

Transfer-Expanded Graphs for On-Demand Multimodal Transit Systems

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Abstract. This paper considers a generalization of the network design problem for On-Demand Multimodal Transit Systems (ODMTS). An ODMTS consists of a selection of hubs served by high frequency buses, and passengers are connected to the hubs by on-demand shuttles which serve the first and last miles. This paper generalizes prior work by including three additional elements that are critical in practice. First, different frequencies are allowed throughout the network. Second, additional modes of transit (e.g., rail) are included. Third, a limit on the number of transfers per passenger is introduced. Adding a constraint to limit the number of transfers has a significant negative impact on existing Benders decomposition approaches as it introduces non-convexity in the subproblem. Instead, this paper enforces the limit through transfer-expanded graphs, i.e., layered graphs in which each layer corresponds to a certain number of transfers. A real-world case study is presented for which the generalized ODMTS design problem is solved for the city of Atlanta. The results demonstrate that exploiting the problem structure through transfer-expanded graphs results in significant computational improvements.

Keywords: Combinatorial optimization · Multimodal transportation · Benders decomposition · Transfer-expanded graphs.

1 Introduction

This paper is motivated by the design and implementation of an On-Demand Multimodal Transit System (ODMTS) for the city of Atlanta. The share of public transit in Atlanta (about 2–3%) is very low compared to other American cities (e.g., about 15% in Boston) and Atlanta is also the 8th most congested city in the world. There is thus a strong need for a modern transit systems that leverages the train and bus infrastructure of the city and complements it with innovative mobility concepts.

This paper considers the design of an ODMTS for Atlanta that combines a network of trains and buses with on-demand multimodal shuttles that act as feeders to/from the bus/rail network and serve local demand. ODMTS address the first/last mile problem that plagues transit systems, while mitigating congestion on high-density corridors and leveraging economy of scale. ODMTS and

their design challenge was introduced in [10], which also presents an overview of related work. The main contribution of this paper is to generalize prior work by including three additional elements that are critical for ODMTS in large cities such as Atlanta. First, different frequencies are allowed throughout the network. Second, additional modes of transit (e.g., rail) are included. Third, a limit on the number of transfers per passenger is introduced. Adding a constraint to limit the number of transfers has a significant negative impact on existing Benders decomposition approaches as it introduces non-convexity in the subproblem. Instead, this paper enforces the limit through transfer-expanded graphs, i.e., layered graphs in which each layer corresponds to a certain number of transfers. A real-world case study is presented for which the generalized ODMTS design problem is solved for the city of Atlanta. The results demonstrate that exploiting the problem structure through transfer-expanded graphs results in significant computational improvements.

2 The Generalized ODMTS Design Problem

This section presents the generalized ODMTS design problem that enhances the model from [10] along several dimensions: The choice of bus frequencies, additional transportation modes and, most importantly, a constraint on the number of transfers. The Benders decomposition approach in [10] exploits a natural decomposition of the ODMTS design problem. The network design is determined by the *master problem*, while the routing of the passengers for a given design is determined by the *subproblem*. A major benefit of this decomposition is that the subproblem can be solved for each trip independently. The same decomposition is used in this paper.

2.1 The Master Problem For Network Design

Consider a directed multigraph $G = (V, A)$, with vertices $V = \{1, \dots, n\}$ and arc set A . Let F be the set of possible frequencies, i.e., the total number of vehicles during the time horizon, let M be the set of possible transportation modes, which may include shuttles, and let K be the total number of arcs that each passenger may travel. By definition, K is equal to the maximum number of transfers plus one. In the multigraph G , each arc $a \in A$ is uniquely defined by the quadruple $a = (i, j, m, f) \in V \times V \times M \times F$, $i \neq j$. Using arc a means traveling from i to j with mode m , which departs with frequency f . For a given arc $a \in A$, these elements are referred to as $i(a)$, $j(a)$, $m(a)$, and $f(a)$, respectively.

Designing a generalized ODMTS amounts to deciding which arcs $a \in A$ are made available to passengers. Let the binary variable $z_a \in \mathbb{B}$ be equal to one if arc a is made available, and zero otherwise. The cost of enabling arc a is given by the parameter β_a . It is assumed that $\beta_a \geq 0$ for all $a \in A$.

For a given design, a cost is incurred due to passengers traveling through the network. This cost $\Phi(z)$ is a function of the values of the z -variables that define

$$\begin{aligned}
\min \quad & \sum_{a \in A} \beta_a z_a + \Phi(z), \tag{1a} \\
\text{s.t.} \quad & \sum_{a \in \delta^+(i, m)} f(a) z_a - \sum_{a \in \delta^-(i, m)} f(a) z_a = 0 \quad \forall i \in V, m \in M, \tag{1b} \\
& \sum_{f \in F | (i, j, m, f) \in A} z_{(i, j, m, f)} \leq 1 \quad \forall i \in V, j \in V, m \in M, \tag{1c} \\
& z_a \in \mathbb{B} \quad \forall a \in A. \tag{1d}
\end{aligned}$$

Fig. 1. Formulation for the generalized ODMTS design problem.

the design. The value of $\Phi(z)$ can be found by solving the subproblem, which is discussed in Section 2.2. If the subproblem is not feasible, then $\Phi(z) = \infty$.

A formulation for the master problem is presented in Figure 1. For convenience, $\delta^+(i)$ is defined as the set of all arcs going out of $i \in V$. Similarly, the set $\delta^+(i, m)$ is defined as the set of all arcs with mode $m \in M$ going out of $i \in V$. The sets $\delta^-(i)$ and $\delta^-(i, m)$ are defined analogously for the incoming arcs.

Objective (1a) minimizes the cost of the design plus the cost of routing the passengers through the network. Constraints (1b) ensure that the frequencies for each mode are balanced at each vertex. For example, if three buses arrive during the time horizon, then three buses should also depart. Constraints (1c) enforce that only one frequency can be selected for a given connection and a given mode. Equations (1d) state the integrality requirements.

2.2 The Subproblem: Routing Passengers Through the Network

For a given design, the passenger trips are routed through the network at minimum cost. Let T be the set of all passenger trips, and let each trip $r \in T$ be defined by an origin $o(r)$, a destination $d(r)$, and a number of passengers $p(r)$. If trip $r \in T$ is routed through arc $a \in A$, then a cost of γ_a^r is incurred. The total cost of routing all passenger trips, $\Phi(z)$, is the sum over the costs per trip. It is assumed that $\gamma_a^r > 0$ for every arc $a \in A$ and trip $r \in T$, such that the optimal route is a simple path from $o(r)$ to $d(r)$.

Solving the subproblem amounts to solving a shortest path problem from $o(r)$ to $d(r)$ for each trip $r \in T$, with the additional restriction that the number of arcs in the path is at most K . This problem is known as the cardinality-constrained shortest path problem (CSP) [6]. Note that the cardinality constraint follows from the limit on the number of transfers. Without this limit, the subproblem is an (unconstrained) shortest path problem (SP), as is the case in [10].

It is well-known that SP possesses total unimodularity and can be solved by linear programming (LP). Adding an additional constraint, however, typically destroys this structure [1]. This is indeed the case when a cardinality constraint is added to the subproblem formulation in [10]. As a result, the cost function $\Phi(z)$ would change from convex to non-convex, which negatively impacts Benders decomposition approaches (see Section 3).

To remedy this limitation, this paper presents a new formulation for the subproblem that enforces the transfer limit without destroying total unimodularity. This formulation uses transfer-expanded graphs, i.e., layered graphs with a each layer for each number of transfers. Transfer-expanded graphs encode the transit constraints directly, making it possible to use shortest-path algorithms.

2.3 Transfer-Expanded Graphs

Transfer-expanded graphs shares some similarities with time-expanded networks, where each vertex has multiple copies for different periods of time. This is the case, for example, for modern algorithms for evacuation planning and scheduling [8,12,13]. Reference [11] also uses a layered network to solve the dynamic generalized assignment problem. As a result, some of the side-constraints do not need to be handled explicitly. See [3] for a recent literature review on time-expanded graphs.

Let $\bar{G}^r = (\bar{V}^r, \bar{A}^r)$ be the transfer-expanded graph for a given trip $r \in T$. This graph contains multiple copies of the original arcs and vertices, organized in $K+1$ layers. It is assumed that $K \geq 2$, as the subproblem is trivial for $K = 1$. A vertex $\bar{v} = (i, k) \in \bar{V}^r$ in the transfer-expanded graph is defined by a vertex $i \in V$ in the original graph and by a layer $k \in \{1, \dots, K+1\}$. Similarly, the definition of an arc is extended to $\bar{a} = (a, k, l)$, in which $a \in A$ is the original arc, $k \in \{1, \dots, K\}$ is the layer of the starting vertex of \bar{a} and $l \in \{2, \dots, K+1\}$ is the layer of the ending vertex.

The transfer-expanded graph is constructed as follows. For convenience, Figure 2 provides an example for $K = 3$. First, the vertex set \bar{V}^r is defined. For the origin and the destination of the trip, introduce the vertices $(o(r), 1)$ and $(d(r), K+1)$. For the other vertices $i \in V \setminus \{o(r), d(r)\}$ of the original graph, add the copies (i, k) for $k \in \{2, \dots, K\}$ to the transfer-expanded graph. The arc set \bar{A}^r is constructed based on the arcs of the original graph, as follows:

1. For each arc starting in the origin, i.e., $a \in \delta^+(o(r))$, add the arc $(a, 1, 2)$ if $j(a) \neq d(r)$, or the arc $(a, 1, K+1)$ if $j(a) = d(r)$.
2. For each arc not adjacent to the origin or the destination, i.e., $a \in A$ and $i(a), j(a) \notin \{o(r), d(r)\}$, add the arcs $(a, k, k+1)$ for all $k \in \{2, \dots, K-1\}$.
3. For each arc ending in the destination that does not start in the origin, i.e., $a \in \delta^-(d(r))$, $i(a) \neq o(r)$, add the arcs $(a, k, K+1)$ for all $k \in \{2, \dots, K\}$.

By construction, it follows that solving CSP on the original graph is equivalent to solving SP on the transfer-expanded graph. Figure 3 formulates the subproblem as a collection of SPs on transfer-expanded graphs. Let $y_{\bar{a}}^r \in \mathbb{B}$ be the flow on arc $\bar{a} \in \bar{A}^r$ of trip $r \in T$. For convenience, define $\bar{\delta}_r^+(\bar{v})$ to be the set of all arcs in \bar{A}^r coming out of $\bar{v} \in \bar{V}^r$. Similarly, let $\bar{\delta}_r^-(\bar{v})$ be the set of incoming arcs.

Objective (2a) minimizes the cost of all trips. Constraints (2b) state that passengers can only use arcs available in the design. Constraints (2c) enforce flow conservation, and Equations (2d) define the variables. Due to total unimodularity of the SPs, no integrality conditions are required.

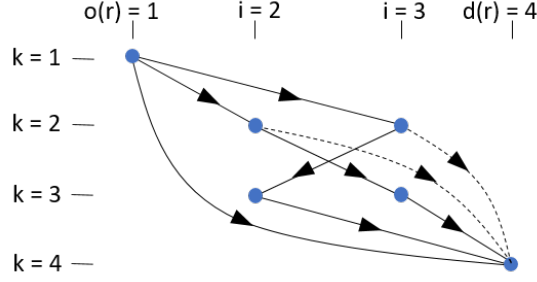


Fig. 2. A transfer-expanded graph for $K = 3$, $|M| = 1$, $|F| = 1$, for a complete graph as the original graph. The dotted arcs are removed when the triangle inequality holds.

$$\begin{aligned}
 \Phi(z) = \min \quad & \sum_{r \in T} \sum_{\bar{a} = (a, k, l) \in \bar{A}^r} \gamma_{\bar{a}}^r y_{\bar{a}}^r & (2a) \\
 \text{s.t.} \quad & y_{\bar{a}}^r \leq z_a \quad \forall r \in T, \bar{a} = (a, k, l) \in \bar{A}^r, & (2b) \\
 \sum_{\bar{a} \in \delta_r^+(\bar{v})} y_{\bar{a}}^r - \sum_{\bar{a} \in \delta_r^-(\bar{v})} y_{\bar{a}}^r = \begin{cases} 1 & \text{if } \bar{v} = (o(r), 1) \\ -1 & \text{if } \bar{v} = (d(r), K+1) \\ 0 & \text{else} \end{cases} \quad \forall r \in T, \bar{v} \in \bar{V}^r, & (2c) \\
 y_{\bar{a}}^r \geq 0 \quad & \forall r \in T, \bar{a} \in \bar{A}^r. & (2d)
 \end{aligned}$$

Fig. 3. Formulation for the subproblem on transfer-expanded graphs.

The main advantage of using transfer-expanded graphs is that the limit on the number of transfers can be enforced without destroying total unimodularity. A potential downside is that the number of variables and constraints in the subproblem increases with K . In public transit, however, the number of transfers that passengers are willing to take, and therefore the value of K , is typically very low. Furthermore, a larger subproblem does not necessarily mean that the subproblem is more difficult to solve, as algorithms may benefit from the fact that the transfer-expanded graph is acyclic. When the z -variables are integers, for example, the acyclic subproblem for each trip can be solved in linear time through topological sorting [5].

Finally, it is worth pointing out that if $o(r)$ and $d(r)$ are only served by shuttles, and shuttles satisfy the triangle inequality, then some arcs may be removed from the transfer-expanded graph without sacrificing optimality. Specifically, using a shuttle on the path $(o(r), 1) \rightarrow (i, 2) \rightarrow (d(r), K+1)$ for $i \in V$ is always dominated by using a direct shuttle from $(o(r), 1)$ to $(d(r), K+1)$. It follows that the shuttle arcs between $(i, 2)$ and $(d(r), K+1)$ may be removed for all $i \in V$, as also indicated in Figure 2. For $K \leq 3$, it then follows that the transfer-expanded graph does not require more edges than the original graph.

3 Benders Decomposition

Following [10], a Benders decomposition approach is presented for the generalized ODMTS design problem. The goal is to solve the master problem (1), which is complicated by the fact that $\Phi(z)$ is defined implicitly. To apply Benders decomposition, replace $\Phi(z)$ in Objective (1a) by a new variable $\theta \in \mathbb{R}$, and add the constraint $\theta \geq \Phi(z)$. Note that this does not change the problem, as $\theta = \Phi(z)$ in any optimal solution. In Benders decomposition, the constraint $\theta \geq \Phi(z)$ is enforced through *Benders cuts*. For subproblem (2), these cuts are

$$\theta \geq \Phi(\bar{z}) + \sum_{r \in T} \sum_{a \in A} \sum_{k=1}^K \lambda_a^{rk}(\bar{z})(z_a - \bar{z}_a), \quad (3)$$

with $\lambda_a^{rk}(z)$ the dual values of Constraints (2b) and \bar{z} any feasible solution to the LP relaxation of the master problem [2]. For the case study in this paper, the subproblem is always feasible. If this assumption is not satisfied, *Benders feasibility cuts*, which are similar to (3), may also be included [2].

The Benders decomposition approach is implemented in C++ and Gurobi 8.1.1. The master problem is the main model, and the Benders cuts (3) are separated in both the MIP solution callback (in case the z -variables are integer) and in the MIP node callback (in case the z -variables are fractional). The subproblem for each trip is also solved with Gurobi, and dual simplex is used to ensure that the basis remains feasible when the subproblem is solved for different values of z . To prevent excessive calls to the subproblem, feasibility heuristics are disabled. The number of cut separation rounds in the root node is set to the maximum value to get the best possible bound. Finally, the 2ϵ -trick is used to stabilize the master problem [7]. This stabilization uses $\epsilon = 0.00001$ and the trivial core point obtained by assigning $z_a = \frac{1}{4}$ to every bus arc.

Without transfer-expanded graphs, the subproblem is not totally unimodular and $\Phi(z)$ is not convex (see Section 2.2). In that case, Benders decomposition cannot be applied directly. Instead, $\theta \geq \Phi(z)$ may be enforced by adding *combinatorial Benders cuts* in the MIP solution callback and Benders cuts for the LP relaxation of the subproblem in both callbacks [4,9]. However, it is well-known that relying too much on combinatorial Benders cuts may result in slow algorithmic progress, and many cuts may be necessary to find the optimal solution.

4 Atlanta as a Case Study

The generalized ODMTS design problem was solved for the city of Atlanta. In Atlanta, the Metropolitan Atlanta Rapid Transit Authority (MARTA) operates two modes: bus and rail. The case study added on-demand shuttles and the bus system was redesigned accordingly. More precisely, define the three modes $M = \{S, B, R\}$ for shuttle, bus, and rail respectively. Shuttle arcs are introduced to connect from origins to hubs and from hubs to destinations, as well as to serve the local demand. The corresponding z_a variables are fixed to one, as shuttles are

always available. Following [10], the cost of using a shuttle is a weighted sum of cost and convenience, controlled by the parameter $\alpha \in [0, 1]$. Let d_a and t_a be the travel distance and the travel time of arc $a \in A$, respectively. The parameter c^S is the cost of using a shuttle per person per unit of distance. The cost of traversing arc $a \in A$ for trip $r \in T$ is then defined as $\gamma_a^r = p(r) \left((1 - \alpha)c^S d_a + \alpha t_a \right)$.

Bus arcs are defined between the potential hub locations and between each hub and the three nearest rail stations. The cost of enabling bus arc $a \in A$ is given by $\beta_a = (1 - \alpha)c^B f(a)d_a$. That is, the distance is multiplied by the cost per unit distance and the number of buses over the time horizon. The cost of traversing a bus arc is given by $\gamma_a^r = \alpha \left(t_a + L + \frac{H}{2f(a)} \right)$. Here L is the fixed time required for a transfer, H is the time horizon, and $\frac{H}{2f(a)}$ is the expected waiting time before the next bus arrives, which depends on the frequency. Rail arcs are defined between all rail stations that are connected by the same rail line. The costs of traversing an arc is defined in the same way as for the buses. For each rail arc $a \in A$, the variable z_a is fixed to one, which makes the cost of enabling an arc irrelevant.

The case study uses the following data and parameters to create a realistic instance and evaluate the computational benefit of transfer-expanded graphs. It uses passenger trip data provided by MARTA for March 16, 2018, between 6am and 10am. Connecting trips have been chained together to obtain origin and destination pairs. This resulted in 2588 unique trips, with 7167 passengers in total. There are 5563 bus stops and rail stations in total, and their locations were also provided by MARTA. Eleven hubs were selected manually on the map. For the distances d_a , great-circle distances are used. To estimate travel times t_a , the distances are divided by a constant speed of 30 mph. The cost parameters are set to $c^S = 5$ and $c^B = 1$. The fixed transfer time is chosen to be five minutes, i.e., $L = 5$ minutes, and the time horizon is set to four hours, i.e., $H = 240$ minutes. To balance cost and convenience, $\alpha = 0.5$ is used. The rail frequency is assumed to be fixed to six per hour, i.e., $f(a) = 6 \times 4 = 24$, and bus frequencies are determined by the model to be either three per hour or six per hour. At most two transfers are allowed, i.e., $K = 3$.

Figure 4a presents the result of solving the generalized ODMTS problem using transfer-expanded graphs. In total, it took 122 seconds to obtain the optimal solution and prove optimality, with a minimum objective value of 131,905. Without transfer-expanded graphs, i.e., when adding combinatorial Benders cuts, it was not possible to obtain an optimal solution in reasonable time. Instead, the evaluation considered a relaxation in which the combinatorial Benders cuts were ignored and only the Benders cuts for the LP relaxation of the subproblem were added. Solving this relaxation to optimality took 3.8 hours. Keep in mind that this relaxation explores routes that may require many transfers. To evaluate the quality of the design obtained by the relaxation, the passengers were routed through the transfer-expanded formulation with the z -variables fixed to their values found in the relaxation. The result is presented in Figure 4b and has an objective value of 131,965. Solving the relaxed problem results in a smaller

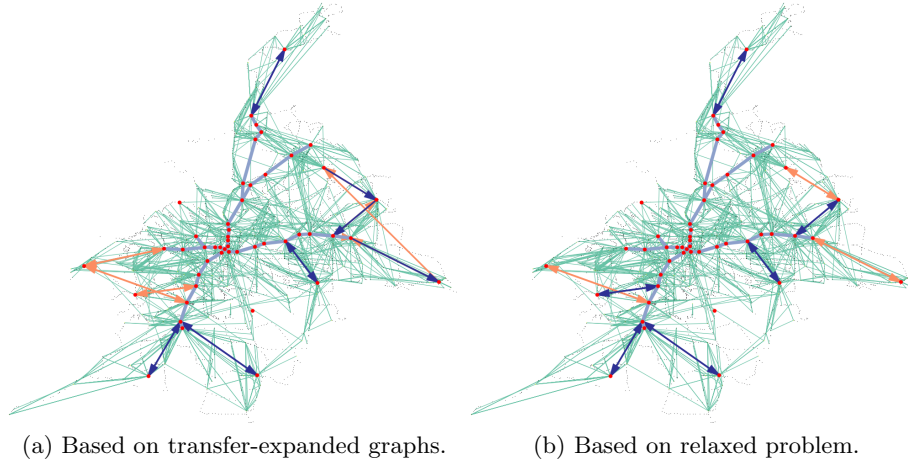


Fig. 4. Network designs for Atlanta showing shuttles (thin lines), rail (thick lines), and buses (arrows, orange/light for low frequency and purple/dark for high frequency).

public transit network because the relaxation does not completely enforce the transfer limit.

In summary, the main benefit of the transfer-expanded formulation is the significant computational benefits it provides in capturing the transfer limit. Without transfer-expanded paths, it can be optimal to fractionally select long paths that do not adhere to this constraints. These longer fractional paths likely play a role in the difference of computational performance.

5 Conclusion

This paper presented a generalization of the ODMTS design problem that introduces three critical elements in practice: different frequencies, additional transit modes, and a limit on the number of transfers. Transfer-expanded graphs are introduced to handle the transfer limit without negatively impacting existing Benders decomposition approaches. The Atlanta case study demonstrates that this approach is very effective, as transfer-expanded graphs significantly improve computational performance. Exploiting the problem structure through transfer-expanded graphs opens the door to designing increasingly realistic networks in the future. One possible extension is to incorporate the capacity of the on-demand shuttles. As capacity of these shuttles is typically small, expanded networks could also be used to model capacity efficiently.

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