# ON THE RESTRICTED $k$-STEINER TREE PROBLEM* 

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#### Abstract

Given a set $P$ of $n$ points in $\mathbb{R}^{2}$ and an input line $\gamma$ in $\mathbb{R}^{2}$, we present an algorithm that runs in optimal $\Theta(n \log n)$ time and $\Theta(n)$ space to solve a restricted version of the 1-Steiner tree problem. Our algorithm returns a minimum-weight tree interconnecting $P$ using at most one Steiner point $s \in \gamma$, where edges are weighted by the Euclidean distance between their endpoints. We then extend the result to $j$ input lines. Following this, we show how the algorithm of Brazil et al. [10] that solves the $k$-Steiner tree problem in $\mathbb{R}^{2}$ in $O\left(n^{2 k}\right)$ time can be adapted to our setting. For $k>1$, restricting the (at most) $k$ Steiner points to lie on an input line, the runtime becomes $O\left(n^{k}\right)$. Next we show how the results of Brazil et al. [10] allow us to maintain the same time and space bounds while extending to some non-Euclidean norms and different tree cost functions. Lastly, we extend the result to $j$ input curves.


## 1 Introduction

Finding the shortest interconnecting network for a given set of points is an interesting optimization problem with various applications for anyone seeking connectivity while at the same time concerned with conserving resources. Sometimes we are able to introduce new points into the point set such that the sum of the lengths of the edges in the interconnecting network is reduced. These extra points are called Steiner points. However, selecting optimal positions for these Steiner points and how many to place is NP-hard [13, 14, 27, 37], and so a natural question is: What is the shortest spanning network that can be constructed by adding only $k$ Steiner points to the given set of points? This is the " $k$-Steiner tree" problem.

Consider a set $P$ of $n$ points in $\mathbb{R}^{2}$, which are also called terminals in the Steiner tree literature. The Minimum Spanning Tree (MST) problem is to find the minimum-weight tree interconnecting $P$, where edges are weighted by the Euclidean distance between their

[^0]endpoints. Let $\operatorname{MST}(P)$ be a Euclidean minimum spanning tree on $P$ and let $|\operatorname{MST}(P)|$ be the sum of its edge-weights (also called the length of the tree). Imagine we are given another set $S$ of points in $\mathbb{R}^{2}$. $S$ is the set of Steiner points that we may use as intermediate nodes in addition to the points of $P$ to compute a minimum-weight interconnection of $P$. An MST on the union of the terminals $P$ with some subset of Steiner points $S^{\prime} \subseteq S$, i.e., $\operatorname{MST}\left(P \cup S^{\prime}\right)$, is a Steiner tree. In the Euclidean Minimum Steiner Tree (MStT) problem, the goal is to find a subset $S^{\prime} \subseteq S$ such that $\left|\operatorname{MST}\left(P \cup S^{\prime}\right)\right|$ is at most $|\operatorname{MST}(P \cup X)|$ for any $X \subseteq S$. Such a minimum-weight tree is an MStT. For our restricted $k$-Steiner tree problem, we are given an input line $\gamma$ in $\mathbb{R}^{2}$; the line $\gamma=S$ and the cardinality of $S^{\prime}$ is at most $k$.

As 3-D printing enters the mainstream, material-saving and time-saving printing algorithms are becoming more relevant. Drawing on the study of MStTs, Vanek et al. [39] presented a geometric heuristic to create support-trees for 3-D-printed objects where the forking points in these trees are solutions to a constrained Steiner tree problem. Inspired by the work of Vanek et al. as well as solutions for the 1-Steiner and $k$-Steiner tree problems in the 2-D Euclidean plane [10, 14, 28], we present an efficient algorithm to compute a solution for the 1-Steiner tree problem where the placement of the Steiner point is constrained to lie on an input line. We present another motivating example. Imagine we have a set $V$ of wireless nodes that must communicate by radio transmission. To transmit a longer distance to reach more distant nodes requires transmitting at a higher power. The MST of $V$ can be used to model a connected network that spans the nodes of $V$ while minimizing total power consumption. Suppose that an additional wireless node is available to be added to $V$, but that the new node's position is restricted to lie on a road $\gamma$ on which it will be delivered on a vehicle. Where on $\gamma$ should the additional node be positioned to minimize the total transmission power of the new network?

We refer to our problem as a 1-Steiner tree problem restricted to a line. For our purposes, let an optimal Steiner point be a point $s \in S$ such that $|\operatorname{MST}(P \cup\{s\})| \leq$ $|\operatorname{MST}(P \cup\{u\})|$ for all $u \in S$.

## 1-Steiner Tree Problem Restricted to a Line

Given a set $P$ of $n$ points in $\mathbb{R}^{2}$ and a line $\gamma$ in $\mathbb{R}^{2}$, select a point $s \in \gamma \cup \emptyset$ that minimizes $|\operatorname{MST}(P \cup\{s\})|$.

A restricted version of our problem has been studied for the case when the input point set $P$ lies to one side of the given input line and a point from the line must be chosen. Chen and Zhang gave an $O\left(n^{2}\right)$-time algorithm to solve this problem [17]. Similar problems have also been studied by Li et al. [33] building on the research of Holby [32]. The two settings they study are: (a) the points of $P$ lie anywhere in $\mathbb{R}^{2}$ and must connect to the input line using any number of Steiner points, and any part of the input line used in a spanning tree does not count towards its length; and (b) the same problem, but the optimal line to minimize the network length is not given and must be computed. Li et al. provide 1.214-approximation ${ }^{1}$ algorithms for both (a) and (b) in $O(n \log n)$ and $O\left(n^{3} \log n\right)$ time

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Figure 1: Here we have $\gamma$ as the $x$-axis with points $a=(0.489,0.237), b=(1.865,-0.114)$, $c=(3.26,0.184)$, and $d=(4.75,-0.141)$. Depicted are the MST of $\{a, b, c, d\}$ in red dashed line segments and its length; the input line $\gamma$; a spanning tree of $\{a, b, c, d\}$ connecting each point to $\gamma$ and the length of this spanning tree for the setting of Holby [32] and Li et al. [33]. The numbers have been rounded to three decimal places.
respectively. Our problem differs from the problems of Chen and Zhang [17], Li et al. [33], and Holby [32] in the following ways: (a) we have no restriction on the placement of the points of $P$ with respect to the input line $\gamma ;(\mathrm{b})$ our problem does not require connecting to $\gamma$; and (c) travel in our network has the same cost on $\gamma$ as off of it. For example, if the points of the set $P$ were close to the line but far from each other, then the solution of Li et al. [33] would connect the points to the line and get a tree with much lower weight/length than even the MStT. Such an example is shown in Fig. 1: the MStT of points $\{a, b, c, d\}$ is the same as its MST since all triples form angles larger than $\frac{2 \pi}{3}[14,29]$. In our setting, the MST is the best solution for this point set, whereas in the setting of Holby [32] and Li et al. [33], the best solution connects each input point directly to $\gamma$ to form a spanning tree between the points using pieces of $\gamma$. The length of the MST is significantly larger than the length of the other solution since in their setting, only the edges connecting the points to $\gamma$ contribute to the length of the spanning tree.

In our algorithm we use a type of Voronoi diagram whose regions are bounded by rays and segments. We make a general position assumption that $\gamma$ is not collinear with any ray or segment in the Voronoi diagrams. In other words, the intersection of the rays and segments of these Voronoi diagrams with $\gamma$ is either empty or a single point. We also assume that the edges of $\operatorname{MST}(P)$ have distinct weights. In this paper we show the following theorem.

Theorem 1. Given a set $P$ of $n$ points in the Euclidean plane and a line $\gamma$, Algorithm 1 computes in optimal $\Theta(n \log n)$ time and optimal $\Theta(n)$ space a minimum-weight tree connecting all points in $P$ using at most one extra point $s \in \gamma$.

Section 2 reviews the tools and properties we will need for our algorithm. Section 3 presents Algorithm 1 and the proof of Theorem 1. Section 4 shows how to adapt the existing $k$-Steiner tree algorithm of Brazil et al. [10] to solve the problem when the set of at most $k$ Steiner points is restricted to lie on $j$ input lines, as well as how the results of Brazil et al. [10] allow us to generalize our algorithms to norms other than Euclidean and cost functions for the tree other than minimizing the sum of edge-weights, giving the following corollary. The definitions of norm, Restrictions 2, 3, 4, and 5, and symmetric $\ell_{1}$-optimizable
cost functions are found in Section 4. Informally, Restriction 2 is that we can quickly solve a $k$-Steiner tree problem with a prespecified topology; and Restrictions 3, 4, and 5 impose properties on the given norm, like the quick computations of intersections and bisectors, that make it behave somewhat like the familiar Euclidean norm.

Corollary 5. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;
- a symmetric $\ell_{1}$-optimizable cost function $\alpha$;
- $j$ lines $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$

By running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j n \log n)$ time and $O(n+j)$ space a minimumweight tree with respect to $\alpha$ and $\|\cdot\|$ is computed that connects all points in $P$ using at most one extra point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

By running the algorithm of Section 4.1 for a constant integer $k>1$ under Restriction 2, the restricted $k$-Steiner tree problem is solved in $O\left((j n)^{k}\right)$ time and $O(j n)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

Lastly, in Section 4.4 we show how to adapt the running time and space of the algorithms when given $j$ input curves of a restricted class, giving the following corollary. The variables $\mu, t, g, c$, and $q$ are defined in Section 4.4 and correspond to the complexities of certain primitive operations and the complexity of the zone of the curves in an arrangement of lines.

Corollary 7. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;
- a symmetric $\ell_{1}$-optimizable cost function $\alpha$;
- $j$ input curves $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$ with maximum space complexity $\gamma_{s p}$

By running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j(n \log n+\mu+t(g+c)))$ time and $O(n+$ $\left.j \gamma_{s p}+\mu_{s p}+t_{s p}\right)$ space a minimum-weight tree with respect to $\alpha$ and $\|\cdot\|$ is computed that connects all points in $P$ using at most one extra point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

For a constant integer $k>1$, the algorithm of Section 4.1 solves the restricted $k$ Steiner tree problem in $O\left((j(g+c))^{k} q+j \mu+n \log n\right)$ time and $O\left(n+j \gamma_{s p}+j \mu_{s p}+q_{s p}\right)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

In this paper we assume we can compute to any fixed precision in constant time and space the derivative and roots of a function that can be written using a constant number of the following operators on one variable and the real numbers: $+,-, *, /$, and $\sqrt[c]{ }$. for a constant $c$.

## 2 Relevant Results

There has been a lot of research on Steiner trees in various dimensions, metrics, norms, and under various constraints. See the surveys by Brazil et al. [9] and Brazil and Zachariasen [14] for a good introduction. In the general Euclidean case it has been shown that Steiner points that reduce the length of the MST have degree three or four [36]. There are results for building Steiner trees when the terminal set is restricted to zig-zags [4, 24], curves [37], ladders [21], and checkerboards [8, 11, 12]; for when the angles between edges are constrained $[13,14]$; for obstacle-avoiding Steiner trees [40, 41, 42, 43, 44] (which include geodesic versions where the terminals, Steiner points, and tree are contained in polygons); and for $k$-Steiner trees with $k$ as a fixed constant where you can use at most $k$ Steiner points (for terminals and Steiner points in various normed planes including the 2-D Euclidean plane, there is an $O\left(n^{2 k}\right)$-time algorithm) [10, 14, 28].

### 2.1 Tools

Without loss of generality, we consider the positive $x$-axis to be the basis for measuring angles, so that 0 radians is the positive $x$-axis, $\frac{\pi}{3}$ radians is a counterclockwise rotation of the positive $x$-axis about the origin by $\frac{\pi}{3}$ radians, etc.
Observation 1. Given a point set $V \subset \mathbb{R}^{2}$, if we build $\operatorname{MST}(V)$, each point $v \in V$ will have at most six neighbours in the MST. This is because, due to the sine law, for any two neighbours $w$ and $z$ of $v$ in $\operatorname{MST}(V)$ the angle $\angle w v z$ must be at least $\frac{\pi}{3}$ radians. These potential neighbours can be found by dividing the plane up into six interior-disjoint cones of angle $\frac{\pi}{3}$ all apexed on $v$. The closest point of $V$ to $v$ in each cone is the potential neighbour of $v$ in the MST in that cone.

Without loss of generality, suppose the input line $\gamma$ passes through the origin of the Euclidean plane with slope 0 . This line can be parametrized by $x$-coordinates. Let an interval on $\gamma$ be the set of points on $\gamma$ in between and including two fixed $x$-coordinates, called the endpoints of the interval. Our approach will be to divide the input line into $O(n)$ intervals using a special kind of Voronoi diagram outlined below. The intervals have the property that for any given interval $I$, if we compute $\operatorname{MST}(P \cup\{s\})$ for any $s \in I$, the subset of possible neighbours of $s$ in the MST is constant. For example, Fig. 2 shows a set $V$ of input points with the blue points labelled $p_{i}$ for $1 \leq i \leq 6$, the input line $\gamma$, and a green interval $I$. The plane is divided into six cones of $\frac{\pi}{3}$ radians, all apexed on the red point $x \in I$. In $\operatorname{MST}(V \cup\{x\})$, if $x$ connects to a point in cone $i$, it connects to $p_{i}$. The green interval $I$ has the property that this is true anywhere we slide $x$ and its cones in $I$.

### 2.1.1 Oriented Voronoi Diagrams

The 1-Steiner tree algorithm of Georgakopoulos and Papadimitriou (we refer to this algorithm as $G P A$ ) [28] works by subdividing the plane into regions defined by the cells of the Overlaid Oriented Voronoi Diagram (overlaid OVD). ${ }^{2}$ They show that the complexity of

[^2]

Figure 2: Every point along the green interval $I$ of $\gamma$ (i.e., between the $\ell$ endpoint and the $r$ endpoint) has the same potential MST neighbour (the blue points) in the same cone.
this diagram is $\Theta\left(n^{2}\right)$. Refer to the cone $\mathbb{K}$ defining an OVD as an $\boldsymbol{O V D}$-cone. Let $\mathbb{K}_{v}$ be a copy of the OVD-cone whose apex coincides with point $v \in \mathbb{R}^{2}$. OVDs are a type of Voronoi diagram made up of oriented Voronoi regions (OVRs) where the OVR of a site $p \in P$ is the set of points $w \in \mathbb{R}^{2}$ for which $p$ is the closest site in $\mathbb{K}_{w} \cap P$ [28]. If $\mathbb{K}_{w} \cap P=\varnothing$ we say $w$ belongs to an OVR whose site is the empty set. These notions are illustrated in Fig. 3.

Chang et al. [15] show that the OVD for a given OVD-cone of angle $\frac{\pi}{3}$ (e.g., the OVD in Fig. 3) can be built in $O(n \log n)$ time using $O(n)$ space. The OVD is comprised of segments and rays that are subsets of bisectors and cone boundaries which bound the OVRs. The size of the OVD is $O(n)$.

Since by Observation 1 a vertex of the MST has a maximum degree of six, by overlaying the six OVDs for the six cones of angle $\frac{\pi}{3}$ that subdivide the Euclidean plane (i.e., each of the six cones defines an orientation for a different OVD) the GPA partitions the plane into $O\left(n^{2}\right)$ regions. Each of these regions has the property that if we place a Steiner point $s$ in the region, the points of $P$ associated with this region (up to six possible points) are the only possible neighbours of $s$ in the MStT (similar to the example in Fig. 2). The GPA then iterates over each of these regions. In region $R$, the GPA considers each subset of possible neighbours associated with $R$. For each such subset it then computes the optimal location for a Steiner point whose neighbours are the elements of the subset, and then computes the length of the MStT using that Steiner point, keeping track of the best solution seen. The generalized algorithm for placing $k$ Steiner points [10, 14] essentially does the same thing $k$ times (by checking the topologies of the MStT for all possible placements of $k$ points), but is more complicated (checking the effects that multiple Steiner points have on the MStT is more complex).

### 2.1.2 Updating Minimum Spanning Trees

In order to avoid actually computing each of the candidate MSTs on the set of $P$ with the addition of our candidate Steiner points, we instead compute the differences in length be-


Figure 3: An example of an OVD for six points defined by the OVD-cone $\mathbb{K}$ with bounding rays oriented towards 0 and $\frac{\pi}{3}$. The six sites (i.e., the points) are the blue top-right points of the coloured OVRs. When intersected with $\gamma$, the OVD creates intervals along $\gamma$. Each interval corresponds to exactly one OVR, but an OVR may create multiple intervals (for example, the light-blue OVR creates the two orange intervals). The site corresponding to an interval outside of a coloured OVR is a special site represented by the empty set.
tween $\operatorname{MST}(P)$ and the candidate MStTs. Georgakopoulos and Papadimitriou [28] similarly avoid repeated MST computations by performing $O\left(n^{2}\right)$ preprocessing to allow them to answer queries of the following type in constant time: given that the edges $a b_{1}, a b_{2}, \ldots, a b_{j}$ are decreased by $\delta_{1}, \delta_{2}, \ldots, \delta_{j}$ for constant $j$, what is the new MST? They then use these queries to find the length of the MStT for each candidate Steiner point. Refer to [28] for details. Brazil et al. also perform some preprocessing in time $O\left(n^{2}\right)$ for $k=1$ and $O\left(n^{3}\right)$ otherwise [10]. However, using an approach involving an auxiliary tree and lowest common ancestor (LCA) queries, we can compute what we need in $o\left(n^{2}\right)$ time. We first compute $\operatorname{MST}(P)$, build an auxiliary tree in $O(n \log n)$ time, and process the auxiliary tree in $O(n)$ time [31] to support LCA queries in $O(1)$ time [7, 34]. This preprocessing helps us determine the benefit a given Steiner point provides by computing in $O(1)$ time the edges of the MST that disappear when the Steiner point is added. Details are outlined in the next section.

## 3 Algorithm

In this section we present Algorithm 1 and prove Theorem 1. Algorithm 1 computes OVDs for the six cones of angle $\frac{\pi}{3}$ that divide up the Euclidean plane (i.e., each of the six cones defines an orientation for a different OVD). Though they can be overlaid in $O\left(n^{2}\right)$ time, we do not need to overlay them. As mentioned in Section 2.1.1, each OVD has $O(n)$ size and is therefore comprised of $O(n)$ rays and segments. As illustrated in Fig. 3, intersecting any given OVD with a line $\gamma$ carves $\gamma$ up into $O(n)$ intervals since we have $O(n)$ rays and


Figure 4: The union of the trees in Figs. 4a and 4 b gives the graph in Fig. 4c with cycles $(s, b, d, a),(s, a, e, c)$, and $(s, b, d, a, e, c)$ whose longest edges excluding $s$ are $(d, a)$ and $(a, e)$.
segments, each of which intersects a line at most once. ${ }^{3}$ Each interval corresponds to an intersection of $\gamma$ with exactly one OVR of the OVD since OVDs are planar, but multiple non-adjacent intervals may be defined by the same OVR, as in Fig. 3. Therefore each interval $I$ is a subset of an OVR, and for every pair of points $u_{1}, u_{2} \in I$ the closest point in $\mathbb{K}_{u_{1}} \cap P$ is the same as in $\mathbb{K}_{u_{2}} \cap P$, where $\mathbb{K}$ is the OVD-cone of the OVD being considered.

If we do this with all six OVDs, $\gamma$ is subdivided into $O(n)$ intervals. As in Fig. 2, each interval $I$ has the property that for any point $u \in I$, if we were to build $\operatorname{MST}(P \cup\{u\})$, the ordered set of six potential neighbours is a constant set of constant size. ${ }^{4}$ Each element of this ordered set is defined by a different OVD and corresponds to the closest point in $\mathbb{K}_{u} \cap P$. In each interval we solve an optimization problem to find the optimal placement for a Steiner point in that interval (i.e., minimize the sum of distances of potential neighbours to the Steiner point) which takes $O(1)$ time since each of these $O(1)$ subproblems has $O(1)$ size.

Once we have computed an optimal placement for a Steiner point for each computed interval of our input line $\gamma$, we want to compute which one of these $O(n)$ candidates produces the MStT, i.e., the candidate $s$ that produces the smallest length of the $\operatorname{MST}(P \cup\{s\})$. Let $T^{*}$ be the union of $\operatorname{MST}(P)$ and $\operatorname{MST}(P \cup\{s\})$, as in Fig. 4. For a candidate Steiner point $s$, the savings gained by including $s$ in the MST are calculated by computing the sum $\Delta_{s}$ of the lengths of the longest edge on each cycle of $T^{*}$ (excluding the edges incident to $s$ ), and then subtracting the sum $\sigma_{s}$ of the lengths of the edges incident to $s$ in $\operatorname{MST}(P \cup\{s\})$. For example, in Fig. 4c, the candidate edges on the left cycle are $(b, d)$ and $(d, a)$, and on the right cycle they are $(a, e)$ and $(e, c)$; we sum the lengths of the longest candidate edge from each cycle, i.e., $(d, a)$ and $(a, e)$, and subtract the sum of the lengths of edges $(s, a),(s, b)$, and $(s, c)$ to calculate the savings we get from choosing $s$ as the solution Steiner point. Note that the longest edge on the cycle $(s, b, d, a, e, c)$ is either $(d, a)$ or $(a, e)$. As will be seen in the proof of Theorem 1, the sum $\sigma_{s}$ is computed when determining $s$. What remains to find are the lengths of the longest edges of $\operatorname{MST}(P)$ on the cycles of $T^{*}$ which will then be used to compute $\Delta_{s}$.

[^3]Lemma 1 (Bose et al. 2004 [7, paraphrased Theorem 2]). A set of $n$ points in $\mathbb{R}^{2}$ can be preprocessed in $O(n \log n)$ time into a data structure of size $O(n)$ such that the longest edge on the path between any two points in the MST can be computed in $O(1)$ time.

Lemma 1 by Bose et al. [7] tells us that with $O(n \log n)$ preprocessing of $\operatorname{MST}(P)$, we can compute the sum $\Delta_{s}$ in $O(1)$ time for each candidate Steiner point. ${ }^{5}$ First an auxiliary binary tree is computed whose nodes correspond to edge lengths and whose leaves correspond to points of $P$. This tree has the property that the LCA of two leaves is the longest edge on the path between them in $\operatorname{MST}(P)$. Using $O(n)$ preprocessing time and space on the auxiliary tree they perform $O(1)$-time LCA queries $[3,31,38]$.

It was shown by Rubinstein et al. [36] that the addition of a Steiner point of degree five cannot appear in a least-cost planar network, meaning we can restrict our search to Steiner points of degree three or four in the Euclidean norm. Let a simple cycle ${ }^{6}$ through points $s^{\prime}, s, s^{\prime \prime}$ be denoted $\mathcal{C}_{s^{\prime} s s^{\prime \prime}}$. In the next two lemmas, when referring to cycle $\mathcal{C}_{s^{\prime} s s^{\prime \prime}}$, we are referring to the cycle in $T^{*}$ that passes through the Steiner point $s$ and its two terminalneighbours $s^{\prime}$ and $s^{\prime \prime}$. When we refer to the longest edge on the cycle we mean the longest edge on the path between $s^{\prime}$ and $s^{\prime \prime}$ in $\operatorname{MST}(P)$.

Lemma 2 (Brazil et al. 2015 [10], rephrased Theorem 11). Let the Steiner point $s$ in $T^{*}$ have three neighbours $a, b$, and $c$. Using $O(1)$-time $L C A$ queries, removing the longest edges on the $\binom{3}{2}$ cycles $\mathcal{C}_{\text {bsa }}, \mathcal{C}_{\text {bsc }}$, and $\mathcal{C}_{\text {csa }}$ results in a tree which can be computed in $O(1)$ additional time and space.

Lemma 3 (Brazil et al. 2015 [10], rephrased Theorem 11). Let the Steiner point $s$ in $T^{*}$ have four neighbours $a, b, c$, and $d$. Using $O(1)$-time $L C A$ queries, removing the longest edges on the $\binom{4}{2}$ cycles $\mathcal{C}_{b s a}, \mathcal{C}_{c s b}, \mathcal{C}_{d s c}, \mathcal{C}_{c s a}, \mathcal{C}_{d s b}$, and $\mathcal{C}_{d s a}$ results in a tree which can be computed in $O(1)$ additional time and space.

By Lemmas 2 and 3 from Brazil et al. [10], removing the longest edges on the cycles between the neighbours of $s$ in $T^{*}$ results in a tree (i.e., one connected component).

We are now ready to finish proving Theorem 1. Though stated for input lines, it is clear that the same result applies for rays and line segments. Rather than creating pointlocation data structures in the OVDs to locate the endpoints and thus determine the correct labels (e.g., in the case where $\gamma$ is a segment that lies completely in OVRs and has no intersections with the OVD boundaries), we notice that the intersections with the OVDs give us endpoints for intervals on $\gamma$. We can extend the line through $\gamma$ and run Algorithm 1 on this extension, adding special interval-delimiting points corresponding to the endpoints of the ray or line segment, and then only process intervals within the interval-delimiting points. These special interval-delimiting points do not affect the labels of the intervals on either side of them.

Theorem 1. Given a set $P$ of $n$ points in the Euclidean plane and a line $\gamma$, Algorithm 1 computes in optimal $\Theta(n \log n)$ time and optimal $\Theta(n)$ space a minimum-weight tree connecting all points in $P$ using at most one extra point $s \in \gamma$.

[^4]```
Algorithm 1: Restricted 1-Steiner
    input : set \(P \subset \mathbb{R}^{2}\) of \(n\) points, a line \(\gamma\)
    output: the MStT for \(P\) using at most one Steiner point \(s \in \gamma\)
    \(s=(\infty, \infty)\);
    \(T=\operatorname{MST}(P)\);
    \(T^{\prime}=\) longest-edge auxiliary tree built from \(T\);
    \(\mathbb{L}=\emptyset ;\)
    for \(i=0, i \leq 5,++i\) do
        Compute \(\mathrm{OVD}_{i}\) and augment the edges with labels corresponding to the
        sites of the two adjacent OVRs;
        \(\mathbb{L}=\mathbb{L} \cup\) set of rays and segments defining the edges of \(\mathrm{OVD}_{i}\);
    end
    for each \(\ell \in \mathbb{L}\) do
        Compute the intersections of \(\ell\) with \(\gamma\) adding the label of \(\ell\) to the
        appropriate intersection points (and implicitly the intervals);
    end
    Sort the intersection points along \(\gamma\) to create the list of labelled intervals
    produced by these intersections;
    for each interval I along \(\gamma\) do
        for each subset of the labels of \(I\) of sizes 3 and 4 do
            \(u=\) an optimal Steiner point along \(\gamma\) for the subset considered;
                /* This computation of an optimal point also gives us the sum
                \(\sigma_{u}\)
                        */
                if \(|\operatorname{MST}(P \cup\{u\})|<|\operatorname{MST}(P \cup\{s\})|\) then
                /* The test in this condition is performed using the sum
                \(\sigma_{u}\) and using \(T^{\prime}\) to compute the sum \(\Delta_{u} \quad\) */
                \(s=u ;\)
                end
        end
    end
    if \(|\operatorname{MST}(P \cup\{s\})|<|T|\) then
        return \(\operatorname{MST}(P \cup\{s\})\);
    end
    else
        return \(T\);
    end
```

Proof. The tree $T=\operatorname{MST}(P)$ and its length are computed in $O(n \log n)$ time and $O(n)$ space by computing the Delaunay triangulation in $O(n \log n)$ time and $O(n)$ space and then computing the MST of the Delaunay triangulation in $O(n)$ time and space [18, 35]. By Lemma 1, in $O(n \log n)$ time and $O(n)$ space we compute the longest-edge auxiliary tree $T^{\prime}$ and preprocess it to answer LCA queries in $O(1)$ time. Each of the six OVDs is then computed in $O(n \log n)$ time and $O(n)$ space $[10,15,19]$. In $O(n)$ time and space we extract $\mathbb{L}$, the set of rays and segments defining each OVR of each OVD. While computing the OVDs, in $O(n)$ time we add labels to the boundary rays and segments describing which OVD-cone defined them and the two sites corresponding to the two OVRs they border.

Since $\gamma$ is a line, it intersects any element of $\mathbb{L}$ at most once. Therefore, computing the intersections of $\gamma$ with $\mathbb{L}$ takes $O(n)$ time and space. Assume without loss of generality that $\gamma$ is the $x$-axis. Given our $O(n)$ intersection points, we can make a list of the $O(n)$ intervals they create along $\gamma$ in $O(n \log n)$ time and $O(n)$ space by sorting the intersection points by $x$-coordinate and then walking along $\gamma$. During this process we also use the labels of the elements of $\mathbb{L}$ to label each interval with its six potential neighbours described above in $O(1)$ time per interval.

An optimal Steiner point has degree greater than two by the triangle inequality. Rubinstein et al. [36] showed that an optimal Steiner point has degree no more than four. Therefore an optimal Steiner point has degree three or four. Once we have computed our labelled intervals, we loop over each interval looking for the solution by finding the optimal placement of a Steiner point in the interval for a constant number of fixed topologies. Consider an interval $I$ and its set of potential neighbours $P^{\prime} \subset P$ of size at most six. For each subset $\mathbb{P}$ of $P^{\prime}$ of size three and four (of which there are $O(1)$ such subsets), we compute a constant number of candidate optimal Steiner points in $\gamma$. Note that $\gamma$ is actually a polynomial function, $\gamma(x)$. Our computation is done using the following function $d_{\mathbb{P}}(x)$, where $a_{x}$ and $a_{y}$ are the $x$ - and $y$-coordinates of point $a$ respectively, and $\gamma(x)$ is the evaluation of $\gamma$ at $x: d_{\mathbb{P}}(x)=\sum_{a \in \mathbb{P}} \sqrt{\left(a_{x}-x\right)^{2}+\left(a_{y}-\gamma(x)\right)^{2}}$. We then take the derivative of this function and solve for the global minima by finding the roots within the domain specified by the endpoints of $I$. Since the size of $\mathbb{P}$ is bounded by a constant and since the degree of the polynomial $\gamma$ is a constant, this computation takes $O(1)$ time and $O(1)$ space and the number of global minima is a constant. Note that the value of this function at a particular $x$ for a particular $\mathbb{P}$ tells us the sum $\sigma_{u}$ of edge lengths from the point $u=(x, \gamma(x))$ to the points in $\mathbb{P}$. We associate this value with $u$. Out of the $O(1)$ candidate points, we choose the one for which $d_{\mathbb{P}}(x)$ is minimum. We can break ties arbitrarily, since a tie means the points offer the same amount of savings to the MST since they both have the same topology in the MST (meaning they have the same cycles in $\operatorname{MST}(P) \cup \operatorname{MST}(P \cup\{u\})$ ), and since the value of $d_{\mathbb{P}}(x)$ being the same means that the sum of adjacent edges is the same.

Once we have our $O(1)$ candidate optimal Steiner points for $I$, we need to compare each one against our current best solution $s$. In other words, for each candidate $u$ we need to compare $|\operatorname{MST}(P \cup\{u\})|$ with $|\operatorname{MST}(P \cup\{s\})|$. We take advantage of the following: if we compute the union of $\operatorname{MST}(P)$ and $\operatorname{MST}(P \cup\{u\})$ we get at most $\binom{4}{2}=6$ simple cycles through $u$. Let this connected set of cycles be $Q$. We have $|\operatorname{MST}(P \cup\{u\})|=$ $|\operatorname{MST}(P)|+\sigma_{u}-\Delta_{u}$, where $\Delta_{u}$ is the sum of the longest edge in each cycle of $Q$ excluding from consideration the edges incident to $u$. By Lemma 1, we can compute $\Delta_{u}$ in $O(1)$ time
using $T^{\prime}$. By Lemmas 2 and 3, removing the longest edge from each cycle of $Q$ results in a tree. If $|\operatorname{MST}(P \cup\{u\})|<|\operatorname{MST}(P \cup\{s\})|$ we set $s=u$.

Finally, we check if $|\operatorname{MST}(P \cup\{s\})|<|T|$. If so, we return $\operatorname{MST}(P \cup\{s\})$. Otherwise we return $T$.

Now we show the space and time optimality. The $\Omega(n)$-space lower bound comes from the fact that we have to read in the input. The $\Omega(n \log n)$-time lower bound comes from a reduction from the closest pair problem ( $\mathbf{C P P}$ ). The CPP is where we are given $n$ points in $\mathbb{R}^{2}$ and we are supposed to return a closest pair with respect to Euclidean distance. The CPP has an $\Omega(n \log n)$-time lower bound [35, Theorem 5.2]. Indeed, given an instance of CPP, we can transform it into our problem in $O(n)$ time by using the points as the input points $P$ and choosing an arbitrary $\gamma$.

Given the solution to our problem, we can find a closest pair in $O(n)$ time by walking over the resulting tree. First, remove the Steiner point (if any) and its incident edges to break our tree up into $O(1)$ connected components. Consider one of these components $\mathbb{C} . \mathbb{C}$ may contain both points of multiple closest pairs, or none. Imagine $\mathbb{C}$ contained both points for exactly one closest pair. Then the edge connecting them will be in $\mathbb{C}$ and it will be the edge with minimum-weight in $\mathbb{C}$; otherwise it contradicts that we had a minimum-weight tree. Imagine $\mathbb{C}$ contained both points for multiple closest pairs. Pick one of the closest pairs. If $\mathbb{C}$ does not contain the edge $e$ connecting the two points of the pair, then there is a path between them in $\mathbb{C}$ consisting of minimum-weight edges (whose weights match $e$ ) connecting other closest pairs; otherwise we contradict the minimality of our tree or that both points were in the same connected component. If no component contains both points of a closest pair, then the path between a closest pair goes through the Steiner point. Once again, choose a closest pair $(a, b)$ and let the edge connecting this closest pair be $e$. Due to the minimality of our tree, the weight of every edge on the path between $a$ and $b$ is no more than that of $e$. However, since no component contains a closest pair, that means that $a$ and $b$ are incident to the Steiner point. Therefore, we get a solution to the CPP by walking over our resulting tree and returning the minimum among a minimum-weight edge connecting neighbours of the Steiner point and a minimum-weight edge seen walking through our tree excluding edges incident to the Steiner point.

Corollary 1. Given a set $P$ of $n$ points in the Euclidean plane and $j$ lines $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$, by running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j n \log n)$ time and $O(n+j)$ space a minimumweight tree is computed that connects all points in $P$ using at most one point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

Until this point, we have implicitly assumed that the $\operatorname{MST}(P)$ is unique. The algorithm still produces a correct result if this is not the case. To see this, consider the case when there are multiple MSTs for $P$. Imagine we have computed a Steiner tree $\tau$ using a MST $T$ of $P$ and let the Steiner point be $s$. Consider the overlay $T^{*}=T \cup \tau$. Now consider a Steiner tree $\tau^{\prime}$ produced using a different MST $T^{\prime}$ of $P$. Let the Steiner point of $\tau^{\prime}$ also be $s$ and let the set of neighbours of $s$ be the same in $\tau$ and $\tau^{\prime}$. Consider the overlay $T^{\boldsymbol{\ell}}=T^{\prime} \cup \tau^{\prime}$. If we compare the differences between $T^{*}$ and $T^{\boldsymbol{\kappa}}$, there are two cases. In the first case, the cycles through $s$ are all the same, in which case the change in the MST did not affect the Steiner tree computation. In the second case, at least one cycle through $s$
and its neighbours is different. This means that for some pair of neighbours of $s$, there is an edge in $T$ on the path between them that is swapped for an edge of equal weight between some other pair of vertices of $P$ in $T^{\prime}$. The weight of the two trees $T$ and $T^{\prime}$ is the same. Since $s$ is connected to the same neighbours, $\sigma_{s}$ is the same for both $\tau$ and $\tau^{\prime}$. So what may change is $\Delta_{s}$. However, the values that compose $\Delta_{s}$ are selected using the auxiliary tree that gives the bottleneck edge between two vertices of the MST. Since a MST on a point set $P$ is transformed into another MST on $P$ by substituting one edge at a time for one of equal weight, if the two neighbours of $s$ have bottleneck edges of different weights in $T$ and $T^{\prime}$, that implies that one of the MSTs is shorter than the other. Thus, $\Delta_{s}$ is the same for both $\tau$ and $\tau^{\prime}$, and $|\tau|=\left|\tau^{\prime}\right|$.

There is another case to consider. Let the Steiner points for $\tau$ and $\tau^{\prime}$ be $s$ and $s^{\prime}$ respectively. They may or may not be the same point, but let them have different sets of neighbours. If $|\tau| \neq\left|\tau^{\prime}\right|$, then the savings for one tree are larger than the other; i.e., without loss of generality, $\Delta_{s}-\sigma_{s}<\Delta_{s^{\prime}}-\sigma_{s^{\prime}}$. However, when computing $\tau$, the algorithm considered $s^{\prime}$ and its neighbour set and so computed the same $\sigma_{s^{\prime}}$. That implies that the sum of bottleneck edges is different between the two trees $T$ and $T^{\prime}$, which (as mentioned above) is a contradiction. Thus, the algorithm produces a $\operatorname{MStT}$ even if $\operatorname{MST}(P)$ is not unique.

## 4 Towards Generalization

In Section 4.1 we show how to adapt the $O\left(n^{2 k}\right)$-time algorithm of Brazil et al. [10] to solve the $k$-Steiner tree problem for $k>1$ in $O\left((j n)^{k}\right)$ time when the Steiner points are restricted to lie on $j$ input lines and when the input abides by the restrictions imposed by Brazil et al. In Sections 4.2 and 4.3 we show that under certain restrictions, we can apply the results of Brazil et al. [10] to our restricted $k$-Steiner tree problem allowing Algorithm 1 and the adapted $k$-Steiner tree algorithm to maintain the same time and space bounds while solving the problem for norms other than Euclidean and tree cost functions other than the sum of the edge-weights. Finally, in Section 4.4 we show that we can solve the restricted $k$-Steiner tree problem when the set of Steiner points is constrained to a restricted class of $j$ input curves rather than lines. The running time comes to depend on $j$, the runtime of certain primitive operations, and the complexity of the zone of the curves in an arrangement of lines.

## $4.1 k$ Steiner Points

Given a set $P$ of $n$ points in the Euclidean plane $\mathbb{R}^{2}$ and a fixed constant positive integer $k$, the algorithm of Brazil et al. [10] solves the $k$-Steiner tree problem in $O\left(n^{2 k}\right)$ time with an $O\left(n^{2}\right)$-time preprocessing step for $k=1$ and an $O\left(n^{3}\right)$-time preprocessing step for $k>1 .{ }^{7}$

[^5]Below we show that straightforward adjustments to the algorithm of Brazil et al. [10] allow their algorithm to be used when Steiner points are constrained to lie on $j$ input lines $\gamma_{1}, \ldots \gamma_{j}$.

Definition 1 (Fixed topology $k$-Steiner tree problem, Brazil et al. 2015 [10] §4). Given a set $A$ of at most $6 k$ embedded terminals, a set $S$ of $k$ free (i.e., non-embedded) Steiner points, and a tree topology $\tau$ spanning $A \cup S$, find the coordinates of the Steiner points (i.e., find the set $S$ ) such that the sum of the edge-weights in the tree is minimized.

Restriction 1 (Brazil et al. 2015 [10]). A solution to the fixed topology $k$-Steiner tree problem is computable within any fixed precision in $f(k)$ time, where $f(k)$ is a function dependent only on $k$.

Since $k$ is a constant, Restriction 1 says the fixed topology $k$-Steiner tree problem can be solved in $O(1)$ time.

Definition 2 (Feasible internal topology, viable forest, minimum $F$-fixed spanning tree, Brazil et al. 2015 [10]). A forest $F$ is said to have a feasible internal topology provided that its node set is $A \cup S$ where $A \subseteq P$ is the set of leaves of $F$ and the Steiner points $S \subseteq \mathbb{R}^{2}$ are the internal nodes. A feasible internal topology with $|S| \leq k$ is called viable if and only if every Steiner point in $S$ has at most six neighbours in $F$. A shortest total-length tree $T_{F}$ on $P \cup S$ such that the set of neighbours of Steiner points in $T_{F}$ is the same as in $F$ is referred to as a minimum $F$-fixed spanning tree.

Brazil et al. [10] compute the overlaid OVD (as does the GPA [28]) which, similar to our intervals from Algorithm 1, has the property that each region has associated with it a set of points: one for each OVD overlaid. For overlaid OVD region $R_{i}$, let this neighbour set be $C_{P}\left(R_{i}\right)$. Under Restriction 1, the algorithm of Brazil et al. works as follows to produce a minimum-weight tree connecting all points in $P$ using at most $k$ points of $\mathbb{R}^{2} \backslash P$ :

1. Compute the overlaid OVD of $P$
2. Compute $T=\operatorname{MST}(P)$
3. Compute the longest edge on the path between every pair of points $x$ and $y$ in $T$
4. Compute a table $H$ whose entry is true for edge $e$ and terminals $y$ and $z$ if and only if $e$ is on the shortest path between $y$ and $z$ in $T$
5. For every $k^{\prime} \leq k$ and each choice (with repetition) of $k^{\prime}$ regions $R_{1}, \ldots, R_{k^{\prime}}$ of the overlaid OVD
(a) Associate the free Steiner point $s_{i}$ with region $R_{i}$
(b) Let $G$ be the graph consisting of the vertices $\cup C_{P}\left(R_{i}\right) \cup\left\{s_{1}, \ldots, s_{k^{\prime}}\right\}$, all edges $\left(s_{i}, s_{j}\right), i \neq j$, and all edges $\left(s_{i}, x\right)$ for every $x \in C_{P}\left(R_{i}\right)$
(c) Let $G^{*}$ be the set of all viable subforests of $G$
(d) For each $\mathcal{F} \in G^{*}$
for $k=1$ in the $L_{1}$ metric, and an $O\left((7 k+1)^{7 k-1} k^{4} \cdot n \log ^{2} n\right)$-time algorithm for the $L_{1}$ and $L_{\infty}$ metrics.
i. Solve the fixed topology $k$-Steiner tree problem for $\mathcal{F}$ to get the forest $F$
ii. Use $T$ and $F$ to compute a minimum $F$-fixed spanning tree $T_{F}$
6. Let $T^{*}$ be a smallest total cost $T_{F}$ produced
7. Let $S$ be the set of Steiner points of $T^{*}$

We now consider the worst-case time and space upper bounds of this algorithm. Most of the time-bounds are discussed in Brazil et al. [10].

- Step 1, computing the overlaid OVD, runs in $O\left(n^{2}\right)$ time and space. ${ }^{8}$
- Step 2, computing $T=\operatorname{MST}(P)$, is done in $O(n \log n)$ time and $O(n)$ space.
- Step 3, computing the longest edge on the path between every pair of terminals in $T$, takes $O\left(n^{2}\right)$ time and space. This is done by computing the MST and doing a depth-first traversal from every terminal [14, §1.4.3 pg. 50].
- Step 4, computing the table $H$ that tells us if a specific edge is on the path in $T$ between two given points, takes $O\left(n^{3}\right)$ time and space. This is only done for $k>1$.
- The loop on step 5, choosing combinations for matching oriented OVD cells with up to $k$ Steiner points, has $O\left(n^{2 k}\right)$ iterations. This follows because:
- each OVD is a linear arrangement of $O(n)$ complexity [15];
- the overlay of a constant number of linear arrangements (each of whose complexity is $O(n))$ has $O\left(n^{2}\right)$ complexity [23, §8.3] [30, §5.4, §28];
- we are checking each topology that may be a solution. Since the solution will have $k^{*}$ Steiner points for $0 \leq k^{*} \leq k$, we iterate through each topology for each choice of up to $k$ Steiner points. This means associating one of the $O\left(n^{2}\right)$ faces of the overlaid OVD with each of our up to $k$ Steiner points. We allow multiple Steiner points to be associated with any given face (i.e., a many-to-one relationship may exist) since Steiner points may be connected to other Steiner points. Therefore, we have a sum of combinations with repetition of the up to $k$ Steiner points choosing from the $O\left(n^{2}\right)$ faces, giving us:

$$
\begin{aligned}
& \sum_{k^{\prime}=0}^{k}\binom{n^{2}+k^{\prime}-1}{k^{\prime}} \\
= & \sum_{k^{\prime}=0}^{k} \frac{\left(n^{2}+k^{\prime}-1\right)!}{k^{\prime}!\left(n^{2}-1\right)!} \\
= & \sum_{k^{\prime}=0}^{k} \frac{n^{2}\left(n^{2}+1\right)\left(n^{2}+2\right) \cdots\left(n^{2}+k^{\prime}-1\right)}{k^{\prime}!}
\end{aligned}
$$

[^6]By Stirling's approximation we have $\left(\frac{k^{\prime}}{e}\right)^{k^{\prime}} \leq k^{\prime}$ !. Thus, we can upper-bound the expression as:

$$
\begin{aligned}
& \sum_{k^{\prime}=0}^{k} \frac{n^{2}\left(n^{2}+1\right)\left(n^{2}+2\right) \cdots\left(n^{2}+k^{\prime}-1\right)}{k^{\prime}!} \\
\leq & \sum_{k^{\prime}=0}^{k} \frac{e^{k^{\prime}} n^{2}\left(n^{2}+1\right)\left(n^{2}+2\right) \cdots\left(n^{2}+k^{\prime}-1\right)}{k^{\prime k^{\prime}}} \\
\leq & \sum_{k^{\prime}=0}^{k} \frac{3^{k^{\prime}}\left(n^{2}+k^{\prime}-1\right)^{k^{\prime}}}{k^{\prime k^{\prime}}} \\
\leq & \sum_{k^{\prime}=0}^{k}\left(\frac{3\left(n^{2}+k^{\prime}\right)}{k^{\prime}}\right)^{k^{\prime}} \\
\leq & \sum_{k^{\prime}=0}^{k}\left(\frac{3 n^{2}}{k^{\prime}}+3\right)^{k^{\prime}} \\
\leq & \sum_{k^{\prime}=0}^{k}\left(\frac{4 n^{2}}{k^{\prime}}\right)^{k^{\prime}} \\
\leq & \sum_{k^{\prime}=0}^{k}\left(\frac{4}{k^{\prime}}\right)^{k^{\prime}} n^{2 k^{\prime}}
\end{aligned}
$$

Since $k^{\prime}$ is a constant, we have:

$$
\begin{aligned}
& \sum_{k^{\prime}=0}^{k}\left(\frac{4}{k^{\prime}}\right)^{k^{\prime}} n^{2 k^{\prime}} \\
\in & \sum_{k^{\prime}=0}^{k} O\left(n^{2 k^{\prime}}\right) \\
\in & O\left(k n^{2 k}\right)
\end{aligned}
$$

Since $k$ is a constant, this leaves us with $O\left(n^{2 k}\right)$.

- Step 5a, associating the Steiner points with an oriented OVD cell, takes $O(k)=O(1)$ time and space.
- Step 5 b , building the complete graph on the Steiner points and adding the edges to the appropriate candidate neighbours from their oriented OVD cells, takes $O\left(k^{2}\right)=O(1)$ time and space.
- Step 5c, creating the set of viable subforests, is mostly listed for exposition. Rather than computing this all at once, to save space it would likely be set up as a function call in the loop condition of step 5d that returns the "next" viable subforest given some index counter, so we will not count the time and space requirements of this step.
- Given $G$, grabbing the next viable subforest of $G^{*}$ in the loop condition of step 5d takes $O(k)$ time and space because a tree on $O(k)$ points has $O(k)$ edges and in the worst case the next enumeration has to change all of them. By Cayley's formula and the fact that each spanning forest is a subgraph of a spanning tree with $O(k)$ edges, the number of iterations of this loop is $O\left(126^{k} \cdot k^{k}\right)=O(1)$ [10].
- By Restriction 1, step 5(d)i takes $f(k)=O(1)$ time and space.
- The subroutine they invoke at step $5(\mathrm{~d})$ ii runs in $O\left(k^{2}\right)$ time and $O(k)$ space for $k=1$ and $O\left(k^{2 k+3} \cdot k!\right)$ time and $O\left(k^{2}\right)$ space otherwise. Either way, it is $O(1)$ time and space.
- Step 6 is also just for exposition. At the end of step $5(\mathrm{~d})$ ii, the cost of the new tree is compared to the cost of the best tree seen which is then updated accordingly. This takes $O(1)$ time and space in each iteration of the loop on step 5 d .
- Step 7, reporting the Steiner points in the solution tree, takes $O(k)=O(1)$ time and space.

We begin our adjustment. Rather than computing the overlaid OVD, we compute $O(j n)$ intervals on $\gamma_{1}, \ldots \gamma_{j}$ as in Algorithm 1. The main for-loop at step 5 becomes: "for each choice (with repetition) of $k^{\prime} \leq k$ intervals, $I_{1}, \ldots I_{k^{\prime}}$, of which there are $O\left((j n)^{k}\right)$ iterations. ${ }^{9}$ Steiner point $s_{i}$ is then associated with interval $I_{i}$, and rather than the "neighbour set of region $i^{\prime \prime}$, we use the set of candidate neighbours of interval $I_{i}$, much like we do in Algorithm 1. The enumeration of the topologies in which we are interested (i.e., the number of iterations of the inner-most for-loop, step 5d) remains the same. We replace their Restriction 1 with one reflecting our problem.

Restriction 2. A solution to the fixed topology $k$-Steiner tree problem where each Steiner point is constrained to lie on its own specified line (not necessarily distinct from the lines of the other Steiner points) is computable within any fixed precision in $f(k)$ time and $f_{s p}(k)$ space, where $f(k)$ and $f_{s p}(k)$ are functions dependent only on $k$.

With Restriction 2, the first step of the inner-most for-loop, step $5(\mathrm{~d}) \mathrm{i}$, runs in $O(1)$ time and space. The second step of the inner-most for-loop, step 5(d)ii, does not change and still runs in $O(1)$ time and space. However, as with our approach for Algorithm 1, by Lemma 1 we are able to use LCA queries to avoid performing the $O\left(n^{2}\right)$ time and space preprocessing in Step 3. Similarly, we are able to use LCA queries to avoid the $O\left(n^{3}\right)$ time and space preprocessing from Step 4. In Brazil et al. [10], for $k>1$, a TRUE/FALSE table is computed in $O\left(n^{3}\right)$ time and space to be able to answer the following query $\mathcal{Q}_{e, y, z}$ in $O(1)$ time: given an edge $e$ of the MST $T$ and two vertices $y$ and $z$, is $e$ on the path between $y$ and $z$ in $T$ ?

[^7]Lemma 4. A given set $P$ of points in $\mathbb{R}^{2}$ can be preprocessed in $O(n \log n)$ time and $O(n)$ space to construct a data structure that supports $\mathcal{Q}_{e, y, z}$ queries in $O(1)$ time.

Proof. As with Theorem 1, we compute $T=\operatorname{MST}(P)$ in $O(n \log n)$ time and $O(n)$ space and then we root $T$ at an arbitrary vertex. We then preprocess $T$ in $O(n)$ time and space, like we did in Section 3, enabling us to perform $O(1)$-time LCA queries [3, 31, 38].

In a tree, there is a unique path between two vertices. LCA queries help us answer $\mathcal{Q}_{e, y, z}$ as follows. We have two cases to consider. In the first case, one of the two query vertices is an ancestor of the other in the rooted tree $T$; and in the other case, $\mathrm{LCA}(y, z)=r$ (i.e., the LCA of $y$ and $z$ is a third vertex $r$ ). Let us consider the first case. Without loss of generality, let $z$ be an ancestor of $y$ (in which case $\operatorname{LCA}(y, z)=z$ ). To traverse $T$ from $y$ to $z$, we follow parent-pointers until we reach $z$. If $e=(a, b)$ is on the path between $y$ and $z$ in $T$, then it would be seen when following the parent-pointers. This means that, in this case, $e$ is on the path between $y$ and $z$ if and only if $\operatorname{LCA}(y, z)=z, \operatorname{LCA}(y, a)=a$, $\operatorname{LCA}(y, b)=b, \operatorname{LCA}(a, z)=z$, and $\operatorname{LCA}(b, z)=z$.

In the second case, $\operatorname{LCA}(y, z)=r$. If $e=(a, b)$ is on the path between $y$ and $z$ in $T$, then it would be seen either when following the parent-pointers from $y$ to $r$, or when following them from $z$ to $r$. Without loss of generality, assume $e$ is seen when traversing the parent-pointers from $z$ to $r$. This means that, in this case, $e$ is on the path between $y$ and $z$ if and only if $\operatorname{LCA}(y, z)=r, \operatorname{LCA}(y, a)=r, \operatorname{LCA}(y, b)=r, \operatorname{LCA}(a, z)=a$, and $\operatorname{LCA}(b, z)=b$.

The approaches of Lemmas 1 and 4 using LCA queries apply both in our setting as well as to the results of Brazil et al. [10], reducing the time and space of steps 3 and 4 from $O\left(n^{2}\right)$ for $k=1$ and $O\left(n^{3}\right)$ for $k>1$ to $O(n \log n)$ time and $O(n)$ space for all $k$, thus also reducing their overall space usage to the $O\left(n^{2}\right)$ space used to build the overlaid OVD. Making the appropriate substitutions in their analysis we get the following corollary for our scenario.

Corollary 2. Given a set $P$ of $n$ points in the Euclidean plane $\mathbb{R}^{2}$, a constant integer $k>1$, and a set of $j$ input lines $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$, under Restriction 2 the modified MStT algorithm of Brazil et al. [10] solves the restricted $k$-Steiner tree problem in $O\left((j n)^{k}\right)$ time and $O(j n)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

Brazil et al. in [10] actually study what they call the generalized $k$-Steiner tree problem, presenting the algorithm outlined in this section for the $k$-Steiner tree problem allowing norms other than Euclidean and allowing different cost functions for the weight of a tree. ${ }^{10}$ Following the approach in [10], our restricted Steiner problem can also be solved in other norms and cost functions. To achieve the runtime and space bounds of Theorem 1, Algorithm 1 takes advantage of the fact that: in the Euclidean norm, a vertex in $\operatorname{MST}(P)$ has a constant maximum degree; the plane can be partitioned the same way into a constant number of regions around any point $u \in P$ and in each region the number of potential neighbours

[^8]of $u$ in the MST is constant; and the OVDs and MST can be computed in $O(n \log n)$ time and $O(n)$ space.

The results of Brazil et al. [10] allow us to keep the same bounds as Theorem 1 and Corollary 2 with different norms and cost functions. The fact that the algorithm from Corollary 2 works for different norms and cost functions (subject to the restrictions presented below) follows from Brazil et al. [10], the fact that computing the intervals along the lines in $\Gamma$ created by the OVDs does not depend on either the norm or tree cost function, the fact that we assume that the weight of an edge can be calculated in constant time, and the fact that finding the weight of a solution tree can still be done in $O(n)$ time. Below we show that the bounds on Algorithm 1 do not change under the same set of norms and cost functions considered by Brazil et al.

### 4.2 Other Norms

We begin with some definitions and notation from Brazil et al. [10]. Let $\|\cdot\|$ be a given norm on $\mathbb{R}^{2}$, i.e., a function $\|\cdot\|: \mathbb{R}^{2} \rightarrow \mathbb{R}$ that satisfies $\|x\| \geq 0$ for all $x \in \mathbb{R}^{2},\|x\|=0$ if and only if $x=0,\|r x\|=|r| \cdot\|x\|$ for $r \in \mathbb{R}$ with $|\cdot|$ the absolute value function, and $\|x+y\| \leq\|x\|+\|y\|$ for all $x, y \in \mathbb{R}^{2}$. The unit ball $B=\{x:\|x\| \leq 1\}$ is a centrally symmetric convex set. Let $\partial(B)$ denote the boundary of $B$.

We now alter our definition of $\operatorname{MST}(\cdot)$ to use the given norm $\|\cdot\|$ rather than the Euclidean norm when computing the length of the tree. To ensure the MST and required OVDs can still be constructed in $O(n \log n)$ time and $O(n)$ space under the different norms, and that the maximum degree of a vertex in the MST is still a constant, Brazil et al. impose some restrictions [10]. We also assume that computing the distance between two points in the given norm takes constant time.

Restriction 3 (Brazil et al. 2015 [10]). The intersection points of any two translated copies of $\partial(B)$, and the intersection points of any straight line and $\partial(B)$, can be calculated to within any fixed precision in constant time.

Lemma 5 (Brazil et al. 2015 [10], Lemma 3). There exist six points $\left\{y_{i}: i=0, \ldots, 5\right\}$ on $\partial(B)$ such that for any pair of rotationally consecutive ones, say $y_{i}, y_{j}$, we have $\left\|y_{i}-y_{j}\right\|=1$. Moreover, these six points are constructable.

Definition 3 (Brazil et al. 2015 [10]). For two directions $\phi_{i}$ and $\phi_{j}$ in the plane, $\mathbb{K}_{y}\left(\phi_{i}, \phi_{j}\right)$ denotes the cone defined to be the set consisting of all rays emanating from $y$ in direction $\phi$, for $\phi_{i} \leq \phi \leq \phi_{j}$. For each $y_{i}$ from Lemma 5 let $\theta_{i}$ be the direction of the ray starting at the center of $B$ and going through $y_{i}$. We assume that the $\left\{\theta_{i}\right\}$ are ordered in a counterclockwise manner, and two consecutive directions will be denoted by $\theta_{i}$ and $\theta_{i+1}$ (i.e., the mod 6 notation will be omitted).

Lemma 6 (Brazil et al. 2015 [10], Lemma 5). Let $y$ be any point in the plane. Then there exists a $T=\operatorname{MST}(P \cup\{y\})$ with the following property: for each $i=0, \ldots, 5$ there is at most one point of $P$ that is adjacent to $y$ in $T$ and lies in cone $\mathbb{K}_{y}\left(\theta_{i}, \theta_{i+1}\right)$, and this point is a closest terminal to $y$ in the cone.

We conclude from Lemma 6 that six OVDs still suffice for Algorithm 1. The computation of the OVDs in Algorithm 1 must be modified so that the defining cone for $\mathrm{OVD}_{i}$ is $\mathbb{K}\left(\theta_{i}, \theta_{i+1}\right)$ for $0 \leq i \leq 5$.

Definition 4 (Chew and Drysdale 1985 [19], Brazil and Zachariasen 2015 [14]). Any closed convex curve $C$ bounding a region containing the origin o defines a generalized Voronoi diagram [19]. ${ }^{11}$ This curve $C$ represents the boundary of a unit ball. We then get a distance function $\delta_{C}$ (defined by $C$ ) in the plane where the distance of $o$ to a point $v$ is $|o v| /\left|o v_{o}\right|$ where $v_{o}$ is the point at which the ray starting at $o$ going through $v$ intersects $C$ and $|\cdot|$ is the Euclidean norm giving us the Euclidean distance between the two points.

Theorem 2 (Chew and Drysdale 1985 [19], Brazil and Zachariasen 2015 [14]). The Voronoi diagram of $n$ points based on a closed convex shape $C$ can be constructed in $O(n \log n)$ time and $O(n)$ space as long as the following operations can be performed in constant time:

1. Given two points $a$ and $b$, compute the bisector curve between them (i.e., the set of points equidistant from $a$ and $b$ using the $\delta_{C}$ distance function).
2. Given two such bisectors, compute their intersection(s).

Restriction 4 (Brazil et al. 2015 [10]). Let $C^{\prime}$ be any sector of $B$. Then, given any two points $a$ and $b$, we can compute the bisector curve between them (i.e., the set of points equidistant from $a$ and $b$ using the $\delta_{C^{\prime}}$ distance function), and, given two such bisectors, we can compute their intersection. Moreover, these operations can be performed to within any fixed precision in constant time.

We make another general position assumption on the input points, namely that the bisector of two points under the given norm does not contain regions. Since Algorithm 1 assumes each OVD is an arrangement of lines, we also require the next restriction.

Restriction 5 (Brazil et al. 2015 [10]). The shape of $B$ implies that the $i^{\text {th }}$ OVD partition of any set of points is piecewise linear.

The first few steps of Algorithm 1 are:

1. compute $T=\operatorname{MST}(P)$
2. build the longest-edge auxiliary tree on $T$
3. compute the constant number of OVDs as labelled linear arrangements
4. compute the intervals of interest along $\gamma$

Norms that comply with Restrictions 3 and 4 allow us to apply Theorem 2 and compute $T$ and the six OVDs in $O(n \log n)$ time and $O(n)$ space. The size of these OVDs is $O(n)$ [19]. Once we have $T$, building the auxiliary tree needs no special care. Norms that also comply with Restriction 5 produce OVDs that are linear arrangements, so intersecting a line

[^9]$\gamma$ with these piecewise-linear OVDs still creates $O(n)$ intervals and we can still compute them in the time and space it takes to compute and sort the intersection points, i.e., $O(n \log n)$ time and $O(n)$ space. A few examples of norms that satisfy Restrictions 3 to 5 are Euclidean, $L_{1}$, and $L_{\infty}$ [10].

The next step is computing candidate solution points in each interval using our input norm and its distance function. The time to compute optimal Steiner points for an interval is still proportional to both: finding the roots of the derivative of the sum-of-distances function mentioned in Algorithm 1; and computing the distance between a terminal and a point on $\gamma$. For norms that respect our restrictions, this computation takes $O(1)$ time and space when $\gamma$ is a line. As mentioned, for our compliant norms, a Steiner point will have at most one neighbour in each OVD cone. However, when computing the Steiner point for an interval we must now consider neighbour subsets of sizes three to six points (of which there are still $O(1)$ such subsets). The LCA queries are not affected by different norms. Thus, we have the following corollary where the weight of a tree is the sum of its edge-weights, which are equal to their length under our given norm.

Corollary 3. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;
- $j$ lines $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$

By running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j n \log n)$ time and $O(n+j)$ space a minimumweight tree is computed that connects all points in $P$ using at most one extra point $s \in$ $\bigcup_{i=1}^{j} \gamma_{i}$.

By running the algorithm of Section 4.1 for a constant integer $k>1$ under Restriction 2, the restricted $k$-Steiner tree problem is solved in $O\left((j n)^{k}\right)$ time and $O(j n)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

### 4.3 Other Cost Functions

Brazil et al. [10] also showed that we can use our results for different tree cost functions. We first review some definitions from Brazil et al. [10].

Definition 5 (Brazil et al. 2015 [10]). For a MStT $T$ with topology $\tau$ built on $P$ with Steiner set $S$, let $\mathbf{e}_{\tau, P, S}$ be the vector whose components are the edge-lengths of $T$ with respect to the given norm $\|\cdot\|$. Let $E(T)$ be the edge set of $T$ with cardinality $|E(T)|$.
Definition 6 (Brazil et al. 2015 [10]). Let the cost function $\alpha: \mathbb{R}_{+}^{|E(T)|} \rightarrow \mathbb{R}$ be a symmetric function (i.e., independent of the order of the components in the vector on which it acts). The cost of $T$ with respect to $\alpha$ is $\alpha\left(\mathbf{e}_{\tau, P, S}\right)$, and $\min _{\tau, S} \alpha\left(\mathbf{e}_{\tau, P, S}\right)$ is the minimum cost of any tree connecting $P$ using $|S|$ other points. The power- $p$ cost function is $\alpha_{p}\left(\mathbf{e}_{\tau, P, S}\right)=$ $\sum_{i=1}^{|E(T)|}\left\|e_{i}\right\|^{p}$, where $e_{i}$ is the $i^{\text {th }}$ element of $E(T)$, and the cost of the longest edge is $\alpha_{\infty}\left(\mathbf{e}_{\tau, P, S}\right)=\max _{i=1 \ldots|E(T)|}\left\|e_{i}\right\|$.

Definition 7 (Brazil et al. 2015 [10]). We say that a symmetric function $\alpha=\alpha\left(\mathbf{e}_{\tau, P, S}\right)$ is $\ell_{1}$ optimizable if and only if there exist $\tau^{*}$ and $S^{*}$ such that both $\alpha\left(\mathbf{e}_{\tau^{*}, P, S^{*}}\right)=\min _{\tau, S} \alpha\left(\mathbf{e}_{\tau, P, S}\right)$ and $\alpha_{1}\left(\mathbf{e}_{\tau^{*}, P, S^{*}}\right)=\min _{\tau} \alpha_{1}\left(\mathbf{e}_{\tau, P, S^{*}}\right)$. In other words, $\alpha$ is $\ell_{1}$-optimizable if and only if, for any given $P$, there exists a Steiner set $S^{*}$ and tree topology interconnecting $P$ and $S^{*}$ that is both: minimum in cost with respect to $\alpha$; and is also an MST on $P \cup S^{*}$ (i.e., minimum with respect to $\alpha_{1}$ ). The functions $\alpha_{p}$, for $p>0$, and $\alpha_{\infty}$ are $\ell_{1}$-optimizable. So far in this paper we have been using the $\alpha_{1}$ cost function to determine the weight of the MStT.

Definition 8 (Brazil et al. $2015[10]$ ). Consider a set $P$ of $n$ points in $\mathbb{R}^{2}$, a norm $\|\cdot\|$, and a symmetric $\ell_{1}$-optimizable function $\alpha$. Let $S$ be a set of at most $k$ points in $\mathbb{R}^{2}$, and let $\mathbb{T}$ be a spanning tree on $P \cup S$ with topology $\tau$. Let $S$ and $\tau$ be such that $\alpha\left(\mathbf{e}_{\tau, P, S}\right)=$ $\min _{\tau^{\prime}, S^{\prime}} \alpha\left(\mathbf{e}_{\tau^{\prime}, P, S^{\prime}}\right)$. We say such a tree $\mathbb{T}$ is a minimum $k$-Steiner tree on $P$ with respect to $\alpha$.

Corollary 4 (Brazil et al. 2015 [10], Corollary 2). Given a set $P$ of $n$ points in $\mathbb{R}^{2}$, a norm $\|\cdot\|$, and a symmetric $\ell_{1}$-optimizable function $\alpha$, let $\mathbb{T}$ be a minimum $k$-Steiner tree on $P$ with respect to $\alpha$ and let $S \subset \mathbb{R}^{2}$ be the set of at most $k$ Steiner points from $\mathbb{T}$ (i.e., for the vertex set $V(\mathbb{T})$ of $\mathbb{T}, S=V(\mathbb{T}) \backslash P)$. Every MST on $P \cup S$ is a minimum $k$-Steiner tree on $P$ with respect to $\alpha$.

It follows from Theorem 1 that Algorithm 1 computes a tree with topology $\tau$ and set of Steiner points $S^{\star}$ of at most one element such that $\alpha_{1}\left(\mathbf{e}_{\tau \star, P, S} \boldsymbol{*}\right)=\min _{\tau, S} \alpha_{1}\left(\mathbf{e}_{\tau, P, S}\right)$. From the definition of $\ell_{1}$-optimizable, it follows that $\alpha\left(\mathbf{e}_{\tau \uparrow, P, S}\right)=\min _{\tau, S} \alpha\left(\mathbf{e}_{\tau, P, S}\right)$. By Corollary 4, it follows that our MStT is also the solution that minimizes the weight of the solution with respect to cost function $\alpha$. A similar argument holds for the algorithm of Section 4.1 and Corollary 3. This discussion leads to Corollary 5.

Corollary 5. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;
- a symmetric $\ell_{1}$-optimizable cost function $\alpha$;
- $j$ lines $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$

By running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j n \log n)$ time and $O(n+j)$ space a minimumweight tree with respect to $\alpha$ and $\|\cdot\|$ is computed that connects all points in $P$ using at most one extra point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

By running the algorithm of Section 4.1 for a constant integer $k>1$ under Restriction 2, the restricted $k$-Steiner tree problem is solved in $O\left((j n)^{k}\right)$ time and $O(j n)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

### 4.4 Curving $\gamma$

Examining Algorithm 1, we note that there are two parts involving $\gamma$ : computing labelled intervals along $\gamma$, and computing an optimal Steiner point for each interval. By examining the properties of lines that were being exploited to perform certain operations in $O(1)$ time, we realize our results are applicable to a more general set of input constraints than lines. We now let $\gamma$ be more general than a line. Abusing notation, we will refer to $\gamma$ as a curve, even if it has endpoints (in which case it may be closer to an arc). We keep our general position assumption that though $\gamma$ may intersect the rays and segments of the OVDs multiple times, each intersection is a single point.

## Computing Labelled Intervals on $\gamma$

## Compute Intersections of $\gamma$ with Lines/Rays/Segments

The first property we used was the zone theorem which said that our input line intersects an arrangement of $O(n)$ lines (i.e., the OVD) $O(n)$ times. Let $O(g)$ be the number of times $\gamma$ intersects an arrangement of $O(n)$ lines. Since we are intersecting $\gamma$ with six OVDs, the number of intervals created on $\gamma$ is $O(g)$. We also used the fact that each intersection can be computed in $O(1)$ time and space. Let $O(h)$ be the time and $O\left(h_{s p}\right)$ be the space it takes to compute an intersection between $\gamma$ and a line. Then the time to compute all of the intersection points on $\gamma$ is $O(g h)$ and the space is $O\left(g+h_{s p}\right)$.

- $O(g)$ : zone of $\gamma$ in an arrangement of $O(n)$ lines
- $O(h)$ : time to compute the intersection of $\gamma$ and a line
- $O\left(h_{s p}\right)$ : space to compute the intersection of $\gamma$ and a line
- $O(g h)$ : time to compute all of the intersection points on $\gamma$
- $O\left(g+h_{s p}\right)$ : space to compute all of the intersection points on $\gamma$

We use $O(\beta)$ to denote the time it takes to compute the intersection points of $\gamma$ with the OVDs (i.e., $\beta=g h$ ), and we use $O\left(\beta_{s p}\right)$ to denote the space.

## Create Intervals

Those computed intersection points are actually the endpoints of intervals on $\gamma$. Next, we sorted the interval endpoints to construct the intervals using the fact that the line was $x$-monotone. Let $m$ be the time and $m_{s p}$ be the space it takes to break $\gamma$ up into $c$ univariate polynomial functions. For example, we can break a circle into two $x$-monotone components, each starting at the left-most point and going to the rightmost point, but one moving clockwise and the other counterclockwise. Overall, all of these components contain $O(g+c)$ points to be sorted by $x$-coordinate, so we spend $O((g+c) \log (g+c))$ time and $O(g+c)$ space to sort the interval endpoints.
Once we have sorted the points, we need to figure out in which of our $c x$-monotone components each point belongs. Let $w$ be the most time and $w_{s p}$ be the most space required to evaluate one of our $c$ functions at a given $x$-coordinate. Then we create
the $O(g+c)$ intervals in $O(c(g+c) w)$ time and $O\left(g+c+w_{s p}\right)$ space as follows. For each of our $c$ components, we walk along our sorted list of points, evaluate the current component's function at the $x$-coordinate of the next point in the sequence, and compare the result against the $y$-coordinate of the tested point. If it matches, then this point will be an interval endpoint on this component of $\gamma$. In this way, we separate the interval endpoints into the components in which they exist on $\gamma$. Then we walk over the resulting lists to match interval endpoints (i.e., decide where intervals start and end).

- $c$ : number of univariate polynomial functions into which $\gamma$ is decomposed
- $O(m)$ : the time to decompose $\gamma$ into $c$ univariate polynomial functions
- $O\left(m_{s p}\right)$ : the space to decompose $\gamma$ into $c$ univariate polynomial functions
- $O(g+c)$ : the number of intervals into which $\gamma$ is decomposed
- $O((g+c) \log (g+c))$ : the time to sort the interval endpoints
- $O(g+c)$ : the space to sort the interval endpoints
- $O(w)$ : the most time to evaluate one of our $c$ functions at a given $x$-coordinate
- $O\left(w_{s p}\right)$ : the most space to evaluate one of our $c$ functions at a given $x$-coordinate
- $O(c(g+c) w):$ the time to create the intervals
- $O\left(g+c+w_{s p}\right)$ : the space to create the intervals

We use $O(\lambda)$ to denote the time it takes to compute the $O(g+c)$ intervals on $\gamma$ and we use $O\left(\lambda_{s p}\right)$ to denote the space (i.e., $O(m+(g+c) \log (g+c)+c(g+c) w)$ time and $O\left(m_{s p}+g+c+w_{s p}\right)$ space $)$.

## Computing Optimal Steiner Points in the Intervals

Lastly, we took advantage of the fact that in $O(1)$ time and space we could find our candidate points in each interval by computing the zeroes of the derivative of the distance function $d_{\mathbb{P}}(x)$ that sums the distances from the points associated with the interval to the point $x$ on the line.

As before, consider an interval and its set of potential neighbours $P^{\prime} \subset P$ of size at most six. For each subset $\mathbb{P}$ of $P^{\prime}$ of size three to six (of which there are $O(1)$ such subsets), we compute a candidate optimal Steiner point in the interval on $\gamma$. Although we have $c$ functions representing $\gamma$, here we abuse notation and refer to them all as $\gamma$. Recall that $\gamma$ is a function whose parameter corresponds to the 1-dimensional position along $\gamma$. The formula for the distance function will depend on the norm. For example, with the Euclidean norm, the sum-of-distances function looks like it did before: $d_{\mathbb{P}}(x)=\sum_{a \in \mathbb{P}} \sqrt{\left(a_{x}-x\right)^{2}+\left(a_{y}-\gamma(x)\right)^{2}}$. Over all intervals on $\gamma$, let $t$ be the most time and $t_{s p}$ be the most space used to find a candidate Steiner point for a given interval. This involves finding the zeroes of the derivative of $d_{\mathbb{P}}(x)$ and thus depends on the degree of the function $\gamma(x)$ and the norm being used. We can get a simple upper bound on the amount of time and space used to compute the candidate optimal Steiner points and the sums of the edge-lengths for edges incident to those Steiner
points in the MStT containing those candidates: $O(t(g+c))$ time and $O\left(t_{s p}+g+c\right)$ space. Let $q$ be the most time and $q_{s p}$ be the most space used to perform step 5(d)i when restricting the Steiner points to lie on $\gamma$ (i.e., solve the fixed topology $k$-Steiner tree problem). Note that step $5(\mathrm{~d})$ ii (i.e., computing the minimum $F$-fixed spanning tree) is performed using operations on weighted graphs and so is not affected by changing $\gamma$ to a curve.

Corollary 6. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;
- a symmetric $\ell_{1}$-optimizable cost function $\alpha$;
- a curve $\gamma$ with space complexity $\gamma_{s p}$

Algorithm 1 computes in $O(n \log n+\beta+\lambda+t(g+c))$ time and $O\left(n+\gamma_{s p}+\beta_{s p}+\lambda_{s p}+t_{s p}\right)$ space a minimum-weight tree with respect to $\alpha$ and $\|\cdot\|$ that connects all points in $P$ using at most one extra point $s \in \gamma$.

For a constant integer $k>1$, the algorithm of Section 4.1 solves the restricted $k$ Steiner tree problem in $O\left((g+c)^{k} q+\beta+\lambda+n \log n\right)$ time and $O\left(n+\gamma_{s p}+\beta_{s p}+\lambda_{s p}+q_{s p}\right)$ space with a Steiner set $S$ of at most $k$ points from $\gamma$.

This means, for example, that when $\gamma$ is a curve such as a conic (e.g., an ellipse, or a hyperbola), or a constant-degree polynomial function, or a curve that can be split into $O(1)$ constant-degree polynomial functions and $O(1)$ loops/self-intersections that can be decomposed in $O(1)$ time and space into $O(1)$ univariate functions described by polynomials of constant degree (such as the Folium of Descartes), Algorithm 1 still runs in $O(n \log n)$ time ${ }^{12}$ and $O(n)$ space with a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5 , such as the Euclidean norm, and an $\ell_{1}$-optimizable cost function $\alpha$.

We can also extend our result to the problem in which we are given a set of $j$ curves $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$. Let the runtime for computing the intervals on $\gamma_{i} \in \Gamma$ be $O\left(\beta_{i}+\lambda_{i}\right)$ and the space be $O\left(\beta_{s p, i}+\lambda_{s p, i}\right)$, let $\mu$ be $\max _{i}\left(\beta_{i}+\lambda_{i}\right)$, and let $\mu_{s p}$ be $\max _{i}\left(\beta_{s p, i}+\lambda_{s p, i}\right)$. Let $(g+c)$ be the maximum number of intervals into which any $\gamma_{i}$ is decomposed. Let $t$ be the most time and $t_{s p}$ be the most space used to find a candidate Steiner point over all intervals of all $\gamma_{i} \in \Gamma$. Let $q$ be the most time and $q_{s p}$ be the most space used to solve the fixed topology $k$-Steiner tree problem when restricting the Steiner points to lie on curves in $\Gamma$.

Corollary 7. Given:

- a set $P$ of $n$ points in $\mathbb{R}^{2}$;
- a norm $\|\cdot\|$ that is compliant to Restrictions 3, 4, and 5;

[^10]- a symmetric $\ell_{1}$-optimizable cost function $\alpha$;
- $j$ input curves $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$ with maximum space complexity $\gamma_{s p}$

By running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j(n \log n+\mu+t(g+c)))$ time and $O(n+$ $j \gamma_{s p}+\mu_{s p}+t_{s p}$ ) space a minimum-weight tree with respect to $\alpha$ and $\|\cdot\|$ is computed that connects all points in $P$ using at most one extra point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

For a constant integer $k>1$, the algorithm of Section 4.1 solves the restricted $k$ Steiner tree problem in $O\left((j(g+c))^{k} q+j \mu+n \log n\right)$ time and $O\left(n+j \gamma_{s p}+j \mu_{s p}+q_{s p}\right)$ space with a Steiner set $S$ of at most $k$ points from $\bigcup_{i=1}^{j} \gamma_{i}$.

## 5 Conclusion

We showed that given a set $P$ of $n$ points in $\mathbb{R}^{2}$ and a line $\gamma$ in $\mathbb{R}^{2}$, Algorithm 1 computes in optimal $\Theta(n \log n)$ time and optimal $\Theta(n)$ space a minimum-weight tree connecting all points in $P$ using at most one extra point $s \in \gamma$. We noted that this result can extend to a set $\Gamma=\left\{\gamma_{1}, \ldots \gamma_{j}\right\}$ of $j$ lines, and that by running Algorithm 1 for each $\gamma \in \Gamma$, in $O(j n \log n)$ time and $O(n+j)$ space a minimum-weight tree is computed that connects all points in $P$ using at most one extra point $s \in \bigcup_{i=1}^{j} \gamma_{i}$.

Next we reviewed the algorithm to solve the $k$-Steiner tree problem presented by Brazil et al. [10] that runs in $O\left(n^{2 k}\right)$ time and $O\left(n^{2}\right)$ space for $k=1$, and $O\left(n^{3}\right)$ space for $k>1$. We showed that we can replace one of their $O\left(n^{2}\right)$ time and space steps and one of their $O\left(n^{3}\right)$ time and space steps with preprocessing steps that require $O(n \log n)$ time and $O(n)$ space. This replacement does not change the asymptotic time and space complexities for the 1-Steiner tree algorithms of Georgakopoulos and Papadimitriou [28] or Brazil et al. [10], nor the time complexity of the $k$-Steiner tree algorithm of Brazil et al. [10] for $k>1$, but it lowers the space complexity from $O\left(n^{3}\right)$ to $O\left(n^{2}\right)$ for $k>1$. We then showed how to adapt the $k$-Steiner tree algorithm to our setting (i.e., where the Steiner points can only be chosen from a set of $j$ input lines), allowing us to solve our restricted $k$-Steiner tree problem in $O\left((j n)^{k}\right)$ time and $O(j n)$ space.

We then pointed out that the more general results of Brazil et al. [10] also apply to our scenario, thereby extending our results to the same norms and cost functions as the algorithm of Brazil et al., abiding by the restrictions they laid out.

Lastly, we showed that our results apply when the set of input restrictions is not a set of lines, but a set of $j$ curves. In this case, the running time and space bounds come to depend on $j$, the runtime and space of certain primitive operations, and the complexity of the zone of the curves in an arrangement of lines.

It is an open question whether or not the unrestricted 1-Steiner tree problem in $\mathbb{R}^{2}$ can be solved in $o\left(n^{2}\right)$ time by using multiple constraint lines and the results of Algorithm 1 to guide a search.

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[^1]:    ${ }^{1}$ This means the length of their tree is at most 1.214 times the length of the optimal solution. Here they take advantage of the result of Chung and Graham [20] showing that the MST is a 1.214 -approximation (to three decimals) of the MStT.

[^2]:    ${ }^{2}$ Georgakopoulos and Papadimitriou [28] refer to the overlaid OVD as Overlaid Oriented Dirichlet Cells.

[^3]:    ${ }^{3}$ This follows from the zone theorem [16, 23, 25, 26].
    ${ }^{4}$ In other words, each $u \in I$ has the same constant-sized set of fixed candidate sub-topologies incident to $u$ that could be the result of $\operatorname{MST}(P \cup\{u\})$.

[^4]:    ${ }^{5}$ A similar result was shown in Monma and Suri [34, Lemma 4.1, pg. 277].
    ${ }^{6}$ A simple cycle is a cycle where no vertex is repeated except the first vertex.

[^5]:    ${ }^{7}$ As summarized in the survey by Brazil and Zachariasen [14], for the bottleneck $k$-Steiner tree problem, i.e., the $k$-Steiner tree problem where the goal is to minimize the length of the longest edge of the resultant tree for the given norm, Bae et al. [2] presented a $\Theta(n \log n)$-time and $O\left(n^{2}\right)$-time algorithm for $k=1$ and $k=2$ respectively in the Euclidean plane, while Bae et al. [1] presented an $O\left(\left(k^{5 k} 2^{O(k)}\right)\left(n^{k}+n \log n\right)\right)$-time algorithm for the $L_{p}$ metric for a fixed rational $p$ with $1<p<\infty$, as well as an $O(n \log n)$-time algorithm

[^6]:    ${ }^{8}$ In the pseudocode for this algorithm in [10] there is a typo saying this is $O(n \log n)$ time, but the correct time bound, $O\left(n^{2}\right)$, is stated in other parts of the paper.

[^7]:    ${ }^{9}$ Note that since Steiner points may lie on different input lines, we cannot simply run the basic algorithm $j$ times as was done in Corollary 1.

[^8]:    ${ }^{10}$ For the reader trying to get a better understanding of the material from Brazil et al. [10], the same material is presented a bit differently in the survey by Brazil and Zachariasen [14].

[^9]:    ${ }^{11} \mathrm{We}$ can even allow $o$ to lie on $C$, but this means some points are infinitely far from $o$ [14, §4.3.1 pg. 258].

[^10]:    ${ }^{12}$ Once again, we assume that, using the univariate polynomial functions into which we have decomposed $\gamma(x)$ in the equations outlined in the previous sections, the derivatives and roots required can be computed in constant time and space.

