

# Optimising Monotone Chance-Constrained Submodular Functions Using Evolutionary Multi-Objective Algorithms

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**Abstract.** Many real-world optimisation problems can be stated in terms of submodular functions. A lot of evolutionary multi-objective algorithms have recently been analyzed and applied to submodular problems with different types of constraints. We present a first runtime analysis of evolutionary multi-objective algorithms for chance-constrained submodular functions. Here, the constraint involves stochastic components and the constraint can only be violated with a small probability of  $\alpha$ . We show that the GSEMO algorithm obtains the same worst case performance guarantees as recently analyzed greedy algorithms. Furthermore, we investigate the behavior of evolutionary multi-objective algorithms such as GSEMO and NSGA-II on different submodular chance constrained network problems. Our experimental results show that this leads to significant performance improvements compared to the greedy algorithm.

## 1 Introduction

Evolutionary algorithms have been widely applied to solve complex optimisation problems. They are well suited for broad classes of problems and often achieve good results within a reasonable amount of time. The theory of evolutionary computation aims to explain such good behaviours and also point out the limitations of evolutionary computing techniques. A wide range of tools and techniques have been developed in the last 25 years and we point the reader to [7,27,2,14] for comprehensive presentations.

Stochastic components play a crucial role in many real-world applications and chance constraints allow to model constraints that can only be violated with a small probability. A chance constraint involves random components and it is required that the constraint is violated with a small probability of at most  $\alpha$ . We consider chance constraints where the weight  $W(S)$  of a possible solution  $S$  can violate a given constraint bound  $C$  with probability at most  $\alpha$ , i.e.  $\Pr[W(S) > C] \leq \alpha$  holds.

Evolutionary algorithms have only recently been considered for chance constrained problems and we are pushing forward this area of research by providing a first runtime analysis for submodular functions. In terms of the theoretical understanding and the applicability of evolutionary algorithms, it is desirable

to be able to analyse them on a broad class of problems and design appropriate evolutionary techniques for such classes. Submodular functions model a wide range of problems where the benefit of adding solution components diminishes with the addition of elements. They have extensively been studied in the literature [24,25,33,18,3,21,9,13] and allow to model a variety of real-world applications [19,20,12,22]. In recent years, the design and analysis of evolutionary algorithms for submodular optimisation problems has gained increasing attention. We refer to the recent book of Zhou et al. [37] for an overview. Such studies usually study evolutionary algorithms in terms of their runtime and approximation behaviour and evaluate the performance of the designed algorithms on classical submodular combinatorial optimisation problems.

To our knowledge, there is so far no runtime analysis of evolutionary algorithms for submodular optimisation with chance constraints and the runtime analysis of evolutionary algorithms for chance constrained problems has only started recently for very special cases of chance-constrained knapsack problem [26]. Chance constraints are in general hard to evaluate exactly, but well-known tail inequalities such as Chernoff bounds and Chebyshev’s inequality may be used to estimate the probability of a constraint violation. We provide a first runtime analysis by analysing GSEMO together with multi-objective formulations that use a second objective taking the chance constraint into account. These formulations based on tail inequalities are motivated by some recent experimental studies of evolutionary algorithms for the knapsack problem with chance constraints [34,35,1]. The GSEMO algorithm has already been widely studied in the area of runtime analysis in the field of evolutionary computation [10] and more broadly in the area of artificial intelligence where the focus has been on submodular functions and Pareto optimisation [31,30,29,32]. We analyse this algorithm in the chance constrained submodular optimisation setting investigated in [6] in the context of greedy algorithms. Our analyses show that GSEMO is able to achieve the same approximation guarantee in expected polynomial time for uniform IID weights and the same approximation quality in expected pseudo-polynomial time for independent uniform weights having the same dispersion.

Furthermore, we study GSEMO experimentally on the influence maximization problem in social networks and the maximum coverage problem. Our results show that GSEMO significantly outperforms the greedy approach [6] for the considered chance constrained submodular optimisation problems. Furthermore, we use the multi-objective problem formulation in a standard setting of NSGA-II. We observe that NSGA-II is outperformed by GSEMO in most of our experimental settings, but usually achieves better results than the greedy algorithm.

The paper is structured as follows. In Section 2, we introduce the problem of optimising submodular functions with chance constraints, the GSEMO algorithm and tail inequalities for evaluating chance constraints. In Section 3, we provide a runtime analysis for submodular functions where the weights of the constraints are either identically uniformly distributed or are uniformly distributed and have the same dispersion. We carry out experimental investigations that compare the

performance of greedy algorithms, GSEMO, and NSGA-II in Section 4 and finish with some concluding remarks.

## 2 Preliminaries

Given a set  $V = \{v_1, \dots, v_n\}$ , we consider the optimization of a monotone submodular function  $f: 2^V \rightarrow \mathbb{R}_{\geq 0}$ . We call a function monotone iff for every  $S, T \subseteq V$  with  $S \subseteq T$ ,  $f(S) \leq f(T)$  holds. We call a function  $f$  submodular iff for every  $S, T \subseteq V$  with  $S \subseteq T$  and  $x \notin T$  we have

$$f(S \cup \{x\}) - f(S) \geq f(T \cup \{x\}) - f(T).$$

Here, we consider the optimization of a monotone submodular function  $f$  subject to a chance constraint where each element  $s \in V$  takes on a random weight  $W(s)$ . Precisely, we examine constraints of the type

$$\Pr[W(S) > C] \leq \alpha.$$

where  $W(S) = \sum_{s \in S} w(s)$  is the sum of the random weights of the elements and  $C$  is the given constraint bound. The parameter  $\alpha$  specifies the probability of exceeding the bound  $C$  that can be tolerated for a feasible solution  $S$ .

The two settings, we investigate in this paper assume that the weight of an element  $s \in V$  is  $w(s) \in [a(s) - \delta, a(s) + \delta]$ ,  $\delta \leq \min_{s \in V} a(s)$ , is chosen uniformly at random. Here  $a(s)$  denotes the expected weight of items  $s$ . For our investigations, we assume that each item has the same dispersion  $\delta$ . We call a feasible solution  $S$  a  $\gamma$ -approximation,  $0 \leq \gamma \leq 1$ , iff  $f(S) \geq \gamma \cdot f(OPT)$  where  $OPT$  is an optimal solution for the given problem.

### 2.1 Chance Constraint Evaluation based on Tail Inequalities

As the probability ( $\Pr(W(X) > C)$ ) used in the objective functions is usually computational expensive to evaluate exactly, we use the approach taken in [34] and compute an upper bound on this probability using tail inequalities [23]. We assume that  $w(s) \in [a(s) - \delta, a(s) + \delta]$  holds for each  $s \in V$  which allows to use Chebyshev's inequality and Chernoff bounds.

The approach based on (one-sided) Chebyshev's inequality used in [34] upper bounds the probability of a constraint violation by

$$\hat{\Pr}(W(X) > C) \leq \frac{\delta^2 |X|}{\delta^2 |X| + 3(C - E_W(X))^2} \quad (1)$$

The approach based on Chernoff bounds used in [34] upper bounds the probability of a constraint violation by

$$\hat{\Pr}[W(X) > C] \leq \left( \frac{e^{-\frac{C - E_W(X)}{\delta |X|}}}{\left( \frac{\delta |X| + C - E_W(X)}{\delta |X|} \right)^{\frac{\delta |X| + C - E_W(X)}{\delta |X|}}} \right)^{\frac{1}{2} |X|} \quad (2)$$

We use  $\hat{\Pr}(W(X) > C)$  instead of  $\Pr(W(X) > C)$  for our investigations using multi-objective models of the problem.

## 2.2 Multi-Objective Formulation

Following the approach of Yue et al. [34] for the chance constrained knapsack problem, we evaluate a set  $X$  by the multi-objective fitness function  $g(X) = (g_1(X), g_2(X))$  where  $g_1$  measures the tightness in terms of the constraint and  $g_2$  measures the quality of  $X$  in terms of the given submodular function  $f$ .

We define

$$g_1(X) = \begin{cases} E_W(X) - C & \text{if } (C - E_W(X))/(\delta \cdot |X|) \geq 1 \\ \hat{\Pr}(W(X) > C) & \text{if } (E_W(X) < C) \wedge (C - E_W(X))/(\delta |X|) < 1 \\ 1 + (E_W(X) - C) & \text{if } E_W(X) \geq C \end{cases} \quad (3)$$

and

$$g_2(X) = \begin{cases} f(X) & \text{if } g_1(X) \leq \alpha \\ -1 & \text{if } \hat{\Pr}(W(X) > C) > \alpha \end{cases} \quad (4)$$

where  $E_W(X) = \sum_{s \in X} a(s)$  denotes the expected weight of the solution. The term  $(C - E_W(X))/(\delta \cdot |X|) \geq 1$  in  $g_1$  implies that a set  $X$  of cardinality  $|X|$  has probability 0 of violating the chance constraint due to the upper bound on the intervals.

We say a solution  $Y$  dominates a solution  $X$  (denoted by  $Y \succ X$ ) iff  $g_1(Y) \leq g_1(X) \wedge g_2(Y) \geq g_2(X)$ . We say that  $Y$  strongly dominates  $X$  (denoted by  $Y \succ X$ ) iff  $Y \succ X$  and  $g(Y) \neq g(X)$ . The dominance relation also translates to the corresponding search points used in GSEMO. Comparing two solutions, the objective function guarantees that a feasible solution strongly dominates every infeasible solution. The objective function  $g_1$  ensures that the search process is guided towards feasible solutions and that trade-offs in terms of the probability of a constraint violation and the function value of the submodular function  $f$  are computed for feasible solutions.

## 2.3 Global SEMO

Our multi-objective approach is based on a simple multi-objective evolutionary algorithm called Global Simple Evolutionary Multi-Objective Optimizer (GSEMO, see Algorithm 1) [11]. The algorithm encodes sets as bitstrings of length  $n$  and the set  $X$  corresponding to a search point  $x$  is given as  $X = \{v_i \mid x_i = 1\}$ . We use  $x$  when referring to the search point in the algorithm and  $X$  when referring to the set of selected elements and use applicable fitness measure for both notations in an interchangeable way. GSEMO starts with a random search point  $x \in \{0, 1\}^n$ . In each iteration, an individual  $x \in P$  is chosen uniformly at random from the current population  $P$ . In the mutation step, it flips each bit with a probability  $1/n$  to produce an offspring  $y$ .  $y$  is added to the

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**Algorithm 1:** Global SEMO
 

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1 Choose  $x \in \{0, 1\}^n$  uniformly at random;
2  $P \leftarrow \{x\}$ ;
3 repeat
4   | Choose  $x \in P$  uniformly at random;
5   | Create  $y$  by flipping each bit  $x_i$  of  $x$  with probability  $\frac{1}{n}$ ;
6   | if  $\nexists w \in P : w \succ y$  then
7   |   |  $S \leftarrow (P \cup \{y\}) \setminus \{z \in P \mid y \succ z\}$ ;
8 until stop;
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population if it is not strongly dominated by any other search point in  $P$ . If  $y$  is added to the population, all search points dominated by  $y$  are removed from the population  $P$ .

We analyze GSEMO in terms of its runtime behaviour to obtain a good approximation. The expected time of the algorithm required to achieve a given goal is measured in terms of the number of iterations of the repeat loop until a feasible solution with the desired approximation quality has been produced for the first time.

### 3 Runtime Analysis

In this section, we provide a runtime analysis of GSEMO which shows that the algorithm is able to obtain a good approximation for important settings where the weights of the constraint are chosen according to a uniform distribution with the same dispersion.

#### 3.1 Uniform IID Weights

We first investigate the case of uniform identically distributed (IID) weights. Here each weight is chosen uniformly at random in the interval  $[a - \delta, a + \delta]$ ,  $\delta \leq a$ . The parameter  $\delta$  is called the dispersion and models the uncertainty of the weight of the items.

**Theorem 1.** *Let  $k = \min\{n + 1, \lfloor C/a \rfloor\}$  and assume  $\lfloor C/a \rfloor = \omega(1)$ . Then the expected time until GSEMO has computed a  $(1 - o(1))(1 - 1/e)$ -approximation for a given monotone submodular function under a chance constraint with uniform iid weights is  $O(nk(k + \log n))$ .*

*Proof.* Every item has expected weight  $a$  and uncertainty  $\delta$ . This implies  $g_1(X) = g_1(Y)$  iff  $|X| = |Y|$  and  $E_W(X) = E_W(Y) < C$ . As GSEMO only stores for each fixed  $g_1$ -value one single solution, the number of solutions with expected weight less than  $C$  is at most  $k = \min\{n + 1, C/a\}$ . Furthermore, there is at most one individual  $X$  in the population  $g_2(X) = -1$ . Hence, the maximum population that GSEMO encounters during the run of the algorithm is at most  $k + 1$ .

We first consider the time until GSEMO has produced the bitstring  $0^n$ . This is the best individual with respect to  $g_1$  and once included will always stay in the population. The function  $g_1$  is strictly monotone decreasing with the size of the solution. Hence, selecting the individual in the population with the smallest number of elements and removing one of them leads to a solution with less elements and therefore with a smaller  $g_1$ -value. Let  $\ell = |x|_1$  be the number of elements of the solution  $x$  with the smallest number of elements in  $P$ . Then flipping one of the 1-bits corresponding to these elements reduces  $k$  by one and happens with probability at least  $\ell/(en)$  once  $x$  is selected for mutation. The probability of selecting  $x$  is at least  $1/(k+1)$  as there are at most  $k+1$  individuals in the population. Using the methods of fitness-based partitions, the expected time to obtain the solution  $0^n$  is at most

$$\sum_{\ell=1}^n \left( \frac{\ell}{e(k+1)n} \right)^{-1} = O(nk \log n).$$

Let  $k_{opt} = \lfloor C/a \rfloor$ , the maximal number of elements that can be included in the deterministic version of the problem.

The function  $g_1$  is strictly monotonically increasing with the number of elements and each solution with same number of elements has the same  $g_1$ -value.

We consider the solution  $X$  with the largest  $k$  for which

$$f(X) \geq (1 - (1 - 1/k_{opt})^k) \cdot f(OPT)$$

holds in the population and the mutation which adds an element with the largest marginal increase  $g_2(X \cup \{x\}) - g_2(X)$  to  $X$ . The probability for such a step picking  $X$  and carrying the mutation with the largest marginal gain is  $\Omega(1/kn)$  and its waiting time is  $O(kn)$ .

This leads to a solution  $Y$  for which

$$f(X) \geq (1 - (1 - 1/k_{opt})^{k+1}) \cdot f(OPT)$$

holds. The maximal number of times such a step is required after having included the search point  $0^n$  into the population is  $k$  which gives the runtime bound of  $O(k^2n)$ .

For the statement on the approximation quality, we make use of the lower bound on the maximal number of elements that can be included using the Chernoff bound and Chebyshev's inequality given in [6].

Using Chebyshev's inequality (Equation 1) at least

$$k_1^* = \max \left\{ k \mid k + \frac{\sqrt{(1-\alpha)k\delta^2}}{\sqrt{3\alpha a}} \leq k_{opt} \right\}$$

elements can be included and when using Chernoff bound (Equation 2), at least

$$k_2^* = \max \left\{ k \mid k + \frac{\sqrt{3\delta k \ln(1/\alpha)}}{a} \leq k_{opt} \right\}$$

elements can be included.

Including  $k^*$  elements in this way least to a solution  $X^*$  with

$$f(X^*) \geq (1 - (1 - 1/k_{opt})^{k^*}) \cdot f(OPT).$$

As shown in [6], both values of  $k_1^*$  and  $k_2^*$  yield  $(1 - o(1))(1 - 1/e) \cdot f(OPT)$  if  $\lfloor C/a \rfloor = \omega(1)$  which completes the proof.  $\square$

### 3.2 Uniform Weights with the Same Dispersion

We now assume that the expected weights do not have to be the same, but still require the same dispersion for all elements, i.e.  $w(s) \in [a(s) - \delta, a(s) + \delta]$  holds for all  $s \in V$ .

We consider the (to be minimized) objective function  $\hat{g}_1(X) = E_W(X)$  (instead of  $g_1$ ) together with the previously defined objective function  $g_2$  and evaluate a set  $X$  by  $\hat{g}(X) = (\hat{g}_1(X), g_2(X))$ . We have  $Y \succeq X$  iff  $\hat{g}_1(Y) \leq \hat{g}_1(X)$  and  $g_2(Y) \geq g_2(X)$ .

Let  $a_{\max} = \max_{s \in V} a(s)$  and  $a_{\min} = \min_{s \in V} a(s)$ , and  $\delta \leq a_{\min}$ . The following theorem shows that GSEMO is able to obtain a  $(1/2 - o(1))(1 - 1/e)$ -approximation if  $\omega(1)$  elements can be included in a solution.

**Theorem 2.** *If  $C/a_{\max} = \omega(1)$  then GSEMO obtains a  $(1/2 - o(1))(1 - 1/e)$ -approximation for a given monotone submodular function under a chance constraint with uniform weights having the same dispersion in expected time  $O(P_{\max} \cdot n(C/a_{\min} + \log n + \log(a_{\max}/a_{\min})))$ .*

*Proof.* We first consider the time until the search point  $0^n$  is included in the population. We always consider the individual  $x$  with the smallest  $\hat{g}_1$ -value. Flipping every 1-bit of  $x \neq 0^n$  leads to an individual with a smaller  $\hat{g}_1$ -value and is therefore accepted. Furthermore, the total weight decrease of these 1-bit flips is  $\hat{g}_1(x)$  which also equals the total weight decrease of all single bit flip mutation when taking into account that 0-bit flips give decrease of the  $\hat{g}_1$ -value of zero. A mutation carrying out a single bit flip happens each iteration with probability at least  $1/e$ . The expected decrease in  $\hat{g}_1$  is therefore at least by a factor of  $(1 - 1/(P_{\max}en))$  and the expected minimal  $\hat{g}_1$ -value in the next generation is at most

$$(1 - 1/(P_{\max} \cdot en)) \cdot \hat{g}_1(x).$$

We use drift analysis, to upper bound the expected time until the search point  $0^n$  is included in the population. As  $a_{\min} \leq \hat{g}_1(x) \leq na_{\max}$  holds for any search point  $x \neq 0^n$ , the search point  $0^n$  is included in the population after an expected number of  $O(P_{\max}n(\log n + \log(a_{\max}/a_{\min})))$  steps.

After having include the search point  $0^n$  in the population, we follow the analysis of POMC for subset selection with general deterministic cost constraints [29] and always consider the individual  $x$  with the largest  $\hat{g}_1$ -value for which

$$g_2(x) \geq \left[ 1 - \prod_{k=1}^n \left( 1 - \frac{a(k)x_k}{C} \right) \right] \cdot f(OPT).$$

Note that the search point  $0^n$  meets this formula. Furthermore, we denote by  $\hat{g}_1^*$ , the maximal  $\hat{g}_1$ -value for which  $\hat{g}_1(x) \leq \hat{g}_1^*$  and

$$g_2(x) \geq \left[ 1 - \left( 1 - \frac{\hat{g}_1^*}{Cr} \right)^r \right] \cdot f(OPT).$$

for some  $r$ ,  $0 \leq r \leq n-1$ , holds. We use  $\hat{g}_1^*$  to track the progress of the algorithm and it has been shown in [29] that  $\hat{g}_1^*$  does not decrease during the optimisation process of GSEMO.

Choosing  $x$  for mutation and flipping the 0-bit of  $x$  corresponding to the largest marginal gain in terms of  $g_2/\hat{g}_1$  gives a solution  $y$  for which

$$\begin{aligned} g_2(y) &\geq \left[ 1 - \left( 1 - \frac{a_{\min}}{C} \right) \cdot \left( 1 - \frac{\hat{g}_1^*}{Cr} \right)^r \right] \cdot f(OPT) \\ &\geq \left[ 1 - \left( 1 - \frac{\hat{g}_1^* + a_{\min}}{C(r+1)} \right)^{r+1} \right] \cdot f(OPT) \end{aligned}$$

holds and  $\hat{g}_1^*$  increases by at least  $a_{\min}$ . The  $\hat{g}_1^*$ -value for the considered solution, can increase at most  $C/a_{\min}$  times and therefore, once having included the search point  $0^n$ , the expected time until such improvements have occurred is  $O(P_{\max}nC/a_{\min})$ .

Let  $x^*$  be the feasible solution of maximal cost included in the population after having increased the  $\hat{g}_1^*$  at most  $C/a_{\min}$  times as described above. Furthermore, let  $v^*$  be the element with the largest  $g_2$ -value not included in  $x^*$  and  $\hat{x}$  be the solution containing the single element with the largest  $g_2$ -value.  $\hat{x}$  is produced from the search point  $0^n$  in expected time  $O(P_{\max}n)$ .

Let  $r$  be the number of elements in a given solution. According to [6], the maximal  $\hat{g}_1^*$  deemed as feasible is at least

$$C_1^* = C - \sqrt{\frac{(1-\alpha)r\delta^2}{3\alpha}}$$

when using Chebyshev's inequality (Equation 1) and at least

$$C_2^* = C - \sqrt{3\delta r \ln(1/\alpha)}$$

when using the Chernoff bound (Equation 2). For a fixed  $C^*$ -value, we have

$$\left[ 1 - \left( 1 - \frac{C^*}{C(r+1)} \right)^{r+1} \right] \cdot f(OPT)$$

We have  $\hat{g}_1(x^*) + a(v^*) > C_1^*$  when working with Chebyshev's inequality and  $\hat{g}_1(x^*) + a(v^*) > C_2^*$  when using Chernoff bound. In addition,  $f(\hat{x}) \geq f(v^*)$  holds. According to [6],  $x^*$  or  $\hat{x}$  is therefore a  $(1/2 - o(1))(1 - 1/e)$ -approximation which completes the proof.  $\square$

For the special case of uniform IID weights, we have  $a = a_{\max} = a_{\min}$  and  $P_{max} \leq C/a + 1$ . Furthermore, the solution  $x^*$  already gives a  $(1 - o(1))(1 - 1/e)$ -approximation as the element with the largest  $f$ -value is included in construction of  $x^*$ . This gives a bound on the expected runtime of  $O(nk(k + \log n))$  to obtain a  $(1 - o(1))(1 - 1/e)$ -approximation for the uniform IID case when working with the function  $\hat{g}_1$  instead of  $g_1$ . Note that this matches the result given in Theorem 1.

## 4 Experimental Investigations

In this section, we investigate the GSEMO and the NSGA-II algorithm on important submodular optimisation problems with chance constraints and compare them to the greedy approach given in [6].

### 4.1 Experimental Setup

We examine GSEMO and NSGA-II for constraints with expected weights 1 and compare them to the greedy algorithm (GA) given in [6]. Our goal is to study different chance constraint settings in terms of the constraint bound  $C$ , the dispersion  $\delta$ , and the probability bound  $\alpha$ . We consider different benchmarks for chance constrained versions of the maximum influence problems and the maximum coverage problem.

For each benchmark set, we study the performance of the GSEMO and the NSGA-II algorithms for different budgets. We consider  $C = 20, 50, 100$  for influence maximization and  $C = 10, 15, 20$  for maximum coverage. We consider all combinations of  $\alpha = 0.1, 0.001$ , and  $\delta = 0.5, 1.0$  for the experimental investigations of the algorithms and problems. Chebyshev's inequality leads to better results when  $\alpha$  is relatively large and the Chernoff bounds gives better results for small  $\alpha$  (see [34,6]). Therefore, we use Equation 1 for  $\alpha = 0.1$  and Equation 2 for  $\alpha = 0.001$  when computing the upper bound on the probability of a constraint violation. We allow 5 000 000 fitness evaluations for each evolutionary algorithm run. We run NSGA-II with parent population size 20, offspring size 10, crossover probability 0.90 and standard bit mutation for 500 000 generations. For each tested instance, we carry out 30 independent runs and report the minimum, maximum, and average results. In order to test the statistical significance of the results, we use the Kruskal-Wallis test with 95% confidence in order to measure the statistical validity of our results. We apply the Bonferroni post-hoc statistical procedure, that is used for multiple comparison of a control algorithm, to two or more algorithms [5].  $X^{(+)}$  is equivalent to the statement that the algorithm in the column outperformed algorithm  $X$ .  $X^{(-)}$  is equivalent to the statement that  $X$  outperformed the algorithm given in the column. If algorithm  $X$  does not appear, then no significant difference was determined between the algorithms.

**Table 1.** Results for Influence Maximization with uniform chance constraints.

$C$	$\alpha$	$\delta$	GA (1)	GSEMO (2)					NSGA-II (3)				
				mean	min	max	std	stat	mean	min	max	std	stat
20	0.1	0.5	51.51	<b>55.75</b>	54.44	56.85	0.5571	1 <sup>(+)</sup>	55.66	54.06	56.47	0.5661	1 <sup>(+)</sup>
	0.1	1.0	46.80	<b>50.65</b>	49.53	51.68	0.5704	1 <sup>(+)</sup>	50.54	49.61	52.01	0.6494	1 <sup>(+)</sup>
50	0.1	0.5	90.55	<b>94.54</b>	93.41	95.61	0.5390	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	92.90	90.75	94.82	1.0445	1 <sup>(+)</sup> , 2 <sup>(-)</sup>
	0.1	1.0	85.71	<b>88.63</b>	86.66	90.68	0.9010	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	86.89	85.79	88.83	0.8479	1 <sup>(+)</sup> , 2 <sup>(-)</sup>
100	0.1	0.5	144.16	<b>147.28</b>	145.94	149.33	0.8830	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	144.17	142.37	146.18	0.9902	2 <sup>(-)</sup>
	0.1	1.0	135.61	<b>140.02</b>	138.65	142.52	0.7362	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	136.58	134.80	138.21	0.9813	2 <sup>(-)</sup>
20	0.001	0.5	48.19	<b>50.64</b>	49.10	51.74	0.6765	1 <sup>(+)</sup>	50.33	49.16	51.25	0.5762	1 <sup>(+)</sup>
	0.001	1.0	39.50	<b>44.53</b>	43.63	45.55	0.4687	1 <sup>(+)</sup>	44.06	42.18	45.39	0.7846	1 <sup>(+)</sup>
50	0.001	0.5	75.71	<b>80.65</b>	78.92	82.19	0.7731	1 <sup>(+)</sup>	80.58	79.29	81.63	0.6167	1 <sup>(+)</sup>
	0.001	1.0	64.49	69.79	68.89	71.74	0.6063	1 <sup>(+)</sup>	<b>69.96</b>	68.90	71.05	0.6192	1 <sup>(+)</sup>
100	0.001	0.5	116.05	<b>130.19</b>	128.59	131.51	0.7389	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	127.50	125.38	129.74	0.9257	1 <sup>(+)</sup> , 2 <sup>(-)</sup>
	0.001	1.0	96.18	<b>108.95</b>	107.26	109.93	0.6466	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	107.91	106.67	110.17	0.7928	1 <sup>(+)</sup> , 2 <sup>(-)</sup>

## 4.2 The Influence Maximization Problem

The influence maximization problem (IM) (see [16,21,29,36] detailed descriptions) is a key problem in the area of social influence analysis.

IM aims to find the set of the most influential users in a large-scale social network. The primary goal of IM is to maximize the spread of influence through a given social network i.e. a graph of interactions and relationships within a group of users [4,15]. However, the problem of influence maximization has been studied subject to a deterministic constraint which limits the cost of selection [29].

The social network is modeled as a directed graph  $G = (V, E)$  where each node represents a user, and each edge  $(u, v) \in E$  has been assigned an edge probability  $p_{u,v}$  that user  $u$  influences user  $v$ . The aim of the IM problem is to find a subset  $X \subseteq V$  such that the expected number of activated nodes  $E[I(X)]$  of  $X$  is maximized. Given a cost function  $c: V \rightarrow \mathbb{R}^+$  and a budget  $C \geq 0$ , the corresponding submodular optimization problem under chance constraints is given as

$$\arg \max_{X \subseteq V} E[I(X)] \text{ s.t. } \Pr[c(X) > C] \leq \alpha.$$

For influence maximization, we consider uniform cost constraints where each node has expected cost 1. The expected cost of a solution is therefore  $E_W(X) = |X|$ .

In order to evaluate the algorithms on the chance constrained influence maximization problem, we use a synthetic data set with 400 nodes and 1594 edges [29].

Table 1 shows the results obtained by GA, GSEMO, and NSGA-II for the combinations of  $\alpha$  and  $\delta$ . The results show that GSEMO obtains the highest mean values compared to the results obtained by GA and NSGA-II. Furthermore, the statistical tests show that for most of the combinations of  $\alpha$  and  $\delta$  GSEMO

and NSGA-II significantly outperform GA. The solutions obtained by GSEMO have significantly better performance than NSGA-II in the case of a high budget i.e. for  $C = 100$ . A possible explanation for this is that the relatively small population size of NSGA-II does not allow one to construct solutions in a greedy fashion, as is possible for GA and GSEMO.

### 4.3 The Maximum Coverage Problem

The maximum coverage problem [17,8] is an important NP-hard submodular optimisation problem. We consider the chance constrained version of the problem. Given a set  $U$  of elements, a collection  $V = \{S_1, S_2, \dots, S_n\}$  of subsets of  $U$ , a cost function  $c: 2^V \rightarrow \mathbb{R}^+$ , and a budget  $C$ , the goal is to find

$$\arg \max_{X \subseteq V} \{f(X) = |\cup_{S_i \in X} S_i| \text{ s.t. } \Pr(c(X) > C) \leq \alpha\}.$$

We consider linear cost functions. For the uniform case each set  $S_i$  has an expected cost of 1 and we have  $E_W(X) = |\{i \mid S_i \in X\}|$ .

For our experiments, we investigate maximum coverage instances based on graphs. The  $U$  elements consist of the vertices of the graph and for each vertex, we generate a set which contains the vertex itself and its adjacent vertices. For the chance constrained maximum coverage problem, we use the graphs frb30-15-01 (450 nodes, 17 827 edges) and frb35-17-01 (595 nodes and 27 856 edges) from [28].

The experimental results are shown in Table 2. It can be observed that GSEMO obtains the highest mean value for each setting. Furthermore, GSEMO statistically outperforms GA for most of the settings. For the other settings, there is no statistically significant difference in terms of the results for GSEMO and GA. NSGA-II is outperforming GA for most of the examined settings and the majority of the results are statistically significant. However, NSGA-II performs worse than GA for frb35-17-01 when  $C = 20$  and  $\alpha = 0.1$ .

## 5 Conclusions

Chance constraints involve stochastic components and require a constraint only to be violated with a small probability. We carried out a first runtime analysis of evolutionary algorithms for the optimisation of submodular functions with chance constraints. Our results show that GSEMO using a multi-objective formulation of the problem based on tail inequalities is able to achieve the same approximation guarantee as recently studied greedy approaches. Furthermore, our experimental results show that GSEMO computes significantly better solutions than the greedy approach and often outperforms NSGA-II.

For future work, it would be interesting to analyse other probability distributions for chance constrained submodular functions. A next step would be to examine uniform weights with a different dispersion and obtain results for uniform weights with the same dispersion when using the fitness function  $g$  instead of  $\hat{g}$ .

**Table 2.** Results for Maximum Coverage with uniform chance constraints for graphs frb30-15-01 (rows 1-12) and frb35-17-01 dataset (rows 13-24).

$C$	$\alpha$	$\delta$	GA (1)					GSEMO (2)					NSGA-II (3)					
			mean	min	max	std	stat	mean	min	max	std	stat	mean	min	max	std	stat	
10	0.1	0.5	371.00	<b>377.23</b>	371.00	379.00	1.8323	1 <sup>(+)</sup>	376.00	371.00	379.00	2.5596	1 <sup>(+)</sup>					
	0.1	1.0	321.00	<b>321.80</b>	321.00	325.00	1.5625	1 <sup>(+)</sup>	321.47	321.00	325.00	1.2521						
15	0.1	0.5	431.00	<b>439.60</b>	435.00	442.00	1.7340	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	437.57	434.00	441.00	1.7555	1 <sup>(+)</sup> , 2 <sup>(-)</sup>					
	0.1	1.0	403.00	<b>411.57</b>	408.00	414.00	1.7750	1 <sup>(+)</sup>	410.67	404.00	414.00	2.5098	1 <sup>(+)</sup>					
20	0.1	0.5	446.00	<b>450.07</b>	448.00	451.00	0.8277	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	448.27	445.00	451.00	1.3113	1 <sup>(+)</sup> , 2 <sup>(-)</sup>					
	0.1	1.0	437.00	<b>443.87</b>	441.00	446.00	1.2794	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	441.37	438.00	444.00	1.6914	1 <sup>(+)</sup> , 2 <sup>(-)</sup>					
10	0.001	0.5	348.00	<b>352.17</b>	348.00	355.00	2.4081	1 <sup>(+)</sup>	350.80	348.00	355.00	2.8935	1 <sup>(+)</sup>					
	0.001	1.0	321.00	<b>321.67</b>	321.00	325.00	1.5162	1 <sup>(+)</sup>	321.33	321.00	325.00	1.0613						
15	0.001	0.5	414.00	<b>423.90</b>	416.00	426.00	2.4824	1 <sup>(+)</sup>	422.67	419.00	426.00	2.2489	1 <sup>(+)</sup>					
	0.001	1.0	371.00	<b>376.77</b>	371.00	379.00	1.8134	1 <sup>(+)</sup>	376.33	371.00	379.00	2.6824	1 <sup>(+)</sup>					
20	0.001	0.5	437.00	<b>443.53</b>	440.00	445.00	1.1958	1 <sup>(+)</sup> , 3 <sup>(+)</sup>	440.23	437.00	443.00	1.6955	1 <sup>(+)</sup> , 2 <sup>(-)</sup>					
	0.001	1.0	414.00	<b>424.00</b>	420.00	426.00	1.7221	1 <sup>(+)</sup>	422.50	417.00	426.00	2.5291	1 <sup>(+)</sup>					
10	0.1	0.5	448.00	<b>458.80</b>	451.00	461.00	3.3156	1 <sup>(+)</sup>	457.97	449.00	461.00	4.1480	1 <sup>(+)</sup>					
	0.1	1.0	376.00	<b>383.33</b>	379.00	384.00	1.7555	1 <sup>(+)</sup>	382.90	379.00	384.00	2.0060	1 <sup>(+)</sup>					
15	0.1	0.5	559.00	<b>559.33</b>	555.00	562.00	2.0057	3 <sup>(+)</sup>	557.23	551.00	561.00	2.4309	1 <sup>(-)</sup> , 2 <sup>(-)</sup>					
	0.1	1.0	503.00	<b>507.80</b>	503.00	509.00	1.1567	1 <sup>(+)</sup>	507.23	502.00	509.00	1.8323	1 <sup>(+)</sup>					
20	0.1	0.5	587.00	<b>587.20</b>	585.00	589.00	1.2149	3 <sup>(+)</sup>	583.90	580.00	588.00	1.9360	1 <sup>(-)</sup> , 2 <sup>(-)</sup>					
	0.1	1.0	569.00	<b>569.13</b>	566.00	572.00	1.4559	3 <sup>(+)</sup>	565.30	560.00	569.00	2.1520	1 <sup>(-)</sup> , 2 <sup>(-)</sup>					
10	0.001	0.5	413.00	<b>423.67</b>	418.00	425.00	1.8815	1 <sup>(+)</sup>	422.27	416.00	425.00	2.6121	1 <sup>(+)</sup>					
	0.001	1.0	376.00	<b>383.70</b>	379.00	384.00	1.1492	1 <sup>(+)</sup>	381.73	377.00	384.00	2.6514	1 <sup>(+)</sup>					
15	0.001	0.5	526.00	<b>527.97</b>	525.00	532.00	2.1573	1 <sup>(+)</sup>	527.30	520.00	532.00	2.7436						
	0.001	1.0	448.00	<b>458.87</b>	453.00	461.00	2.9564	1 <sup>(+)</sup>	457.10	449.00	461.00	4.1469	1 <sup>(+)</sup>					
20	0.001	0.5	568.00	<b>568.87</b>	565.00	572.00	1.5025	3 <sup>(+)</sup>	564.60	560.00	570.00	2.7618	1 <sup>(-)</sup> , 2 <sup>(-)</sup>					
	0.001	1.0	526.00	<b>528.03</b>	525.00	530.00	1.8843	1 <sup>(+)</sup>	527.07	522.00	530.00	2.2427						

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