Two-Dimensional Codes

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Extended Abstract

The theory of codes of strings takes its origin in the theory of information developed by Shannon in the 1950s. Since then, it has evolved in several different directions. Among them, we can mention the theory of entropy, a branch of probability theory, and the theory of error-correcting codes, more related to commutative algebra. Due to the nature of the involved concepts, the theory of codes exhibits both theoretical and practical features. The former are related to combinatorics on words, automata theory and formal languages (see [12] for some classical references), while the latter apply in finding efficient methods for transmitting and storing data.

In the 1960s, the interest in image processing, pattern recognition and pattern matching motivated the research on families of matrices with entries taken from a finite alphabet, as a two-dimensional counterpart of strings. Significant work has been done to transfer formalisms and results from formal language theory to a two-dimensional setting (see for example [2,7,13,19–21,30]). Several classes of two-dimensional objects have been introduced and investigated, namely polyominoes, labelled polyominoes, directed polyominoes, and rectangular labelled polyominoes. This paper focusses on this last kind of polyominoes, which we will refer to as *pictures*.

Recently, we are seeing a renewed interest in two-dimensional languages in different frameworks (see for example [15–18,23,28,29,31]). The motivations of this paper are mainly theoretical. Nevertheless, as formal language theory had very significant impact in several applications, we do not exclude that theoretical results on two-dimensional languages may be exploited for practical applications. Besides researchers who investigate open questions in the aforementioned fields of image processing, pattern recognition and pattern matching, researchers in other scientific areas are interested in the investigation on pictures. Please note that some families of picture languages are of particular meaning in physics; as a matter of fact, they represent the evolution of noteworthy discrete systems (see [25]). Moreover, the recognizability of picture languages by finite models is connected to the study of some properties of symbolic dynamical systems (see [24]).

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Extending results from formal language theory to two dimensions is often a non-trivial task, and sometimes a very challenging one. The two-dimensional structure may give rise to new problems, even in some basic concepts (see [3, 20]). As an example, the generalization of the classical operation of string concatenation to two dimensions, leads to the definition of two different operations between pictures, the horizontal and the vertical concatenations. Differently from the string case, these concatenations are partial operations and, as a very remarkable aspect, they do not induce a monoid structure on the set of all pictures over a given alphabet. Please note that the monoid structure of the set of all strings over a given alphabet has played an important role in the theory of formal languages. Another basic definition on strings, the one of prefix of a string, opens new scenarios when generalized to pictures. It can be extended to pictures in a natural way; a prefix of a picture is a rectangular portion in the top-left corner of the picture. Nevertheless, if a prefix is removed from a picture, the remaining part is no longer a picture. On the contrary, if a prefix is deleted from a string, the remaining part is still a string and this is a big advantage.

In the literature, we find several attempts to generalize the notion of code to two-dimensional objects. A set C of two-dimensional objects over a given alphabet Σ, is a code if every two-dimensional object over Σ can be tiled without holes or overlaps in at most one way with copies of elements of C. Most of the results show that we lose important properties when moving from one to two dimensions. A major result due to D. Beauquier and M. Nivat states that the problem whether a finite set of polyominoes is a code is undecidable; the same result also holds for dominoes (see [11]). Some particular cases have been studied in [1]. Codes of directed polyominoes with respect to some concatenation operation are considered in [22]; some special decidable cases are detected. Codes of labeled polyominoes, called bricks, as well as codes of directed figures are studied in [27] and in [26], respectively. In these papers, further undecidability results are proved. Doubly-ranked monoids are introduced in [14] with the aim of extending syntactic properties to two dimensions; in this framework a notion of picture code is introduced and studied. More recently, non-overlapping codes have been considered in [10].

In this paper, we consider the definition of *picture code* that has been introduced in [3]. Here, the pictures are composed using the operation of tiling star defined in [30]. The tiling star of a set of pictures X is the set X^{**} of all pictures that are tilable by elements of X, i.e. all pictures that can be covered by pictures of X without holes or overlapping. Then, X is a code if any picture in X^{**} is tilable in a unique way. Again, it is not decidable whether a finite language of pictures is a code. Consequently, it is a goal to find decidable subclasses of picture codes. Taking the families of codes of strings as starting point, many families of two-dimensional codes have been introduced (see [3–6, 8, 9]). The investigation mainly focussed on their combinatorial properties, their construction, their decidability, and the notions of maximality and completeness in this setting.

The generalization of the notion of prefix code of strings to two-dimensions leads to two different definitions of codes, namely the prefix and the strong prefix codes of pictures. In these definitions, pictures must be considered in relation to a given scanning direction; e.g. from the top-left corner toward the bottom-right one. The families of prefix and strong prefix picture codes inherit several properties from the family of prefix string codes. For example, the results in [5] present a recursive procedure to construct all finite maximal strong prefix codes of pictures, starting from the singleton pictures, i.e. those which contain only one alphabet symbol. This construction generalizes the tree representation of prefix codes of strings (cf. [12]). Subsequently, the codes of pictures with finite deciphering delay have been introduced, in analogy to the case of strings. More recently, three new classes of picture codes have been introduced in [8,9]: the comma-free, the cylindric and the toroidal codes. They extend the notions of comma-free and circular code of strings to two dimensions. Again, the generalization can be achieved in more than one way. Notably, these definitions share the property to be "non-oriented", in the sense that they do not require to set a specific scanning direction.

In this paper, we shall introduce some of the aforementioned families of picture codes and discuss their properties. Classes of two-dimensional codes will be compared each other and with their one-dimensional counterpart. The results highlight new scenarios which also help one in understanding some hidden features in the one-dimensional setting.

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