames in the standard manner to define models for probabilistic and doxastic fuzzy modal logic. First we need to specify the modal language:

Definition 6. *Let Var be a countably infinite set of propositional variables and* A *a nonempty set of agents. By* $\mathscr{S}_A$ *we denote the propositional signature of the logic* ŁΠ *expanded by the unary modalities* P_a *for all* $a \in A$.

Thus, e.g., $(p \mathbin{\&} \mathrm{P}_a \triangle q) \Rightarrow \triangle \mathrm{P}_a \mathrm{P}_b (p \mathbin{\&} q)$ is a well-formed formula of $\mathscr{S}_{\{a,b\}}$.

Definition 7. *A* (multi-agent) fuzzy doxastic model *is a pair* $\boldsymbol{M} = (\boldsymbol{F}, e)$*, where* $\boldsymbol{F} = (W, \boldsymbol{L}, A, \nu)$ *is a multi-agent fuzzy probabilistic frame and* $e\colon \mathit{Var} \times W \to \boldsymbol{L}$ *is an* $\boldsymbol{L}$*-evaluation of propositional variables in each world.*

If $\boldsymbol{F}$ *is a fuzzy probabilistic frame, we speak of a* fuzzy probabilistic model.

As usual in intensional possible-world semantics, the semantic value of a formula φ in a model $\boldsymbol{M}$ is identified with its *intension* $\|\varphi\|\colon W \to \boldsymbol{L}$. The value of the intension $\|\varphi\|$ in a given world $w \in W$, i.e., the degree $\|\varphi\|(w) \in \boldsymbol{L}$, is called the *extension* of φ in w and denoted by $\|\varphi\|_w$. Note that in fuzzy doxastic frames, intensions of formulae are events and extensions are the degrees of the event's occurrence in particular worlds. Their values in fuzzy doxastic models are defined in a standard manner by recursive Tarski conditions (cf. [8,21]):

Definition 8. *The intensions* $\|\varphi\|\colon W \to \boldsymbol{L}$ *and extensions* $\|\varphi\|_w = \|\varphi\|(w)$ *of* $\mathscr{S}_A$*-formulae* φ *in the fuzzy doxastic model* $\boldsymbol{M}$ *are defined inductively as follows:*

$$\begin{aligned} \|p\|_w &= e(p, w) \\ \|c(\varphi_1, \dots, \varphi_n)\|_w &= c^{\boldsymbol{L}}(\|\varphi_1\|_w, \dots, \|\varphi_n\|_w) \\ \|\mathrm{P}_a \varphi\|_w &= \nu_{a,w}(\|\varphi\|) \end{aligned}$$

for all worlds $w \in W$*, agents* $a \in A$*, propositional variables* $p \in \mathit{Var}$*, all* $\mathscr{S}_A$*-formulae* $\varphi_1, \dots, \varphi_n, \varphi$*, and each* n*-ary connective* c *of* ŁΠ*, where* $c^{\boldsymbol{L}}$ *is the truth function of* c *in the* ŁΠ*-algebra* $\boldsymbol{L}$*.*

It can be observed that standard doxastic frames (see Sect. 2) are special cases of fuzzy doxastic frames over the two-element ŁΠ-algebra $\{0, 1\}$, when taking the maxitive two-valued measures $\beta_{a,w}$ of (1) for $\nu_{a,w}$.

Although the language $\mathscr{S}_A$ contains just a single graded probabilistic or doxastic modality P_a for each agent $a \in A$, various ranges and comparisons of probabilities (or certainty degrees) used in bivalent probabilistic logics are expressible by means of the connectives of ŁΠ. For instance, the qualitative probabilistic conditional $\varphi \succeq \psi$ of [12], "φ is at least as probable as ψ", is expressed for any agent a as $\triangle(\mathrm{P}_a \psi \Rightarrow \mathrm{P}_a \varphi)$. Similarly, the bivalent statement that "the probability assigned to φ by a is in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$" can be expressed by the $\mathscr{S}_A$-formula $\triangle(\mathrm{P}_a \varphi \oplus \mathrm{P}_a \varphi \oplus \mathrm{P}_a \varphi) \wedge \neg(\mathrm{P}_a \varphi \otimes \mathrm{P}_a \varphi)$. In general, by Proposition 1 below, any rational interval of a's probabilities for φ is expressible by an $\mathscr{S}_A$-formula. Note that this includes the threshold probabilistic modalities $\mathrm{P}_a^{\geq r} \varphi$, "$a$ believes that the probability of φ is at least r", for $r \in \mathbb{Q} \cap [0, 1]$, used, e.g., in [11]. Since all infinite linear ŁΠ-algebras embed $\mathbb{Q} \cap [0, 1]$, we can formulate the proposition more generally than just for standard models:

Proposition 1. *Let* $\boldsymbol{L}$ *be a linear* ŁΠ*-algebra,* $\boldsymbol{M} = (W, \boldsymbol{L}, A, \nu, e)$ *a fuzzy doxastic model,* $a \in A$*,* $w \in W$*, and* $r, s \in \mathbb{Q} \cap [0, 1]$*.*

1. *There exist* ŁΠ*-formulae* $\chi_r, \chi_{\geq r}$ *in one propositional variable such that:*

$$\chi_r^{\boldsymbol{L}}(x) = \begin{cases} 1 & \text{if } x = r \\ 0 & \text{otherwise} \end{cases} \qquad \chi_{\geq r}^{\boldsymbol{L}}(x) = \begin{cases} 1 & \text{if } x \geq r \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_r^{\boldsymbol{L}}, \chi_{\geq r}^{\boldsymbol{L}}$ *are the truth functions of* $\chi_r, \chi_{\geq r}$ *in* $\boldsymbol{L}$.

2. $\|\triangle(\mathrm{P}_a\psi \Rightarrow \mathrm{P}_a\varphi)\|_w = \begin{cases} 1 & \text{if } \nu_{a,w}(\|\varphi\|) \geq \nu_{a,w}(\|\psi\|) \\ 0 & \text{otherwise.} \end{cases}$

3. $\left\|\chi_{\geq r}(\mathrm{P}_a\varphi) \wedge \chi_{\geq 1-s}({\sim}\mathrm{P}_a\varphi)\right\|_w = \begin{cases} 1 & \text{if } \nu_{a,w}(\|\varphi\|) \in [r, s] \\ 0 & \text{otherwise.} \end{cases}$

Proof. 1. If $r = 0$, take $\chi_r \equiv_{\mathrm{df}} \neg p$ and $\chi_{\geq r} \equiv_{\mathrm{df}} p \Rightarrow p$. If $r = 1$, take $\chi_r, \chi_{\geq r} \equiv_{\mathrm{df}} \triangle p$. If $r = \frac{m}{n} \in (0, 1)$, $m, n \in \mathbb{N}$, then let $\varphi \equiv_{\mathrm{df}} p \wedge {\sim}p$; $\psi \equiv_{\mathrm{df}} \varphi \oslash (\varphi \oplus \varphi)$; $\vartheta \equiv_{\mathrm{df}} \bigodot_{i=1}^{\lceil \log_2(\max(m,n)) \rceil} \psi$; $\eta \equiv_{\mathrm{df}} \left(\bigoplus_{i=1}^{m} \vartheta\right) \oslash \left(\bigoplus_{i=1}^{n} \vartheta\right)$; $\chi_r \equiv_{\mathrm{df}} \triangle(\eta \Leftrightarrow p) \wedge \neg\triangle p$; and $\chi_{\geq r} \equiv_{\mathrm{df}} \triangle(\eta \Rightarrow p) \wedge \neg\neg p$. If $x \in \{0, 1\}$, then it is easy to verify that $\eta^{\boldsymbol{L}}(x) = 1$, and thus $\chi_r^{\boldsymbol{L}}(0) = \chi_r^{\boldsymbol{L}}(1) = \chi_{\geq r}^{\boldsymbol{L}}(0) = 0$ and $\chi_{\geq r}^{\boldsymbol{L}}(1) = 1$ as desired. If $x \in \boldsymbol{L} \setminus \{0, 1\}$, then $\varphi^{\boldsymbol{L}}(x) \leq \frac{1}{2}$, so $\psi^{\boldsymbol{L}}(x) = \frac{1}{2}$, $\vartheta^{\boldsymbol{L}}(x) = 2^{-\lceil \log_2(\max(m,n)) \rceil} \leq \max\left(\frac{1}{m}, \frac{1}{n}\right)$, thus $\eta^{\boldsymbol{L}}(x) = \frac{m}{n}$, and then it is straightforward to verify that $\chi_r^{\boldsymbol{L}}(x), \chi_{\geq r}^{\boldsymbol{L}}(x)$ have the desired values.

Claims 2 and 3 follow directly from Claim 1 by Definition 8 and the semantics of propositional connectives in linear ŁΠ-algebras. □

4 Fuzzy Probabilistic and Doxastic Logic

The notions of truth, validity, tautologicity, and (local) entailment w.r.t. (classes of) fuzzy doxastic models are defined as usual in (fuzzy) intensional semantics (cf. [3,6,8]):

Definition 9. *Let* $\boldsymbol{M} = (W, \boldsymbol{L}, A, \nu, e)$ *be a fuzzy doxastic model. We say that an* $\mathscr{S}_A$*-formula* φ *is* true *in* $w \in W$ *if* $\|\varphi\|_w = 1^{\boldsymbol{L}}$. *We say that* φ *is* valid *in* $\boldsymbol{M}$ *if* $\|\varphi\|_w = 1^{\boldsymbol{L}}$ *for all* $w \in W$. *We say that* $\mathscr{S}_A$*-formulae* $\varphi_1, \ldots, \varphi_n$ *locally entail an* $\mathscr{S}_A$*-formula* φ *in* $\boldsymbol{M}$ *if* φ *is true in all worlds where all* $\varphi_1, \ldots, \varphi_n$ *are true.*

Let $\mathbf{K}$ *be a class of fuzzy doxastic models for* $\mathscr{S}_A$. *We say that an* $\mathscr{S}_A$*-formula* φ *is a* $\mathbf{K}$*-tautology, written* $\models_{\mathbf{K}} \varphi$, *if* φ *is valid in all models* $\boldsymbol{M} \in \mathbf{K}$. *We say that* $\mathscr{S}_A$*-formulae* $\varphi_1, \ldots, \varphi_n$ *locally entail an* $\mathscr{S}_A$*-formula* φ *in* $\mathbf{K}$, *written* $\varphi_1, \ldots, \varphi_n \models_{\mathbf{K}} \varphi$, *if* $\varphi_1, \ldots, \varphi_n$ *locally entail* φ *in every model* $\boldsymbol{M} \in \mathbf{K}$.

If $\mathbf{K}$ *is the class of* all *fuzzy doxastic models for* $\mathscr{S}_A$, *we denote* $\mathbf{K}$*-tautologies and entailment by* $\models_{\mathrm{FDL}_A}$ *and speak of* (multi-agent) fuzzy doxastic logic FDL_A. *Similarly if* $\mathbf{K}$ *is the class of all fuzzy* probabilistic *models for* $\mathscr{S}_A$, *we use* $\models_{\mathrm{FPL}_A}$ *and speak of* (multi-agent) fuzzy probabilistic logic FPL_A. *(For a generic set* A *of agents, we may drop the subscript and write just* FDL *or* FPL*).*

A sound and complete axiomatization, or at least an axiomatic approximation sufficiently strong for formalizing typical probabilistic or doxastic arguments, of

FDL and FPL in their own modal language $\mathscr{S}_A$ is a future work. Nevertheless, there is a standard translation (cf. [6, Prop. 4.18]) into higher-order ŁΠ, which provides a syntactic verification method for laws valid in FDL and FPL:

Definition 10. *The second-order predicate language $\mathcal{L}^2_A$ corresponding to the modal language $\mathscr{S}_A$ of Definition 6 consists of countably many monadic predicate symbols $P_1, P_2, \dots$, one for each $p_1, p_2, \dots \in Var$; individual variables $x, y, z, \dots$; and a second-order monadic predicate symbol $\mathcal{N}_a$ for each $a \in A$.*

Let x be an individual variable of $\mathcal{L}^2_A$. The standard translation *of an $\mathscr{S}_A$-formula φ of* FDL *or* FPL *into an $\mathcal{L}^2_A$-formula $\mathrm{tr}_x(\varphi)$ of* $Ł\Pi^\omega$ *is defined recursively as follows:*

$$\mathrm{tr}_x(p_i) = P_i(x)$$
$$\mathrm{tr}_x\big(c(\varphi_1, \dots, \varphi_n)\big) = c\big(\mathrm{tr}_x(\varphi_1), \dots, \mathrm{tr}_x(\varphi_n)\big)$$
$$\mathrm{tr}_x(\mathrm{P}_a\varphi) = \mathcal{N}_a(\{x \mid \mathrm{tr}_x(\varphi)\})$$

for each $p_i \in Var$, each n-ary connective c of ŁΠ*, and each $a \in A$.*

It can be observed that every fuzzy doxastic model $\boldsymbol{M} = (\boldsymbol{F}, e)$ over a fuzzy doxastic frame $\boldsymbol{F} = (W, \boldsymbol{L}, A, \nu)$ can be regarded as an $\boldsymbol{L}$-valued $Ł\Pi^\omega$-model $\boldsymbol{M'} = (W, \boldsymbol{L}, I)$ with the domain W and the interpretation I of $\mathcal{L}^2_A$ such that $I(P_i) = e(p_i)$ for each $p_i \in Var$ and $I(\mathcal{N}_a)(w) = \nu_{a,w}$ for each $a \in A$ and $w \in W$. Vice versa, every $\boldsymbol{L}$-valued $Ł\Pi^\omega$-model $\boldsymbol{M'} = (W, \boldsymbol{L}, I)$ for $\mathcal{L}^2_A$ can be regarded as a fuzzy doxastic model $\boldsymbol{M} = ((W, \boldsymbol{L}, A, \nu), e)$, where $e(p_i) = I(P_i)$ and $\nu_{a,w} = I(\mathcal{N}_a)(w)$. The correspondence between the models is clearly one to one and the translation preserves the truth values of formulae:

Lemma 1. *Let $\boldsymbol{M}, \boldsymbol{M'}$ be as above. Then $\|\varphi\|^{\boldsymbol{M}}_w = \|\mathrm{tr}_x(\varphi)\|^{\boldsymbol{M'}}_{x \mapsto w}$, where $x \mapsto w$ denotes any $\boldsymbol{M'}$-evaluation η such that $\eta(x) = w$.*

Proof. Straightforward by definitions, analogously to [3, Th. 5.9].

Proposition 2. *Let $\varphi_1, \dots, \varphi_n, \psi$ be $\mathscr{S}_A$-formulae and x an individual variable of* $Ł\Pi^\omega$*. Then:*

1. $\varphi_1, \dots, \varphi_n \models_{\mathrm{FDL}} \psi$ *iff* $\mathrm{tr}_x(\varphi_1), \dots, \mathrm{tr}_x(\varphi_n) \models_{Ł\Pi^\omega} \mathrm{tr}_x(\psi)$ *iff*
$$\models_{Ł\Pi^\omega} \Big(\bigwedge_{i=1}^n \triangle \mathrm{tr}_x(\varphi_i)\Big) \Rightarrow \mathrm{tr}_x(\psi).$$
2. $\varphi_1, \dots, \varphi_n \models_{\mathrm{FPL}} \psi$ *iff* $\pi, \mathrm{tr}_x(\varphi_1), \dots, \mathrm{tr}_x(\varphi_n) \models_{Ł\Pi^\omega} \mathrm{tr}_x(\psi)$ *iff*
$$\models_{Ł\Pi^\omega} \Big(\pi \wedge \bigwedge_{i=1}^n \triangle \mathrm{tr}_x(\varphi_i)\Big) \Rightarrow \mathrm{tr}_x(\psi),$$
where π is the $Ł\Pi^\omega$*-formalization of the fuzzy probability axioms of Definition 2,*
$$\pi \equiv_{\mathrm{df}} (\forall A)(\forall B) \bigwedge_{a \in A} \triangle\big[\neg\mathcal{N}_a(\underline{0}) \wedge \big(\mathcal{N}_a(\setminus A) \Leftrightarrow {\sim}\mathcal{N}_a(A)\big) \wedge$$
$$\big(\neg\mathcal{N}_a(A \Cap B) \Rightarrow \big(\mathcal{N}_a(A \Cup B) \Leftrightarrow (\mathcal{N}_a(A) \oplus \mathcal{N}_a(B))\big)\big)\big].$$

Proof. By inspection and easy modification of the proofs of [3, Th. 5.10, Cor. 5.12] and [3, Rem. 5.14]. □

5 Probabilistic and Doxastic Logic in Strategic Games

In this section we illustrate the apparatus of fuzzy probabilistic logic by applying it to formalization of uncertain doxastic reasoning in matrix games. In order to do so, we first need to have the game, determined by its payoff matrix, represented by formulae of fuzzy logic.

Logical representation of matrix games with only two strategies of each player and only two payoffs (*Boolean games*) was done in [14]. In [17], the representation was extended to finite strategic games with payoff values in finite MV-chains (*Łukasiewicz games*). A representation of a fairly broad class of matrix games in suitable fuzzy logics (including ŁΠ) was obtained in [2]. The latter representation covers all finite matrix games with rational payoffs, and also all n-player matrix games with strategies that can be mapped into rationals or reals from $[0, 1]$ and with each payoff function ŁΠ-representable. In this representation, the players' strategies and utilities (payoff values) are all encoded as elements of the standard ŁΠ-algebra $[0, 1]_{Ł\Pi}$ and the payoff function of a player a is expressed by an ŁΠ-formula υ_a. It has been shown in [2] that for finite matrix games with rational payoffs, such game-theoretic concepts as the Nash equilibria in pure or mixed (i.e., probabilistic) strategies are expressible by ŁΠ-formulas.

Let us consider a finite matrix game $\mathbf{G}$ with a set $A = \{a_1, \dots, a_n\}$ of players, where each player a_i is assigned a finite set of strategies S_{a_i} and a payoff function $u_{a_i} \colon \prod_{a \in A} S_a \to \mathbb{Q}$. By [2], the ŁΠ-representation of $\mathbf{G}$ encodes the strategies by elements of the standard ŁΠ-algebra $\boldsymbol{L} = [0, 1]_{Ł\Pi}$; without loss of generality we can assume that $|S_{a_i}| = m_i > 1$ and $S_{a_i} = \{\frac{j-1}{m_i-1} \mid 1 \leq j \leq m_i\}$ for each player $a_i \in A$. As shown in [2], the payoff functions u_a are affinely representable by ŁΠ-formulas: i.e., for each $a \in A$ there is an ŁΠ-formula υ_a in n variables such that $\upsilon_a^{\boldsymbol{L}}(x_1, \dots, x_n) = f(u_a(x_1, \dots, x_n))$ whenever $x_i \in S_{a_i}$ for all $i \leq n$, where $\upsilon_a^{\boldsymbol{L}}$ is the truth function of υ_a in the standard ŁΠ-algebra $\boldsymbol{L}$ and f is an affine function.

To model probabilistic beliefs of the players of $\mathbf{G}$, we will use a fuzzy probabilistic model $\boldsymbol{M} = (W, \boldsymbol{L}, A, \nu, e)$ over the standard ŁΠ-algebra $\boldsymbol{L}$. The events of interest are the players' chosen strategies; these will be represented by propositional variables $c^{a_1}, \dots, c^{a_n} \in \mathit{Var}$. To ensure that $e(c^{a_i}, w) \in S_{a_i}$, we will characterize the strategies of $\mathbf{G}$ by a finite propositional theory $\mathit{\Gamma}_{\mathbf{G}} = \{\mathit{\Gamma}^1_{\mathbf{G}}, \dots, \mathit{\Gamma}^n_{\mathbf{G}}\}$ in ŁΠ. The language of $\mathit{\Gamma}_{\mathbf{G}}$ consists of the variables c^{a_i} and additional variables $s^{a_i}_j$, representing the j-th strategy of player a_i (for all $i \leq n$ and $j \leq m_i$). The formulas $\mathit{\Gamma}^i_{\mathbf{G}}$ of $\mathit{\Gamma}_{\mathbf{G}}$ fix the values of $s^{a_i}_j$ as the elements of S_{a_i} and ensure that c^{a_i} are evaluated in S_{a_i}:

$$\mathit{\Gamma}^i_{\mathbf{G}} \equiv_{\mathrm{df}} \bigwedge_{j=1}^{m_i} \chi_{\frac{j-1}{m_i-1}}(s^{a_i}_j) \wedge \bigvee_{j=1}^{m_i} (c^{a_i} \Leftrightarrow s^{a_i}_j),$$

where $\chi_{\frac{j-1}{m_i-1}}$ is the formula from Proposition 1. Since the values of s^a_j are fixed by $\Gamma_{\mathbf{G}}$ as elements of S_a (i.e., the $\boldsymbol{L}$-codes of a's strategies), by a slight abuse of language we will use s^a_j to refer directly to the elements of S_a and write, e.g., $s^a_j \in S_a$.

For every $a \in A$ and $j \le m_i$, let c^a_j denote the formula $\triangle(c^a \Leftrightarrow s^a_j)$. The evaluation $e(c^a_j, w) \in \{0,1\} \subseteq \boldsymbol{L}$ indicates for each world w whether the player a chose the strategy s^a_j in w or not. The event $\|c^a_j\| \subseteq W$ is thus the (crisp) set of worlds where the player a chose to deploy the strategy $s^a_j \in S_a$. Player b's subjective probabilities (in w) of these events (i.e., player b's probabilistic beliefs about player a's choice of strategy) are the values $\|\mathrm{P}_a c^b_j\|_w \in \boldsymbol{L}$.

By the ŁΠ-representation of $\mathbf{G}$, the (affinely scaled) payoff of a player $a \in A$ is the value of the ŁΠ-formula $\upsilon_a(c^{a_1}, \dots, c^{a_n})$. Given the choices $c^{a_i}_{j_i}$ (of the strategies $s^{a_i}_{j_i}$) by all players $a_i \in A$, the latter payoff formula is equivalent to $\upsilon_a(s^{a_1}_{j_1}, \dots, s^{a_n}_{j_n})$. Thus, in each world $w \in \bigcap_{i=1}^n \|c^{a_i}_{j_i}\|$, the player a's payoff value is $\|\upsilon_a(s^{a_1}_{j_1}, \dots, s^{a_n}_{j_n})\|_w \in \boldsymbol{L}$.

For simplicity, in the rest of the section we assume $A = \{a, b\}$.

Definition 11. *The* expected value *of a's i-th strategy $s^a_i \in S_a$ according to a's beliefs in w is the sum of a's payoffs weighted by a's probabilities for b's strategy choices, expressed by the $\mathscr{S}_A$-formula*

$$\eta_a(s^a_i) \equiv_{\mathrm{df}} \bigoplus_{j=1}^{|S_b|} \bigl(\upsilon_a(s^a_i, s^b_j) \odot \mathrm{P}_a c^b_j\bigr). \tag{2}$$

Observe that a player a's best-value strategy is indicated by the formula:

$$\sigma_a(s^a_m) \equiv_{\mathrm{df}} \triangle\Bigl(\bigl(\bigvee_{i=1}^{|S_a|} \eta_a(s^a_i)\bigr) \Rightarrow \eta_a(s^a_m)\Bigr) \tag{3}$$

Thus, for the optimal play according to the player's probabilistic beliefs about the other player's choice of strategy, the player a should choose the strategy s^a_m only in those worlds w where $\|\sigma_a(s^a_m)\|_w = 1$.

As an illustrative case study generalizable to any finite matrix game, the following example provides an analysis of strategy choices in the well known game of Stag Hunt.

Example 1 (Stag Hunt). The Stag Hunt game with 2 players ($\mathbf{SH}_2$) is specified as follows. To catch a stag, the two hunters $\{a, b\}$ need to cooperate (i.e., deploy the strategies $s^a_{\mathbf{C}}$ and $s^b_{\mathbf{C}}$); a hunter can also go for less valuable hares instead, i.e., defect ($s^a_{\mathbf{D}}$ or $s^b_{\mathbf{D}}$). The payoffs are given by the following payoff matrices:

u_a	$s^b_{\mathbf{C}}$	$s^b_{\mathbf{D}}$
$s^a_{\mathbf{C}}$	3	0
$s^a_{\mathbf{D}}$	2	1

u_b	$s^b_{\mathbf{C}}$	$s^b_{\mathbf{D}}$
$s^a_{\mathbf{C}}$	3	2
$s^a_{\mathbf{D}}$	0	1

The game has two pure Nash equilibria: either both players cooperate or both defect. Of the two, mutual defection is risk dominant (i.e., less risky), while the other is payoff dominant (i.e., yields better payoffs). Consequently, the more uncertainty about the other player's cooperation, the better to defect; however, if the player considers the other player's cooperation sufficiently probable to make it worth the risk, cooperation has a better expected value.

By [2], $\mathbf{SH}_2$ can be encoded in the logic ŁΠ as described above (e.g., with the payoff values $0, \frac{1}{3}, \frac{2}{3}, 1$). In fuzzy probabilistic logic FPL of Sect. 4 over ŁΠ, the expected values of a's strategies $s \in \{s^a_{\mathbf{C}}, s^a_{\mathbf{D}}\}$ and a's best-value strategy are expressed by the formulas $\eta_a(s)$ and $\sigma_a(s)$ as in (2) and (3).

The following examples of $\mathscr{S}_A$-formulae are valid in FPL with $\Gamma_{\mathbf{SH}_2}$. In each world, they suggest the best-value strategy in $\mathbf{SH}_2$ under particular first- and higher-order beliefs of the player:

$$\triangle\big(\mathrm{P}_a c^b_{\mathbf{D}} \Rightarrow (\mathrm{P}_a c^b_{\mathbf{C}} \otimes \mathrm{P}_a c^b_{\mathbf{C}})\big) \Rightarrow \sigma_a(s^a_{\mathbf{C}}) \tag{4}$$

$$\big(\triangle\mathrm{P}_a(\triangle\mathrm{P}_b c^a_{\mathbf{D}} \Rightarrow c^b_{\mathbf{D}}) \otimes \triangle\mathrm{P}_a\triangle\mathrm{P}_b c^a_{\mathbf{D}}\big) \Rightarrow \sigma_a(s^a_{\mathbf{D}}) \tag{5}$$

The first formula says that a should cooperate (i.e., $s^a_{\mathbf{C}}$ is optimal) in worlds w where a believes that b will defect ($c^b_{\mathbf{D}}$) with probability at most $\frac{1}{3}$. Formula (5) says that if a believes that b plays rationally and that b believes that a is going to defect, then a's best-value strategy is to defect. It is straightforward to verify that the standard translations of both formulae by Definition 10 are indeed derivable in $\text{Ł}\Pi^\omega$ from the standard translation of $\Gamma_{\mathbf{SH}_2}$.

6 Conclusions

In this contribution, we proposed a logic for modeling probabilistic and doxastic multi-agent reasoning (Definition 9). The main feature of our approach is a parsimony of the presented formalism. We rendered propositions in the fuzzy logic ŁΠ, which is expressively rich enough to provide the basic apparatus for formalizing game-theoretical notions therein.

In Sect. 3, we formulated a technical result related to a syntactic verification method for probabilistic and doxastic laws (Proposition 2). An open question remains the axiomatization of FDL and FPL in the modal language $\mathscr{S}_A$ itself, or at least an axiomatic approximation sufficiently strong for formalizing common probabilistic or doxastic reasoning.

Further, we showed in Sect. 5 that in the proposed logic, various important game-theoretic concepts (such as expected values and best strategy choices under uncertainty) can be expressed by formulas and derived by logical deduction. Similarly as the Stag Hunt game (Example 1), the framework can formalize uncertain reasoning in other simple matrix games such as the Prisoner's Dilemma, Chicken, Matching Pennies, Paper–Rock–Scissors, etc. The framework also naturally accommodates higher-order beliefs and, being based on fuzzy logic, also various graded concepts in games (such as the strength of a player's hand in Poker).

In Sect. 5 we assumed that the agents' beliefs are governed by the axioms of fuzzy probability (i.e., that $\nu_{a,w}$ are fuzzy probability measures). In future work, we want to model also agents with incoherent probability assignments, in order to formalize how to exploit their irrationality in games by Dutch-book strategies.

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