# Internal Quasiperiod Queries

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#### Abstract

Internal pattern matching requires one to answer queries about factors of a given string. Many results are known on answering internal period queries, asking for the periods of a given factor. In this paper we investigate (for the first time) internal queries asking for covers (also known as quasiperiods) of a given factor. We propose a data structure that answers such queries in  $\mathcal{O}(\log n \log \log n)$  time for the shortest cover and in  $\mathcal{O}(\log \log (\log n)^2)$  time for a representation of all the covers, after  $\mathcal{O}(n \log n)$  time and space preprocessing.

### **1** Introduction

A cover (also known as a quasiperiod) is a weak version of a period. It is a factor of a text T whose occurrences cover all positions in T; see Fig. 1. The notion of cover is well-studied in the off-line model. Linear-time algorithms for computing the shortest cover and all the covers of a string of length n were proposed in [2] and [23, 24], respectively. Moreover, linear-time algorithms for computing shortest and longest covers of all prefixes of a string are known; see [6] and [22], respectively. Covers were also studied in parallel [5, 7] and streaming [13] models of computation. Definitions of other variants of quasiperiodicity can be found in the survey [12]. In this work we introduce covers to the internal pattern matching model [20].

T: abaababaababaababa

Figure 1: MINCOVER(T) = aba is the shortest cover of T and MINCOVER(T[2..13]) = baababa is the shortest cover of its suffix of length 12.

In the internal pattern matching model, a text T of length n is given in advance and the goal is to answer queries related to factors of the text. One of the basic internal queries in texts are *period queries*, that were introduced in [19] (actually, internal primitivity queries were considered even earlier [9, 10]). A period query requires one to compute all the periods of a given factor of T. It is known that they can be expressed as  $\mathcal{O}(\log n)$  arithmetic sequences. The fastest known algorithm answering period queries is from [20]. It uses a data structure of  $\mathcal{O}(n)$  size that can be constructed in  $\mathcal{O}(n)$  expected time and answers period queries in  $\mathcal{O}(\log n)$  time (a deterministic construction of this data structure was given in [16]). A special case of period queries are *two-period queries*, which ask for the shortest period of a factor that is known to be periodic. In [20] it was shown that two-period queries can be answered in constant time after  $\mathcal{O}(n)$ -time preprocessing. Another algorithm for answering such queries was proposed in [3].

Let us denote by MINCOVER(S) and ALLCOVERS(S), respectively, the length of the shortest cover and the lengths of all covers of a string S. Similarly as in the case of periods, it can be shown that the set ALLCOVERS(S) can be expressed as a union of  $\mathcal{O}(\log |S|)$  pairwise disjoint arithmetic sequences. We consider data structures that allow to efficiently answer these queries in the internal model.

> INTERNAL QUASIPERIOD QUERIES
> Input: A text T of length n
> Query: For any factor S of T, compute MINCOVER(S) or ALLCOVERS(S) after efficient preprocessing of the text T

Recently [11] we have shown how to compute the shortest cover of each cyclic shift of a string T of length n, that is, the shortest cover of each length-|T| factor of  $T^2$ , in  $\mathcal{O}(n \log n)$  total time. This work can be viewed as a generalization of [11] to computing covers of any factor of a string. It also generalizes the earlier works on computing covers of prefixes of a string [6, 22].

**Our results.** We show that MINCOVER and ALLCOVERS queries can be answered in  $\mathcal{O}(\log n \log \log n)$  time and  $\mathcal{O}(\log n (\log \log n)^2)$  time, respectively, with a data structure that uses  $\mathcal{O}(n \log n)$  space and can be constructed in  $\mathcal{O}(n \log n)$  time. In particular, the time required to answer an ALLCOVERS query is slower by only a poly log log n factor from optimal. Moreover, we show that any m MINCOVER or ALLCOVERS queries can be answered off-line in  $\mathcal{O}((n+m) \log n)$  and  $\mathcal{O}((n+m) \log n \log \log n)$  time, respectively, and  $\mathcal{O}(n+m)$  space. In particular, the former matches the complexity of the best known solution for computing shortest covers of all cyclic shifts of a string [11], despite being far more general. We assume the word RAM model of computation with word size  $\Omega(\log n)$ .

**Our approach.** Our main tool are *seeds*, a known generalization of the notion of cover. A seed is defined as a cover of a superstring of the text [14]. A representation of all seeds of a string T, denoted here SeedSet(T), can be computed in linear time [17]. We will frequently extract individual seeds from SeedSet(T); each time such an auxiliary query needs  $\mathcal{O}(\log \log n)$  time. Consequently,  $\log \log n$  is a frequent factor in our query times related to internal covers.

We construct a tree-structure (static range tree) of so-called *basic factors* of a string. For each basic factor F we store a compact representation of the set SeedSet(F). The crucial point is that the total length of all these factors is  $\mathcal{O}(n \log n)$  and every other factor can be represented, using the tree-structure, as a concatenation of  $\mathcal{O}(\log n)$  basic factors. Representations of seed-sets of basic factors are precomputed. Then, upon an internal query related to a specific factor S, we decompose S into concatenation of basic factors  $F_1, F_2, \ldots, F_k$ . Intuitively, the representation of the set of covers or (in easier queries) the shortest cover will be computed as a "composition" of  $SeedSet(F_1), SeedSet(F_2), \ldots, SeedSet(F_k)$ , followed by adjusting it to border conditions using internal pattern matching. To get efficiency, when quering about covers of a factor S, we do not compute the whole representation of SeedSet(S) (these representations are only precomputed for basic factors).

Finally, several stringology tools related to properties of covers and string periodicity are used to improve polylog *n*-factors in the query time that would result from a direct application this approach.

## 2 Preliminaries

We consider a text T of length n over an integer alphabet  $\{0, \ldots, n^{\mathcal{O}(1)}\}$ . If this is not the case, its letters can be sorted and renumbered in  $\mathcal{O}(n \log n)$  time, which does not influence the preprocessing time of our data structure.

For a string S, by |S| we denote its length and by S[i] we denote its *i*th letter (i = 1, ..., |S|). By S[i...j] we denote the string S[i] ... S[j] called a factor of S; it is a prefix if i = 1 and a suffix if j = |S|. A factor that occurs both as a prefix and as a suffix of S is called a border of S. A factor is proper if it is shorter than the string itself. A positive integer p is called a period of S if S[i] = S[i + p] holds for all i = 1, ..., |S| - p. By per(S) we denote the smallest period of S. A string S is called a cyclic shift of S. We use the following simple fact related to covers.

**Observation 2.1.** Let A, B, C be strings such that |A| < |B| < |C|.

- (a) If A is a cover of B and B is a cover of C, then A is a cover of C.
- (b) If B is a border of C and A is a cover of C, then A is a cover of B.

Below we list several algorithmic tools used later in the paper.

### 2.1 Queries Related to Suffix Trees and Arrays

A range minimum query on array A[1..n] requires to compute min $\{A[i], \ldots, A[j]\}$ .

**Lemma 2.2** ([4]). Range minimum queries on an array of size n can be answered in  $\mathcal{O}(1)$  time after  $\mathcal{O}(n)$ -time preprocessing.

By lcp(i, j) (lcs(i, j)) we denote the length of the longest common prefix of T[i ...n] and T[j ...n] (longest common suffix of T[1...i] and T[1...j], respectively). Such queries are called longest common extension (LCE) queries. The following lemma is obtained by using range minimum queries on suffix arrays.

**Lemma 2.3** ([4, 15]). After  $\mathcal{O}(n)$ -time preprocessing, one can answer LCE queries for T in  $\mathcal{O}(1)$  time.

The suffix tree of T, denoted as  $\mathcal{T}(T)$ , is a compact trie of all suffixes of T. Each implicit or explicit node of  $\mathcal{T}(T)$  corresponds to a factor of T, called its *string label*. The *string depth* of a node of  $\mathcal{T}(T)$  is the length of its string label.

We use weighted ancestor (WA) queries on a suffix tree. Such queries, given an explicit node v and an integer value  $\ell$  that does not exceed the string depth of v, ask for the highest explicit ancestor u of v with string depth at least  $\ell$ .

**Lemma 2.4** ([1, 17]). Let  $\mathcal{T}(T)$  be the suffix tree of T. WA queries on  $\mathcal{T}(T)$  can be answered in  $\gamma_n = \mathcal{O}(\log \log n)$  time after  $\mathcal{O}(n)$ -time preprocessing. Moreover, any m WA queries on  $\mathcal{T}(T)$  can be answered off-line in  $\mathcal{O}(n+m)$  time.

### 2.2 Internal Pattern Matching (IPM)

The data structure for IPM queries is built upon a text T and allows efficient location of all occurrences of one factor X of T inside another factor Y of T, where  $|Y| \leq 2|X|$ .

**Lemma 2.5** ([20]). The result of an IPM query is a single arithmetic sequence. After linear-time preprocessing one can answer IPM queries for T in  $\mathcal{O}(1)$  time.

A period query, for a given factor X of text T, returns a compact representation of all the periods of X (as a set of  $\mathcal{O}(\log n)$  arithmetic sequences).

**Lemma 2.6** ([20]). After  $\mathcal{O}(n)$  time and space preprocessing, for any factor of T we can answer a period query in  $\mathcal{O}(\log n)$  time.

The data structures of Lemmas 2.5 and 2.6 are constructed in  $\mathcal{O}(n)$  expected time. These constructions were made worst-case in [16].

#### 2.3 Static Range Trees

A basic interval is an interval  $[a ... a + 2^i)$  such that  $2^i$  divides a - 1. We assume w.l.o.g. that n is a power of two. We consider a static range tree structure whose nodes correspond to basic subintervals of [1 ... n] and a non-leaf node has children corresponding to the two halves of the interval. (See e.g. [18]). The total number of basic intervals is  $\mathcal{O}(n)$ . Using the tree, every interval [i ... j] can be decomposed into  $\mathcal{O}(\log n)$  pairwise disjoint basic intervals. The decomposition can be computed in  $\mathcal{O}(\log n)$  time by inspecting the paths from the leaves corresponding to i and j to their lowest common ancestor. A basic factor of T is a factor that corresponds to positions from a basic interval.

### 2.4 Seeds

We say that a string S is a seed of a string U if S is a factor of U and S is a cover of a string U' such that U is a factor of U'; see Fig. 2. The second point of the lemma below follows from Lemma 2.4.

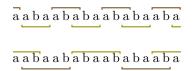


Figure 2: The strings *aba*, *abaab* are seeds of the given string (as well as strings *abaaba*, *abaababa*, *abaababaa*).

- **Lemma 2.7** ([17]). (a) All the seeds of T can be represented as a collection of a linear number of disjoint paths in the suffix tree  $\mathcal{T}(T)$ . Moreover, this representation can be computed in  $\mathcal{O}(n)$  time if T is over an integer alphabet.
- (b) After  $\mathcal{O}(n)$  time preprocessing we can check if a given factor of T is a seed of T in  $\mathcal{O}(\gamma_n)$  time.

Our main data structure is a static range tree SeedSets(T) which stores all seeds of every basic factor of T represented as a collection of paths in its suffix tree. Actually, only seeds of length at most half of a string will be of interest; see Fig. 3.

The sum of lengths of basic factors in T is  $\mathcal{O}(n \log n)$ . Consequently, due to Lemma 2.7, the tree SeedSets(T) has total size  $\mathcal{O}(n \log n)$  and can be computed in  $\mathcal{O}(n \log n)$  time. (To use Lemma 2.7(a) we renumber letters in basic factors of T via bucket sort so that the letters of a basic factor S are from  $\{0, \ldots, |S|^{\mathcal{O}(1)}\}$ .)

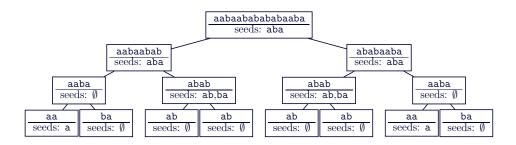


Figure 3: A schematic view of tree *SeedSets* of T (in the real data structure, seeds are stored on suffix trees of basic factors). For example, **ba** is a seed of T[5..12] since it is a seed of basic factors T[5..8] and T[9..12] and its occurrence covers T[8..9] (Lemma 3.3).

## **3** Internal Cover of a Given Length

In this section we show how to use SeedSets(T) to answer internal queries related to computing the longest prefix of a factor S of T that is covered by its length- $\ell$  prefix. We start with the following, easier queries.

COVER OF A GIVEN LENGTH QUERY (ISCOVER $(\ell, S)$ ) Input: A factor S of T and a positive integer  $\ell$ Query: Does S have a cover of length  $\ell$ ?

The following three lemmas provide the building blocks of the data structure for answering ISCOVER queries.

**Lemma 3.1** (Seed of a basic factor). After  $\mathcal{O}(n \log n)$ -time preprocessing, for any factor C and basic factor B of T such that  $2|C| \leq |B|$ , we can check if C is a seed of B in  $\mathcal{O}(\gamma_n)$  time.

*Proof.* Let |C| = c and  $B = T[a \dots b]$ . We first ask an IPM query to find an occurrence of C inside  $T[a \dots a + 2c-1]$ . If such an occurrence does not exist, then C cannot be a seed of  $T[a \dots b]$  as it is already not a seed of  $T[a \dots a + 2c-1]$  (there must be a full occurrence to cover the middle letter, and  $a + 2c - 1 \leq b$ ). Otherwise, we can use the occurrence to check if C is a seed of B with Lemma 2.7(b).

For strings C and S, by Cov(C, S) we denote the set of positions of S that are covered by occurrences of C.

**Lemma 3.2** (Covering short factors). After  $\mathcal{O}(n)$ -time preprocessing, for any two factors C and F of T such that  $|F|/|C| = \mathcal{O}(1)$ , the set Cov(C, F), represented as a union of maximal intervals, can be computed in  $\mathcal{O}(1)$  time.

*Proof.* We ask IPM queries for pattern C on length-2|C| factors of F with step |C|. Each IPM query returns an arithmetic sequence of occurrences that corresponds to an interval of covered positions (possibly empty). It suffices to compute the union of these intervals.

**Lemma 3.3** (Seeds of strings concatenation). After  $\mathcal{O}(n)$ -time preprocessing, for any three factors C,  $F_1 = T[i ... j]$  and  $F_2 = T[j + 1... k]$  of T such that  $2|C| \leq |F_1|, |F_2|$  and C is a seed of both  $F_1$  and  $F_2$ , we can check if C is also a seed of  $F_1F_2$  in constant time.

*Proof.* For a string C of length c being a seed of both  $T[i \dots j]$  and  $T[j + 1 \dots k]$  to be a seed of  $T[i \dots k]$ , it is enough if its occurrences cover the string  $U = T[j - c + 1 \dots j + c]$ . We can check this condition if we apply Lemma 3.2 for C and  $F = T[j - 2c + 1 \dots j + 2c]$ .

**Lemma 3.4.** After  $\mathcal{O}(n \log n)$  time and space preprocessing of T, a query  $\text{ISCOVER}(\ell, S)$  can be answered in  $\mathcal{O}(\log(|S|/\ell) \gamma_n + 1)$  time.

*Proof.* Let  $S = T[i \dots j]$ , |S| = s and  $C = T[i \dots i + \ell - 1]$ .

We consider a decomposition of S into basic factors, but we are only interested in basic factors of length at least  $2\ell$  in the decomposition. Let  $F_1, \ldots, F_k$  be those factors and  $T[i \ldots i'], T[j' \ldots j]$  be the remaining prefix and suffix of length  $\mathcal{O}(\ell)$ . Note that  $k = \mathcal{O}(\log(s/\ell))$ . Moreover, this decomposition can be computed in  $\mathcal{O}(k+1)$  time by starting from the leftmost and rightmost basic factors of length  $2^b$ , where  $b = \lceil \log \ell \rceil + 1$ , that are contained in S.

If C is a cover of S, it must be a seed of each of the basic factors  $F_1, \ldots, F_k$ . We can check this condition by using Lemma 3.1 in  $\mathcal{O}(k\gamma_n)$  total time.

Next we check if C is a seed of  $F_1 \cdots F_k$  in  $\mathcal{O}(k)$  total time using Lemma 3.3. Finally, we use IPM queries to check if occurrences of C cover all positions in each of the strings  $T[i \dots i' + c - 1]$ ,  $T[j' - c + 1 \dots j]$  and if C is a suffix of  $T[i \dots j]$ , using Lemma 3.2. This takes  $\mathcal{O}(1)$  time.

The total time complexity is  $\mathcal{O}(k\gamma_n+1)$ .

As we will see in the next section, ISCOVER queries immediately imply a slower,  $O(\log^2 n \gamma_n)$ -time algorithm for answering MINCOVER queries. However, they are also used in our algorithm for answering ALLCOVERS queries. In the efficient algorithm for MINCOVER queries we use the following generalization of ISCOVER queries.

LONGEST COVERED PREFIX QUERY (COVEREDPREF $(\ell, S)$ ) **Input:** A factor S of T and a positive integer  $\ell$ **Query:** The longest prefix P of S that is covered by  $S[1..\ell]$ 

To answer these queries, we introduce an intermediate problem that is more directly related to the range tree containing seeds representations.

SEEDEDBASICPREF $(C, \ell, S)$  QUERY

**Input:** A length- $\ell$  factor C of T and a factor S being a concatenation of basic factors of T of length  $2^p$ , where  $p = \min\{q \in \mathbb{Z} : 2^q \ge 2\ell\}$ 

**Output:** The length m of the longest prefix of S which is a concatenation of basic factors of length  $2^p$  such that C is a seed of this prefix

In other words, we consider only blocks of S which are basic factors of length  $2^p = \Theta(\ell)$ . Everything starts and ends in the beginning/end of a basic factor of length  $2^p$ . The number of such blocks in the prefix returned by SEEDEDBASICPREF is  $\mathcal{O}(\operatorname{result}'/\ell)$ , where  $\operatorname{result}' = \operatorname{SEEDEDBASICPREF}(C, \ell, S)$ , and, as we show in Lemma 3.6, it can be computed in  $\mathcal{O}(\log(\operatorname{result}'/\ell)\gamma_n + 1)$  time. This is how we achieve  $\mathcal{O}(\log(\operatorname{result}/\ell)\gamma_n + 1)$  time for COVEREDPREF $(\ell, S)$  queries. In a certain sense the computations behind Lemma 3.6 can work in a pruned range tree  $\operatorname{SeedSets}(T)$ .

**Lemma 3.5.** After  $\mathcal{O}(n)$ -time preprocessing, a COVEREDPREF $(\ell, S)$  query reduces in  $\mathcal{O}(1)$  time to a SEEDEDBASICPREF $(C, \ell, S')$  query with  $|S'| \leq |S|$ .

*Proof.* First, let us check if the answer to COVEREDPREF $(\ell, S)$  is small, i.e. at most  $4\ell$ , using Lemma 3.2. Otherwise, let p be defined as in a SEEDEDBASICPREF query,  $C = S[1 \dots \ell]$  and S' be the maximal factor of S that is composed of basic factors of length  $2^p$  (S' can be the empty string, if  $|S| < 3 \cdot 2^p$ ). Let  $S = T[i \dots j]$  and  $S' = T[i' \dots j']$ . Then

 $|(i' + \text{SEEDEDBASICPREF}(C, \ell, S')) - (i + \text{COVEREDPREF}(\ell, S))| < 2^p;$ 

see Fig. 4. Hence, knowing  $d = \text{SEEDEDBASICPREF}(C, \ell, S')$ , we check in  $\mathcal{O}(1)$  time, using Lemma 3.2 in a factor  $T[i' + d - 2^p \dots i' + d + 2^p - 1]$  of length  $2^{p+1}$ , what is the exact value of  $\text{COVEREDPREF}(\ell, S)$ .

We compute p using the formula  $p = 1 + \lceil \log \ell \rceil$ . Then the endpoints of S' can be computed from the endpoints of S in  $\mathcal{O}(1)$  time using simple modular arithmetic. The  $\mathcal{O}(n)$  preprocessing is due to Lemma 3.2.

To answer SEEDEDBASICPREF queries we use our range tree which stores seeds of every basic factor. Recall that for each basic factor  $T[i \dots j]$  we can check if C is a seed of this factor in  $\mathcal{O}(\gamma_n)$  time (Lemma 3.1); we denote this test SeededBasic(C, i, j).

Also for any two neighboring factors T[i ... j], T[j + 1 ... k], for which C is a seed, we can check in  $\mathcal{O}(1)$  time if C is a seed of the composite factor T[i ... k] (Lemma 3.3); we denote this test TestConcat(C, i, j, k).

**Lemma 3.6.** After  $\mathcal{O}(n \log n)$  time and space preprocessing of T, a query SEEDEDBASICPREF $(C, \ell, S)$  can be answered in  $\mathcal{O}(\log(\operatorname{result}/\ell) \gamma_n + 1)$  time, where  $\operatorname{result} = |\operatorname{SEEDEDBASICPREF}(C, \ell, S)|$ .

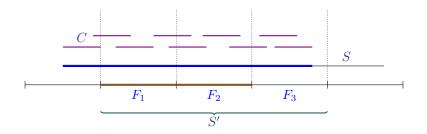


Figure 4:  $F_1$ ,  $F_2$ ,  $F_3$  are basic factors of length  $2^p$ . The answers to COVEREDPREF $(\ell, S)$  and SEEDEDBASICPREF $(C, \ell, S')$  queries are shown in bold. Note that C is a seed of  $F_1$  and  $F_2$  and that it could be the case that C is also a seed of  $F_3$ , even though it has no further full occurrence.

*Proof.* Let us define

 $rank(i) = \max\{k : [i \dots i + 2^k) \text{ is a basic interval}\}.$ 

All rank values for i = 1, ..., n can be computed in  $\mathcal{O}(n)$  time from the basic intervals.

**Observation 3.7.** If  $rank(i) \ge k$ , then  $rank(i+2^k) \ge k$ .

We introduce a Boolean function that is applied only if  $rank(j+1) \ge k$ :

$$TestExtend(i, j, k) \Leftrightarrow SeededBasic(C, j + 1, j + 2^k) \land TestConcat(C, i, j, j + 2^k).$$

We can then use the following Algorithm 1 to compute the result of a query. The algorithm implicitly traverses the static range tree. For an illustration, see Fig. 5, where the *Doubling Phase* corresponds to ascending the tree, and the *Binary Search Phase* corresponds to descending the tree. Intuitively, if  $rank(i) \ge k$ , then the basic interval  $[i . . i + 2^k)$  is the left child of its parent in the tree if and only if rank(i) > k.

Variable k is incremented in every second step of the Doubling Phase. At the conclusion of the phase, we know that C is a seed of T[start..last], where  $last - start + 1 \ge 2^k$ , and the final output will be T[start..last'], where  $(last' - last) \in [0..2^k)$ . Intuitively, we already have an approximation and use the Binary Search Phase to compute the actual result.

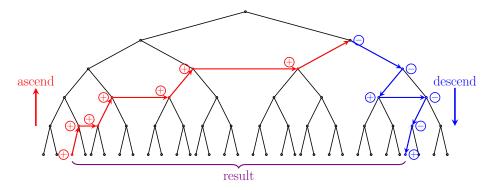


Figure 5: Interpretation of Algorithm 1 on the range tree. Basic factors correspond to the nodes of the tree. If the basic interval in the currently queried node corresponds to S[i ... j], then in this moment we know that C is a seed of S[1 ... i - 1]. The query asks whether C is a seed of the basic factor S[i ... j] (in  $\mathcal{O}(\gamma_n)$  time) and whether the concatenation of S[1 ... i - 1] and this basic factor is seeded by C (constant time). If "yes", then the known seeded prefix is extended and ends at j.

Each of the phases makes at most  $\mathcal{O}(\log(\operatorname{result}/\ell))$  iterations and uses  $\mathcal{O}(\gamma_n)$  time for each iteration. Thus, we have arrived at the required complexity.

Algorithm 1: Compute SEEDEDBASICPREF $(C, \ell, S)$ 

// Doubling Phase: Let  $S = T[start \dots end]$  and  $p = 1 + \lceil \log \ell \rceil$ ;  $last := start + 2^p - 1; k := p;$ repeat // Invariant: C is a seed of T[start ... last],  $last - start + 1 \ge 2^k$  and  $rank(last + 1) \ge k$ . if last > end then return S; if not TestExtend(start, last, k) then break;  $last := last + 2^k$ ; if rank(last + 1) > k then k := k + 1; // Binary Search Phase: repeat k := k - 1;if k < p then break; if  $last \ge end$  then return S: if TestExtend(start, last, k) then  $last := last + 2^k;$ return  $T[start \dots last];$ 

As a corollary of Lemmas 3.5 and 3.6, we obtain the following result.

**Lemma 3.8.** After  $\mathcal{O}(n \log n)$  time and space preprocessing of T, a query  $\text{COVEREDPREF}(\ell, S)$  can be answered in  $\mathcal{O}(\log(\text{result}/\ell) \gamma_n + 1)$  time, where  $\text{result} = |\text{COVEREDPREF}(\ell, S)|$ .

## 4 Internal Shortest Cover Queries

For a string S, by Borders(S) we denote a decomposition of the set of all border lengths of S into  $\mathcal{O}(\log |S|)$  arithmetic sequences  $A_1, \ldots, A_k$  such that each sequence  $A_i$  is either a singleton or, if p is its difference, then the borders with lengths in  $A_i \setminus {\min(A_i)}$  are periodic with the shortest period p. Moreover,  $\max(A_i) < \min(A_{i+1})$  for every  $i \in [1 \ldots k - 1]$ . See e.g. [8]. The following lemma is shown by applying a period query (Lemma 2.6).

**Lemma 4.1** ([16, 20]). For any factor S of T, Borders(S) can be computed in  $\mathcal{O}(\log n)$  time after  $\mathcal{O}(n)$ -time preprocessing.

## 4.1 Simple Algorithm with $O(\log^2 n \gamma_n)$ Query Time

Let us start with a much simpler but slower algorithm for answering MINCOVER queries using ISCOVER queries. We improve it in Theorem 4.3 by using COVEREDPREF queries and applying an algorithm for computing shortest covers that resembles, to some extent, computation of the shortest cover from [2].

**Proposition 4.2.** Let T be a string of length n. After  $\mathcal{O}(n \log n)$ -time preprocessing, for any factor S of T we can answer a MINCOVER(S) query in  $\mathcal{O}(\log^2 n \log \log n)$  time.

*Proof.* Using Lemma 4.1 we compute the set  $Borders(S) = A_1, \ldots, A_k$  in  $\mathcal{O}(\log n)$  time. Let us observe that the shortest cover of a string is aperiodic. This implies that from each progression  $A_i$  only the border of length  $\min(A_i)$  can be the shortest cover of S. We use Lemma 3.4 to test each of the  $\mathcal{O}(\log n)$  candidates in  $\mathcal{O}(\log n \gamma_n)$  time.

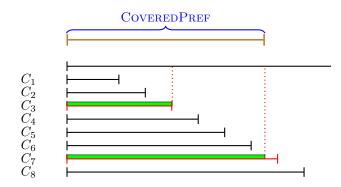


Figure 6: If  $C_3$  is an active border, then the next active one is  $C_7$ . We skip  $C_4, C_5, C_6$  as candidates for the shortest cover.

#### 4.2 Faster Queries

**Theorem 4.3.** Let T be a string of length n. After  $\mathcal{O}(n \log n)$ -time preprocessing, for any factor S of T we can answer a MINCOVER(S) query in  $\mathcal{O}(\log n \log \log n)$  time.

*Proof.* Again we use Lemma 4.1 we compute the set  $Borders(S) = A_1, \ldots, A_k$ , in  $\mathcal{O}(\log n)$  time. Let us denote the border of length  $\min(A_i)$  by  $C_i$  and  $C_{k+1} = S$ . We assume that  $C_i$ 's are sorted in increasing order of lengths. Then we proceed as shown in Algorithm 2. See also Fig. 6.

Algorithm 2: MINCOVER(S) query.

$$\begin{split} i &:= 1; \\ \textbf{while true do} \\ & // \text{ Invariant: } C_1, \dots, C_{i-1} \text{ are not covers of } S \\ & // C_i \text{ is an active border} \\ & P &:= \text{COVEREDPREF}(|C_i|, S); \\ \textbf{if } P &= S \text{ then return } |C_i|; \\ \textbf{while } |C_i| \leq |P| \text{ do} \\ & i &:= i+1; \end{split}$$

To argue for the correctness of the algorithm it suffices to show the invariant. The proof goes by induction. The base case is trivial. Let us consider the value of i at the beginning of a step of the while-loop. If P = S, then by the inductive assumption  $C_i$  is the shortest cover of S and can be returned. Otherwise,  $C_i$  is not a cover of S.

Moreover, for each j such that  $|C_i| < |C_j| \le |P|$ , since  $C_j$  is a prefix of P,  $C_i$  is a seed of  $C_j$ . Moreover, both  $C_i$  and  $C_j$  are borders of S, so  $C_i$  is a border of  $C_j$ . Consequently,  $C_j$  cannot be a cover of T, as then  $C_i$  would also be a cover of T by Observation 2.1. This shows that the inner while-loop correctly increases i.

The algorithm stops because at each point  $|P| \ge |C_i|$  and *i* is increased.

Let  $c_1, \ldots, c_p$  be equal to the length of an active border in the algorithm at the start of subsequent outer while-loop iterations and let  $c_{p+1} = |S|$ .

Let us note that, for all j = 1, ..., p,  $|\text{COVEREDPREF}(c_j, S)| \le c_{j+1}$ . By Lemma 3.8, the total complexity of answering longest covered prefix queries in the algorithm is at most

$$\mathcal{O}\left(p+\gamma_n\sum_{j=1}^p\log\frac{c_{j+1}}{c_j}\right)=\mathcal{O}(\log n+\gamma_n(\log c_{p+1}-\log c_1))=\mathcal{O}(\log n\,\gamma_n).$$

The preprocessing of Lemmas 3.8 and 4.1 takes  $\mathcal{O}(n \log n)$  time. The conclusion follows.

If MINCOVER queries are to be answered in a batch, we can use off-line WA queries of Lemma 2.4 to save the  $\gamma_n$ -factor. We can also avoid storing the whole data structure *SeedSets* by using an approximate version of COVEREDPREF queries.

**Theorem 4.4.** For a string T of length n, any m queries MINCOVER(T[i..j]) can be answered in  $O((n + m)\log n)$  time and O(n + m) space.

*Proof.* The  $\gamma_n$  factor from the query complexity of Theorem 4.3 stems from using on-line weighted-ancestor queries to access *SeedSets*. In the off-line setting it suffices to answer such queries in a batch; see Lemma 2.4.

The only data structure from Theorem 4.3 which takes  $\omega(n)$  space is the *SeedSets* tree. However each level of the tree, corresponding to basic factors of the same length, takes only  $\mathcal{O}(n)$  space and can be constructed independently in  $\mathcal{O}(n)$  time. We will modify the algorithm for answering a MINCOVER query so that it will access the levels of *SeedSets* in the order of increasing lengths of basic factors. Then we will be able to answer all MINCOVER queries simultaneously in  $\mathcal{O}(n + m)$  space.

Approximate CoveredPref queries. The building block of the data structure for answering MINCOVER queries are COVEREDPREF queries. In Lemma 3.5, a COVEREDPREF $(\ell, S)$  query is either answered in  $\mathcal{O}(1)$ time via IPM queries, or reduced to a SEEDEDBASICPREF $(C, \ell, S')$  query for S' being a maximal factor of S that is a concatenation of basic factors of length  $\Delta \geq \lfloor 2\ell \rfloor$  and  $C = S \lfloor 1 \dots \ell \rfloor$ .

Let result' =  $|\text{SEEDEDBASICPREF}(C, \ell, S')|$ . Note that if result' < 5 $\Delta$ , then a SEEDEDBASICPREF query can be answered naively via IPM queries in  $\mathcal{O}(1)$  time. Otherwise, the algorithm for answering SEEDEDBASICPREF queries (Lemma 3.6) consists of two phases; the first phase considers basic factors of non-decreasing lengths, but the second phase considers them according to non-increasing lengths.

This implementation does not satisfy the condition that the algorithm visits *SeedSets* level by level. However, we will show that the result of the first phase yields a constant factor approximation of the result =  $|\text{COVEREDPREF}(\ell, S)|$ . Indeed, after the first phase a prefix U of S' is computed such that C is a seed of U and  $|U| \ge \frac{1}{2}$  result'. If S = LS'R, this means that C is a cover of a prefix P of LU of length at least  $|L| + \frac{1}{2}$  result' -  $(\ell - 1)$ .

We obviously have  $|P| \leq \text{result}$ . Moreover, we have  $\text{result} \leq |L| + \text{result}' + 2\ell - 1$ . Hence,

$$|3|P| > |L| + \frac{3}{2}$$
 result  $' - 3\ell + 1 \ge |L| +$ result  $' + 5\ell - 3\ell + 1 \ge$ result,

so indeed |P| is a 3-approximation of result. Let us call the resulting routine APPROXCOVEREDPREF.

Simultaneous calls to IsCover. We use the approximate routine instead of COVEREDPREF to compute P in Algorithm 2. Then all candidates  $C_i$  that are eliminated in the inner while-loop are eliminated correctly. The only issue is with correctness of the if-statement, since P is only a lower bound for the result. To address this issue, for each active border in the algorithm we start running IsCOVER( $|C_i|, S$ ). According to Lemma 3.4, this requires  $\mathcal{O}(1)$ -time checks using IPM queries on the edges of S and checking if  $C_i$  is a seed of concatenation of basic factors  $F_1, \ldots, F_k$ , each of length at least  $2\ell$ . To this end, we use Lemmas 3.1 and 3.3.

Let us recall that the sequence of lengths of basic factors  $F_1, \ldots, F_k$  is first increasing and then decreasing, and it can be computed step by step in  $\mathcal{O}(k)$  time. Hence, we can answer queries for  $F_i$  starting from the ends of the sequence simultaneously with computing APPROXCOVEREDPREF.

At the conclusion of the latter query, if  $|P| < \frac{1}{3}|S|$ , then we know that  $ISCOVER(|C_i|, S)$  would return false and we can discard the computations. Otherwise we continue computing  $ISCOVER(|C_i|, S)$  for this active border in the subsequent steps of the outer while-loop. This may yield several ISCOVER queries that are to be answered in parallel. However, the number of such queries is only  $\mathcal{O}(1)$ , since  $|C_{i+2}| > \frac{3}{2}|C_i|$ (otherwise  $C_{i+1}$  would have been in the same arithmetic sequence as one of  $C_i$ ,  $C_{i+2}$ ).

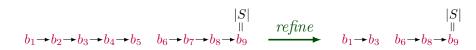


Figure 7: There is an arrow from  $b_i$  to  $b_{i+1}$  iff  $S[1 \dots b_i]$  is a cover of  $S[1 \dots b_{i+1}]$ . Note that all elements in the last chain  $b_6, b_7, b_8, b_9$  are cover lengths of  $S, b_5$  is not, but some prefix of  $b_1, b_2, b_3, b_4$  may be.

## 5 Internal All Covers Queries

In this section we refer to ALLCOVERS(S) as to the set of lengths of all covers of S. This set consists of a logarithmic number of arithmetic sequences since the same is true for all borders. In each sequence of borders we show that it is needed only to check  $\mathcal{O}(1)$  borders to be a cover of S. Hence we start with an algorithm testing any sequence of  $\mathcal{O}(\log n)$  candidate borders.

### 5.1 Verifying $\mathcal{O}(\log n)$ Candidates

Assume that B is an increasing sequence  $b_1, \ldots, b_k$  of lengths of borders of a given factor S (not necessarily all borders), with  $b_k = |S|$ . A *chain* in B is a maximal subsequence  $b_i, \ldots, b_j$  of consecutive elements of B such that  $S[1 \dots b_t]$  is a cover of  $S[1 \dots b_{t+1}]$  for each  $t \in [i \dots j)$ . From Observation 2.1 we get the following.

**Observation 5.1.** The set of elements of a chain that belong to ALLCOVERS(S) is a prefix of this chain. Moreover, if the last element of a chain is not |S|, then it is not a cover of S.

We denote by chains(B) and covers(B), respectively, the partition of B into chains and the set of elements  $b \in B$  such that  $S[1 \dots b]$  is a cover of S. For  $b \in B$  by prev(b) we denote the previous element in its chain (if it exists). Moreover, for  $C \subseteq B$  by  $next_C(b)$  we denote the smallest  $c \in C$  such that c > b.

**Lemma 5.2.** Let T be a string of length n. After  $\mathcal{O}(n \log n)$ -time preprocessing, for any factor S of T and a sequence B of  $\mathcal{O}(\log n)$  borders of S we can compute covers(B) in  $\mathcal{O}(\log n \log \log n \gamma_n)$  time.

*Proof.* We introduce two operations and use them in a recursive Algorithm 3.

- refine(B): removes the last element of each chain in B and every second element of each chain, except |S| (see Fig. 7). Note that  $|refine(B)| \le |B|/2 + 1$ .
- **computeUsing**(B, C): Assuming that we know the set C of all covers of S among refine(B), for each element b of  $B \setminus refine(B)$  we add it to C if  $prev(b) \in C$  and  $S[1 \dots b]$  is a cover of  $S[1 \dots next_C(b)]$ . The set of all elements that satisfy this condition together with C is returned as covers(B).

Algorithm 3: covers(B)	
Compute $chains(B)$ ;	
if B is a single chain (ending with $ S $ ) then return B;	
B' := refine(B);	$ /  B'  \le  B /2 + 1$
C := covers(B');	
return $computeUsing(B,C);$	

If  $B = (b_1, \ldots, b_k)$ , then chains(B) can be constructed in  $\mathcal{O}(\sum_{i=1}^{k-1} (\log \frac{b_{i+1}}{b_i} \gamma_n + 1)) = \mathcal{O}(\log n \gamma_n)$  time using Lemma 3.4. Similarly, operation computeUsing(B, C) requires  $\mathcal{O}(\log n \gamma_n)$  time since the intervals  $[b, next_C(b)]$  for  $b \in B \setminus refine(B)$  such that  $prev(b) \in C$  are pairwise disjoint. The depth of recursion of Algorithm 3 is  $\mathcal{O}(\log \log n)$ . This implies the required complexity.

### 5.2 Computing Periodic Covers

Our tool for periodic covers are (as usual) runs. A run (also known as a maximal repetition) is a periodic factor  $R = T[a \dots b]$  which can be extended neither to the left nor to the right without increasing the period p = per(R), i.e.,  $T[a-1] \neq T[a+p-1]$  and  $T[b-p+1] \neq T[b+1]$  provided that the respective positions exist. The following observation is well-known.

**Observation 5.3.** Two runs in T with the same period p can overlap on at most p-1 positions.

The exponent  $\exp(S)$  of a string S is  $|S|/\operatorname{per}(S)$ . The Lyndon root of a string S is the minimal cyclic shift of  $S[1..\operatorname{per}(S)]$ .

If S = T[a..b] is periodic, then by run(S) we denote the run R with the same period that contains S. We say that S is *induced* by R. A periodic factor of T is induced by exactly one run [10]. The run-queries are essentially equivalent to two-period queries. By  $\mathcal{R}(T)$  we denote the set of all runs in a string T.

**Lemma 5.4** ([3, 10, 21]). (a)  $|\mathcal{R}(T)| \leq n$  and  $\mathcal{R}(T)$  can be computed in  $\mathcal{O}(n)$  time.

(b) After  $\mathcal{O}(n)$ -time preprocessing,  $\operatorname{run}(S)$  queries can be answered in  $\mathcal{O}(1)$  time.

(c) The runs from  $\mathcal{R}(T)$  can be grouped by their Lyndon roots in  $\mathcal{O}(n)$  time.

The following lemma implies that indeed for any string S, ALLCOVERS(S) can be expressed as a union of  $\mathcal{O}(\log |S|)$  arithmetic sequences. It also shows a relation between periodic covers and runs in S.

**Lemma 5.5.** Let S be a string,  $A \in Borders(S)$  be an arithmetic sequence with difference  $p, A' = A \setminus \{\min(A)\}$  and  $a' = \min(A')$ . Moreover, let x be the minimal exponent of a run in S with Lyndon root being a cyclic shift of S[1 ... p].

(a) If  $a' \notin ALLCOVERS(S)$ , then  $A' \cap ALLCOVERS(S) = \emptyset$ .

(b) Otherwise, there exists  $c \in ((x-2)p, xp] \cap A'$  such that  $A' \cap ALLCOVERS(S) = \{a', a' + p, \dots, c\}$ .

*Proof.* Part (a) follows from Observation 2.1. Indeed, assume that S has a cover of length  $b \in A'$ , with b > a'. As  $S[1 \dots a']$  is a cover of  $S[1 \dots b]$ , we would have  $a' \in ALLCOVERS(S)$ .

We proceed to the proof of part (b). Let c be the maximum element of A' such that C := S[1 .. c] is a cover of S. By the same argument as before, we have that  $A' \cap ALLCOVERS(S) = A' \cap [1, c]$ . It suffices to prove the bounds for c.

Let L be the minimum cyclic shift of S[1..p]. We consider all runs  $R_1, \ldots, R_k$  in S with Lyndon root L. Each occurrence of C in S is induced by one of them. Each of the runs must hold an occurrence of C. Indeed, by Observation 5.3, no two of the runs overlap on more than p-1 positions, so the pth position of each run cannot be covered by occurrences of C that are induced by other runs. The shortest of the runs has length xp, so  $c \leq xp$ .

Furthermore, let C' = S[1 .. c'] be a prefix of S of length c' = c + p. If  $p \cdot \exp(R_i) \ge c' + p - 1$ , then  $R_i$  induces an occurrence of C' and  $Cov(C', R_i) = Cov(C, R_i)$ . Hence, if  $px \ge c' + p - 1$  would hold, C' would be a cover of S, which contradicts our assumption. Therefore, px < c' + p - 1 = c + 2p - 1, so c > (x - 2)p.

Lemma 5.7 transforms Lemma 5.5 into a data structure. We use static dictionaries.

**Lemma 5.6** (Rui [25]). A static dictionary of n integers that supports  $\mathcal{O}(1)$ -time lookups can be stored in  $\mathcal{O}(n)$  space and constructed in  $\mathcal{O}(n(\log \log n)^2)$  time. The elements stored in the dictionary may be accompanied by satellite data.

### Lemma 5.7 (Computing $\mathcal{O}(\log n)$ Candidates).

For any factor S of T we can compute in  $\mathcal{O}(\log n)$  time  $\mathcal{O}(\log n)$  borders of S which are candidates for covers of S. After knowing which of these candidates are covers of S, we can in  $\mathcal{O}(\log n)$  time represent (as  $\mathcal{O}(\log n)$ arithmetic progressions) all borders which are covers of S. The preprocessing time is  $\mathcal{O}(n(\log \log n)^2)$  and the space used is  $\mathcal{O}(n)$ . *Proof.* It is enough to show that for any factor S of T and a single arithmetic sequence  $A \in Borders(S)$  we can compute in  $\mathcal{O}(1)$  time up to four candidate borders. Then, after knowing which of them are covers of S, we can in  $\mathcal{O}(1)$  time represent (as a prefix subsequence of A) all borders in A which are covers of S. We first describe the data structure and then the query algorithm.

**Data structure.** Let  $T[a_1...b_1],...,T[a_k...b_k]$  be the set of all runs in T with Lyndon root L, with  $a_1 < \cdots < a_k$  (and  $b_1 < \cdots < b_k$ ). The part of the data structure for this Lyndon root consists of an array  $A_L$  containing  $a_1,...,a_k$ , an array  $E_L$  containing the exponents of the respective runs, as well as a dictionary on  $A_L$  and a range-minimum query data structure on  $E_L$ . Formally, to each Lyndon root we assign an integer identifier in [1, n] that is retained with every run with this Lyndon root and use it to index the data structures. We also store a dictionary of all the runs. The data structure takes  $\mathcal{O}(n)$  space and can be constructed in  $\mathcal{O}(n(\log \log n)^2)$  time by Lemmas 2.2, 5.4 and 5.6 (RMQ, computing runs and grouping runs by Lyndon roots, and static dictionary, respectively). We also use LCE-queries on T (Lemma 2.3).

Queries. Let us consider a query for  $S = T[i \dots j]$  and  $A \in Borders(S)$ . If |A| = 1, we have just one candidate. Otherwise, A is an arithmetic sequence with difference p. Let  $a = \min(A)$ ,  $A' = A \setminus \{a\}$ , and  $a' = \min(A')$ . We select borders of length a and a' as candidates. If  $a' \notin ALLCOVERS(S)$ , then Lemma 5.5(a) implies that  $A \cap ALLCOVERS(S) \subseteq \{a\}$ . We also select borders of lengths in  $A \cap ((x-2)p, xp]$  as candidates, where x is defined as in Lemma 5.5. Note that there are at most two of them. Let c be the maximum candidate which turned out to be a cover of S. Then  $A \cap ALLCOVERS(S) = A \cap [1, c]$  by Lemma 5.5(b).

What is left is to compute x, that is, the minimum exponent of a run in S with Lyndon root L that is a cyclic shift of S[1..p]. Since  $|A| \ge 2$ , S has a prefix run with Lyndon root L. Then  $\ell = \min(p + d, |S|)$ , where  $d = \mathsf{lcp}(i, i + p)$ , is the length of the run. If  $\ell = |S|$ , then  $x = \ell/p$  and we are done. Otherwise, let i' = i + p + d. We make the following observation.

**Claim 5.8.** If  $a' \in ALLCOVERS(S)$ , then  $T[i' \dots i' + p]$  is contained in a run in T with Lyndon root L.

*Proof.* Runs in T with Lyndon root L must cover  $S = T[i \dots j]$ , since  $S[1 \dots a']$  is a periodic cover of S and each of its occurrences is induced by a run. The prefix run in S corresponds to a run ending at position i' - 1 in T. By Observation 5.3, the run with Lyndon root L containing the position i' must end after position i' + p.

We identify the run T[a..b] with period p containing T[i'..i'+p] by asking lcp(i',i'+p) and lcs(i',i'+p) queries. This lets us recover the identifier of its Lyndon root L. Similarly we compute the suffix run with Lyndon root L in S and the previous run T[a'..b'] with Lyndon root L in T. Using the dictionary on  $A_L$ , we recover the range in the array that corresponds to elements from a to a'. This lets us use a range minimum query on this range in  $E_L$  and use it together with the exponents of the prefix and suffix runs of S to compute x. All the operations in a query are performed in  $\mathcal{O}(1)$  time.

### 5.3 Main Query Algorithm

The main result of this section follows from Lemma 5.2 and Lemma 5.7.

**Theorem 5.9.** Let T be a string of length n. After  $\mathcal{O}(n \log n)$ -time preprocessing, for any factor S of T we can answer a query ALLCOVERS(S), with output represented as a union of  $\mathcal{O}(\log n)$  pairwise disjoint arithmetic sequences, in  $\mathcal{O}(\log n (\log \log n)^2)$  time.

The transformation to the off-line model is similar as in Theorem 4.4.

**Corollary 5.10.** For a string T of length n, any m queries ALLCOVERS(T[i..j]) can be answered in  $\mathcal{O}((n+m)\log n\log \log n)$  time and  $\mathcal{O}(n+m)$  space.

*Proof.* As in the proof of Theorem 4.4, we use off-line WA queries and we need to transform the query algorithm to make sure that the *SeedSets* tree is processed level by level. The data structure counterpart of Lemma 5.7 uses only  $\mathcal{O}(n)$  space and does not require any transformations. Lemma 5.2 for computations on the *SeedSets* tree uses only ISCOVER queries, which are simpler than COVEREDPREF queries.

Each single such query can be naturally implemented by traversing the tree level by level. Moreover, in each recursive call of the algorithm of Lemma 5.2, all subsequent calls to ISCOVER in computing chains(B) and in the routine computeUsing(B, C) visit the tree level by level.

Hence, it suffices to simultaneously process the first recursive calls of all ALLCOVERS queries, similarly for the second recursive calls etc, where the maximum depth of recursion is  $\mathcal{O}(\log \log n)$ . Thus we will construct the *SeedSets* tree, level by level,  $\mathcal{O}(\log \log n)$  times, each time in  $\mathcal{O}(n \log n)$  time and  $\mathcal{O}(n)$  space.

## 6 Final Remarks

We showed an efficient data structure for computing internal covers. However, a similar problem for seeds, which are another well-studied notion in quasiperiodicity, seems to be much harder. We pose the following question.

#### Open problem.

Can one answer internal queries related to seeds in  $\mathcal{O}(\text{polylog } n)$  time after  $\mathcal{O}(n \text{ polylog } n)$  time preprocessing?

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