Abstract models for systems identification

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In this paper we try to give some contributions, in order to solve the problem of system identification (or minimal realization), for various classes of systems. In order to give some more insight into this problem, we shall define some abstract versions of the minimal realization process. The models we discuss are used for the study of subsets (or subcategories)in which there exists the minimal realization and it is unique.

In section 1 we recall the minimal realization problem and some of its properties.

Section 2 develops the first model, which is based on equivalence relations. The sets which support minimal realization are in connection with the systems of representants of an equivalence relation. This model corresponds to the external behaviour point of view.

In section 3 we give a model based on ordering relations. This version corresponds to the qualificative "minimal". Both models in sections 2 and 3 include systems identification. This is not however, the only example ; one can find other, as the "integer part" function and the "congruence modulo n".

The most powerful model seems to be the categorical one, which is introduced in section 4. The admissible subcategories are those one for which the inclusion functor admits an adjoint.

Some conclusions and further developments of the subject are discussed in the last section.

1 - Introduction

The problem of system identification (or minimal realization) is of great importance in studying various classes of systems. Usually this problem states as follows ; given some behaviour, one looks for a system (in some class) which have this behaviour and whose characteristics are the best ones, in some sense to be specified. The term "minimal", for example, denotes some "optimality" of structure", of the system under consideration.

This problem was put first for linear systems, and then generalized for deterministic dynamic systems (see [2]).

Later on, the categorical approach to systems theory permitted to include a broad class of systems, as probabilistic or fuzzy systems (see [1]) Someminimal realization theorems were proved, such that in some special conditions the minimal realization exists (see [3], [6]).

In recent years (see [9]) there was proved that an equivalence exists between the category of reachable systems with a given behaviour, and some category of equivalence relations on the input space. In this way, the minimal realization corresponds to the Nerode equivalence (a wellknown result). The new fact is that this equivalence is the supremum of all other relations in that category.

Thinking at the minimal realization at someprocess of "best approximation", we can restate this problem and generalize it. If the class of all dynamical systems is given, we shall look for the identification of a system in a given subclass which will be the "best one".

In the next sections we shall describe three models of general system identification. These models will include classical minimal realization, but also other situations, as rings of equivalence classes, for example.

We shall briefly sketch the minimal realization problem for dynamic deterministic systems.

Such a system will be a complex

$$\mathcal{S} = \left\{ X, U, Y, \delta, \beta, x_0 \right\}$$

where X, U, Y are arbitrary sets called, respectively, the state-space, input-space and output-space.

The dynamics δ is a map $\delta : X \times U \longrightarrow X$, and the output function is $\beta : X \longrightarrow Y$.

The system $\mathcal{S}$ is initialized, and $x_0 \in X$ is its initial state. We can build the category of dynamical systems, denoted by S_{ys}. The objects of S_{ys} will be systems $\mathcal{S}$ as above. A morfism between two systems S_1 and S_2 will be a triple $\Phi = (u, v, w)$. More explicitely, if $S_i = \left\{ X_i, U_i, Y_i, \delta_i, \beta_i, x_0^i \right\}$, $i = 1, 2$, then $u : X_1 \longrightarrow X_2$, $v : U_1 \longrightarrow U_2$, $w : Y_1 \longrightarrow Y_2$ are usual functions, such that the diagrams

$$\begin{array}{ccccc} X_1 \times U_1 & \xrightarrow{\delta_1} & X_1 & \xrightarrow{\beta_1} & Y_1 \\ \downarrow u \times v & & \downarrow u & & \downarrow w \\ X_2 \times U_2 & \xrightarrow{\delta_2} & X_2 & \xrightarrow{\beta_2} & Y_2 \end{array}$$

are comutative, and $u(x_0^1) = x_0^2$

Briefly, morphisms between systems, must comute with dynamics and output maps, and preserve the initial states.

Usualy the dynamics δ of a system $\mathcal{S}$ is extended to an action of the free monoid U^* on the state space X. This extension is making recursively, by 1) $\delta(x, \Lambda) = x$, $(\forall)\ x \in X$

2) $\delta(x, \theta\theta') = \delta(\delta(x, \theta)\ \theta')$, $(\forall)\ x \in X, \theta, \theta' \in U^*$

We can build the reachability map δ_{x_0} :

$\delta_{x_0} : U^* \longrightarrow X$, $\delta_{x_0}(\theta) = \delta(x_0, \theta)$

and the response map from initial state x_0 :

$f_{x_0} : U^* \longrightarrow Y$, $fx_0 = \beta \circ \delta_{x_0}$

Here x_0 is a fixed in X ; thus generally, we have a family of responses $(f_x)\ x \in X$.

The reachability map gives all states the system can reach, after receiving inputs, starting from x_0.

The response map gives the output of the system, which starts in x_0. It is also called the external-behaviour map.

A system $\mathcal{S}$ is reachable (from x_0) if δ_{x_0} is surjective.

A system $\mathcal{S}$ is observable if the map $x \longmapsto f_x$ is injective. Thus, observing the output, we can rediscover the initial state.

What we have sketched above, is called the passage from the internal description of a dynamical system, to the external description.

Thinking at a system as a model of some physical process, one may say that obtaining f_{x_0} means a simulation of that process.

In practice however, we have merely given an external behaviour, and want to build a system. We shall refer to this problem as to modelling.

The system we are looking for must be, of course, connected with the given external behaviour. The first condition is that the unknown system must have the same behaviour as that given one, starting from some initial state. This condition is however discutable, since we may look for a system with a behaviour "very closed" to the given one. The next condition is that we look for a system which must have some "optimality of structure". In precise mathematical terms, this optimality is achieved by looking for reachability and observability. This condition is also discutable because, at least for complex systems, it would be better to ask for some weak reachability (see [12] for details).

We shall, however, describe here the classical minimal realization, which gives a rechable and observable system (see [2]). This problem is also reffered to as system identification (see [13]).

Suppose given a function $f : U^* \longrightarrow Y$ which describes the external behaviour of some process (i.e. input-output relationship). We have then :

Theorem : There exists a dynamic system

$$\mathcal{S}_f = \left\{ X_f, U, Y, \delta_f, \beta_f, x_0 \right\}$$

such that

1) $f_{x_0} = f$ (i.e. the behaviour of $\mathcal{S}_f$ is f)

2) $\mathcal{S}_f$ is reachable from x_0 and observable

Proof : The proof can be found in [2] and will be omitted here. We mention, however, that X_f is obtained via the Nerode equivalence in U^* :

$$\theta_1 N \theta_2 \Longleftrightarrow f(\theta_1 \theta) = f(\theta_2 \theta), (\forall) \theta \in U^*$$

and $X_f = U^*/N$, the quotient set.

We remember again that system $\mathcal{S}_f$ is "the best one" with the behaviour f. The above result can be put into a categorical framework, considering the category of behaviours and the category of systems. In this way, a deepest result says that there is a pair of adjoint functors between these categories (see [7]).

In the next sections we shall generalize minimal realization. We shall describe some abstract models for the process of obtaining special system. These models are based on equivalence relations (section 2), on orderings (section 3), and on category theory (section 4). These are some points of view of identifying systems in a given class.

2 - Relational models

We shall describe in this section, a relational model for system identification. This model, as its name says, is based upon equivalence relations. Our version starts with Zadeh's definition of a system identifica tion : " the determination on the basis of input and output, of a system within a specified class of systems, to which the system under test is <u>equivalent</u> " (see [13]).

The formulation below is also connected with a paper by Gaines [5] .

Let us suppose a pair (X, R) where X is a set, and R an equivalence relation in X.

<u>Definition</u> : An <u>admissible subset</u> is $A \subseteq X$ with the property :

$$(\forall)\ x \in X \Longrightarrow (\exists)\ \alpha \in A,\ a\ R\ x$$

We shall denote by $\mathfrak{A}(X)$ the set of all admissible subsets of X.

In which follows, we shall try to characterize the admissible sets, i.e. elements of $\mathfrak{A}(X)$. First, some simple remarks :

1) $X \in \mathfrak{A}(X)$

2) $A \in \mathfrak{A}(X),\ B \supseteq A \Longrightarrow B \in \mathfrak{A}(X)$

We shall prove now that each admissible set contains an admissible subset, which is the "best one", in some sense :

<u>Theorem</u> : For each $A \in \mathfrak{A}(X)$ there exists $A_0 \in \mathfrak{A}(X)$, $A_0 \subseteq A$ with the property $a, b \in A_0 \Longrightarrow a \not R\ b$

Proof : Let us consider A / R, the quotient set (in fact, R is replaced by the equivalence induced by R on A). We shall select one and only one element from each equivalence class $\hat{a} \in A / R$. The collection of the obtained elements will be denoted by A_0, and, of course, $A_0 \subseteq A$. It is also obvious that $(\forall)$ $a, b \in A_0 \Longrightarrow a \not R b$, since a and b belong to different classes. We must prove that $A_0 \in \mathcal{A}(X)$. If $x \in X$, there is $a \in A$, $x R a$, thus $x \in \hat{a}$, $\hat{a} \in A/R$. But in A_0 we already have an element from $\hat{a}$, say $a_0 \in A_0$. It results that $x, a_0 \in \hat{a}$, thus $x R a_0$, and the proof ends. We shall give now two examples of this abstract identification model. The first one will be, of course, the minimal realization for deterministic systems.

Example 1. Let us suppose that Sys is the class of all deterministic systems, as in section 1. For $\mathcal{S} \in$ Sys, we shall denote by $f_{\mathcal{S}}$ its external behaviour, from initial state of $\mathcal{S}$. Let us note that we do not start from a behaviour, and look for a system. Our identification problem will be : starting from a system, to find an "optimal system" with the same behaviour. Let us consider the pair (Syst, R) where R is the equivalence relation defined by :

$$\mathcal{S}, \mathcal{S}' \in \text{Sys}, \qquad \mathcal{S} R \mathcal{S}' \Longleftrightarrow f_{\mathcal{S}} = f_{\mathcal{S}'}$$

Let us denote now, by Sys (r, σ) the subclass of reachable and observable systems, Sys $(r, 0) \subset$ Sys. We think that $\mathcal{S} \in$ Sys $(r, 0)$ is reachable from its initial state. The classical minimal realization theory says now, that Sys $(r, 0)$ is an admissible subclass of Sys. In other words.

$(\forall) \mathcal{S} \in \text{Sys} \Longrightarrow (\exists)\ \mathcal{S}_m \in \text{Sys}(r, 0)\ f_{\mathcal{S}} = f_{\mathcal{S}_m}$ and $\mathcal{S}_m$ is the minimal realization of $\mathcal{S}$.

We mention that each class of systems $\mathcal{C}$, $\text{Sys}(r, 0) \subset \mathcal{C} \subset \text{Sys}$, will be also admissible ; we can think, for example, at some weak concepts of reachability and observability.

Example 2 : This example will be different in nature, and it will tell us in a way, about the limits of this identification model.

We shall consider $\mathbb{Z}$ the set of integers, and R the congruence modulo n, $n \geq 2$: $p, q \in \mathbb{Z}$, $p \equiv q \pmod n \Longleftrightarrow n \mid p-q$

An admissible subset is $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of natural mumbers. The set $\mathbb{N}^* = \{1, 2, \ldots\}$ is not admissible.

We can apply the above theorem, in order to obtain "minimal" admissible subsets. For example, $\mathbb{N}_0 = \{0, 1, 2, \ldots, n-1\} \subset \mathbb{N}$ is admissible, and $p, q \in \mathbb{N}_0 \Longrightarrow p \not\equiv q \pmod n$. Another admissible is $\mathbb{N}_1 = \{n, n+1, \ldots 2n+1\}$. It is wellknown that the elements of $\mathbb{N}_0$ are a system of representants for the equivalence $\equiv \pmod n$, and $\mathbb{Z} / \equiv \pmod n = \mathbb{Z}_n = \{\hat{0}\ \hat{1}, \ldots, \widehat{n-1}\}$

Corollary : For $A \in \mathcal{A}(X)$ and A_0 as in the above theorem, we have

$$(\forall) \quad x \in X \Longrightarrow (\exists) \quad a \in A_0, \text{ unique, } x R a$$

Proof : obvious

We may call this unique a, the minimal realization of x.

We shall prove now a theorem which characterizes admissible subsets. It will relate $A \in \mathcal{A}(X)$ and the sections of the canonical map $X \longrightarrow X/R$.

We say that a function $M \xrightarrow{f} N$ is sectionable, if there is a function $N \xrightarrow{g} M$ such that $f \circ g = 1_N$; we shall call g a section of f.

It is clear that a section is a right inverse for f.

We shall denote by $\mathcal{A}_0(X) \subset \mathcal{A}(X)$ the "minimal" admissible sets i.e.

$$\mathcal{A}_0(X) = \{A \subseteq X \mid (\forall) \quad x \in X \Longrightarrow (\exists)\ a \in A, \text{ unique, } x R a\}$$

Theorem : There is a bijection between $\mathcal{A}_0(X)$ and the set of all sections of $X \xrightarrow{\varphi} X/R$.

Proof : Let us denote by $\mathrm{Sec} = \{s \mid s: X/R \longrightarrow X,\ \varphi \circ s = 1_{X/R}\}$.

We shall build two maps :

$$\mathcal{R}_0(X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Sec}$$

such that $\Phi \circ \Psi = id$, $\Psi \circ \Phi = id$

If $A \in \mathcal{R}_0(X)$, it is clear that $\hat{x} \cap A$ contains a single element, for each $\hat{x} \in X/R$. Let $\hat{x} \cap A = \{a_x\}$. We define $\Phi(A) = s_A$, $s_A : X/R \longrightarrow X$, $S_A(\hat{x}) = a_x$.

As $\varphi(s_A(\hat{x})) = \varphi(a_x) = \hat{a}_x = \hat{x}$, it results that Φ is well defined.

Let now $s \in \mathrm{Sec}$, we set $\Psi(s) = \mathrm{Im}\, s = s(X/R)$.

We prove first that $\Psi(s) \in \mathcal{R}_0(X)$; for each $x \in X$, as $\hat{x} \in X/R$, $s(\hat{x}) = a_x \in \Psi(s)$.

Since s is a section of φ, we have $\varphi(s(\hat{x})) = \hat{x} \Longrightarrow \varphi(a_x) = \hat{x} \Longrightarrow \hat{a}_x = \hat{x} \Longrightarrow a_x R_x$.

This a_x is also unique , for example, if $a_x R x$, $a'_x R x$ with a_x, $a'_x \in \Psi(s)$; we have $a_x = s(\hat{x}_1)$, $a'_x = s(\hat{x}_2)$.

But $\hat{a}_x = \hat{a}'_x \Longrightarrow \varphi(s(\hat{x}_1)) = \varphi(s(\hat{x}_2)) \Longrightarrow \hat{x}_1 = \hat{x}_2 \Longrightarrow a_x = a'_x$.

We must prove now that $\Phi \circ \Psi = id$, $\Psi \circ \Phi = id$. We have :

$$\Phi(\Psi(s)) = \Phi(\mathrm{Im}\, s) = s_{\mathrm{Im}\, s}.$$

As $\hat{x} \cap \mathrm{Ims} = \{s(x)\}$, it is clear that $s_{\mathrm{Ims}} = s$, and thus $\Phi \circ \Psi = id$.

Now $\Psi(\Phi(A)) = \Psi(s_A) = \mathrm{Im}\, S_A = A$, and thus $\Psi \circ \Phi = id$, and the proof ends. We shall see in the next sections that almost all models for system identification are related to such "right inverses".

Our relational model here is good to encompass with minimal realization and also with other situations in mathematics. However, its structure is too poor, in order to obtain deep results.

3 - Ordering models

We shall develop here a model for system identification, which is based upon order relations. We shall not obtain, as a particular case, classical minimal realization.

The motivation of introducing an ordering to replace the equivalence relation in section 2, is that we may look now for systems which "approximate" a given system, for example. This model will be well suited for identification of nondeterministic systems, where the concept of an equivalence is too powerful.

Such an ordering can also be thought as complexity , with preference for the less complex system (see [5]).

Hoping that no confusion can arise, our terminology here will be the same as in the previous section.

Let $(X, \leqslant)$ be a pair, with X a set, and $\leqslant$ an ordering on X.

Definition : An admissible subset of X is $A \subseteq X$ such that

1) $(\forall)\ x \in X \implies (\exists)\ a_x \in A,\ \ x \geqslant a_x$

2) $(\forall)\ a \in A,\ x \geqslant a \implies a_x \geqslant a.$

Remark : For each $x \in X$, the a_x is unique, after the definition above.

It is clear that $ax = \sup \{ a \in A / a \leqslant x \} \in A$.

We shall see later that such admissible subsets can exist, even if X is not complete lattice.

Let us denote by $\mathcal{A}(X)$ the collection of admissible subsets of X.

We may, of course, call $a_x \in A$ (for each $x \in X$), the minimal realization of x.

At this level it is difficult to give conditions under which there exist admissible subsets. It is also difficult to determinate whether "minimal realizations" exist or not.

We shall however, give a theorem which relates $\mathcal{A}(X)$ to the existence of retracts for the inclusion map $A \longrightarrow X$.

A function $M \xrightarrow{f} N$ is <u>rectractible</u>, if there is a function $N \xrightarrow{g} M$, such that $g \circ f = 1_M$; we call g a <u>rectract</u> of f. It is clear that a retract is a <u>left inverse</u> for f.

<u>Theorem</u> : The following statements are equivalent :

1) $A \in \mathcal{A}(X)$

2) there exists an isotone retract $A \underset{\varphi}{\overset{i}{\rightleftarrows}} X$, with $i \circ \varphi \leqslant 1_X$

<u>Proof</u> : 1) $\Longrightarrow$ 2). From definition of $A \in \mathcal{A}(X)$, let us set $\varphi : X \longrightarrow A$, $\varphi(x) = a_x$. It is clear that $\varphi \circ i = 1_A$, and $(i \circ \varphi)(x) = a_x \leqslant x$, thus $i \circ \varphi \leqslant 1_X$. Now φ is also isotone, since $x \leqslant y \Longrightarrow a_x \leqslant x \leqslant y$. But $a_x \leqslant y$, and $a_x \in A$ $\Longrightarrow a_x \leqslant a_y$, i. e. $\varphi(x) \leqslant \varphi(y)$.

2) $\Longrightarrow$ 1). Let us suppose 2) true. If $x \in X$, we set $a_x = \varphi(x) \in A$. From $i \circ \varphi \leqslant 1_X$ it results that $a_x \leqslant x$.
Let us prove that $a \in A$, $a \leqslant x \Longrightarrow a \leqslant a_x$.
As φ is isotone, $a \leqslant x \Longrightarrow \varphi(a) \leqslant \varphi(x)$, but $\varphi(a) = \varphi(i(a)) = a$ thus $a \leqslant a_x$, and the proof ends.

This theorem, as it will be seen later, reflects an adjoint property between a pair of functors. The importance of such results resides in the fact that we can give global conditions in order to characterize admissible sets. These admissible sets are important, since they are subclasses in which "optimal models" exist.

Example 1 : This example is theoretical, and will reflect the problem of finding admissible subsets. Let $X = \mathbb{R}$, the set of real numbers, and $\leqslant$ the usual ordering. Then $\mathbb{Z} \subset \mathbb{R}$, the set of integers, is an admissible subset According to the above theorem, the retract φ is the "integer part function" :

$$\varphi : \mathbb{R} \longrightarrow \mathbb{Z}, \quad \varphi(x) = [x]$$

we have denoted by $[x]$ the greatest integer which is $\leqslant$ to the given $x \in \mathbb{R}$. It easy to see that, if X is sup-complete, then each subset, which is sup-complete, is an admissible subset. This is obvious, since for such $A \subset X$, $x \in X$, we have $a_x = \sup\{a \in A \,/\, a \leqslant x\} \in A$.

Example 2 : Let $X = \mathcal{F}(M) = \{f \,/\, f : M \longrightarrow [0, 1]\}$, the set of all fuzzy subsets of M. The ordering is $\leqslant$, defined by

$$f, g \in \mathcal{F}(M),\ f \leqslant g \Longleftrightarrow f(m) \leqslant g(m), \quad (\forall)\ m \in M$$

We choose $A = \mathcal{P}(M) = \{f \,/\, f : M \longrightarrow \{0, 1\}\}$, the set of all subsets of M (identified with their characteristic functions).

As $\mathcal{F}(M)$ is a complete lattice, and $\mathcal{P}(M)$ a complete sublattice, it results that $\mathcal{P}(M)$ is admissible.

If we denote for each $f \in \mathcal{F}(M)$, by $A_f = \{m \in M / f(m) = 1\}$, then the "best approximation" for each $f \in \mathcal{F}(M)$, is $f_{Af} \in \mathcal{P}(M)$ (the characteristic function of A_f).

This is easy to be proved, according to :

$f \in \mathcal{F}(M)$, $\sup\{g \,/\, g \in \mathcal{P}(M),\ g \leqslant f\} = f_{Af}$. This example can be related to the problem of approximating fuzzy sets (see [10], [11]). We shall now give an example, in order to apply this abstract identification model to systems theory. Usually is quite difficult to define an ordering in the class of systems. The relations of " complexity" or "approximation" as given in [5], are preordrings. Of course, the theory above can be restated in terms of preorderings we loose in that cas the unicity.

Example 3 : Let us consider X = Sys, the class of systems. The ordering will be the "inclusion" of systems. More exactly, if $\mathcal{S} = \{X, U, Y, \delta, \beta\}$, $\mathcal{S}' = \{X', U', Y', \delta', \beta'\}$, we shall say that $\mathcal{S}$ is a subsystem of $\mathcal{S}'$ if :

$$X \subseteq X', U \subseteq U', Y \subseteq Y'; \ \delta'|_{X \times U} = \delta, \beta'|_X = \beta.$$

We shall write $\mathcal{S} \subseteq \mathcal{S}'$, if $\mathcal{S}$ is a subsystem of $\mathcal{S}'$. It is clear that $\subseteq$ is an ordering in Sys.

Let us prove that the class of reachable systems Sys(r) $\subset$ Sys is admissible.

If $\mathcal{S} \in$ Sys, $\mathcal{S} = \{X, U, Y, \delta, \beta\}$, let us consider the reachability map $\delta_{x_0} : U^* \longrightarrow X$. We denote by $X_0 = \mathrm{Im}\ \delta_{x_0} = \delta_{x_0}(U^*)$, and consider the system $\mathcal{S}_0 = \{X_0, U, Y, \delta_0, \beta_0\}$. The definitions of the dynamics δ_0, and the output map β_0 are as follows :

$$\delta_0 : X_0 \times U \longrightarrow X_0, \ \delta_0(x,u) = \delta(x,u)$$

$$\beta_0 : X_0 \longrightarrow Y, \ \beta_0(x) = \beta(x)$$

and it is clear that these definitions are correct.

We also see that $\mathcal{S}_0 \in$ Sys(r), and $\mathcal{S}_0 \subseteq \mathcal{S}$.

Now, if $\mathcal{S}' \in$ Sys(r), and $\mathcal{S}' \subseteq \mathcal{S}$, it is easy to prove that $\mathcal{S}' \subseteq \mathcal{S}_0$.

Thus $\mathcal{S}_0$ is the "best approximation" of the system $\mathcal{S}$ by a reachable system. It is clear that this example does not contain very much information ; it simply proves that our model of system identification works in some cases.

This example also proves that among admissible classes of systems, we can find those with important structural properties, such as reachability. This is in some sense, coherent with the model in section 2, even if we cannot call the above $\mathcal{S}_0$ the minimal realization of $\mathcal{S}$.

4 - Categorical models

We shall describe in this section the last model for system identification. This model is based on some concepts of category theory.

Our point of view will be again to consider minimal realization of a dynamical system. We shall not speak about external behaviour, but we shall think at the "optimal structure model" assigned to a given model. In this way we shall manipulate only internal descriptions.

Let us remember first, some facts of category theory, which will be needed later.

If $\mathcal{C}$ and $\mathcal{C}'$ are two categories, a (covariant) functor from $\mathcal{C}$ into $\mathcal{C}'$ is $F : \mathcal{C} \longrightarrow \mathcal{C}'$. This means an assignment $|\mathcal{C}| \to |\mathcal{C}'|$ (on objects), and, for each $A, B \in |\mathcal{C}|$, an assignment $\mathcal{C}(A,B) \longrightarrow \mathcal{C}'(FA, FB)$ (on morphisms).

The following axioms are supposed :

1) $F(1_A) = 1_{FA}$, for each $A \in |\mathcal{C}|$

2) $F(v \circ u) = F(v) \circ F(u)$

We suppose the reader already familiar with such concepts.

If $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{C}'$ are two functors, a natural transformation from F to G is $\Phi : F \longrightarrow G$, a collection of morphisms $\Phi = (\varphi_A)_{A \in |\mathcal{C}|}$, $\varphi_A : FA \longrightarrow GA$. It is supposed that, for each $A, B \in |\mathcal{C}|$ and $u \in \mathcal{C}(A, B)$, the following diagram commutes :

$$\begin{tikzcd} FA \arrow[r, "\varphi_A"] \arrow[d, "Fu"'] & GA \arrow[d, "Gu"] \\ FB \arrow[r, "\varphi_B"'] & GB \end{tikzcd}$$

Such concepts as functors (transformations between categories), and natural transformations (transformations between functors), arise very naturally in many problems.

We shall denote by Sets the category of sets.

Two functors $\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}'$ are <u>isomorphic</u>, denoted $F \simeq G$, if there exists a natural transformation $\Phi : F \longrightarrow G$, $\Phi = (\varphi_A)_{A \in |\mathcal{C}|}$, such that φ_A is an isomorphism, for each $A \in |\mathcal{C}|$.

We shall speak now about adjoint functors, which will play an important rôle in the development of system identification models.

If $\mathcal{C}$, $\mathcal{C}'$ are categories, we can build the product category $\mathcal{C} \times \mathcal{C}'$ whose objects are pairs (A,B), $A \in |\mathcal{C}|$, $B \in |\mathcal{C}'|$. The morphisms are defined obviously.

If $\mathcal{C}$ is a category, we shall denote its opposite (or dual) category by $\mathcal{C}^{op}$.

The objects of $\mathcal{C}^{op}$ are the same as those of $\mathcal{C}$, but the arrows (morphisms) are reversed.

Let now $\mathcal{C}$ and $\mathcal{C}'$ be two categories, and a pair of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}'$$

We can build two functors :

$$\mathcal{C}(\cdot, G\cdot) : \mathcal{C}^{op} \times \mathcal{C}' \longrightarrow \text{Sets}$$
$$\mathcal{C}'(F\cdot, \cdot) : \mathcal{C}^{op} \times \mathcal{C}' \longrightarrow \text{Sets}$$

such that, for example, $\mathcal{C}(\cdot, G\cdot)(A,B) = \mathcal{C}(A, GB)$. These functors are defined on morphisms in an obvious way.

<u>Definition.</u> We say that G is a <u>right adjoint</u> to F (or F is a <u>left ajoint</u> to G), if the functors $\mathcal{C}(\cdot, G\cdot)$ and $\mathcal{C}'(F\cdot, \cdot)$ are isomorphic.

We shall denote by

$$G \text{ r.a } F \Longleftrightarrow \mathcal{C}(\cdot, G\cdot) \simeq \mathcal{C}'(F\cdot, \cdot)$$

Examples of adjoint functors (or adjoint pairs) arise in many problems ; one may argue that each "natural construction" gives an adjoint pair of functors. We shall describe later, in an example, such a pair. For much more on category theory than we described here, the reader may successfully consult [8] .

Our categorical model for system identification will be built by distinguishing some subcategories of a given category $\mathcal{C}$.

Definition : A realization subcategory of $\mathcal{C}$ will be $\mathcal{R} \subseteq \mathcal{C}$, such that the inclusion functor $\mathcal{R} \xrightarrow{F} \mathcal{C}$ has a left adjoint $\mathcal{C} \xrightarrow{G} \mathcal{R}$.

In the pair $\mathcal{R} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}$, we shall call G the realization functor ; for each $X \in |\mathcal{C}|$, $GX \in |\mathcal{R}|$ will be called the minimal realization of X.

We shall later see that classical minimal realization of systems can be recaptured in this way.

Example : Let us restrict our attention to reachable systems. After section 3 we have seen that a system, even if not reachable, contains a reachable subsystem (its reachable part).

We shall consider Sys(r), the category of reachable systems. A morphism $\mathcal{S} \longrightarrow \mathcal{S}'$ will be a triple (u, v, w), with second component v (which operates on input spaces) being surjective. More exactly, if U, U' are respectively, the input spaces of $\mathcal{S}$ and $\mathcal{S}'$, then $v : U \longrightarrow U'$ is an epimorphism.

Let Sys(r,o) be the subcategory of Sys(r), which contains reachable and observable systems. We shall prove that Sys (r, o) is a realization subcategory of Sys(r).

We must build the functor $G : \text{Sys}(r) \longrightarrow \text{Sys}(r,o)$ which will be a left adjoint to the inclusion functor $F : \text{Sys}(r,o) \longrightarrow \text{Sys}(r)$.

For this purpose, let $\mathcal{S} \in |\mathrm{Sys}\,(r)|$: $\mathcal{S} = \{X, U, Y, \delta, \beta, x_0\}$.

We shall define in X the equivalence relation :

$$x, x' \in X, \quad x \sim x' \iff f_x = f_{x'}$$

It is simple to prove that the system

$$\mathcal{S}_m = \{X/\sim, U, Y, \delta_m, \beta_m, \hat{x}_0\}$$

with $\delta_m : X/\sim \times U \longrightarrow X/\sim$, $\delta_m(\hat{x}, u) = \widehat{\delta(x, u)}$

$\beta_m : X/\sim \longrightarrow Y$, $\beta_m(\hat{x}) = \beta(x)$

is reachable and observable.

Thus $\mathcal{S}_m$ is the minimal realization of $\mathcal{S}$ (all such minimal realizations of $\mathcal{S}$ are isomorphic).

The functor G will be defined by $G(\mathcal{S}) = \mathcal{S}_m$.

Let us remark that there is a morphism :

$$\Phi : \mathcal{S} \longrightarrow \mathcal{S}_m, \Phi = (\varphi, 1_U, 1_Y)$$

where $\varphi : X \longrightarrow X/\sim$ is the canonical map (see also section 1). Since 1_U is surjective, Φ is indeed a morphism in $\mathrm{Sys}(r)$.

We may prove that the following universality property holds ; for each system $\mathcal{S}' \in |\mathrm{Sys}(r, \gamma)|$ and for each morphism $\mathcal{S} \xrightarrow{\alpha} \mathcal{S}'$, there exists an unique morphism $\mathcal{S}_m \xrightarrow{\beta} \mathcal{S}'$ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Phi"] \arrow[dr, "\alpha"'] & \mathcal{S}_m \arrow[d, dashed, "\beta"] \\ & \mathcal{S}' \end{tikzcd}$$

To prove this, let $\alpha = (f, g, h)$, $\beta = (\bar{f}, \bar{g}, \bar{h})$, we set $\bar{g} = g$, $\bar{h} = h$ and $\bar{f}$ results from the diagram :

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & X/\sim \arrow[d, dashed, "\bar{f}"] \\ & X' \end{tikzcd}$$

$$\bar{f}(\hat{x}) = f(x), \ (\forall)\ \hat{x} \in X/\sim .$$

To see that $\bar{f}$ is <u>well-defined</u>, let us prove that $x \sim x' \Longrightarrow f(x) = f(x')$.
As the equivalence in X means the equality of behaviours, we shall prove that

$f_x = f_{x'} \Longrightarrow f'_{f(x)} = f'_{f(x')}$

(here f_x, $f_{x'}$ are behaviours of $\mathcal{S}$, while $f'_{f(x)}$, $f'_{f(x)}$ are behaviours of $\mathcal{S}'$). According to the definition of morphisms in the category Sys(r) (see also section 1), we can easily obtain the following commutative diagram.

$$\begin{array}{ccc} U^* & \xrightarrow{f_x} & Y \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle h} \\ U'^* & \xrightarrow[f'_{f(x)}]{} & Y' \end{array}$$

By g^* we have denoted the extension of $g : U \longrightarrow U'$ at the free monoids :
$g^* : U^* \longrightarrow U'^*$, $g^*(u_1u_2\ldots u_p) = g(u_1)g(u_2)\ldots g(u_p)$
Thus $h \circ f_x = f'_{f(x)} \circ g^*$, and, in the same way, we obtain : $h \circ f_{x'} = f'_{f(x')} \circ g^*$.
It results that $f_x = f_{x'} \Longrightarrow f'_{f(x)} \circ g^* = f'_{f(x')} \circ g^*$.
Since g is epimorphism, it results that g^* is also epimorphism and, from well-known property we have :

$$f'_{f(x)} \circ g^* = f'_{f(x')} \circ g^* \Longrightarrow f'_{f(x)} = f'_{f(x')}$$

Now, to end the proof, we remember that the system $\mathcal{S}'$ is observable, thus

$$f'_{f(x)} = f'_{f(x')} \Longrightarrow f(x) = f(x')$$

From the above universality property, it is simple to prove that functors
Sys (r,o) $\underset{G}{\overset{F}{\rightleftarrows}}$ Sys(r) form an adjoint pair.
<u>Remark</u> : We may say, after this example, that minimal realization is <u>left adjoint</u> to inclusion, restricting our attention to reachable systems, and morphisms whose input component is surjective.

There can be given, of course, a lot of examples of realization subcategories ; we shall not insist here on this aspect.

The inclusion functor in the above example can also be thought as one which "forgets" the observability property.

5 - Conclusions

We have sketched in this paper three models for systems identification. The most significant one is the categorical model.

This model applies to system theory in such a way that if we can prove that an inclusion functor has a left adjoint, we prove the existence of minimal realization. Such minimal realization theorems can be redescovered, for example, for fuzzy systems (see [10]), by defining reachability and observability in an apropriate way.

All the presented models were deeply connected with system theory, either in which realization of behaviours is concerned, or in the problem of approximating a system by a simpler one.

We shall underline here a property which is neither new in mathematics, nor in system theory ; in order to obtain the "optimality of structure", we must build an adjoint pair of functors.

This idea works also for the model in section 3, since we are faced there with an adjointness property, too.

It is to be expected that these results can be generalized in many ways. A first step will be to give sufficient conditions for a subcategory to be a realization one. This will imply sufficient conditions for existence of minimal realization.

The second idea, which may be more fruitful, is to think that feedback is in some sense adjoint, or dual (see [4]) to dynamics or behaviour. We shall speak about these problems in a next paper.

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