Ackermannian Goodstein Sequences of Intermediate Growth

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Abstract. The original Goodstein process proceeds by writing natural numbers in nested exponential k-normal form, then successively raising the base to $k+1$ and subtracting one from the end result. Such sequences always reach zero, but this fact is unprovable in Peano arithmetic. In this paper we instead consider notations for natural numbers based on the Ackermann function. We define two new Goodstein processes, obtaining new independence results for ACA_0' and ACA_0^+, theories of second order arithmetic related to the existence of Turing jumps.

Keywords: Goodstein sequences · Independence proofs · Ordinal notation systems

1 Introduction

Goodstein's principle [6] is arguably the oldest example of a purely number-theoretic statement known to be independent of PA, as it does not require the coding of metamathematical notions such as Gödel's provability predicate [4]. The proof proceeds by transfinite induction up to the ordinal ε_0 [5]. PA does not prove such transfinite induction, and indeed Kirby and Paris later showed that Goodstein's principle is unprovable in PA [8].

Goodstein's original principle involves the termination of certain sequences of numbers. Say that m is in *nested (exponential) base-k normal form* if it is written in standard exponential base k, with each exponent written in turn in base k. Thus for example, 20 would become $2^{2^2}+2^2$ in nested base-2 normal form. Then, define a sequence $(g_k(0))_{m\in\mathbb{N}}$ by setting $g_0(m) = m$ and defining $g_{k+1}(m)$ recursively by writing $g_k(m)$ in nested base-$(k+2)$ normal form, replacing every occurrence of $k+2$ by $k+3$, then subtracting one (unless $g_k(m) = 0$, in which case $g_{k+1}(m) = 0$).

In the case that $m = 20$, we obtain

$$
\begin{aligned}
g_0(20) &= 20 = 2^{2^2} + 2^2\\
g_1(20) &= 3^{3^3} + 3^3 - 1 = 3^{3^3} + 3^2\cdot 2 + 3\cdot 2 + 2\\
g_2(20) &= 4^{4^4} + 4^2\cdot 2 + 4\cdot 2 + 2 - 1 = 4^{4^4} + 4^2\cdot 2 + 4\cdot 2 + 1,
\end{aligned}
$$

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and so forth. At first glance, these numbers seem to grow superexponentially. It should thus be a surprise that, as Goodstein showed, for every m there is k^* for which $g_{k^*}(m) = 0$.

By coding finite Goodstein sequences as natural numbers in a standard way, Goodstein's principle can be formalized in the language of arithmetic, but this formalized statement is unprovable in PA. Independence can be shown by proving that the Goodstein process takes at least as long as stepping down the *fundamental sequences* below ε_0; these are canonical sequences $(\xi[n])_{n<\omega}$ such that $\xi[n] < \xi$ for all ξ and for limit ξ, $\xi[n] \to \xi$ as $n \to \infty$. For standard fundamental sequences below ε_0, PA does not prove that the sequence $\xi > \xi[1] > \xi[1][2] > \xi[1][2][3] \ldots$ is finite.

Exponential notation is not suitable for writing very big numbers (e.g. Graham's number [7]), in which case it may be convenient to use systems of notation which employ faster-growing functions. In [2], T. Arai, S. Wainer and the authors have shown that the Ackermann function may be used to write natural numbers, giving rise to a new Goodstein process which is independent of the theory ATR_0 of *arithmetical transfinite recursion;* this is a theory in the language of second order arithmetic which is much more powerful than PA. The main axiom of ATR_0 states that for any set X and ordinal α, the α-Turing jump of X exists; we refer the reader to [13] for details.

The idea is, for each $k \geq 2$, to define a notion of Ackermannian normal form for each $m \in \mathbb{N}$. Having done this, we can define Ackermannian Goodstein sequences analogously to Goodstein's original version. The normal forms used in [2] are defined using an elaborate 'sandwiching' procedure first introduced in [14], approximating a number m by successive branches of the Ackermann function. In this paper, we consider simpler, and arguably more intuitive, normal forms, also based on the Ackermann function. We show that these give rise to two different Goodstein-like processes, independent of ACA_0' and ACA_0^+, respectively. As was the case for ATR_0, these are theories of second order arithmetic which state that certain Turing jumps exist. ACA_0' asserts that, for all $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$, the n-Turing jump of X exists, while ACA_0^+ asserts that its ω-jump exists; see [13] for details. The proof-theoretic ordinal of ACA_0' is ε_ω [1], and that of ACA_0^+ is $\varphi_2(0)$ [9]; we will briefly review these ordinals later in the text, but refer the reader to standard texts such as [10,12] for a more detailed treatment of proof-theoretic ordinals.

2 Basic Definitions

Let us fix $k \geq 2$ and agree on the following version of the Ackermann function.

Definition 1. *For $a, b \in \mathbb{N}$ we define $A_a(k, b)$ by the following recursion.*

1. $A_0(k, b) := k^b$,
2. $A_{a+1}(k, 0) := A_a^k(k, \cdot)(0)$,
3. $A_{a+1}(k, b + 1) := A_a^k(k, \cdot)(A_{a+1}(k, b))$.

Here, the notation $A_a^k(k,\cdot)$ refers to the k-fold composition of the function $x \mapsto A_a(k,x)$. It is well known that for every fixed a, the function $b \mapsto A_a(k,b)$ is primitive recursive and the function $a \mapsto A_a(k,0)$ is not primitive recursive. We use the Ackermann function to define k normal forms for natural numbers. These normal forms emerged from discussions with Toshiyasu Arai and Stan Wainer, which finally led to the definition of a more powerful normal form defined in [14] and used to prove termination in [2].

Lemma 1. *Let $k \geq 2$. For all $c > 0$, there exist unique $a, b, m, n \in \mathbb{N}$ such that*

1. $c = A_a(k,b) \cdot m + n$,
2. $A_a(k,0) \leq c < A_{a+1}(k,0)$,
3. $A_a(k,b) \leq c < A_a(k,b+1)$, *and*
4. $n < A_a(k,b)$.

We write $c =_{\mathrm{NF}} A_a(k,b) \cdot m + n$ in this case. This means that we have in mind an underlying context fixed by k and that for the number c we have uniquely associated the numbers a, b, m, n. Note that it could be possible that $A_{a+1}(k,0) = A_a(k,b)$, so that we have to choose the right representation for the context; in this case, item 2 guarantees that a is chosen to take the maximal possible value.

By rewriting iteratively b and n in such a normal form, we arrive at the Ackermann k-normal form of c. If we also rewrite a iteratively, we arrive at the nested Ackermann k-normal form of c. The following properties of normal forms are not hard to prove from the definitions.

Lemma 2. *1. $A_a^\ell(k,0)$ is in k-normal form for every ℓ such that $0 < \ell < k$.*
2. if $A_a(k,b)$ is in k-normal form, then for every $\ell < b$, the number $A_a(k,\ell)$ is also in k-normal form.

In the sequel we work with standard notations for ordinals. We use the function $\xi \mapsto \varepsilon_\xi$ to enumerate the fixed points of $\xi \mapsto \omega^\xi$. With $\alpha, \beta \mapsto \varphi_\alpha(\beta)$ we denote the binary Veblen function, where $\beta \mapsto \varphi_\alpha(\beta)$ enumerates the common fixed points of all $\varphi_{\alpha'}$ with $\alpha' < \alpha$. We often omit parentheses and simply write $\varphi_\alpha\beta$. Then $\varphi_0\xi = \omega^\xi$, $\varphi_1\xi = \varepsilon_\xi$, $\varphi_2 0$ is the first fixed point of the function $\xi \mapsto \varphi_1\xi$, $\varphi_\omega 0$ is the first common fixed point of the function $\xi \mapsto \varphi_n\xi$, and Γ_0 is the first ordinal closed under $\alpha, \beta \mapsto \varphi_\alpha\beta$. In fact, not much ordinal theory is presumed in this article; we almost exclusively work with ordinals less than $\varphi_2 0$, which can be written in terms of addition and the functions $\xi \mapsto \omega^\xi$, $\xi \mapsto \varepsilon_\xi$. For more details, we refer the reader to standard texts such as [10,12].

3 Goodstein Sequences for ACA_0'

In this section we define a Goodstein process that is independent of ACA_0'. We do so by working with unnested Ackermannian normal forms. Such normal forms give rise to the following notion of base change.

Definition 2. *Given $k \geq 2$ and $c \in \mathbb{N}$, define $c[k \leftarrow k+1]$ by:*

1. $0[k \leftarrow k+1] := 0$.
2. $c[k \leftarrow k+1] := A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1]$ *if* $c =_{\text{NF}} A_a(k,b) \cdot m + n$.

With this, we may define a new Goodstein process, based on unnested Ackermannian normal forms.

Definition 3. *Let* $\ell < \omega$. *Put* $b_0(\ell) := \ell$. *Assume recursively that* $b_k(\ell)$ *is defined and* $b_k(\ell) > 0$. *Then* $b_{k+1}(\ell) = b_k(\ell)[k+2 \leftarrow k+3] - 1$. *If* $b_k(\ell) = 0$, *then* $b_{k+1}(\ell) := 0$.

We will show that for every ℓ there is i with $b_i(\ell) = 0$. In order to prove this, we first establish some natural properties of the base-change operation.

Lemma 3. *Fix* $k \geq 2$ *and let* $c, d \in \mathbb{N}$. *Then:*

1. $c \leq c[k \leftarrow k+1]$.
2. *If* $c < d$, *then* $c[k \leftarrow k+1] < d[k \leftarrow k+1]$.

Proof. The first assertion is proved by induction on c. It clearly holds for $c = 0$. If $c =_{\text{NF}} A_a(k,b) \cdot m + n$ then the induction hypothesis yields $c = A_a(k,b) \cdot m + n \leq A_a(k, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1] = c[k \leftarrow k+1]$.

The second assertion is harder to prove. The proof is by induction on d with a subsidiary induction on c. The assertion is clear if $c = 0$. Let $c =_{\text{NF}} A_a(k,b) \cdot m + n$ and $d =_{\text{NF}} A_{a'}(k, b') \cdot m' + n'$. We distinguish cases according to the position of a relative to a', the position of b relative to b', etc.

Case 1 ($a < a'$). We sub-divide into two cases.
Case 1.1 ($A_{a+1}(k,0) < d$). Then, the induction hypothesis applied to $c < A_{a+1}(k,0)$ yields $c[k \leftarrow k+1] < A_{a+1}(k+1,0) < A_{a'}(k+1, b'[k \leftarrow k+1]) \cdot m' + n'[k \leftarrow k+1] = d[k \leftarrow k+1]$.
Case 1.2 ($A_{a+1}(k,0) = d$). In this case, $a+1 = a'$, $b' = 0$, $m' = 1$, and $n' = 0$. We have $A_a(k,b) \leq c < A_{a+1}(k,0) = A_a(k, A_a^{k-1}(k,\cdot)(0))$. For $\ell < k$ we have that $A_a^{\ell}(k,0)$ is in k-normal form by Lemma 2. Thus the induction hypothesis yields $b[k \leftarrow k+1] < A_a^{k-1}(k+1,\cdot)(0)$. The number $A_a(k,b)$ is in k-normal form and so the induction hypothesis applied to $n < A_a(k,b)$ yields $n[k \leftarrow k+1] < A_a(k+1, b[k \leftarrow k+1])$. Moreover we have that $m < A_{a+1}(k,0)$. This yields

$$\begin{aligned}
c[k \leftarrow k+1] &= A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1] \\
&\leq A_a(k+1, A_a^{k-1}(k+1,\cdot)(0)) \cdot A_{a+1}(k,0) + A_a(k+1, A_a^{k-1}(k+1,\cdot)(0)) \\
&\leq (A_a^k(k+1,\cdot)(0))^2 + A_a^k(k+1,\cdot)(0) \\
&\leq A_a(k+1, A_a^k(k+1,\cdot)(0)) = A_{a+1}(k+1,0),
\end{aligned}$$

where the second inequality follows from

$$A_{a+1}(k,0) = A_a^k(k,\cdot)(0) \leq A_a^k(k+1,\cdot)(0)$$

and the last from

$$A_a(k+1,x) \geq A_0(k+1,x) \geq 3^x \geq x^2 + x. \tag{1}$$

Case 2 ($a' < a$). This case does not occur since then $d < A_{a'+1}(k,0) \leq A_a(k,0) \leq c$.
Case 3 ($a = a'$ and $b < b'$). The induction hypothesis yields $b[k \leftarrow k+1] < b'[k \leftarrow k+1]$ and $n[k \leftarrow k+1] < A_a(k+1, b[k \leftarrow k+1])$. Now, consider two sub-cases.
Case 3.1 ($A_a(k, b{+}1) < d$). Since d is in k-normal form and $b{+}1 \leq b'$ we see that $A_a(k, b+1)$ is in k-normal form by Lemma 2. Then, the induction hypothesis yields $c[k \leftarrow k+1] < A_a(k+1, (b+1)[k \leftarrow k+1]) \leq A_a(k+1, b'[k \leftarrow k+1]) \leq d[k \leftarrow k+1]$.
Case 3.2 ($A_a(k, b{+}1) = d$). We know that $c = A_a(k,b) \cdot m + n < A_a(k, b{+}1) = d$. Consider two further sub-cases.
Case 3.2.1 ($a = 0$). This means that $c = k^b \cdot m + n < k^{b+1} = d$, $m < k$, and $n < k^b$, where d has k-normal form k^{b+1}. The induction hypothesis yields $b[k \leftarrow k+1] < (b+1)[k \leftarrow k+1]$ and $n[k \leftarrow k+1] < (k+1)^{b[k \leftarrow k+1]}$. We then have that $c[k \leftarrow k+1] = (k+1)^{b[k \leftarrow k+1]} \cdot m + n[k \leftarrow k+1] < (k+1)^{b[k \leftarrow k+1]+1} \leq (k+1)^{(b+1)[k \leftarrow k+1]} = d$.
Case 3.2.2 ($a > 0$). Then,

$$
\begin{aligned}
c[k \leftarrow k+1] &= A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1] \\
&\leq A_a(k+1, b[k \leftarrow k+1])) \cdot A_a(k, b+1) + A_a(k+1, b[k \leftarrow k+1]) \\
&\leq (A^k_{a-1}(k+1, \cdot)(A_a(k+1, b[k \leftarrow k+1])))^2 \\
&+ A^k_{a-1}(k+1, \cdot)(A_a(k+1, b[k \leftarrow k+1])) \\
&< A_a(k+1, b'[k \leftarrow k+1]) \text{ by (1)},
\end{aligned}
$$

where the second inequality uses

$$
A_a(k, b+1) = A^k_{a-1}(k, \cdot)(A_a(k,b)) \leq A^k_{a-1}(k+1, \cdot)(A_a(k+1, b[k \leftarrow k+1])).
$$

Case 4 ($a = a'$ and $b' < b$). This case does not appear since otherwise $d \leq A_a(k, b'+1) \leq c$.
Case 5 ($a = a'$ and $b' = b$ and $m < m'$). Then the induction hypothesis yields

$$
\begin{aligned}
c[k \leftarrow k+1] &= A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1] \\
&< A_a(k+1, b[k \leftarrow k+1])) \cdot m + A_a(k+1, b[k \leftarrow k+1]) \\
&\leq A_a(k+1, b[k \leftarrow k+1])) \cdot m' \leq d[k \leftarrow k+1].
\end{aligned}
$$

Case 6 ($a = a'$ and $b' = b$ and $m' < m$). This case is not possible given the assumptions.
Case 7 ($a = a'$ and $b' = b$ and $m' = m$). Then $n < n'$ and the induction hypothesis yields

$$
\begin{aligned}
c[k \leftarrow k+1] &= A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1] \\
&< A_a(k+1, b[k \leftarrow k+1]) \cdot m + n'[k \leftarrow k+1] = d[k \leftarrow k+1].
\end{aligned}
$$

Thus, the base-change operation is monotone. Next we see that it also preserves normal forms.

Lemma 4. *If $c = A_a(k,b) \cdot m + n$ is in k-normal form, then $c[k\leftarrow k+1] = A_a(k+1, b[k\leftarrow k+1]) \cdot m + n[k\leftarrow k+1]$ is in $k+1$ normal form.*

Proof. Assume that $c =_{\rm NF} A_a(k,b)\cdot m+n$. Then, $c < A_{a+1}(k,0)$, $c < A_a(k,b+1)$, and $n < A_a(k,b)$. Clearly, $A_a(k+1,0) \le c[k\leftarrow k+1]$. By Lemma 2, $A_{a+1}(k,0)$ is in k-normal form, so that by Lemma 3, $c < A_{a+1}(k,0)$ yields $c[k\leftarrow k+1] < A_{a+1}(k+1,0)$. Since $A_a(k,b)$ is in k-normal form, Lemma 3 yields $n[k\leftarrow k+1] < A_a(k+1, b[k\leftarrow k+1])$. It remains to check that we also have $c[k\leftarrow k+1] < A_a(k+1, b[k\leftarrow k+1]+1)$.

If $a = 0$, then $c =_{\rm NF} A_a(k,b) \cdot m + n$ means that $c = k^b \cdot m + n$ with $m < k$ and $n < k^b$. Then, $m < k+1$ and $n[k\leftarrow k+1] < (k+1)^{b[k\leftarrow k+1]}$. Thus $c[k\leftarrow k+1] = (k+1)^{b[k\leftarrow k+1]} \cdot m + n[k\leftarrow k+1] < (k+1)^{b[k\leftarrow k+1]+1}$ and thus $c[k\leftarrow k+1] =_{\rm NF} (k+1)^{b[k\leftarrow k+1]} \cdot m + n[k\leftarrow k+1]$. In the remaining case, we have for $a > 0$ that

$$\begin{aligned}
c[k\leftarrow k+1] &= A_a(k+1, b[k\leftarrow k+1]) \cdot m + n[k\leftarrow k+1] \\
&< A_a(k+1, b[k\leftarrow k+1]) \cdot A_a(k, b+1) + A_a(k+1, b[k\leftarrow k+1]) \\
&\le A_a(k+1, b[k\leftarrow k+1]) \cdot A_a(k, b[k\leftarrow k+1]+1) + A_a(k+1, b[k\leftarrow k+1]) \\
&\le (A^k_{a-1}(k,\cdot)A_a(k+1, b[k\leftarrow k+1]))^2 + A^k_{a-1}(k,\cdot)A_a(k+1, b[k\leftarrow k+1]) \\
&< A^{k+1}_{a-1}(k+1,\cdot)A_a(k+1, b[k\leftarrow k+1]) \text{ by (1)} \\
&= A_a(k+1, b[k\leftarrow k+1]+1).
\end{aligned}$$

So $A_a(k+1, b[k\leftarrow k+1]) \cdot m + n[k\leftarrow k+1]$ is in $k+1$-normal form.

These Ackermannian normal forms give rise to a new Goodstein process. In order to prove that this process is terminating, we must assign ordinals to natural numbers, in such a way that the process gives rise to a decreasing (hence finite) sequence. For each k, we define a function $\psi_k \colon \mathbb{N} \to \Lambda$, where Λ is a suitable ordinal, in such a way that $\psi_k m$ is computed from the k-normal form of m. Unnested Ackermannian normal forms correspond to ordinals below $\Lambda = \varepsilon_\omega$, as the following map shows.

Definition 4. *For $k \ge 2$, define $\psi_k \colon \mathbb{N} \to \varepsilon_\omega$ as follows:*

1. $\psi_k 0 := 0$.
2. $\psi_k c := \omega^{\varepsilon_a + \psi_k b} \cdot m + \psi_k n$ *if* $c =_{\rm NF} A_a(k,b) \cdot m + n$.

Lemma 5. *If $c < d < \omega$ then $\psi_k c < \psi_k d$.*

Proof. Proof by induction on d with subsidiary induction on c. The assertion is clear if $c = 0$. Let $c =_{\rm NF} A_a(k,b) \cdot m + n$ and $d =_{\rm NF} A_{a'}(k,b') \cdot m' + n'$. We distinguish cases according to the position of a relative to a', the position of b relative to b', etc.

Case 1 ($a < a'$). We have $n < c < A_{a+1}(k,0) \le A_{a'}(k,0)$ and, since $A_{a'}(k,0) \le d$, the induction hypothesis yields $\psi_k n < \omega^{\varepsilon_{a'} + \psi_k 0} = \varepsilon_{a'}$. We have $b < c < A_{a+1}(k,0) \le A_{a'}(k,0)$ and the induction hypothesis yields $\psi_k b < \omega^{\varepsilon_{a'} + \psi_k 0} = \varepsilon_{a'}$. It follows that $\varepsilon_a + \psi_k b < \varepsilon_{a'}$, hence $\psi_k c = \omega^{\varepsilon_a + \psi_k b} \cdot m + \psi_k n < \varepsilon_{a'} \le \psi_k d$.

Case 2 ($a > a'$). This case is not possible since this would imply that $d < A_{a'+1}(k,0) \leq A_a(k,0) \leq c < d$.
Case 3 ($a = a'$). We consider several sub-cases.
Case 3.1 ($b < b'$). The induction hypothesis yields $\psi_k b < \psi_k b'$. Hence $\omega^{\varepsilon_a+\psi_k b} < \omega^{\varepsilon_a+\psi_k b'}$. We have $n < A_a(k,b)$, and the subsidiary induction hypothesis yields $\psi_k n < \omega^{\varepsilon_a+\psi_k b} < \omega^{\varepsilon_a+\psi_k b'}$. Putting things together we see $\psi_k c = \omega^{\varepsilon_a+\psi_k b} \cdot m + \psi_k n < \omega^{\varepsilon_a+\psi_k b'} \leq \psi_k d$.
Case 3.2 ($b > b'$). This case is not possible since this would imply $d < A_a(k, b'+1) \leq A_a(k,b) \leq c < d$.
Case 3.3 ($b = b'$). This case is divided into further sub-cases.
Case 3.3.1 ($m < m'$). We have $n < A_a(k,b)$ and the subsidiary induction hypothesis yields $\psi_k n < \omega^{\varepsilon_a+\psi_k b}$. Hence $\psi_k c = \omega^{\varepsilon_a+\psi_k b} \cdot m + \psi_k n < \omega^{\varepsilon_a+\psi_k b'} \cdot m' \leq \psi_k d$.
Case 3.3.2 ($m > m'$). This case is not possible since this would imply $d = A_a(k,b) \cdot m' + n' \leq A_a(k,b) \cdot m \leq c < d$.
Case 3.3.3 ($m = m'$). The inequality $c < d$ yields $n < n'$ and the induction hypothesis yields $\psi_k n < \psi_k n'$. Hence $\psi_k c = \omega^{\varepsilon_a+\psi_k b} \cdot m + \psi_k n < \omega^{\varepsilon_a+\psi_k b} \cdot m + \psi_k n' = \psi_k d$.

Our ordinal assignment is invariant under base change, in the following sense.

Lemma 6. $\psi_{k+1}(c[k \leftarrow k+1]) = \psi_k c$.

Proof. Proof by induction on c. The assertion is clear for $c = 0$. Let $c =_{\text{NF}} A_a(k,b) \cdot m + n$. Then, $c[k \leftarrow k+1] =_{\text{NF}} A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1]$, and the induction hypothesis yields

$$\begin{aligned}\psi_{k+1}(c[k \leftarrow k+1]) &= \psi_{k+1}(A_a(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1]) \\ &= \omega^{\varepsilon_a+\psi_{k+1}(b[k \leftarrow k+1])} \cdot m + \psi_{k+1}(n[k \leftarrow k+1]) \\ &= \omega^{\varepsilon_a+\psi_k b} \cdot m + \psi_k n = \psi_k c.\end{aligned}$$

It is well-known that the so-called slow-growing hierarchy at level $\varphi_\omega 0$ matches up with the Ackermann function, so one might expect that the corresponding Goodstein process can be proved terminating in $\mathsf{PA} + \mathrm{TI}(\varphi_\omega 0)$. This is true but, somewhat surprisingly, much less is needed here. We can lower $\varphi_\omega 0$ to $\varepsilon_\omega = \varphi_1 \omega$.

Theorem 1. *For all $\ell < \omega$, there exists a $k < \omega$ such that $b_k(\ell) = 0$. This is provable in $\mathsf{PA} + \mathrm{TI}(\varepsilon_\omega)$.*

Proof. Define $o(\ell,k) := \psi_{k+2} b_k(\ell)$. If $b_k(\ell) > 0$, then, by the previous lemmata,

$$\begin{aligned}o(\ell,k+1) &= \psi_{k+3} b_{k+1}(\ell) = \psi_{k+3}(b_k(\ell)[k \leftarrow k+1] - 1) \\ &< \psi_{k+3}(b_k(\ell)[k \leftarrow k+1]) = \psi_{k+2}(b_k(\ell)) = o(\ell,k).\end{aligned}$$

Since $(o(\ell,k))_{k<\omega}$ cannot be an infinite decreasing sequence of ordinals, there must be some k with $o(\ell,k) = 0$, yielding $b_k(\ell) = 0$.

Now we are going to show that for every $\alpha < \varepsilon_\omega$, $\mathsf{PA}+\mathrm{TI}(\alpha) \nvdash \forall\ell\exists k\, b_k(\ell) = 0$. This will require some work with fundamental sequences.

Definition 1. *Let Λ be an ordinal. A* system of fundamental sequences on Λ *is a function $\cdot[\cdot]\colon \Lambda\times\mathbb{N} \to \Lambda$ such that $\alpha[n] \leq \alpha$ with equality holding if and only if $\alpha = 0$, and $\alpha[n] \leq \alpha[m]$ whenever $n \leq m$. The system of fundamental sequences is* convergent *if $\lambda = \lim_{n\to\infty} \lambda[n]$ whenever λ is a limit, and has the* Bachmann property *if whenever $\alpha[n] < \beta < \alpha$, it follows that $\alpha[n] \leq \beta[1]$.*

It is clear that if Λ is an ordinal then for every $\alpha < \Lambda$ there is n such that $\alpha[1][2]\ldots[n] = 0$, but this fact is not always provable in weak theories. The Bachmann property that will be useful due to the following.

Proposition 1. *Let Λ be an ordinal with a system of fundamental sequences satisfying the Bachmann property, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of elements of Λ such that, for all n, $\xi_n[n+1] \leq \xi_{n+1} \leq \xi_n$. Then, for all n, $\xi_n \geq \xi_0[1][2]\ldots[n]$.*

Proof. Let $\preceq_k$ be the reflexive transitive closure of $\{(\alpha[k],\alpha) : \alpha < \varphi_2(0)\}$. We need a few properties of these orderings. Clearly, if $\alpha \preceq_k \beta$, then $\alpha \leq \beta$. It can be checked by a simple induction and the Bachmann property that, if $\alpha[n] \leq \beta < \alpha$, then $\alpha[n] \preceq_1 \beta$. Moreover, $\preceq_k$ is monotone in the sense that if $\alpha \preceq_k \beta$, then $\alpha \preceq_{k+1} \beta$, and if $\alpha \preceq_k \beta$, then $\alpha[k] \preceq_k \beta[k]$ (see, e.g., [11] for details).

We claim that for all n, $\xi_n \succeq_n \xi_0[1]\ldots[n]$, from which the desired inequality immediately follows. For the base case, we use the fact that $\succeq_0$ is transitive by definition. For the successor, note that the induction hypothesis yields $\xi_0[1]\ldots[n] \preceq_n \xi_n$, hence $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_n[n+1]$. Then, consider three cases.

Case 1 ($\xi_{n+1} = \xi_n$). By transitivity and monotonicity, $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_0[1]\ldots[n] \preceq_n \xi_n = \xi_{n+1}$ yields $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_{n+1}$.
Case 2 ($\xi_{n+1} = \xi_n[n+1]$). Then, $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_n[n+1] = \xi_{n+1}$.
Case 3 ($\xi_n[n+1] < \xi_{n+1} < \xi_n$). The Bachmann property yields $\xi_n[n+1] \preceq_1 \xi_{n+1}$, and since $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_n[n+1]$, monotinicity and transitivity yield $\xi_0[1]\ldots[n+1] \preceq_{n+1} \xi_{n+1}$.

Let $\omega_0(\alpha) := \alpha$ and $\omega_{k+1}(\alpha) = \omega^{\omega_k(\alpha)}$. Let us define the standard fundamental sequences for ordinals less than $\varphi_2 0$ as follows.

1. If $\alpha = \omega^\beta + \gamma$ with $0 < \gamma < \alpha$, then $\alpha[k] := \omega^\beta + \gamma[k]$.
2. If $\alpha = \omega^\beta > \beta$, then we set $\alpha[k] := 0$ if $\beta = 0$, $\alpha[k] := \omega^\gamma \cdot k$ if $\beta = \gamma + 1$, and $\alpha[k] := \omega^{\beta[k]}$ if $\beta \in \mathrm{Lim}$.
3. If $\alpha = \varepsilon_\beta > \beta$, then $\alpha[k] := \omega_k(1)$ if $\beta = 0$, $\alpha[k] := \omega_k(\varepsilon_\gamma + 1)$ if $\beta = \gamma + 1$, and $\alpha[k] := \varepsilon_{\beta[k]}$ if $\beta \in \mathrm{Lim}$.

This system of fundamental sequences enjoys the Bachmann property [11].

In view of Proposition 1, the following technical lemma will be crucial for proving our main independence result for ACA_0'.

Lemma 7. *Given $k, c < \omega$ with $k \geq 2$, $\psi_{k+1}(c[k\leftarrow k+1] - 1) \geq (\psi_k c)[k]$.*

Proof. We prove the claim by induction on c. Let $c =_{\text{NF}} A_a(k,b) \cdot m + n$.

CASE 1 ($n > 0$). Then the induction hypothesis and Lemma 5 yield

$$\begin{aligned}\psi_{k+1}(c[k\leftarrow k+1]-1) &= \omega^{\varepsilon_a+\psi_{k+1}(b[k\leftarrow k+1])} \cdot m + \psi_{k+1}(n[k\leftarrow k+1]-1)\\ &\geq \omega^{\varepsilon_a+\psi_k(b)} \cdot m + (\psi_k(n))[k] = (\omega^{\varepsilon_a+\psi_k(b)} \cdot m + \psi_k(n))[k]\\ &= (\psi_k(A_a(k,b)\cdot m+n))[k] = (\psi_k c)[k].\end{aligned}$$

CASE 2 ($n = 0$ and $m > 1$). Then the induction hypothesis and Lemma 5 yield

$$\begin{aligned}&\psi_{k+1}(c[k\leftarrow k+1]-1)\\ &= \psi_{k+1}(A_a(k+1,b[k\leftarrow k+1])\cdot(m-1)+\psi_{k+1}(A_a(k+1,b[k\leftarrow k+1])-1)\\ &\geq \psi_k(A_a(k,b)\cdot(m-1)) + (\psi_k(A_a(k,b)))[k] = (\psi_k(A_a(k,b)\cdot m))[k] = (\psi_k c)[k].\end{aligned}$$

CASE 3 ($n = 0$ and $m = 1$). We consider several sub-cases.
CASE 3.1 ($a > 0$ and $b > 0$). The induction hypothesis yields

$$\begin{aligned}\psi_{k+1}(c[k\leftarrow k+1]-1) &= \psi_{k+1}(A_a(k+1,b[k\leftarrow k+1])-1)\\ &\geq \psi_{k+1}(A_a(k+1,(b[k\leftarrow k+1])-1)\cdot k) = \omega^{\varepsilon_a+\psi_{k+1}(b[k\leftarrow k+1]-1)} \cdot k\\ &\geq \omega^{\varepsilon_a+(\psi_k(b))[k]} \cdot k \geq (\omega^{\varepsilon_a+\psi_k(b)})[k] = (\psi_k c)[k],\end{aligned}$$

since $A_a(k+1,(b[k\leftarrow k+1])-1)\cdot k$ is in $k+1$ normal form by Lemma 2 and Lemma 4.
CASE 3.2 ($a > 0$ and $b = 0$). Then, the induction hypothesis yields

$$\begin{aligned}\psi_{k+1}(c[k\leftarrow k+1]-1) &= \psi_{k+1}(A_a(k+1,0)-1) = \psi_{k+1}(A_{a-1}^{k+1}(k,\cdot)(0)-1)\\ &= \psi_{k+1}(A_{a-1}(k+1,A_{a-1}^{k}(k+1,\cdot)(0)-1))\\ &\geq \psi_{k+1}(A_{a-1}^{k}(k+1,\cdot)(0)) = \omega^{\varepsilon_{a-1}+\psi_{k+1}((A_{a-1}^{k-1}(k+1,\cdot)(0)))}\\ &\geq \omega^{\psi_{k+1}((A_{a-1}^{k-1}(k+1,\cdot)(0)))} \geq \omega^{\omega_{k-1}(\varepsilon_{a-1}+1)}\\ &= (\varepsilon_a)[k] = (\psi_k(A_a(k,0)))[k] = (\psi_k c)[k],\end{aligned}$$

since $A_{a-1}^{\ell}(k+1,\cdot)(0)$ is in $k+1$ normal form for $\ell \leq k$ by Lemma 2 and Lemma 4.
CASE 3.3 ($a = 0$ and $b > 0$). Then the induction hypothesis yields similarly as in CASE 3.1:

$$\begin{aligned}\psi_{k+1}(c[k\leftarrow k+1]-1) &= \psi_{k+1}(A_0(k+1,b)-1)\\ &= \psi_{k+1}((k+1)^{(b[k\leftarrow k+1]-1)} \cdot k + \cdots + (k+1)^0 \cdot k)\\ &\geq \psi_{k+1}((k+1)^{(b[k\leftarrow k+1]-1)} \cdot k)\\ &\geq \omega^{\psi_{k+1}(b[k\leftarrow k+1]-1)} \cdot k \geq \omega^{(\psi_k b)[k]} \cdot k \geq (\psi_k c)[k],\end{aligned}$$

since $(k+1)^{(b[k\leftarrow k+1]-1)} \cdot k$ is in $k+1$ normal form.
CASE 3.4 ($a = 0$ and $b = 0$). The assertion follows trivially since then $c = 1$.

Theorem 2. *Let* $\alpha < \varepsilon_\omega$. *Then* $\mathsf{PA} + \mathrm{TI}(\alpha) \nvdash \forall \ell \exists k\ b_k(\ell) = 0$. *Hence* $\mathsf{ACA}_0' \nvdash \forall \ell \exists k b_k(\ell) = 0$.

Proof. Assume for a contradiction that $\mathsf{PA} + \mathrm{TI}(\alpha) \vdash \forall \ell \exists k\ b_k(\ell) = 0$. Then $\mathsf{PA} + \mathrm{TI}(\alpha) \vdash \forall \ell \exists k\ b_k(A_\ell(2,0)) = 0$. Recall that $o(A_\ell(2,0),k) = \psi_{k+2}(b_k(A_\ell(2,0)))$. We have $o(A_\ell(2,0),0) = \varepsilon_n$. Lemma 7 and Lemma 5 yield $o(A_\ell(2,0),k)[k+1] \leq o(A_\ell(2,0),k+1) < o(A_\ell(2,0),k)$, hence Proposition 1 yields $o(A_\ell(2,0),k) \geq o(A_\ell(2,0))[1]\ldots[k]$. So the least k such that $b_k(A_\ell(2,0)) = 0$ is at least as big as the least k such that $\varepsilon_\ell[1]\ldots[k] = 0$. But by standard results in proof theory [3], $\mathsf{PA} + \mathrm{TI}(\alpha)$ does not prove that this k is always defined as a function of ℓ. This contradicts $\mathsf{PA} + \mathrm{TI}(\alpha) \vdash \forall \ell \exists k\ b_k(A_\ell(2,0))) = 0$.

4 Goodstein Sequences for ACA_0^+

In this section, we indicate how to extend our approach to a situation where the base change operation can also be applied to the first argument of the Ackermann function. The resulting Goodstein principle will then be independent of ACA_0^+. The key difference is that the base-change operation is now performed recursively on the first argument, as well as the second.

Definition 5. *For* $k \geq 2$ *and* $c \in \mathbb{N}$, *define* $c[k \leftarrow k+1]$ *by:*

1. $0[k \leftarrow k+1] := 0$
2. $c[k \leftarrow k+1] := A_{a[k \leftarrow k+1]}(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1]$ *if* $c =_{\mathrm{NF}} A_a(k,b) \cdot m + n$.

Note that in this section, $c[k \leftarrow k+1]$ will always indicate the operation of Definition 5. We can then define a Goodstein process based on this new base change operator.

Definition 6. *Let* $\ell < \omega$. *Put* $c_0(\ell) := \ell$. *Assume recursively that* $c_k(\ell)$ *is defined and* $c_k(\ell) > 0$. *Then,* $c_{k+1}(\ell) = c_k(\ell)[k+2 \leftarrow k+3] - 1$. *If* $c_k(\ell) = 0$, *then* $c_{k+1}(\ell) := 0$.

Termination and independence results can then be obtained following the same general strategy as before. We begin with the following lemmas, whose proofs are similar to those for their analogues in Sect. 3.

Lemma 8. *If* $c < d$ *and* $k \geq 2$, *then* $c[k \leftarrow k+1] < d[k \leftarrow k+1]$.

Lemma 9. *If* $c = A_a(k,b) \cdot m + n$ *is in* k*-normal form, then* $c[k \leftarrow k+1] = A_{a[k \leftarrow k+1]}(k+1, b[k \leftarrow k+1]) \cdot m + n[k \leftarrow k+1]$ *is in* $k+1$ *normal form.*

It is well-known that the so-called slow-growing hierarchy at level Γ_0 matches up with the functions which are elementary in the Ackermann function, so one might expect that the corresponding Goodstein process can be proved terminating in $\mathsf{PA} + \mathrm{TI}(\Gamma_0)$. This is true but, somewhat surprisingly, much less is needed here. Indeed, nested Ackermannian normal forms are related to the much smaller ordinal $\varphi_2(0)$ by the following mapping.

Definition 7. *Given $k \geq 2$, define a function $\chi_k \colon \mathbb{N} \to \varphi_2(0)$ given by:*

1. $\chi_k 0 := 0$.
2. $\chi_k c := \omega^{\varepsilon_{\chi_k a} + \chi_k b} \cdot m + \psi_k n$ *if* $c =_{\mathrm{NF}} A_a(k, b) \cdot m + n$.

As was the case for the mappings ψ_k, the maps χ_k are strictly increasing and invariant under base change, as can be checked using analogous proofs to those in Sect. 3.

Lemma 10. *Let $c, d, k < \omega$ with $k \geq 2$.*

1. *If $c < d$, then $\chi_k c < \chi_k d$.*
2. $\chi_{k+1}(c[k \leftarrow k+1]) = \chi_k c$.

Theorem 3. *For all $\ell < \omega$, there exists a $k < \omega$ such that $c_k(\ell) = 0$. This is provable in* $\mathsf{PA} + \mathrm{TI}(\varphi_2 0)$.

Next, we show that for every $\alpha < \varphi_2 0$, $\mathsf{PA} + \mathrm{TI}(\alpha) \nvdash \forall \ell \exists k \; c_k(\ell) = 0$. For this, we need the following analogue of Lemma 7.

Lemma 11. $\chi_{k+1}(c[k \leftarrow k+1] - 1) \geq (\chi_k c)[k]$.

Proof. We proceed by induction on c. Let $c =_{\mathrm{NF}} A_a(k, b) \cdot m + n$. Let us concentrate on the critical case $m = 1$ and $n = 0$, where $a > 0$ and $b = 0$.

The induction hypothesis yields

$$
\begin{aligned}
&\chi_{k+1}(c[k \leftarrow k+1] - 1) = \chi_{k+1}(A_a(k+1, 0) - 1) \\
&\quad = \chi_{k+1}(A^{k+1}_{a[k \leftarrow k+1]-1}(k+1, \cdot)(0) - 1) \geq \chi_{k+1}(A^{k}_{a[k \leftarrow k+1]-1}(k+1, \cdot)(0)) \\
&\quad = \omega^{\varepsilon_{\chi_{k+1}(a[k \leftarrow k+1]-1)} + \omega^{\chi_{k+1}(A^{k-1}_{a[k \leftarrow k+1]-1}(k+1, \cdot)(0))}} \geq \omega_k(\varepsilon_{\chi_{k+1}(a[k \leftarrow k+1]-1)} + 1) \\
&\quad \geq \omega_k(\varepsilon_{(\chi_k a)[k]} + 1) \geq (\varepsilon_{\chi_k a})[k] = (\chi_k(A_a(k, 0)))[k],
\end{aligned}
$$

since $A^{k}_{a[k \leftarrow k+1]-1}(k+1, \cdot)(0)$ is in $k+1$ normal form.

The remaining details of the proof of the theorem can be carried out similarly as before.

Theorem 4. *For every $\alpha < \varphi_2 0$, $\mathsf{PA} + \mathrm{TI}(\alpha) \nvdash \forall \ell \exists k \; c_k(\ell) = 0$. Hence $\mathsf{ACA}_0^+ \nvdash \forall \ell \exists k c_k(\ell) = 0$.*

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