Nilpotent Quotients of Associative $\mathbb{Z}$-Algebras and Augmentation Quotients of Baumslag-Solitar Groups

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Abstract. We describe the functionality of the package *zalgs* for the computer algebra system GAP. The package contains an implementation of the nilpotent quotient algorithm for finitely presented associative $\mathbb{Z}$-algebras described in [3]. As an application of this algorithm we calculate augmentation quotients, i.e. successive quotients of powers of the augmentation ideal $I(G)$ of the integral group ring $\mathbb{Z}G$, where G is a finitely presented group. We apply these methods to obtain conjectures for augmentation quotients of the Baumslag-Solitar groups $BS(m,n)$ with $|m-n|$ equal to $0, 1$ or a prime p.

Keywords: Associative algebras · Augmentation quotients · Computer algebra · Group theory · Nilpotent quotient algorithm

1 Introduction

An associative $\mathbb{Z}$-algebra A is called *nilpotent of class* $c \in \mathbb{N}$ if its series of power ideals has the form $A = A^1 > A^2 > \ldots > A^c > A^{c+1} = \{0\}$. The power ideal A^i, $i \in \mathbb{N}$ is the ideal in A generated by all products of length i. In [3] we introduced so called *nilpotent presentations* to describe such algebras in a way that exhibits their nilpotent structure. We also introduced a nilpotent quotient algorithm, which computes a nilpotent presentation for the class-c quotient A/A^{c+1} for a given finitely presented associative $\mathbb{Z}$-algebra A and a non-negative integer c. An implementation of this algorithm is available in the package *zalgs* [5] for the computer algebra system GAP [4].

The purpose of this paper is to describe the functionality of the *zalgs* package and to exhibit applications of the nilpotent quotient algorithm. In particular, we apply the algorithm in the calculation of *augmentation quotients*, i.e. the quotients $Q_k(G) = I^k(G)/I^{k+1}(G)$, where G is a finitely presented group and $I(G)$ denotes the augmentation ideal of the integral group ring $\mathbb{Z}G$. The augmentation ideal $I(G)$ is defined as the kernel of the augmentation map

$$\varepsilon\colon \mathbb{Z}G \to \mathbb{Z},\ \sum_i a_i g_i \mapsto \sum_i a_i,$$

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where $a_i \in \mathbb{Z}$ and $g_i \in G$. The augmentation quotients are interesting objects studied in the integral representation theory of groups. We present conjectures on the augmentation quotients of certain Baumslag-Solitar groups, which are based on computer experiments using the *zalgs* package.

2 Nilpotent Presentations and Nilpotent Quotient Systems

As all algebras considered in this paper will be associative $\mathbb{Z}$-algebras, we will simply refer to them as algebras. For completeness we recall several important definitions from [3].

Definition 1. *Let A be a finitely generated algebra of class c and $s \in \mathbb{N}$. We call $(b_1, \ldots, b_s)$ a* weighted generating sequence *for A with* powers $(e_1, \ldots, e_s)$ *and* weights $(w_1, \ldots, w_s)$ *if*

(a) $A = \langle b_1, \ldots, b_s \rangle$, i.e. A is the $\mathbb{Z}$-span of $b_1, \ldots, b_s$.
(b) $b_i b_j \in \langle b_{\max\{i,j\}}, \ldots, b_s \rangle$ for $1 \leq i, j \leq s$.
(c) e_i is minimal in $\mathbb{N}$ with respect to the property that $e_i b_i \in \langle b_{i+1}, \ldots, b_s \rangle$, or $e_i = 0$, if such an $e_i \in \mathbb{N}$ does not exist.
(d) The elements $b_i + A^{k+1}$ with $1 \leq i \leq s$ such that $w_i = k$ generate A^k/A^{k+1} for $1 \leq k \leq c$.

Definition 2. *A* consistent weighted nilpotent presentation *for a finitely generated nilpotent algebra A is given by a weighted generating sequence $(b_1, \ldots, b_s)$ with* powers $(e_1, \ldots, e_s)$, weights $(w_1, \ldots, w_s)$ *and relations of the following form:*

(a) $e_i b_i = x_{i,i+1} b_{i+1} + \ldots + x_{i,s} b_s$ for all $1 \leq i \leq s$ where $e_i > 0$.
(b) $b_i b_j = y_{i,j,l+1} b_{l+1} + \ldots + y_{i,j,s} b_s$ for $1 \leq i, j \leq s$ and $l = \max\{i, j\}$.
(c) The $x_{i,k}$ and $y_{i,j,k}$ are integers with $0 \leq x_{i,k}, y_{i,j,k} < e_k$ if $e_k > 0$.

We note that every finitely generated nilpotent algebra has a consistent weighted nilpotent presentation, see [3, Theorem 7]. In an algebra A given by a consistent weighted nilpotent presentation, we can determine a normal form for each $a \in A$, i.e. there are uniquely determined $z_i \in \mathbb{Z}$ with

$$a = z_1 b_1 + \ldots + z_s b_s$$

and $0 \leq z_i < e_i$ if $e_i > 0$.

Definition 3. *Let $A = \langle x_1, \ldots, x_n \mid R_1, \ldots, R_t \rangle$ be a finitely presented algebra, $c \in \mathbb{N}$ and let $\varphi \colon A \to A/A^{c+1}$ be the natural homomorphism. A* nilpotent quotient system *describes φ using the following data:*

(a) A consistent weighted nilpotent presentation for A/A^{c+1} with generators $(b_1, \ldots, b_s)$, powers $(e_1, \ldots, e_s)$, weights $(w_1, \ldots, w_s)$, multiplication relations for $b_i b_j$ and power relations $e_i b_i$ if $e_i > 0$.

(b) Images $\varphi(x_i)$ *for* $1 \leq i \leq n$ *given in normal form.*
(c) Definitions $(d_1, \ldots, d_s)$*, with* d_i *being an integer or a pair of integers, s.t.*
  – *If* d_i *is an integer, then* $w_i = 1$ *and* $b_i = \varphi(x_{d_i})$*.*
  – *If* $d_i = (k, j)$*, then* $b_i = b_k b_j$*, where* $w_k = 1$ *and* $w_j = w_i - i$*.*

The description of φ using this data is very useful for computational purposes and usually the output of our calculations will be in the form of nilpotent quotient systems. The following example shall illustrate the definition of nilpotent quotient systems.

Example 1. Consider the finitely presented algebra given by

$$A = \langle x_1, x_2 \mid 2x_1, x_2^2, x_1^2 - x_1 x_2 \rangle.$$

Then a nilpotent quotient system for $\varphi\colon A \to A/A^2$ consists of:

– generators (b_1, b_2) with powers $(2, 0)$ and weights $(1, 1)$,
– the power relation $2b_1 = 0$ and the multiplication relations $b_i b_j = 0$ for $1 \leq i, j \leq 2$,
– images $\varphi(x_1) = b_1$ and $\varphi(x_2) = b_2$, and
– definitions $(1, 2)$.

A nilpotent quotient system for $\varphi\colon A \to A/A^3$ consists of:

– generators (b_1, b_2, b_3, b_4) with powers $(2, 0, 2, 2)$ and weights $(1, 1, 2, 2)$,
– the power relations $2b_1 = 2b_3 = 2b_4 = 0$ and the multiplication relations $b_1 b_1 = b_3$, $b_1 b_2 = b_3$, $b_2 b_1 = b_4$ and $b_i b_j = 0$ for all other $1 \leq i, j \leq 4$,
– images $\varphi(x_1) = b_1$ and $\varphi(x_2) = b_2$, and
– definitions $(1, 2, (1, 2), (2, 1))$.

3 Functionality

The central functionality provided by the *zalgs* package is the function

▷ `NilpotentQuotientFpZAlgebra(A, c)`,

which takes as input a finitely presented associative $\mathbb{Z}$-algebra A and a non-negative integer c. The output is a nilpotent quotient system for $\varphi\colon A \to A/A^{c+1}$.

Example 2. The following is an example calculation of a nilpotent quotient system in GAP for the class-2 quotient of the associative $\mathbb{Z}$-algebra considered in Example 1 above, i.e.

$$A = \langle x_1, x_2 \mid 2x_1, x_2^2, x_1^2 - x_1 x_2 \rangle.$$

To carry out the computation, we start by defining A as the quotient of the free associative $\mathbb{Z}$-algebra on two generators by the given relations.

```
gap> F := FreeAssociativeAlgebra(Integers, 2);;
gap> x1 := F.1;; x2 := F.2;;
gap> A := F / [2*x1, x2^2, x1^2-x1*x2];;
```

We then call `NilpotentQuotientFpZAlgebra(A, 2)` to compute the class-2 quotient. The output contains lists for the definitions `dfs`, the powers `pows`, the weights `wgs` and an integer `dim` indicating the dimension of the quotient. The entries for the images `imgs`, power relations `ptab` and multiplication relations `mtab` are to be interpreted as coefficients of normal forms. For computational purposes there is an additional entry `rels` in the output.

```
gap> NilpotentQuotientFpZAlgebra(A, 2);
rec( dfs := [1,2,[1,2],[2,1]],
     dim := 4,
     imgs := [ [1,0,0,0], [0,1,0,0] ],
     mtab := [ [[0,0,1,0], [0,0,1,0], [0,0,0,0], [0,0,0,0]],
               [[0,0,0,1], [0,0,0,0], [0,0,0,0], [0,0,0,0]],
               [[0,0,0,0], [0,0,0,0], [0,0,0,0], [0,0,0,0]],
               [[0,0,0,0], [0,0,0,0], [0,0,0,0], [0,0,0,0]] ],
     pows := [2,0,2,2],
     ptab := [ [0,0,0,0], [0,0,0,0], [0,0,0,0], [0,0,0,0] ],
     rels := [ <- omitted -> ],
     wgs := [1,1,2,2] )
```

In [3, Section 5], we describe how to obtain a presentation P for an algebra, such that $I(G)/I^{c+1}(G)$ is isomorphic to the class-c quotient of P. The nilpotent quotient algorithm can now be applied to determine this nilpotent quotient. The following functions are available to calculate the class-c quotient of the augmentation ideal of integral group rings.

▷ `AugmentationQuotientFpGroup(G, c)`,
▷ `AugmentationQuotientPcpGroup(G, c)`,

which take as input a finitely presented group or a polycyclically presented group G, respectively, and a non-negative integer c. The output in both cases is a nilpotent quotient system for $I(G)/I^{c+1}(G)$. Note that the augmentation quotients $Q_n(G)$ for $n \leq c$ can be read off from this.

4 Augmentation Quotients of Baumslag-Solitar Groups

In [3, Section 5], we describe how to obtain, for a given finitely presented group G, a presentation for an algebra A, such that $I(G)/I^{c+1}(G)$ is isomorphic to the class-c quotient of A. We apply these methods to compute augmentation quotients of the *Baumslag-Solitar groups* $BS(m,n)$, which for $m,n \in \mathbb{Z} \setminus \{0\}$ are given by the presentations

$$BS(m,n) = \langle a, b \mid ba^m b^{-1} = a^n \rangle.$$

These one-relator groups form an interesting set of groups with applications in combinatorial and geometric group theory, e.g. the group $BS(1,1)$ is the free abelian group on two generators and $BS(1,-1)$ arises as the fundamental group of the Klein bottle. The Baumslag-Solitar groups were introduced in [2] as examples of non-Hopfian groups and the isomorphism problem for these groups has been considered in [6].

We carried out computer experiments to gain some insight into the structure of the augmentation quotients $Q_k(BS(m,n))$ for $|m|, |n| \leq 10$ and small values of k. Our computations suggest the following conjectures for certain special cases:

Conjecture 1. Let p be a prime.

(a) If $|m-n| = 0$ and $|m| = |n| = p$, then

$$Q_k(BS(m,n)) = \begin{cases} \mathbb{Z}^{k+1} \oplus \mathbb{Z}_p^{A(k,1)}, & \text{if } k \leq p+2, \\ \mathbb{Z}^{k+1} \oplus \mathbb{Z}_p^{A(k,1)-C(k)}, & \text{if } k > p+2, \end{cases}$$

where

$$A(u,v) = \sum_{\ell=0}^{v} (-1)^\ell \binom{u+1}{\ell} (v+1-\ell)^u$$

are the Eulerian numbers and the values $C(k)$ are given by the recursion

$$C(1) = 1, \quad C(k) = C(k-1) + B(k+8) \text{ for all } k \geq 2,$$

where $B(u)$ is the number of 8-element subsets of $\{1, \dots, u\}$ whose elements sum to a triangular number, i.e. a number of the form $T_w = \binom{w+1}{2}$, $w \in \mathbb{N}$.

(b) If $|m-n| = 1$, then for all $k \in \mathbb{N}$:

$$Q_k(BS(m,n)) \cong \mathbb{Z}.$$

(c) If $|m-n| = p$, then for all $k \in \mathbb{N}$:

$$Q_k(BS(m,n)) \cong \mathbb{Z} \oplus \mathbb{Z}_p^{T_k},$$

where $T_k = \binom{k+1}{2}$ is the k-th triangular number.

The behaviour appears to be more complicated if $|m-n|$ contains several (not necessarily distinct) prime factors, as is illustrated in the following example.

Example 3. Let G be the Baumslag-Solitar group $BS(5,-1)$. Then the first few augmentation quotients are as follows:

$$\begin{aligned}
Q_1(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_6 \\
Q_2(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_6^3 \\
Q_3(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_6^6 \\
Q_4(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6^8 \oplus \mathbb{Z}_{18} \\
Q_5(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_6^{10} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{54} \\
Q_6(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_6^{13} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{54}^2 \\
Q_7(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_6^{16} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{54}^2 \oplus \mathbb{Z}_{162} \\
Q_8(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_6^{20} \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{54}^2 \oplus \mathbb{Z}_{162}^2 \\
Q_9(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^{15} \oplus \mathbb{Z}_6^{23} \oplus \mathbb{Z}_{18}^2 \oplus \mathbb{Z}_{54}^2 \oplus \mathbb{Z}_{162}^3 \\
Q_{10}(G) &\cong \mathbb{Z} \oplus \mathbb{Z}_2^{19} \oplus \mathbb{Z}_6^{27} \oplus \mathbb{Z}_{18}^3 \oplus \mathbb{Z}_{54}^2 \oplus \mathbb{Z}_{162}^4
\end{aligned}$$

5 Further Aims

Bachmann and Grünenfelder [1] showed that for finite groups G the sequence $Q_n(G)$ for $n \in \mathbb{N}$ is virtually periodic, i.e. there exist $N \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $Q_n(G) \cong Q_{n+k}(G)$ for all $n \geq N$. It will be interesting to extend our methods to allow the determination of these parameters, which in theory allows to determine all augmentation quotients for a given finite group G.

Furthermore, we plan to extend our algorithms to compute nilpotent presentations for the largest associative $\mathbb{Z}$-algebra on d generators so that every element a of the algebra satisfies $a^n = 0$, i.e. compute $\mathbb{Z}$-algebra analogues of Burnside groups and Kurosh algebras.

References

1. Bachmann, F., Grünenfelder, L.: The periodicity in the graded ring associated with an integral group ring. J. Pure Appl. Algebra **5**, 253–264 (1974)
2. Baumslag, G., Solitar, D.: Some two-generator one-relator non-hopfian groups. Bull. Am. Math. Soc. **68**, 199–201 (1962)
3. Eick, B., Moede, T.: A nilpotent quotient algorithm for finitely presented associative $\mathbb{Z}$-algebras and its application to integral groups rings. Accepted for Mathematics of Computation (2020)
4. The GAP Group: GAP - Groups, Algorithms, and Programming, Version 4.11.0 (2020). https://www.gap-system.org
5. Moede, T.: zalgs - a GAP package for the computation of nilpotent quotients of finitely presented $\mathbb{Z}$-algebras (2020). https://www.tu-braunschweig.de/iaa/personal/moede
6. Moldavanskii, D.: Isomorphism of the Baumslag-Solitar groups. Ukr. Math. J. **43**, 1569–1571 (1991)