# On the Area Requirements of Planar Greedy Drawings of Triconnected Planar Graphs* 

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#### Abstract

In this paper we study the area requirements of planar greedy drawings of triconnected planar graphs. Cao, Strelzoff, and Sun exhibited a family $\mathcal{H}$ of subdivisions of triconnected plane graphs and claimed that every planar greedy drawing of the graphs in $\mathcal{H}$ respecting the prescribed plane embedding requires exponential area. However, we show that every $n$-vertex graph in $\mathcal{H}$ actually has a planar greedy drawing respecting the prescribed plane embedding on an $O(n) \times O(n)$ grid. This reopens the question whether triconnected planar graphs admit planar greedy drawings on a polynomial-size grid. Further, we provide evidence for a positive answer to the above question by proving that every $n$-vertex Halin graph admits a planar greedy drawing on an $O(n) \times O(n)$ grid. Both such results are obtained by actually constructing drawings that are convex and anglemonotone. Finally, we consider $\alpha$-Schnyder drawings, which are angle-monotone and hence greedy if $\alpha \leq 30^{\circ}$, and show that there exist planar triangulations for which every $\alpha$-Schnyder drawing with a fixed $\alpha<60^{\circ}$ requires exponential area for any resolution rule.


## 1 Introduction

Let $(M, d)$ be a geometric metric space, where $M$ is a set of points and $d$ is a metric on $M$. A greedy embedding of a graph $G$ into $(M, d)$ is a function $\phi$ that maps each vertex $v$ of $G$ to a point $\phi(v)$ in $M$ in such a way that, for every ordered pair $(u, v)$ of vertices of $G$, there is a distance-decreasing path from $u$ to $v$ in $G$, i.e., a path $\left(u=w_{1}, w_{2}, \ldots, w_{k}=v\right)$ such that $d\left(\phi\left(w_{i}\right), \phi(v)\right)>d\left(\phi\left(w_{i+1}\right), \phi(v)\right)$, for $i=1, \ldots, k-1$. Greedy embeddings, introduced by Rao et al. [22], support a simple and local routing scheme, called greedy routing, in which a vertex forwards a packet to any neighbor that is closer to the packet's destination than itself. In order for greedy routing to be efficient, a greedy embedding should be succinct, i.e., a polylogarithmic number of bits should be used to store the coordinates of each vertex. A number of algorithms have been proposed to construct succinct greedy embeddings of graphs $[10,12,13,17,24,25]$. Notably, every graph admits a succinct greedy embedding into the hyperbolic plane [10]. A natural choice is the one of considering $M$ to be the Euclidean plane $\mathbb{R}^{2}$ and $d$ to be the Euclidean distance $\ell_{2}$. Within this setting, not every graph [20,21], and not even every binary tree [14,19], admits a greedy embedding; further, there exist trees whose every greedy embedding requires a polynomial number of bits to store the coordinates of some of the vertices [2].

From a theoretical point of view, most research efforts have revolved around two conjectures posed by Papadimitriou and Ratajczak [20,21]. The first one asserts that every 3-connected planar graph admits a greedy drawing, i.e., a straight-line drawing in $\mathbb{R}^{2}$ that induces a greedy embedding into $\left(\mathbb{R}^{2}, \ell_{2}\right)$. This conjecture has been confirmed independently by Leighton and Moitra [14] and by Angelini et al. [3]. The second conjecture, which strengthens the first one, asserts that every 3-connected planar graph admits a greedy drawing that is also convex. While this conjecture is still open, it has been recently proved by the authors of this paper that every 3 -connected planar graph admits a planar greedy drawing [7].

An interesting question is whether succinctness and planarity can be achieved simultaneously. That is, does every 3 -connected planar graph admit a planar, and possibly convex, greedy drawing on a polynomial-size grid? Cao, Strelzoff, and Sun [6] claimed a negative answer by exhibiting a family $\mathcal{H}$ of subdivisions of 3 -connected plane graphs and by showing that, for any $n$-vertex graph in $\mathcal{H}$, any planar greedy drawing that respects the prescribed plane embedding requires $2^{\Omega(n)}$ area and hence $\Omega(n)$ bits for representing the coordinates of some vertices.

[^0]Subsequently to the definition of greedy drawings, a number of more constrained graph drawing standards have been introduced and studied. Analogously to greedy drawings, they all concern straightline drawings in $\mathbb{R}^{2}$. In a self-approaching drawing $[1,8,18]$, for every pair of vertices $u$ and $v$, there is a self-approaching path from $u$ to $v$, i.e., a path $P$ such that $\ell_{2}(a, c)>\ell_{2}(b, c)$, for any three points $a, b$, and $c$ in this order along $P$. In an increasing-chord drawing [ $1,8,18$ ], for every pair of vertices $u$ and $v$, there is a path from $u$ to $v$ which is self-approaching both from $u$ to $v$ and from $v$ to $u$. In an anglemonotone drawing [4,8,16,15], for every pair of vertices $u$ and $v$, there exists a $\beta$-monotone path from $u$ to $v$ for some angle $\beta$, i.e., a path $P=\left(w_{1}=u, w_{2}, \ldots, w_{k}=v\right)$ such that, for each $i=1, \ldots, k-1$, the edge $\left(w_{i}, w_{i+1}\right)$ lies in the closed $90^{\circ}$-wedge centered at $w_{i}$ and bisected by the ray originating at $w_{i}$ with slope $\beta$. Note that an angle-monotone drawing is increasing-chord, an increasing-chord drawing is self-approaching, and a self-approaching drawing is greedy. The first implication was proved in [8], while the other two descend from the definitions. Finally, a notable class of planar straight-line drawings are $\alpha$-Schnyder drawings [18], which are angle-monotone if $\alpha \leq 30^{\circ}$ and will be formally defined later.

Our contributions. We show that every $n$-vertex graph in the family $\mathcal{H}$ defined by Cao et al. [6] actually admits a convex angle-monotone drawing that respects the prescribed plane embedding and that lies on an $O(n) \times O(n)$ grid. This refutes their claim that every planar greedy drawing of an $n$ vertex graph in $\mathcal{H}$ requires $\Omega(n)$ bits for representing the coordinates of some vertices and reopens the question about the existence of succinct planar greedy drawings of 3 -connected planar graphs. Further, we provide an indication that this question might have a positive answer by proving that the $n$-vertex Halin graphs, a notable family of triconnected planar graphs, admit convex angle-monotone drawings on an $O(n) \times O(n)$ grid. Finally, we show that there exist bounded-degree planar triangulations whose every $\alpha$-Schnyder drawing requires exponential area, for any fixed $\alpha<60^{\circ}$. This result was rather surprising to us, as any planar triangulation admits a $60^{\circ}$-Schnyder drawing on an $O(n) \times O(n)$ grid [23]; further, although $30^{\circ}$-Schnyder drawings have been proved to exist for all stacked triangulations, our result shows that they are not the right tool to obtain succinct planar greedy drawings.

## 2 Definitions and Preliminaries

A straight-line drawing of a graph maps each vertex to a point in the plane and each edge to a straight-line segment between its end-points. A drawing is planar if no two edges cross. A planar drawing partitions the plane into connected regions, called faces. The only unbounded face is the outer face; the other faces are internal. Two planar drawings of the same connected planar graph are equivalent if they determine the same circular order of the edges incident to each vertex. A planar embedding is an equivalence class of planar drawings. A plane graph is a planar graph equipped with a planar embedding and a designated outer face. A straight-line drawing is convex if it is planar and every face is delimited by a convex polygon. A grid drawing is such that each vertex is mapped to a point with integer coordinates. The width (resp. height) of a grid drawing is the number of grid columns (rows) intersecting it. We say that a drawing lies on a $W \times H$ grid if it is a grid drawing with width $W$ and height $H$. The area of a graph drawing is usually defined as the area of the smallest axis-parallel rectangle enclosing the drawing (when proving upper bounds) or as the area of the smallest convex polygon enclosing the drawing (when proving lower bounds). Any constraint implying a finite minimum area for a graph drawing is called a resolution rule.

From here on out, we measure angles in radians. In a straight-line drawing of a graph, the slope of an edge $(u, v)$ is the angle spanned by a counter-clockwise rotation around $u$ of a ray originating at $u$ and directed rightwards bringing the ray to overlap with $(u, v)$; hence, the edge slopes are in the range $[0,2 \pi)$. We denote by $(\mathrm{x}(v), \mathrm{y}(v))$ the point in the plane representing a vertex $v$ in a drawing of a graph.

A planar triangulation $G$ is a plane graph whose every face is bounded by a 3 -cycle. Denote by $\left(a_{1}, a_{2}, a_{3}\right)$ the 3-cycle bounding the outer face of $G$. A Schnyder wood $\left(\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{T}_{3}\right)$ of $G$ is an assignment of directions and colors 1,2 and 3 to the internal edges of $G$ such that the following two properties hold; see the figure below and refer to [23]. Let $i-1=3$, if $i=1$, and let $i+1=1$, if $i=3$.

Property (1) Each internal vertex $v$ has one outgoing edge $e_{i}$ of each color $i$, with $i=1,2,3$. The outgoing edges $e_{1}, e_{2}$, and $e_{3}$ appear in this clockwise order at $v$. Further, all the incoming edges of color $i$ appear in the clockwise sector between the edges $e_{i+1}$ and $e_{i-1}$.
Property (2) At the external vertex $a_{i}$, all the internal edges are incoming and of color $i$.


Property (1)


Property (2)

For $0<\alpha \leq \frac{\pi}{3}$, a planar straight-line drawing of $G$ is an $\alpha$-Schnyder drawing if, for each internal vertex $v$ of $G$, its outgoing edge in $\mathcal{T}_{1}$ has direction in [ $\frac{\pi}{2}-\frac{\alpha}{2}, \frac{\pi}{2}+\frac{\alpha}{2}$ ], its outgoing edge in $\mathcal{T}_{2}$ has direction in $\left[\frac{11 \pi}{6}-\frac{\alpha}{2}, \frac{11 \pi}{6}+\frac{\alpha}{2}\right]$, and its outgoing edge in $\mathcal{T}_{3}$ has direction in $\left[\frac{7 \pi}{6}-\frac{\alpha}{2}, \frac{7 \pi}{6}+\frac{\alpha}{2}\right]$. Observe that, by definition, in an $\alpha$-Schnyder drawing, for each internal vertex $v$ of $G$, its incoming edges in $\mathcal{T}_{1}$, if any, have direction in $\left[\frac{3 \pi}{2}-\frac{\alpha}{2}, \frac{3 \pi}{2}+\frac{\alpha}{2}\right]$, its incoming edges in $\mathcal{T}_{2}$, if any, have direction in $\left[\frac{5 \pi}{6}-\frac{\alpha}{2}, \frac{5 \pi}{6}+\frac{\alpha}{2}\right]$, and its incoming edges in $\mathcal{T}_{3}$, if any, have direction in $\left[\frac{\pi}{6}-\frac{\alpha}{2}, \frac{\pi}{6}+\frac{\alpha}{2}\right]$. Fig. 3 b shows the angular widths of an $\alpha$-Schnyder drawing. "Usual" Schnyder drawings [23] are $60^{\circ}$-Schnyder drawings; see, e.g., [9].

## 3 Angle-Monotone Drawings of Cao-Strelzoff-Sun Graphs

Cao et al. [6] defined the following family $\mathcal{H}$ of plane graphs. For every integer $i \geq 1$, the plane graph $\mathfrak{H}_{i} \in \mathcal{H}$ on $3 i+4$ vertices is inductively defined as follows:

- The plane graph $\mathfrak{H}_{1}$ is composed of a cycle $\left(x_{2}, z_{1}, y_{2}, x_{1}, z_{2}, y_{1}\right)$ and of a vertex $x_{0}$ embedded inside such a cycle and adjacent to $x_{1}, y_{1}$, and $z_{1}$; see the left part of the figure.
- For $i \geq 2$, the plane graph $\mathfrak{H}_{i}$ is obtained by embedding in the outer face of $\mathfrak{H}_{i-1}$ the vertices $x_{i+1}$, $y_{i+1}$, and $z_{i+1}$, and the edges of the cycle $\left(x_{i+1}, z_{i}, y_{i+1}, x_{i}, z_{i+1}, y_{i}\right)$, which bounds the outer face of $\mathfrak{H}_{i}$; see the right part of the figure.

In contrast to the result in [6], we prove the following. ${ }^{1}$
Theorem 1. Every $n$-vertex plane graph in $\mathcal{H}$ admits a planar angle-monotone drawing on an $O(n) \times O(n)$ grid that respects the plane embedding.

Proof. In order to prove the statement, we construct, for every $i \geq 1$, a planar straight-line drawing $\Gamma_{i}$ of $\mathfrak{H}_{i}=\left(V_{i}, E_{i}\right)$ satisfying the following properties:

(i) the vertices of $\mathfrak{H}_{i}$ lie on an $(2 i+3) \times(2 i+3)$ grid;
(ii) there exist paths $p_{i}(\alpha)$, with $\alpha \in\left\{\frac{\pi}{2}, \frac{5 \pi}{4}, \frac{7 \pi}{4}\right\}$, originating at $x_{0}$ and each terminating at a distinct vertex in $\left\{x_{i+1}, y_{i+1}, z_{i+1}\right\}$, that are vertex-disjoint except at $x_{0}$, that together span all the vertices in $V_{i}$, and such that all the edges in $p_{i}(\alpha)$ have slope $\alpha$.

Property ii implies that $\Gamma_{i}$ is angle-monotone. Namely, consider any two vertices $u$ and $v$ of $\mathfrak{H}_{i}$. If both $u$ and $v$ belong to the same path $p_{i}(\alpha)$, then the subpath of $p_{i}(\alpha)$ from $u$ to $v$ is either $\alpha$-monotone or $(\pi+\alpha)$-monotone. If $u \in p_{i}(\alpha)$ and $v \in p_{i}(\beta)$, with $\alpha \neq \beta$, then the path $p^{*}$ consisting of the subpath of $p_{i}(\alpha)$ from $u$ to $x_{0}$ and of the subpath of $p_{i}(\beta)$ from $x_{0}$ to $v$ is $\frac{\pi}{2}$-monotone (if $\beta=\frac{\pi}{2}$ ), or $\frac{3 \pi}{2}$-monotone (if $\alpha=\frac{\pi}{2}$ ), or $\pi$-monotone (if $\alpha=\frac{7 \pi}{4}$ and $\beta=\frac{5 \pi}{4}$ ), or 0 -monotone (if $\alpha=\frac{5 \pi}{4}$ and $\beta=\frac{7 \pi}{4}$ ).

Our proof is by induction on $i$.
Base case. If $i=1$, we construct a drawing $\Gamma_{1}$ of $\mathfrak{H}_{1}$ as follows; refer to Fig. 1a. We place the vertex $x_{0}$ at the point $(0,0)$, the vertices $x_{1}, y_{1}$, and $z_{1}$ at the points $(0,1),(1,-1)$, and $(-1,-1)$, respectively, and the vertices $x_{2}, y_{2}$, and $z_{2}$ at the points $(-2,-2),(0,2)$, and $(2,-2)$, respectively, and draw the edges of $\mathfrak{H}_{1}$ as straight-line segments. By construction, $\Gamma_{1}$ is a planar straight-line grid drawing of $\mathfrak{H}_{1}$ on the $5 \times 5$ grid, thus satisfying Property i. Further, the paths $p_{1}\left(\frac{\pi}{2}\right)=\left(x_{0}, x_{1}, y_{2}\right), p_{1}\left(\frac{5 \pi}{4}\right)=\left(x_{0}, z_{1}, x_{2}\right)$, and $p_{1}\left(\frac{7 \pi}{4}\right)=\left(x_{0}, y_{1}, z_{2}\right)$ show that Property ii is satisfied by $\Gamma_{1}$.

[^1]

Fig. 1: Illustrations for the proof of Theorem 1: (a) The drawing $\Gamma_{1}$ of $\mathfrak{H}_{1}$; (b) The drawing $\Gamma_{i}$ of $\mathfrak{H}_{i}$ obtained from the drawing $\Gamma_{i-1}$ of $\mathfrak{H}_{i-1}$, with $i>1$. (d) The convex angle-monotone drawing $\Gamma_{3}^{\prime}$ of $\mathfrak{H}_{3}$ obtained from (c) $\Gamma_{3}$.

Inductive case. If $i>1$, suppose to have inductively constructed a drawing $\Gamma_{i-1}$ of $\mathfrak{H}_{i-1}$ satisfying Properties i and ii. Assume, as in Fig. 1b, that $z_{i}$ is in $p_{i-1}\left(\frac{\pi}{2}\right)$, that $y_{i}$ is in $p_{i-1}\left(\frac{5 \pi}{4}\right)$, and that $x_{i}$ is in $p_{i-1}\left(\frac{7 \pi}{4}\right)$; the other cases can be treated analogously.

We obtain $\Gamma_{i}$ from $\Gamma_{i-1}$ by placing $x_{i+1}$ at the point $\left(\mathrm{x}\left(z_{i}\right), \mathrm{y}\left(z_{i}\right)+1\right), y_{i+1}$ at the point $\left(\mathrm{x}\left(x_{i}\right)+\right.$ $\left.1, \mathrm{y}\left(x_{i}\right)-1\right)$, and $z_{i+1}$ at the point $\left(\mathrm{x}\left(y_{i}\right)-1, \mathrm{y}\left(y_{i}\right)-1\right)$, and by drawing the edges incident to these vertices as straight-line segments. We have the following.

Claim 1. $\Gamma_{i}$ satisfies Properties i and ii.
Proof. First observe that, since $\Gamma_{i-1}$ is a grid drawing, by induction, we have that $x_{i+1}, y_{i+1}$, and $z_{i+1}$ have integer coordinates, hence $\Gamma_{i}$ is a grid drawing as well. By Property ii of $\Gamma_{i-1}$, all the vertices of $\mathfrak{H}_{i-1}$ lie on the straight-line segments connecting $x_{0}$ with $x_{i}, y_{i}$, and $z_{i}$. Hence, $\Gamma_{i-1}$ lies inside the triangle $\Delta_{i-1}$ with vertices $x_{i}, y_{i}$, and $z_{i}$. On the other hand, the edges incident to $x_{i+1}, y_{i+1}$, and $z_{i+1}$ lie in the exterior of $\Delta_{i-1}$ in $\Gamma_{i}$ (except, possibly, for their endpoints); since these edges do not cross each other, we have that $\Gamma_{i}$ is planar.

We prove that $\Gamma_{i}$ satisfies Property i. By construction, $\Gamma_{i}$ intersects two more grid rows and two more grid columns than $\Gamma_{i-1}$, hence it lies on the $(2 i+3) \times(2 i+3)$ grid, since $\Gamma_{i-1}$ lies on the $(2 i+1) \times(2 i+1)$ grid, by induction.

We prove that $\Gamma_{i}$ satisfies Property ii. Define the paths $p_{i}\left(\frac{\pi}{2}\right)=p_{i-1}\left(\frac{\pi}{2}\right) \cup\left(z_{i}, x_{i+1}\right), p_{i}\left(\frac{5 \pi}{4}\right)=$ $p_{i-1}\left(\frac{5 \pi}{4}\right) \cup\left(y_{i}, z_{i+1}\right)$, and $p_{i}\left(\frac{7 \pi}{4}\right)=p_{i-1}\left(\frac{7 \pi}{4}\right) \cup\left(x_{i}, y_{i+1}\right)$. We have that $p_{i}\left(\frac{\pi}{2}\right), p_{i}\left(\frac{5 \pi}{4}\right)$, and $p_{i}\left(\frac{7 \pi}{4}\right)$ originate at $x_{0}$ and are vertex-disjoint except at $x_{0}$, since $p_{i-1}\left(\frac{\pi}{2}\right), p_{i-1}\left(\frac{5 \pi}{4}\right)$, and $p_{i-1}\left(\frac{7 \pi}{4}\right)$ satisfy the same properties, by induction. By construction, each of $p_{i}\left(\frac{\pi}{2}\right), p_{i}\left(\frac{5 \pi}{4}\right)$, and $p_{i}\left(\frac{7 \pi}{4}\right)$ terminates at a distinct vertex in $\left\{x_{i+1}, y_{i+1}, z_{i+1}\right\}$. The paths $p_{i}\left(\frac{\pi}{2}\right), p_{i}\left(\frac{5 \pi}{4}\right)$, and $p_{i}\left(\frac{7 \pi}{4}\right)$ together span all the vertices in $V_{i}$ since $p_{i-1}\left(\frac{\pi}{2}\right)$, $p_{i-1}\left(\frac{5 \pi}{4}\right)$, and $p_{i-1}\left(\frac{7 \pi}{4}\right)$ together span all the vertices in $V_{i-1}$ and $V_{i}=V_{i-1} \cup\left\{x_{i+1}, y_{i+1}, z_{i+1}\right\}$. Finally, the edges of $p_{i}\left(\frac{\pi}{2}\right), p_{i}\left(\frac{5 \pi}{4}\right)$, and $p_{i}\left(\frac{7 \pi}{4}\right)$ have slope $\frac{\pi}{2}, \frac{5 \pi}{4}$, and $\frac{7 \pi}{4}$ in $\Gamma_{i}$, respectively, since by induction the edges of $p_{i-1}\left(\frac{\pi}{2}\right), p_{i-1}\left(\frac{5 \pi}{4}\right)$, and $p_{i-1}\left(\frac{7 \pi}{4}\right)$ have slope $\frac{\pi}{2}, \frac{5 \pi}{4}$, and $\frac{7 \pi}{4}$ in $\Gamma_{i-1}$, respectively, and since, by construction, the edges $\left(z_{i}, x_{i+1}\right),\left(y_{i}, z_{i+1}\right)$, and $\left(x_{i}, y_{i+1}\right)$ have slope $\frac{\pi}{2}, \frac{5 \pi}{4}$, and $\frac{7 \pi}{4}$ in $\Gamma_{i}$, respectively.

Claim 1 concludes the induction and the proof of the theorem.
We note that, for $i \geq 1$, the graph $\mathfrak{H}_{i}$ even admits a convex angle-monotone drawing $\Gamma_{i}^{\prime}$ on an $(2 i+3) \times(2 i+3)$ grid; indeed, $\Gamma_{i}^{\prime}$ can be obtained from the planar angle-monotone drawing $\Gamma_{i}$ of $\mathfrak{H}_{i}$ described in the proof of Theorem 1 by moving $x_{i}$ one unit to the right and one unit down, $y_{i+1}$ and $z_{i+1}$ one unit to the right, and $x_{i+1}$ one unit to the left; see Figs. 1c and 1d. We have the following.

Claim 2. $\Gamma_{i}^{\prime}$ is a convex angle-monotone drawing of $\mathfrak{H}_{i}$ on an $(2 i+3) \times(2 i+3)$ grid.
Proof. It is easy to see that $\Gamma_{i}^{\prime}$ is a convex drawing of $\mathfrak{H}_{i}$ on an $(2 i+3) \times(2 i+3)$ grid. We prove that $\Gamma_{i}^{\prime}$ is angle-monotone. Consider the following three paths:

$$
-P_{1}=p_{i}\left(\frac{\pi}{2}\right) \cup p_{i}\left(\frac{5 \pi}{4}\right) ;
$$

$$
\begin{aligned}
& -P_{2}=p_{i}\left(\frac{\pi}{2}\right) \cup p_{i}\left(\frac{7 \pi}{4}\right) ; \text { and } \\
& -P_{3}=p_{i-1}\left(\frac{5 \pi}{4}\right) \cup p_{i}\left(\frac{7 \pi}{4}\right) .
\end{aligned}
$$

If $u$ and $v$ both belong to the path $P_{1}, P_{2}$, or $P_{3}$, then the subpath of such a path from $u$ to $v$ is

- $\frac{\pi}{2}$-monotone or $\frac{3 \pi}{2}$-monotone,
- $\frac{3 \pi}{4}$-monotone or $\frac{7 \pi}{4}$-monotone, or
- 0-monotone or $\pi$-monotone, respectively.

Note that $u$ and $v$ both belong to one of $P_{1}, P_{2}$, or $P_{3}$, unless one of them, say $u$, is $z_{i+1}$ and the other one, say $v$, belongs to $p_{i}\left(\frac{7 \pi}{4}\right)$. In such a case, a $\beta$-monotone path $P$ from $u$ to $v$ can be defined as follows. If $v=x_{i}$, then $P$ coincides with the edge $\left(z_{i+1}, x_{i}\right)$; if $v=y_{i+1}$, then $P$ coincides with the path $\left(z_{i+1}, x_{i}, y_{i+1}\right)$; in both cases, $P$ is 0 -monotone. Finally, if $v$ belongs to $p_{i-2}\left(\frac{7 \pi}{4}\right)$, then $P$ is defined as the subpath of $p_{i}\left(\frac{5 \pi}{4}\right)$ from $u$ to the only neighbor of $v$ in $p_{i}\left(\frac{5 \pi}{4}\right)$, and from that neighbor to $v$; then $P$ is $\frac{\pi}{4}$-monotone.

He and Zhang [13] pointed out that, although the graphs $\mathfrak{H}_{i}$ 's are not 3-connected, they can be made so by adding the three additional edges $\left(x_{i+1}, y_{i+1}\right),\left(y_{i+1}, z_{i+1}\right)$, and $\left(z_{i+1}, x_{i+1}\right)$. Let $\mathfrak{H}_{i}^{+}$be the resulting graph. We note here that the drawing $\Gamma_{i}$ of $\mathfrak{H}_{i}$ whose construction is described in the proof of Theorem 1 can be turned into a convex angle-monotone drawing $\Gamma_{i}^{+}$of $\mathfrak{H}_{i}^{+}$simply by drawing the edges $\left(x_{i+1}, y_{i+1}\right),\left(y_{i+1}, z_{i+1}\right)$, and $\left(z_{i+1}, x_{i+1}\right)$ as straight-line segments.

## 4 Angle-Monotone Drawings of Halin Graphs

In this section, we show how to construct convex angle-monotone drawings of Halin graphs on a polynomialsize grid.

We denote the number of leaves of a tree $T$ by $\ell(T)$. A tree whose all vertices but one are leaves is a star. A rooted tree $T$ is a tree with one distinguished vertex, called root and denoted by $r(T)$. The height of a rooted tree is the maximum number of edges in any path from the root to a leaf. In a rooted tree $T$, we denote by $T(v)$ the subtree of $T$ rooted at a vertex $v$. An ordered rooted tree is a rooted tree in which the children of each internal vertex $u$ are assigned a left-to-right order $u_{1}, \ldots, u_{k}$; the vertices $u_{1}$ and $u_{k}$ are the leftmost and the rightmost child of $u$, respectively. The leftmost path of an ordered rooted tree $T$ is the path $\left(v_{1}, \ldots, v_{h}\right)$ in $T$ such that $v_{1}$ is the root of $T, v_{i+1}$ is the leftmost child of $v_{i}$, for $i=1, \ldots, h-1$, and $v_{h}$ is a leaf, which is called the leftmost leaf of $T$. The rightmost path and the rightmost leaf of $T$ can be defined analogously.

A Halin graph $G$ is a 3 -connected planar graph that admits a plane embedding $\mathcal{E}$ such that, by removing all the edges incident to the outer face $f_{\mathcal{E}}$ of $\mathcal{E}$, one gets a tree $T_{G}$ whose internal vertices have degree at least 3 and whose leaves are incident to $f_{\mathcal{E}}$. We have the following main result.

Theorem 2. Every $n$-vertex Halin graph $G$ admits a convex angle-monotone drawing on an $O(n) \times O(n)$ grid.

If $T_{G}$ contains one internal vertex, then $G$ is a wheel and a convex angle-monotone drawing on a $3 \times(n-1)$ grid can easily be computed; refer to Fig. 2a(top). In the following, we assume that $T_{G}$ contains at least two internal vertices.

Let $\xi$ be an internal vertex of $T_{G}$ whose every neighbor is a leaf, except for one, which we denote by $\rho$; see Fig. 2b. Such a vertex exists by the above assumption. Further, let $T \subset T_{G}$ be the tree obtained from $T_{G}$ by removing $\xi$ and all its adjacent leaves and by rooting the resulting tree at $\rho$. Also, let $S \subset T_{G}$ be the star obtained from $T_{G}$ by removing the vertices of $T$ and by rooting the resulting tree at $\xi$. We regard $T$ and $S$ as ordered rooted trees such that the left-to-right order of the children of each vertex is the one induced by the plane embedding $\mathcal{E}$ of $G$. For any subtree $T^{\prime} \subseteq T_{G}$, let $G\left[T^{\prime}\right]$ be the subgraph of $G$ induced by the vertices of $T^{\prime}$. In Lemma 1 , we show how to construct a drawing $\Gamma$ of $G[T]$. Then, we will exploit Lemma 1 in order to prove Theorem 2.

Lemma 1. The graph $G[T]$ has a drawing $\Gamma$ satisfying the following properties:
(i) $\Gamma$ is angle-monotone and convex;
(ii) $\Gamma$ lies on a $W_{\Gamma} \times H_{\Gamma}$ grid, where $W_{\Gamma}=2 \ell(T)-1$ and $H_{\Gamma}=\ell(T)$;


Fig. 2: (a) A convex angle-monotone drawing of a wheel on the grid (top) and the base case for the proof of Lemma 1 (bottom). (b) The trees $T$ and $S$ for the proof of Theorem 2. (c) The convex angle-monotone drawing $\Gamma_{G}$ of $G$ constructed from the drawings $\Gamma$ of $G[T]$ and $\overline{\Gamma_{S}}$ of $G[S]$.
(iii) the leaves of $T$ lie at $(0,0),(2,0), \ldots,(2 \ell(T)-2,0)$, where the $i$-th leaf of $T$ lies at $(2 i-2,0)$, for $i=1, \ldots, \ell(T)$; and
(iv) for each vertex $v$ of $T$, the edges of the leftmost path (resp., of the rightmost path) of $T(v)$ have slope $\frac{5 \pi}{4}$ (resp., slope $\frac{7 \pi}{4}$ ).

Proof. Our proof is by induction on the height $h$ of $T$. Recall that $T$ contains at least one internal vertex, hence $h \geq 1$. In the base case, $h=1$, that is, $T$ is a star. Let $v_{1}, \ldots, v_{k}$ be the children of $r(T)$ in left-to-right order and note that $k \geq 2$ since the internal vertices of $T_{G}$ have degree at least 3. Place $v_{1}, \ldots, v_{k}$ at the points $(0,0),(2,0), \ldots,(2 k-2,0)$. Place $r(T)$ at $(k-1, k-1)$. Refer to Fig. 2a(bottom). The resulting straight-line drawing $\Gamma$ of $G[T]$ clearly satisfies Properties i to iv.

Suppose now that $h>1$ and refer to Fig. 2c. Let $T_{1}, \ldots, T_{k}$ be the left-to-right order of the subtrees of $T$ rooted at the children of $r(T)$; as in the base case, we have $k \geq 2$. For each $T_{i}$ which is not a single vertex, assume to have inductively constructed a drawing $\Gamma_{i}$ of $G\left[T_{i}\right]$ satisfying Properties i to iv. For each $T_{i}$ which is a single vertex, let $\Gamma_{i}$ consist of the point $(0,0)$. For $i=1, \ldots, k$, let $W_{i}$ be the width of $\Gamma_{i}$. Place the drawings $\Gamma_{1}, \ldots, \Gamma_{k}$ side by side, so that all their leaves lie on the $x$-axis, so that the leftmost leaf of $T_{1}$ is at $(0,0)$, and so that, for $i=1, \ldots, k-1$, the rightmost leaf of $T_{i}$ is two units to the left of the leftmost leaf of $T_{i+1}$. We conclude the construction of $\Gamma$ by placing $r(T)$ at $(\ell(T)-1, \ell(T)-1)$. We have the following.

Claim 3. $\Gamma$ satisfies Properties ito iv.
Proof. Property iii holds true since it is inductively satisfied by each drawing $\Gamma_{i}$ and since, by construction, the rightmost leaf of $T_{i}$ is two units to the left of the leftmost leaf of $T_{i+1}$, for $i=1, \ldots, k-1$.

Concerning Property ii, we have that $W_{\Gamma}=\sum_{i=1}^{k} W_{\Gamma_{i}}+(k-1)=\sum_{i=1}^{k}\left(2 \ell\left(T_{i}\right)-1\right)+(k-1)=2 \ell(T)-1$, where we exploited $W_{\Gamma_{i}}=2 \ell\left(T_{i}\right)-1$, which is true by induction. Further, by construction and by induction, each vertex of $T_{i}$ has a $y$-coordinate between 0 and $\ell\left(T_{i}\right)-1$. Since $\ell\left(T_{i}\right)<\ell(T)$, the maximum $y$-coordinate of any vertex of $T$ in $\Gamma$ is the one of $r(T)$, hence $H_{\Gamma}=\ell(T)$.

Property iv holds true for each vertex different from $r(T)$ since it is inductively satisfied by each drawing $\Gamma_{i}$. Further, since $W_{\Gamma}=2 \ell(T)-1$, since $r(T)$ lies at $(\ell(T)-1, \ell(T)-1)$, and since the leftmost and rightmost leaves of $T$ lie at $(0,0)$ and $(2 \ell(T)-2,0)$, respectively, the slopes of the segments from $r(T)$ to such leaves are $\frac{5 \pi}{4}$ and $\frac{7 \pi}{4}$, respectively. This implies that the edges of the leftmost path (resp., of the rightmost path) of $T$ have slope $\frac{5 \pi}{4}$ (resp., $\frac{7 \pi}{4}$ ), given that the edges of the leftmost path of $T_{1}$ (resp. of the rightmost path of $T_{k}$ ) have slope $\frac{5 \pi}{4}$ (resp., $\frac{7 \pi}{4}$ ), by induction.

Finally, we prove Property i. We first prove that $\Gamma$ is convex. By induction, each internal face of $\Gamma$ which is also a face of $\Gamma_{i}$, with $i \in\{1, \ldots, k\}$, is delimited by a convex polygon. The outer face of $\Gamma$ is delimited by a triangle, by Properties iii and iv. It remains to prove that each internal face $f$ incident to $r(T)$ is delimited by a convex polygon. Note that $f$ is delimited by the two edges $\left(r(T), r\left(T_{i}\right)\right)$ and $\left(r(T), r\left(T_{i+1}\right)\right)$, for some $i \in\{1, \ldots, k-1\}$, by the rightmost path of $T_{i}$, by the leftmost path of $T_{i+1}$, and by the edge of $G[T]$ connecting the rightmost leaf of $T_{i}$ with the leftmost leaf of $T_{i+1}$.

- The angle of $f$ at $r(T)$ is at most $\frac{\pi}{2}$, by Property iv.
- The angles of $f$ at the internal vertices of the rightmost path of $T_{i}$ or of the leftmost path of $T_{i+1}$ are exactly $\pi$, by Property iv.
- The angle of $f$ at the rightmost leaf of $T_{i}$ (resp., at the leftmost leaf of $T_{i+1}$ ) is $\frac{3 \pi}{4}$ if $T_{i}$ (resp., $T_{i+1}$ ) is not a single vertex or at most $\frac{3 \pi}{4}$ otherwise, by Properties iii and iv.
- The angle of $f$ at $r\left(T_{i}\right)$ is larger than or equal to $\frac{\pi}{2}$ and smaller than $\pi$; namely, the slope of the edge $\left(r(T), r\left(T_{i}\right)\right)$ is in the interval $\left[\frac{5 \pi}{4}, \frac{7 \pi}{4}\right)$, by Property iv and by $i<k$; further, the slope of the edge of the rightmost path of $T_{i}$ incident to $r\left(T_{i}\right)$ is $\frac{7 \pi}{4}$, by Property iv.
- Symmetrically, the angle of $f$ at $r\left(T_{i+1}\right)$ is larger than or equal to $\frac{\pi}{2}$ and smaller than $\pi$.

We now prove that $\Gamma$ is angle-monotone. Let $u$ and $v$ be any two vertices of $T$. If $u$ and $v$ both belong to the same subtree $T_{i}$ of $T$, for some $i \in\{1, \ldots, k\}$, then a $\beta$-monotone path between $u$ and $v$ exists in $\Gamma$ since it exists in $\Gamma_{i}$, by induction. Otherwise, either $u$ and $v$ belong to distinct subtrees $T_{i}$ and $T_{j}$ of $T$, or one of $u$ and $v$ is $r(T)$.

In the former case, suppose w.l.o.g. that $i<j$. Let $P$ be the path from $u$ to $v$ consisting of: (i) the rightmost path $P_{u}$ of $T_{i}(u)$; (ii) the path $P_{u v}$ in $G[T]$ from the rightmost leaf of $T_{i}(u)$ to the leftmost leaf of $T_{j}(v)$ that only passes through leaves of $T$; and (iii) the leftmost path $P_{v}$ of $T_{j}(v)$. Since the edges of $P_{u}$ (which are traversed in the direction of $P_{u}$ ) and those of $P_{v}$ (which are traversed in the direction opposite to the one of $P_{v}$ ) have slope $\frac{7 \pi}{4}$ and $\frac{\pi}{4}$, by Property iv, and the edges of $P_{u v}$ have slope 0 , by Property iii, we have that $P$ is 0 -monotone.

In the latter case, suppose w.l.o.g. that $v=r(T)$ and that $u \in V\left(T_{i}\right)$, for some $i \in\{1, \ldots, k\}$. By Property iv, all the edges of the path from $u$ to $v$ in $T$ have slope in the closed interval $\left[\frac{\pi}{4}, \frac{3 \pi}{4}\right]$, hence such a path is $\frac{\pi}{2}$-monotone.

It follows that $\Gamma$ is angle-monotone; this concludes the proof of the claim.
Claim 3 concludes the proof of the lemma.
We are now ready to prove Theorem 2. We construct a drawing $\Gamma_{G}$ of $G$ as follows; refer to Fig. 2c. First, we initialize $\Gamma_{G}$ to the drawing $\Gamma$ of $G[T]$ obtained by applying Lemma 1. Further, we apply Lemma 1 a second time in order to construct a drawing $\Gamma_{S}$ of $G[S]$. Let $\overline{\Gamma_{S}}$ be the drawing of $G[S]$ obtained by rotating $\Gamma_{S}$ by $\pi$ radians. We translate $\overline{\Gamma_{S}}$ so that $\xi$ lies one unit above $\rho$. Further, we draw the edge $(\rho, \xi)$ as a vertical straight-line segment. Finally, we draw the edge between the leftmost (rightmost) leaf of $S$ and the rightmost (leftmost) leaf of $T$ as a straight-line segment.

We have the following claim, which concludes the proof of Theorem 2.
Claim 4. $\Gamma_{G}$ is a convex angle-monotone drawing of $G$ on an $O(n) \times O(n)$ grid.
Proof. First, Properties ii and iv of Lemma 1 ensure that the width of $\Gamma_{G}$ is equal to $\max (2 \ell(T)-$ $1,2 \ell(S)-1)$ and that the height of $\Gamma_{G}$ is equal to $\ell(T)+\ell(S)$. Both such values are in $O(n)$.

Second, we prove that $\Gamma_{G}$ is convex. Every face of $\Gamma_{G}$ which is also a face of $\Gamma$ or $\overline{\Gamma_{S}}$ is delimited by a convex polygon since $\Gamma$ and $\overline{\Gamma_{S}}$ are convex, by Property i of Lemma 1. Further, by Properties iii and iv of Lemma 1, the outer face of $\Gamma_{G}$ is delimited by an isosceles trapezoid. Finally, consider any face $f$ incident to the edge $(\rho, \xi)$. By Property iv of Lemma 1, the angles of $f$ incident to the internal vertices of the leftmost and rightmost paths of $T$ are equal to $\pi$, hence $f$ is delimited by a quadrilateral $Q$; the angles of $f$ incident to $\rho$ and to $\xi$ are $\frac{3 \pi}{4}$, again by Property iv of Lemma 1 and since the edge $(\rho, \xi)$ is vertical, hence the remaining two angles of $Q$ sum up to $\frac{\pi}{2}$. It follows that $Q$ is convex.

Third, we prove that $\Gamma_{G}$ is angle-monotone. Let $u$ and $v$ be any two vertices of $G$. If $u$ and $v$ both belong to $T$ or both belong to $S$, then a $\beta$-monotone path between $u$ and $v$ exists in $\Gamma_{G}$ since it exists in $\Gamma$ or in $\overline{\Gamma_{S}}$, respectively, by Lemma 1. Otherwise, we can assume that $u$ belongs to $S$ and that $v$ belongs to $T$. Then the path $P$ from $u$ to $v$ in $T_{G}$ is $\frac{3 \pi}{2}$-monotone. Namely, by Property iv of Lemma 1 , the edge of $P$ in $S$, if any, has slope in the interval $\left[\frac{5 \pi}{4}, \frac{7 \pi}{4}\right]$; further, by construction, the edge $(\xi, \rho)$ has slope $\frac{3 \pi}{2}$; finally, again by Property iv of Lemma 1, all the edges of $P$ in $T$, if any, have slope in the interval $\left[\frac{5 \pi}{4}, \frac{7 \pi}{4}\right]$.

## $5 \quad \alpha$-Schnyder Drawings of Plane Triangulations

In this section, we prove an exponential lower bound for the area requirements of $\alpha$-Schnyder drawings of plane triangulations, for any fixed $\alpha<\pi / 3$. For a function $f(n)$ and a parameter $\varepsilon>0$, we write $f(n) \in \Omega_{\varepsilon}(n)$ if $f(n) \geq c_{\varepsilon} n$ for some constant $c_{\varepsilon}>0$ which only depends on $\varepsilon$.


Fig. 3: Illustrations for the proof of Theorem 3: (a) The graph $G_{m}$. (b) The different angular widths of an $\alpha$-Schnyder drawing, with $\alpha<\frac{\pi}{3}$.

Theorem 3. There exists an infinite family $\mathcal{F}$ of bounded-degree planar 3 -trees such that, for any resolution rule, any $n$-vertex graph in $\mathcal{F}$ requires $2^{\Omega_{\varepsilon}(n)}$ area in any $\left(\frac{\pi}{3}-\varepsilon\right)$-Schnyder drawing, for any fixed $0<\varepsilon<\frac{\pi}{3}$.
Proof. We start by defining, for each integer $m>0$, a $3 m$-vertex plane 3 -tree $G_{m}$; refer to Fig. 3a. The plane 3 -tree $G_{1}$ is a cycle $\left(a_{1}, b_{1}, c_{1}\right)$. For any integer $m>1$, the plane 3 -tree $G_{m}$ is obtained from the plane 3 -tree $G_{m-1}$ by embedding a cycle ( $a_{m}, b_{m}, c_{m}$ ) in the outer face of $G_{m-1}$, so that it contains $G_{m-1}$ in its interior, and by inserting the edges $\left(a_{m}, a_{m-1}\right),\left(b_{m}, a_{m-1}\right),\left(b_{m}, b_{m-1}\right),\left(c_{m}, a_{m-1}\right),\left(c_{m}, b_{m-1}\right)$, and $\left(c_{m}, c_{m-1}\right)$.

Since $G_{m}$ is a plane 3 -tree, it has a unique $\operatorname{Schnyder} \operatorname{mood}\left(\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{T}_{3}\right)$; see [5,11]. Assume, w.l.o.g., that all the internal edges incident to $a_{m}$, to $b_{m}$, and to $c_{m}$ belong to $\mathcal{T}_{1}, \mathcal{T}_{2}$, and $\mathcal{T}_{3}$, respectively. We have the following.

Claim 5. For every $1 \leq k<m$, the Schnyder $\operatorname{wood}\left(\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{T}_{3}\right)$ of $G_{m}$ satisfies the following properties:
(a) The edges $\left(a_{k}, b_{k}\right)$ and $\left(a_{k}, c_{k}\right)$ belong to $\mathcal{T}_{1}$ and are directed towards $a_{k}$ and the edge $\left(b_{k}, c_{k}\right)$ belongs to $\mathcal{T}_{2}$ and is directed towards $b_{k}$.
(b) The edge $\left(a_{k}, a_{k+1}\right)$ belongs to $\mathcal{T}_{1}$ and is directed towards $a_{k+1}$, the edges $\left(b_{k+1}, a_{k}\right)$ and $\left(b_{k+1}, b_{k}\right)$ belong to $\mathcal{T}_{2}$ and are directed towards $b_{k+1}$, and the edges $\left(c_{k+1}, a_{k}\right),\left(c_{k+1}, b_{k}\right)$, and $\left(c_{k+1}, c_{k}\right)$ belong to $\mathcal{T}_{3}$ and are directed towards $c_{k+1}$.

Proof. We prove the claim by reverse induction on $k$. If $k=m-1$, then Property (2) of the definition of Schnyder wood directly implies that Property (b) is satisfied. Further, by Property (1) of the definition of Schnyder wood we have that all the edges (including $\left(a_{k}, b_{k}\right)$ and $\left.\left(a_{k}, c_{k}\right)\right)$ that appear after $\left(a_{k}, b_{k+1}\right)$ and before $\left(a_{k}, c_{k+1}\right)$ in clockwise order around $a_{k}$ belong to $\mathcal{T}_{1}$ and are directed towards $a_{k}$; again by the same property, all the edges (including $\left(b_{k}, c_{k}\right)$ ) that appear after $\left(b_{k}, c_{k+1}\right)$ and before $\left(b_{k}, a_{k}\right)$ in clockwise order around $b_{k}$ belong to $\mathcal{T}_{2}$ and are directed towards $b_{k}$, hence Property (a) is satisfied.

Now inductively assume that Properties (a) and (b) are satisfied for some $2 \leq k \leq m-1$; we prove that they are satisfied for $k-1$, as well. By Property (1) of the definition of Schnyder wood we have that the edge $\left(a_{k}, a_{k-1}\right)$ belongs to $\mathcal{T}_{1}$ and is directed towards $a_{k}$, since it appears after ( $a_{k}, b_{k}$ ) and before $\left(a_{k}, c_{k}\right)$ in clockwise order around $a_{k}$. By the same property, we have that the edges $\left(c_{k}, a_{k-1}\right),\left(c_{k}, b_{k-1}\right)$, and $\left(c_{k}, c_{k-1}\right)$ belong to $\mathcal{T}_{3}$ and are directed towards $c_{k}$, since they appear after $\left(c_{k}, a_{k}\right)$ and before $\left(c_{k}, b_{k}\right)$ in clockwise order around $c_{k}$. Again by the same property, the edges ( $b_{k}, a_{k-1}$ ) and ( $b_{k}, b_{k-1}$ ) belong to $\mathcal{T}_{2}$ and are directed towards $b_{k}$, since they appear after $\left(c_{k}, b_{k}\right)$ and before ( $a_{k}, b_{k}$ ) in clockwise order around $b_{k}$. This concludes the proof that Property (b) is satisfied. The proof that Property (a) is satisfied is the same as in the base case.

We now prove that, for any fixed $\alpha=\frac{\pi}{3}-\varepsilon$ with $\varepsilon>0$, any $\alpha$-Schnyder drawing $\Gamma$ of $G_{m}$ (respecting the plane embedding of $G_{m}$ ) requires $2^{\Omega_{\varepsilon}(m)}$ area. To this aim, we exploit the next claim.

Claim 6. In $\Gamma$, the area of the triangle $\left(a_{i}, b_{i}, c_{i}\right)$ is at least $k_{\varepsilon}$ times the area of the triangle $\left(a_{i-1}, b_{i-1}, c_{i-1}\right)$, for any $i=2, \ldots, m-1$, where $k_{\varepsilon}>1$ is a constant only depending on $\varepsilon$.

Proof. Let $A_{i-1}$ and $A_{i}$ denote the areas of the triangles $\Delta_{i-1}=\left(a_{i-1}, b_{i-1}, c_{i-1}\right)$ and $\Delta_{i}=\left(a_{i}, b_{i}, c_{i}\right)$ in $\Gamma$, respectively. We prove that $\frac{A_{i}}{A_{i-1}} \geq k_{\varepsilon}$, where $k_{\varepsilon}>1$ is a constant that depends only on $\varepsilon$; refer to Fig. 4a.

In the following, given a pair of elements $x$ and $y$, each representing either a point or a vertex in $\Gamma$, we will denote by $\overline{x y}$ and by $\ell(x, y)$ the segment connecting $x$ and $y$ in $\Gamma$, and the line passing through both $x$ and $y$ in $\Gamma$, respectively.

The following statements exploit Claim 5 and the fact that $\Gamma$ is an $\alpha$-Schnyder drawing (refer to Fig. 3b):

- the slopes of the edges $\left(b_{i}, a_{i}\right)$ and $\left(c_{i}, a_{i}\right)$ are in the range $\left[\frac{\pi}{2}-\frac{\alpha}{2}, \frac{\pi}{2}+\frac{\alpha}{2}\right] ;$
- the slope of the edge $\left(b_{i}, c_{i}\right)$ is in the range $\left[\frac{5 \pi}{6}-\frac{\alpha}{2}, \frac{5 \pi}{6}+\frac{\alpha}{2}\right]$; and
- the slope of the edge $\left(c_{i}, a_{i-1}\right)$ is in the range $\left[\frac{\pi}{6}-\frac{\alpha}{2}, \frac{\pi}{6}+\frac{\alpha}{2}\right]$.

Next, we reduce the problem of proving $\frac{A_{i}}{A_{i-1}} \geq k_{\varepsilon}$ to the one of determining a lower bound for the ratio between the areas of two triangles whose sides (except for one) have fixed slopes.

Let $o^{\prime}$ be the intersection point between the edge $\left(a_{i}, b_{i}\right)$ and the line $\ell\left(c_{i}, a_{i-1}\right)$. Let $A^{\prime}$ denote the area of the triangle $\Delta^{\prime}=\left(c_{i}, o^{\prime}, b_{i}\right)$; see Fig. 4b. Observe that $\Delta^{\prime}$ strictly encloses $\Delta_{i-1}$; therefore, we have that $A^{\prime}>A_{i-1}$. It follows that in order to prove that $\frac{A_{i}}{A_{i-1}} \geq k_{\varepsilon}$, it suffices to prove that $\frac{A_{i}}{A^{\prime}} \geq k_{\varepsilon}$.

Let $o^{\prime \prime}$ be the intersection point between the edge $\left(a_{i}, b_{i}\right)$ and the line with slope $\frac{\pi}{6}+\frac{\alpha}{2}$ passing through $c_{i}$. Let $A^{\prime \prime}$ denote the area of the triangle $\Delta^{\prime \prime}=\left(c_{i}, o^{\prime \prime}, b_{i}\right)$; see Fig. 4c. Since the slope of the segment $\overline{c_{i} o^{\prime}}$ is in the range $\left[\frac{\pi}{6}-\frac{\alpha}{2}, \frac{\pi}{6}+\frac{\alpha}{2}\right]$ (as it contains the drawing of the edge $\left(c_{i}, a_{i-1}\right)$ ), we have that $\Delta^{\prime \prime}$ encloses $\Delta^{\prime}$; therefore, $A^{\prime \prime} \geq A^{\prime}$. It follows that in order to prove that $\frac{A_{i}}{A^{\prime}} \geq k_{\varepsilon}$, it suffices to prove that $\frac{A_{i}}{A^{\prime \prime}} \geq k_{\varepsilon}$.

Let $o^{\prime \prime \prime}$ be the intersection point between the edge $\left(a_{i}, b_{i}\right)$ and the line with slope $\frac{\pi}{2}-\frac{\alpha}{2}$ passing through $c_{i}$. Let $A^{\prime \prime \prime}$ denote the area of the triangle $\Delta^{\prime \prime \prime}=\left(c_{i}, o^{\prime \prime \prime}, o^{\prime \prime}\right)$; see Fig. 4c. Since the slope of the edge $\left(c_{i}, a_{i}\right)$ is in the range $\left[\frac{\pi}{2}-\frac{\alpha}{2}, \frac{\pi}{2}+\frac{\alpha}{2}\right]$, we have that $\Delta^{\prime \prime \prime}$ is enclosed in the triangle $\Delta^{+}=\left(c_{i}, a_{i}, o^{\prime \prime}\right)$. Let $A^{+}$denote the area of the triangle $\Delta^{+}$. We have that $A^{\prime \prime \prime} \leq A^{+}=A_{i}-A^{\prime \prime}$. Thus, $A_{i} \geq A^{\prime \prime}+A^{\prime \prime \prime}$. It follows that in order to prove that $\frac{A_{i}}{A^{\prime \prime}} \geq k_{\varepsilon}$, it suffices to prove that $\frac{A^{\prime \prime}+A^{\prime \prime \prime}}{A^{\prime \prime}} \geq k_{\varepsilon}$, i.e., $\frac{A^{\prime \prime \prime}}{A^{\prime \prime}} \geq k_{\varepsilon}-1$.

Let $o^{*}$ be the intersection point between the line $\ell\left(a_{i}, b_{i}\right)$ and the line with slope $\frac{5 \pi^{A}}{6}-\frac{\alpha}{2}$ passing through $c_{i}$; see Fig. 4d. Let $A^{*}$ denote the area of the triangle $\Delta^{*}=\left(c_{i}, o^{\prime \prime}, o^{*}\right)$. Since the slope of the edge $\left(c_{i}, b_{i}\right)$ is in the range $\left[\frac{11 \pi}{6}-\frac{\alpha}{2}, \frac{11 \pi}{6}+\frac{\alpha}{2}\right]$, we have that $\Delta^{*}$ encloses $\Delta^{\prime \prime}$; therefore, we have that $A^{*} \geq A^{\prime \prime}$. It follows that in order to prove that $\frac{A^{\prime \prime \prime}}{A^{\prime \prime}} \geq k_{\varepsilon}-1$, it suffices to prove that $\frac{A^{\prime \prime \prime}}{A^{*}} \geq k_{\varepsilon}-1$.

Finally, let $o^{\diamond}$ be the intersection point between the line $\ell\left(c_{i}, o^{\prime \prime \prime}\right)$ and the line passing through $o^{\prime \prime}$ and perpendicular to $\ell\left(c_{i}, o^{\prime \prime}\right)$. Consider the right triangle $\Delta^{\diamond}=\left(c_{i}, o^{\diamond}, o^{\prime \prime}\right)$; see Fig. 4d. We claim that $\Delta^{\prime \prime \prime}$ strictly encloses $\Delta^{\diamond}$, and thus, $A^{\diamond}<A^{\prime \prime \prime}$, which implies that in order to prove that $\frac{A^{\prime \prime \prime}}{A^{*}} \geq k_{\varepsilon}-1$, it suffices to prove the following main inequality

$$
\begin{equation*}
\frac{A^{\diamond}}{A^{*}} \geq k_{\varepsilon}-1 \tag{1}
\end{equation*}
$$

To show that $\Delta^{\prime \prime \prime}$ strictly encloses $\Delta^{\diamond}$, we prove that the internal angle of $\Delta^{\prime \prime \prime}$ at $o^{\prime \prime}$ is larger than $\frac{\pi}{2}$. This immediately descends from the fact that the slope of the segment $\overline{o^{\prime \prime} a_{i}}$ is in the range $\left[\frac{\pi}{2}-\frac{\alpha}{2}, \frac{\pi}{2}+\frac{\alpha}{2}\right]$ (as it overlaps with the edge $\left(b_{i}, a_{i}\right)$ which belongs to $\left.\mathcal{T}_{1}\right)$, that the slope of the segment $\overline{o^{\prime \prime} c_{i}}$ is $\frac{7 \pi}{6}+\frac{\alpha}{2}$, by construction, and that $\alpha<\frac{\pi}{3}$.

We are now ready to compute $A^{\diamond}$ and give an upper bound for $A^{*}$. In the following, we denote by $h$ the length of the segment $\overline{c_{i} O^{\prime \prime}}$; refer to Fig. 4d.
Value of $A^{\diamond}$. Let us denote by $\delta$ the interior angle of $\Delta^{\diamond}$ at $c_{i}$. Since, by construction, the slope of the segments $\overline{c_{i} O^{\circ}}$ and $\overline{c_{i} O^{\prime \prime}}$ are $\frac{\pi}{2}-\frac{\alpha}{2}$ and $\frac{\pi}{6}+\frac{\alpha}{2}$, respectively, we have that $\delta=\frac{\pi}{3}$. Therefore, since $\Delta^{\diamond}$ is a right triangle whose cathetus incident to $\delta$ is the segment $\overline{c_{i} o^{\prime \prime}}$, the following holds

$$
\begin{equation*}
A^{\diamond}=\frac{h^{2} \tan (\delta)}{2}=\frac{\sqrt{3}}{2} h^{2} \tag{2}
\end{equation*}
$$

Upper bound for $A^{*}$. Let $\sigma, \beta$, and $\gamma$ be the internal angles of the triangle $\Delta^{*}$ at $c_{i}, o^{\prime \prime}$, and $o^{*}$, respectively. First, we show that (i) $\sigma=\frac{\pi}{3}+\alpha$, (ii) $\beta \leq \frac{\pi}{3}$, and (iii) $\gamma \geq \frac{\pi}{3}-\alpha$.

Item (i) is due to the facts: 1 . the angle $\sigma^{\prime}$ spanned by a clockwise rotation around $c_{i}$ bringing a ray originating at $c_{i}$ and directed rightward to overlap with $\overline{c_{i} O^{*}}$ is $\frac{\pi}{6}+\frac{\alpha}{2}$, given that the slope of $\overline{c_{i} O^{*}}$ is
$\frac{11 \pi}{6}-\frac{\alpha}{2}$, by construction; 2 . the angle $\sigma^{\prime \prime}$ spanned by a counter-clockwise rotation around $c_{i}$ bringing a ray originating at $c_{i}$ and directed rightward to overlap with $\overline{c_{i} O^{\prime \prime}}$ is $\frac{\pi}{6}+\frac{\alpha}{2}$, given that the slope of $\overline{c_{i} O^{\prime \prime}}$ is $\frac{\pi}{6}+\frac{\alpha}{2}$, by construction; and 3. $\sigma=\sigma^{\prime}+\sigma^{\prime \prime}$.

Item (ii) is due to the following facts: 1 . the angle $\beta^{\prime}$ spanned by a counter-clockwise rotation around $o^{\prime \prime}$ bringing a ray originating at $o^{\prime \prime}$ and directed rightward to overlap with $\overline{o^{\prime \prime} o^{*}}$ is at most $\frac{3 \pi}{2}+\frac{\alpha}{2}$, given that the slope of $\overline{o^{\prime \prime} o^{*}}$ is in the range $\left[\frac{3 \pi}{2}-\frac{\alpha}{2}, \frac{3 \pi}{2}+\frac{\alpha}{2}\right]$ (as it overlaps with the edge $\left(b_{i}, a_{i}\right)$, which belongs to $\mathcal{T}_{1}$ ); 2. the angle $\beta^{\prime \prime}$ spanned by a counter-clockwise rotation around $o^{\prime \prime}$ bringing a ray originating at $o^{\prime \prime}$ and directed rightward to overlap with $\overline{o^{\prime \prime} c_{i}}$ is $\frac{7 \pi}{6}+\frac{\alpha}{2}$, given that the slope of $\overline{o^{\prime \prime} c_{i}}$ is $\frac{7 \pi}{6}+\frac{\alpha}{2}$, by construction; and 3. $\beta=\beta^{\prime}-\beta^{\prime \prime}$.

Item (iii) is due to (i), (ii) and of $\sigma+\beta+\gamma=\pi$.
Let $\kappa$ denote the length of the segment $\overline{c_{i} o^{*}}$. By the law of sines applied to the triangle $\Delta^{*}$, we have that $\frac{h}{\sin (\gamma)}=\frac{\kappa}{\sin (\beta)}$, i.e., $\kappa=h \frac{\sin (\beta)}{\sin (\gamma)}$. Applying Items (ii) and (iii), the following holds

$$
\begin{equation*}
\kappa \leq h \frac{\sin \left(\frac{\pi}{3}\right)}{\sin \left(\frac{\pi}{3}-\alpha\right)}=h \frac{\sqrt{3}}{2 \sin \left(\frac{\pi}{3}-\alpha\right)} \tag{3}
\end{equation*}
$$

Since the segments $\overline{c_{i} o^{\prime \prime}}$ and $\overline{c_{i} o^{*}}$ form the two sides of $\Delta^{*}$ whose angle at $c_{i}$ is $\sigma$, we have that $A^{*}=\frac{h \cdot \kappa \sin (\sigma)}{2}$. Applying Eq. (3) and Item (i), we thus have that the following holds

$$
\begin{equation*}
A^{*} \leq \frac{h^{2} \sqrt{3}}{4} \frac{\sin \left(\frac{\pi}{3}+\alpha\right)}{\sin \left(\frac{\pi}{3}-\alpha\right)} \tag{4}
\end{equation*}
$$

Determining $k_{\varepsilon}$. Combining Eqs. (2) and (4), we have that

$$
\frac{A^{\diamond}}{A^{*}} \geq \frac{2 \sin \left(\frac{\pi}{3}-\alpha\right)}{\sin \left(\frac{\pi}{3}+\alpha\right)}
$$

By the above inequality, and exploiting the fact that $\frac{\pi}{3}-\alpha=\varepsilon$ and $\frac{\pi}{3}+\alpha=\frac{2 \pi}{3}-\varepsilon$, we have that Eq. (1) is satisfied by setting

$$
\begin{equation*}
k_{\varepsilon}=1+\frac{2 \sin (\varepsilon)}{\sin \left(\frac{2 \pi}{3}-\varepsilon\right)} \tag{5}
\end{equation*}
$$

Note that $k_{\varepsilon}$ is a constant, for any fixed $\varepsilon$; further, $k_{\varepsilon}>1$, since $\frac{2 \sin (\varepsilon)}{\sin \left(\frac{2 \pi}{3}-\varepsilon\right)}>0$ with $0<\varepsilon<\frac{\pi}{3}$. This concludes the proof.

Claim 6 immediately implies that the area of the triangle $\left(a_{m-1}, b_{m-1}, c_{m-1}\right)$ is at least $k_{\varepsilon}^{m-2}$ times the area of the triangle $\left(a_{1}, b_{1}, c_{1}\right)$. Since the area of the triangle $\left(a_{1}, b_{1}, c_{1}\right)$ is greater than some constant depending on the adopted resolution rule, we get that the area of $\Gamma$ is in $2^{\Omega_{\varepsilon}(m)}$.

We are now ready to define the family $\mathcal{F}$ of the statement. For any positive integer $m$ and any $n=6 m-2$, we construct the graph $F_{n} \in \mathcal{F}$ from the complete graph $K_{4}$ on 4 vertices, by taking two copies $G_{m}^{\prime}$ and $G_{m}^{\prime \prime}$ of $G_{m}$ and by identifying the vertices incident to the outer face of each copy with the three vertices incident to two distinct triangular faces of the $K_{4}$. Observe that $F_{n}$ is a bounded-degree planar 3-tree. In any $\alpha$-Schnyder drawing $\Gamma$ of $F_{n}$ (in fact in planar drawing of $F_{n}$ ), at least one of the two copies of $G_{m}$, say $G_{m}^{\prime}$, is drawn so that its outer face is delimited by the triangle ( $a_{m}, b_{m}, c_{m}$ ). Since $F_{n}$ has a unique Schnyder wood [5,11], the restriction of such a Schnyder wood to the internal edges of $G_{m}^{\prime}$ satisfies the properties of Claim 5. It follows that the restriction of $\Gamma$ to $G_{m}^{\prime}$ is an $\alpha$-Schnyder drawing of $G_{m}^{\prime}$ (respecting the plane embedding of $G_{m}^{\prime}$ ), and therefore it requires $2^{\Omega_{\varepsilon}(m)}$ area. The proof is concluded by observing that $m \in \Omega(n)$.

## 6 Conclusions and Open Problems

In this paper, we refuted a claim by Cao et al. [6] and re-opened the question of whether 3-connected planar graphs admit planar, and possibly convex, greedy drawings on a polynomial-size grid. Further, we provided some evidence for a positive answer by showing that every $n$-vertex Halin graph admits a convex greedy drawing on an $O(n) \times O(n)$ grid; in fact, our drawings are angle-monotone, which is a
stronger property than greediness. Moreover, we proved that $\alpha$-Schnyder drawings, which are an even more constrained drawing standard, might require exponential area for any fixed $\alpha<\frac{\pi}{3}$.

Several questions remain open in this topic. We mention two of them that seem to be natural next steps. (Q1) Does every 2-outerplanar graph admit a planar, and possibly convex, greedy drawing on a polynomial-size grid? Note that the class of 2-outerplanar graphs is strictly larger than the one of Halin graphs. (Q2) Does every plane 3-tree admit a planar greedy drawing on a polynomial-size grid? We indeed proved a negative answer if "greedy drawing" is replaced by " $\alpha$-Schnyder drawing", for any fixed $\alpha<\frac{\pi}{3}$.

## References

1. Alamdari, S., Chan, T.M., Grant, E., Lubiw, A., Pathak, V.: Self-approaching graphs. In: Didimo, W., Patrignani, M. (eds.) GD '12. LNCS, vol. 7704, pp. 260-271. Springer (2013)
2. Angelini, P., Di Battista, G., Frati, F.: Succinct greedy drawings do not always exist. Networks 59(3), 267-274 (2012)
3. Angelini, P., Frati, F., Grilli, L.: An algorithm to construct greedy drawings of triangulations. Journal of Graph Algorithms and Applications 14(1), 19-51 (2010)
4. Bonichon, N., Bose, P., Carmi, P., Kostitsyna, I., Lubiw, A., Verdonschot, S.: Gabriel triangulations and angle-monotone graphs: Local routing and recognition. In: Hu, Y., Nöllenburg, M. (eds.) GD '16. LNCS, vol. 9801, pp. 519-531. Springer (2016)
5. Brehm, E.: 3-orientations and Schnyder 3-tree decompositions. Ph.D. thesis, Freie Universität Berlin (2000)
6. Cao, L., Strelzoff, A., Sun, J.Z.: On succinctness of geometric greedy routing in euclidean plane. In: ISPAN '09. pp. 326-331. IEEE Computer Society (2009)
7. Da Lozzo, G., D'Angelo, A., Frati, F.: On planar greedy drawings of 3-connected planar graphs. Discrete \& Computational Geometry (2018)
8. Dehkordi, H.R., Frati, F., Gudmundsson, J.: Increasing-chord graphs on point sets. Journal of Graph Algorithms and Applications 19(2), 761-778 (2015)
9. Dhandapani, R.: Greedy drawings of triangulations. Discrete \& Computational Geometry 43(2), 375-392 (2010)
10. Eppstein, D., Goodrich, M.T.: Succinct greedy geometric routing using hyperbolic geometry. IEEE Transactions on Computers 60(11), 1571-1580 (2011)
11. Felsner, S., Zickfeld, F.: On the number of planar orientations with prescribed degrees. Electr. J. Comb. 15(1) (2008)
12. Goodrich, M.T., Strash, D.: Succinct greedy geometric routing in the Euclidean plane. In: Dong, Y., Du, D.Z., Ibarra, O.H. (eds.) ISAAC '09. LNCS, vol. 5878, pp. 781-791. Springer (2009)
13. He, X., Zhang, H.: On succinct greedy drawings of plane triangulations and 3-connected plane graphs. Algorithmica 68(2), 531-544 (2014)
14. Leighton, T., Moitra, A.: Some results on greedy embeddings in metric spaces. Discrete \& Computational Geometry 44(3), 686-705 (2010)
15. Lubiw, A., Mondal, D.: Construction and local routing for angle-monotone graphs. Journal of Graph Algorithms and Applications 23(2), 345-369 (2019)
16. Lubiw, A., O'Rourke, J.: Angle-monotone paths in non-obtuse triangulations. In: Gudmundsson, J., Smid, M.H.M. (eds.) CCCG '17. pp. 25-30 (2017)
17. Muhammad, R.: A distributed geometric routing algorithm for ad hoc wireless networks. In: ITNG '07. pp. 961-963. IEEE (2007)
18. Nöllenburg, M., Prutkin, R., Rutter, I.: On self-approaching and increasing-chord drawings of 3-connected planar graphs. Journal of Computational Geometry 7(1), 47-69 (2016)
19. Nöllenburg, M., Prutkin, R.: Euclidean greedy drawings of trees. Discrete \& Computational Geometry 58(3), 543-579 (2017)
20. Papadimitriou, C.H., Ratajczak, D.: On a conjecture related to geometric routing. Theoretical Computer Science 344(1), 3-14 (2005)
21. Papadimitriou, C.H., Ratajczak, D.: On a conjecture related to geometric routing. In: ALGOSENSORS ' 04. LNCS, vol. 3121, pp. 9-17. Springer (2004)
22. Rao, A., Papadimitriou, C.H., Shenker, S., Stoica, I.: Geographic routing without location information. In: Johnson, D.B., Joseph, A.D., Vaidya, N.H. (eds.) MOBICOM '03. pp. 96-108. ACM (2003)
23. Schnyder, W.: Embedding planar graphs on the grid. In: Johnson, D.S. (ed.) SODA '90. pp. 138-148. SIAM (1990)
24. Sun, Y., Zhang, Y., Fang, B., Zhang, H.: Succinct and practical greedy embedding for geometric routing. Computer Communications 114, 51-61 (2017)
25. Wang, J.J., He, X.: Succinct strictly convex greedy drawing of 3-connected plane graphs. Theoretical Computer Science 532, 80-90 (2014)


Fig. 4: Illustration for the proof of Claim 6. Values in radiants indicate segments' slopes.


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[^1]:    ${ }^{1}$ The flaw in the proof presented in [6] seems to be in the statement "it is not difficult to see that the most economic way (i.e., consuming the minimum area) of stretching $\left[\mathfrak{H}_{i}\right]$ into a greedy embedding is to do it symmetrically [...]".

