# First-Order Tests for Toricity 

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#### Abstract

Motivated by problems arising with the symbolic analysis of steady state ideals in Chemical Reaction Network Theory, we consider the problem of testing whether the points in a complex or real variety with non-zero coordinates form a coset of a multiplicative group. That property corresponds to Shifted Toricity, a recent generalization of toricity of the corresponding polynomial ideal. The key idea is to take a geometric view on varieties rather than an algebraic view on ideals. Recently, corresponding coset tests have been proposed for complex and for real varieties. The former combine numerous techniques from commutative algorithmic algebra with Gröbner bases as the central algorithmic tool. The latter are based on interpreted first-order logic in real closed fields with real quantifier elimination techniques on the algorithmic side. Here we take a new logic approach to both theories, complex and real, and beyond. Besides alternative algorithms, our approach provides a unified view on theories of fields and helps to understand the relevance and interconnection of the rich existing literature in the area, which has been focusing on complex numbers, while from a scientific point of view the (positive) real numbers are clearly the relevant domain in chemical reaction network theory. We apply prototypical implementations of our new approach to a set of 129 models from the BioModels repository.


## 1 Introduction

We are interested in situations where the points with non-zero coordinates in a given complex or real variety form a multiplicative group or, more generally, a coset of such a group. For irreducible varieties this corresponds to toricity [23, 16] and shifted toricity [28, 27, respectively, of both the varieties and the corresponding ideals.

While toric varieties are well established and have an important role in algebraic geometry [23, 16], our principal motivation here to study generalizations of toricity comes from the sciences, specifically chemical reaction networks such as the following model of the kinetics of intra- and intermolecular zymogen activation with formation of an enzyme-zymogen complex [22], which
can also be found as model no. $92^{11}$ in the BioModels database [9]:

$$
\begin{gathered}
\mathrm{Z} \xrightarrow{0.004} \mathrm{P}+\mathrm{E} \\
\mathrm{Z}+\mathrm{E} \xrightarrow[2.1 \mathrm{E}-4]{1000} \mathrm{E}-\mathrm{Z} \xrightarrow{5.4 \mathrm{E}-4} \mathrm{P}+2 \mathrm{E}
\end{gathered}
$$

Here Z stands for zymogen, P is a peptide, E is an enzyme, $\mathrm{E}-\mathrm{Z}$ is the enzyme substrate complex formed from that enzyme and zymogen. The reactions are labelled with reaction rate constants.

Let $x_{1}, \ldots, x_{4}: \mathbb{R} \rightarrow \mathbb{R}$ denote the concentrations over time of the species $\mathrm{Z}, \mathrm{P}, \mathrm{E}, \mathrm{E}-\mathrm{Z}$, respectively. Assuming mass action kinetics one can derive reaction rates and furthermore a system of autonomous ordinary differential equations describing the development of concentrations in the overall network [20, Section 2.1.2]:

$$
\begin{array}{ll}
\dot{x_{1}}=p_{1} / 100000, & p_{1}=-100000000 x_{1} x_{2}-400 x_{1}+21 x_{4}, \\
\dot{x_{3}}=p_{3} / 50000, & p_{2}=-100000000 x_{1} x_{2}+400 x_{1}+129 x_{4}, \\
\dot{x_{2}}=p_{2} / 100000, & p_{3}=200 x_{1}+27 x_{4}, \\
\dot{x_{4}}=p_{4} / 4000, & p_{4}=4000000 x_{1} x_{2}-3 x_{4} .
\end{array}
$$

The chemical reaction is in equilibrium for positive concentrations of species lying in the real variety of the steady state ideal

$$
\left\langle p_{1}, \ldots, p_{4}\right\rangle \subseteq \mathbb{Z}\left[x_{1}, \ldots, x_{4}\right]
$$

intersected with the first orthant of $\mathbb{R}^{4}$.
Historically, the principle of detailed balancing has attracted considerable attention in the sciences. It states that at equilibrium every single reaction must be in equilibrium with its reverse reaction. Detailed balancing was used by Boltzmann in 1872 in order to prove his Htheorem [4], by Einstein in 1916 for his quantum theory of emission and absorption of radiation [15], and by Wegscheider 50] and Onsager [42] in the context of chemical kinetics, which lead to Onsager's Nobel prize in Chemistry in 1968. In the field of symbolic computation, Grigoriev and Weber [29] applied results on binomial varieties to study reversible chemical reactions in the case of detailed balancing.

In particular with the assumption of irreversible reactions, like in our example, detailed balancing has been generalized to complex balancing [19, 20, 33, which has widely been used in the context of chemical reaction networks. Here one considers complexes, like $\mathrm{Z}, \mathrm{P}+\mathrm{E}, \mathrm{Z}+\mathrm{E}$, etc. in our example, and requires for every such complex that the sum of the reaction rates of its inbound reactions equals the sum of the reaction rates of its outbound reactions.

Craciun et al. [11] showed that toric dynamical systems [18, 33, in turn, generalize complex balancing. The generalization of the principle of complex balancing to toric dynamical systems has obtained considerable attention in the last years [44, 24, 11, 40]. Millan, Dickenstein and Shiu in [44] considered steady state ideals with binomial generators. They presented a sufficient linear algebra condition on the stoichiometry matrix of a chemical reaction network in order to test whether the steady state ideal has binomial generators. Conradi and Kahle showed that the sufficient condition is even equivalent when the ideal is homogenous [10, 35, 34. That condition also led to the introduction of MESSI systems [43. Recently, binomiality of steady states ideals was used to infer network structure of chemical reaction networks out of measurement data 49].

Besides its scientific adequacy as a generalization of complex balancing there are practical motivations for studying toricity. Relevant models are typically quite large. For instance, with our comprehensive computations in this article we will encounter one system with 90 polynomials

[^0]in dimension 71. This brings symbolic computation to its limits. Our best hope is to discover systematic occurrences of specific structural properties in the models coming from a specific context, e.g. the life sciences, and to exploit those structural properties towards more efficient algorithms. In that course, toricity could admit tools from toric geometry, e.g., for dimension reduction.

Detecting toricity of varieties in general, and of steady state varieties of chemical reaction networks in particular, is a difficult problem. The first issue in this regard is finding suitable notions to describe the structure of the steady states. Existing work, such as the publications mentioned above, typically focuses on the complex numbers and addresses algebraic properties of the steady state ideal, e.g., the existence of binomial Gröbner bases. Only recently, a group of researchers including the authors of this article have taken a geometric approach, focusing on varieties rather than ideals [27, 28]. Besides irreducibility, the characteristic property for varieties $V$ to be toric over a field $\mathbb{K}$ is that $V \cap\left(\mathbb{K}^{*}\right)^{n}$ forms a multiplicative group. More generally, one considers shifted toricity, where $V \cap\left(\mathbb{K}^{*}\right)^{n}$ forms a coset of a multiplicative group.

It is important to understand that chemical reaction network theory generally takes place in the interior of the first orthant of $\mathbb{R}^{n}$, i.e., all species concentrations and reaction rates are assumed to be strictly positive [20]. Considering $\left(\mathbb{C}^{*}\right)^{n}$ in contrast to $\mathbb{C}^{n}$ resembles the strictness condition, and considering also $\left(\mathbb{R}^{*}\right)^{n}$ in [27] was another step in the right direction.

The plan of the article is as follows. In Section 2 we motivate and formally introduce firstorder characterizations for shifted toricity, which have been used already in [27], but exclusively with real quantifier elimination methods. In Section 3 we put a model theoretic basis and prove transfer principles for our characterizations throughout various classes of fields, with zero as well as with positive characteristics. In Section 4 we employ Hilbert's Nullstellensatz as a decision procedure for uniform word problems and use logic tests also over algebraically closed fields. This makes the link between the successful logic approach from [27] and the comprehensive existing literature cited above. Section 5 clarifies some asymptotic worst-case complexities for the sake of scientific rigor. In Section 6 it turns out that for a comprehensive benchmark set of 129 models from the BioModels database [9] quite simple and maintainable code, requiring only functionality available in most decent computer algebra systems and libraries, can essentially compete with highly specialized and more complicated purely algebraic methods. This motivates in Section 7 a perspective that our symbolic computation approach has a potential to be interesting for researchers in the life sciences, with communities much larger than our own, with challenging applications, not least in the health sector.

## 2 Syntax: First-Order Formulations of Group and Coset Properties

In this section we set up our first-order logic framework. We are going to use interpreted firstorder logic with equality over the signature $\mathcal{L}=(0,1,+,-, \cdot)$ of rings.

For any field $\mathbb{K}$ we denote its multiplicative group $\mathbb{K} \backslash\{0\}$ by $\mathbb{K}^{*}$. For a coefficient ring $Z \subseteq \mathbb{K}$ and $F \subseteq Z\left[x_{1}, \ldots, x_{n}\right]$ we denote by $V_{\mathbb{K}}(F)$, or shortly $V(F)$, the variety of $F$ over $\mathbb{K}$. Our signature $\mathcal{L}$ naturally induces coefficient rings rings $Z=\mathbb{Z} / p$ for finite characteristic $p$, and $Z=\mathbb{Z}$ for characteristic 0 . We define $V(F)^{*}=V(F) \cap\left(\mathbb{K}^{*}\right)^{n} \subseteq\left(\mathbb{K}^{*}\right)^{n}$. Note that the direct product $\left(\mathbb{K}^{*}\right)^{n}$ establishes again a multiplicative group.

Let $F=\left\{f_{1}, \ldots, f_{m}\right\} \subseteq Z\left[x_{1}, \ldots, x_{n}\right]$. The following semi-formal conditions state that $V(F)^{*}$ establishes a coset of a multiplicative subgroup of $\left(\mathbb{K}^{*}\right)^{n}$ :

$$
\begin{equation*}
\forall g, x \in\left(\mathbb{K}^{*}\right)^{n}: \quad g \in V(F) \wedge g x \in V(F) \Rightarrow g x^{-1} \in V(F) \tag{1}
\end{equation*}
$$

$$
\begin{gather*}
\forall g, x, y \in\left(\mathbb{K}^{*}\right)^{n}: \quad g \in V(F) \wedge g x \in V(F) \wedge g y \in V(F) \Rightarrow g x y \in V(F)  \tag{2}\\
V(F) \cap\left(\mathbb{K}^{*}\right)^{n} \neq \emptyset . \tag{3}
\end{gather*}
$$

If we replace (3) with the stronger condition

$$
\begin{equation*}
1 \in V(F) \tag{4}
\end{equation*}
$$

then $V(F)^{*}$ establishes even a multiplicative subgroup of $\left(\mathbb{K}^{*}\right)^{n}$. We allow ourselves to less formally say that $V(F)^{*}$ is a coset or group over $\mathbb{K}$, respectively.

Denote $M=\{1, \ldots, m\}, N=\{1, \ldots, n\}$, and for $(i, j) \in M \times N$ let $d_{i j}=\operatorname{deg}_{x_{j}}\left(f_{i}\right)$. We shortly write $x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right), g=\left(g_{1}, \ldots, g_{n}\right)$. Multiplication between $x, y, g$ is coordinate-wise, and $x^{d_{i}}=x_{1}^{d_{i 1}} \cdots x_{n}^{d_{i n}}$. As a first-order $\mathcal{L}$-sentence, condition (1) yields

$$
\begin{aligned}
& \iota \doteq \forall g_{1} \ldots \forall g_{n} \forall x_{1} \ldots \forall x_{n}\left(\bigwedge_{j=1}^{n} g_{j} \neq 0 \wedge \bigwedge_{j=1}^{n} x_{j} \neq 0 \wedge\right. \\
&\left.\bigwedge_{i=1}^{m} f_{i}\left(g_{1}, \ldots, g_{n}\right)=0 \wedge \bigwedge_{i=1}^{m} f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right)=0 \longrightarrow \bigwedge_{i=1}^{m} x^{d_{i}} f_{i}\left(g_{1} x_{1}^{-1}, \ldots, g_{n} x_{n}^{-1}\right)=0\right) .
\end{aligned}
$$

Here the multiplications with $x^{d_{i}}$ drop the principal denominators from $f_{i}\left(g_{1} x_{1}^{-1}, \ldots, g_{n} x_{n}^{-1}\right)$. This is an equivalence transformation, because the left hand side of the implication constrains $x_{1}, \ldots, x_{n}$ to be different from zero.

Similarly, condition (2) yields a first-order $\mathcal{L}$-sentence

$$
\begin{aligned}
\mu \doteq \forall g_{1} \ldots \forall g_{n} \forall x_{1} \ldots \forall x_{n} \forall y_{1} \ldots \forall y_{n}\left(\bigwedge_{j=1}^{n} g_{j} \neq 0 \wedge\right. & \bigwedge_{j=1}^{n} x_{j} \neq 0 \wedge \\
\bigwedge_{i=1}^{m} f_{i}\left(g_{1}, \ldots, g_{n}\right)=0 \wedge & y_{j} \neq 0 \wedge \\
\bigwedge_{i=1}^{m} f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right)= & 0 \wedge \bigwedge_{i=1}^{m} f_{i}\left(g_{1} y_{1}, \ldots, g_{n} y_{n}\right)=0 \\
& \left.\longrightarrow \bigwedge_{i=1}^{m} f_{i}\left(g_{1} x_{1} y_{1}, \ldots, g_{n} x_{n} y_{n}\right)=0\right)
\end{aligned}
$$

For condition (3) we consider its logical negation $V(F) \cap\left(\mathbb{K}^{*}\right)^{n}=\emptyset$, which gives us an $\mathcal{L}$ sentence

$$
\eta \doteq \forall x_{1} \ldots \forall x_{n}\left(\bigwedge_{i=1}^{m} f_{i}=0 \longrightarrow \bigvee_{j=1}^{n} x_{j}=0\right)
$$

Accordingly, the $\mathcal{L}$-sentence $\neg \eta$ formally states (3).
Finally, condition (4) yields a quantifier-free $\mathcal{L}$-sentence

$$
\gamma \doteq \bigwedge_{i=1}^{m} f_{i}(1, \ldots, 1)=0
$$

## 3 Semantics: Validity of Our First-Order Formulations over Various Fields

Let $p \in \mathbb{N}$ be 0 or prime. We consider the $\mathcal{L}$-model classes $\mathfrak{K}_{p}$ of fields of characteristic $p$ and $\mathfrak{A}_{p} \subseteq \mathfrak{K}_{p}$ of algebraically closed fields of characteristic $p$. Recall that $\mathfrak{A}_{p}$ is complete, decidable, and admits effective quantifier elimination [48, Note 16].

We assume without loss of generality that $\mathcal{L}$-sentences are in prenex normal form $Q_{1} x_{1} \ldots Q_{n} x_{n} \psi$ with $Q_{1}, \ldots, Q_{n} \in\{\exists, \forall\}$ and $\psi$ quantifier-free. An $\mathcal{L}$-sentence is called universal if it is of the form $\forall x_{1} \ldots \forall x_{n} \psi$ and existential if it is of the form $\exists x_{1} \ldots \exists x_{n} \psi$ with $\psi$ quantifier-free. A quantifier-free $\mathcal{L}$-sentence is both universal and existential.

Lemma 1. Let $\varphi$ be a universal $\mathcal{L}$-sentence. Then

$$
\mathfrak{K}_{p} \models \varphi \quad \text { if and only if } \quad \mathfrak{A}_{p} \models \varphi .
$$

Proof. The implication from the left to the right immediately follows from $\mathfrak{A}_{p} \subseteq \mathfrak{K}_{p}$. Assume, vice versa, that $\mathfrak{A}_{p} \models \varphi$, and let $\mathbb{K} \in \mathfrak{K}_{p}$. Then $\mathbb{K}$ has an algebraic closure $\overline{\mathbb{K}} \in \mathfrak{A}_{p}$, and $\overline{\mathbb{K}} \models \varphi$ due to the completeness of $\mathfrak{A}_{p}$. Since $\mathbb{K} \subseteq \overline{\mathbb{K}}$ and $\varphi$ as a universal sentence is persistent under substructures, we obtain $\mathbb{K} \models \varphi$.

All our first-order conditions $\iota, \mu, \eta$, and $\gamma$ introduced in the previous section 2 are universal $\mathcal{L}$-sentences. Accordingly, $\neg \eta$ is equivalent to an existential $\mathcal{L}$-sentence.

In accordance with the our language $\mathcal{L}$ we are going to use polynomial coefficient rings $Z_{p}=$ $\mathbb{Z} / p$ for finite characteristic $p$, and $Z_{0}=\mathbb{Z}$. Let $F \subseteq Z_{p}\left[x_{1}, \ldots, x_{n}\right]$. Then $V(F)^{*}$ is a coset over $\mathbb{K} \in \mathfrak{K}_{p}$ if and only if

$$
\begin{equation*}
\mathbb{K} \models \iota \wedge \mu \wedge \neg \eta \tag{5}
\end{equation*}
$$

Especially, $V(F)^{*}$ is a group over $\mathbb{K}$ if even

$$
\begin{equation*}
\mathbb{K} \models \iota \wedge \mu \wedge \gamma \tag{6}
\end{equation*}
$$

where $\gamma$ entails $\neg \eta$.
Proposition 2. Let $F \subseteq Z_{p}\left[x_{1}, \ldots, x_{n}\right]$, and let $\mathbb{K} \in \mathfrak{K}_{p}$. Then $V(F)^{*}$ is a group over $\mathbb{K}$ if and only if at least one of the following conditions holds:
(a) $\mathbb{K}^{\prime} \models \iota \wedge \mu \wedge \gamma$ for some $\mathbb{K} \subseteq \mathbb{K}^{\prime} \in \mathfrak{K}_{p}$;
(b) $\mathbb{K}^{\prime} \models \iota \wedge \mu \wedge \gamma$ for some $\mathbb{K}^{\prime} \in \mathfrak{A}_{p}$.

Proof. Recall that $V(F)^{*}$ is a group over $\mathbb{K}$ if and only if $\mathbb{K} \models \iota \wedge \mu \wedge \gamma$. If $V(F)^{*}$ is a group over $\mathbb{K}$, then (a) holds for $\mathbb{K}^{\prime}=\mathbb{K}$. Vice versa, there are two cases. In case (a), we can conclude that $\mathbb{K} \models \iota \wedge \mu \wedge \gamma$ because the universal sentence $\iota \wedge \mu \wedge \gamma$ is persistent under substructures. In case (b), we have $\mathfrak{A}_{p} \models \iota \wedge \mu \wedge \gamma$ by the completeness of that model class. Using Lemma 1 we obtain $\mathfrak{K}_{p} \models \iota \wedge \mu \wedge \gamma$, in particular $\mathbb{K} \models \iota \wedge \mu \wedge \gamma$.

Example 3. (i) Assume that $V(F)^{*}$ is a group over $\mathbb{C}$. Then $V(F)^{*}$ is a group over any field of characteristic 0 . Alternatively, it suffices that $V(F)^{*}$ is a group over the countable algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.
(ii) Assume that $V(F)^{*}$ is a group over the countable field of real algebraic numbers, which is not algebraically closed. Then again $V(F)^{*}$ is a group over any field of characteristic 0 .
(iii) Let $\varepsilon$ be a positive infinitesimal, and assume that $V(F)^{*}$ is a group over $\mathbb{R}(\varepsilon)$. Then $V(F)^{*}$ is group also over $\mathbb{Q}$ and $\mathbb{R}$, but not necessarily over $\overline{\mathbb{Q}}$. Notice that $\mathbb{R}(\varepsilon)$ is not algebraically closed.
(iv) Assume that $V(F)^{*}$ is a group over the algebraic closure of $\mathbb{F}_{p}$. Then $V(F)^{*}$ is a group over any field of characteristic $p$. Alternatively, it suffices that $V(F)^{*}$ is a group over the algebraic closure of the rational function field $\mathbb{F}_{p}(t)$, which has been studied with respect to effective computations 36 .

Proposition 4. Let $F \subseteq Z_{p}\left[x_{1}, \ldots, x_{n}\right]$ and let $\mathbb{K} \in \mathfrak{K}_{p}$. Then $V(F)^{*}$ is a coset over $\mathbb{K}$ if and only if $\mathbb{K} \models \neg \eta$ and at least one of the following conditions holds:
(a) $\mathbb{K}^{\prime} \models \iota \wedge \mu$ for some $\mathbb{K} \subseteq \mathbb{K}^{\prime} \in \mathfrak{K}_{p}$;
(b) $\mathbb{K}^{\prime} \models \iota \wedge \mu$ for some $\mathbb{K}^{\prime} \in \mathfrak{A}_{p}$.

Proof. Recall that $V(F)^{*}$ is a coset over $\mathbb{K}$ if and only if $\mathbb{K} \models \iota \wedge \mu \wedge \neg \eta$. If $V(F)^{*}$ is a coset over $\mathbb{K}$, then $\mathbb{K} \models \neg \eta$, and (a) holds for $\mathbb{K}^{\prime}=\mathbb{K}$. Vice versa, we require that $\mathbb{K} \models \neg \eta$ and obtain $\mathbb{K} \models \iota \wedge \mu$ analogously to the proof of Proposition 2
Example 5. (i) Assume that $V(F)^{*}$ is a coset over $\mathbb{C}$. Then $V(F)^{*}$ is a coset over $\mathbb{R}$ if and only if $V(F)^{*} \neq \emptyset$ over $\mathbb{R}$. This is the case for $F=\left\{x^{2}-2\right\}$ but not for $F=\left\{x^{2}+2\right\}$.
(ii) Consider $F=\left\{x^{4}-4\right\}=\left\{\left(x^{2}-2\right)\left(x^{2}+2\right)\right\}$. Then over $\mathbb{R}, V(F)^{*}=\{ \pm \sqrt{2}\}$ is a coset, because $V(F)^{*} / \sqrt{2}=\{ \pm 1\}$ is a group. Similarly over $\mathbb{C}, V(F)^{*}=\{ \pm \sqrt{2}, \pm i \sqrt{2}\}$ is a coset, as $V(F)^{*} / \sqrt{2}=\{ \pm 1, \pm i\}$ is a group.
(iii) Consider $F=\left\{x^{4}+x^{2}-6\right\}=\left\{\left(x^{2}-2\right)\left(x^{2}+3\right)\right\}$. Then over $\mathbb{R}, V(F)^{*}=\{ \pm \sqrt{2}\}$ is a coset, as $V(F)^{*} / \sqrt{2}=\{ \pm 1\}$ is a group. Over $\mathbb{C}$, in contrast, $V(F)^{*}=\{ \pm \sqrt{2}, \pm i \sqrt{3}\}$ is not a coset.

## 4 Hilbert's Nullstellensatz as a Swiss Army Knife

A recent publication [27] has systematically applied coset tests to a large number for real-world models from the BioModels database [9], investigating varieties over both the real and the complex numbers. Over $\mathbb{R}$ it used essentially our first-order sentences presented in Section 2 and applied efficient implementations of real decision methods based on effective quantifier elimination [51, 52, 13, 14, 45, 38 .

Over $\mathbb{C}$, in contrast, it used a purely algebraic framework combining various specialized methods from commutative algebra, typically based on Gröbner basis computations [8, 17. This is in line with the vast majority of the existing literature (cf. the Introduction for references), which uses computer algebra over algebraically closed fields, to some extent supplemented with heuristic tests based on linear algebra.

Generalizing the successful approach for $\mathbb{R}$ and aiming at a more uniform overall framework, we want to study here the application of decision methods for algebraically closed fields to our first-order sentences. Recall that our sentences $\iota, \mu, \eta$, and $\gamma$ are universal $\mathcal{L}$-sentences. Every such sentence $\varphi$ can be equivalently transformed into a finite conjunction of universal $\mathcal{L}$-sentences of the following form:

$$
\widehat{\varphi} \doteq \forall x_{1} \ldots \forall x_{n}\left(\bigwedge_{i=1}^{m} f_{i}\left(x_{1}, \ldots, x_{n}\right)=0 \longrightarrow g\left(x_{1}, \ldots, x_{n}\right)=0\right)
$$

where $f_{1}, \ldots, f_{m}, g \in Z_{p}\left[x_{1}, \ldots, x_{n}\right]$. Such $\mathcal{L}$-sentences are called uniform word problems [3]. Over an algebraically closed field $\overline{\mathbb{K}}$ of characteristic $p$, Hilbert's Nullstellensatz 31 provides a decision procedure for uniform word problems. It states that

$$
\overline{\mathbb{K}} \models \widehat{\varphi} \quad \text { if and only if } \quad g \in \sqrt{\left\langle f_{1}, \ldots, f_{m}\right\rangle} .
$$

Recall that $\mathfrak{A}_{p}$ is complete so that we furthermore have $\mathfrak{A}_{p} \models \widehat{\varphi}$ if and only if $\overline{\mathbb{K}} \models \widehat{\varphi}$.
Our $\mathcal{L}$-sentence $\iota$ for condition (1) can be equivalently transformed into

$$
\begin{aligned}
& \forall g_{1} \ldots \forall g_{n} \forall x_{1} \ldots \forall x_{n}\left(\bigvee_{j=1}^{n} g_{j}=0 \vee \bigvee_{j=1}^{n} x_{j}=0 \vee\right. \\
& \left.\qquad \bigvee_{i=1}^{m} f_{i}\left(g_{1}, \ldots, g_{n}\right) \neq 0 \vee \bigvee_{i=1}^{m} f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right) \neq 0 \vee \bigwedge_{i=1}^{m} x^{d_{i}} f_{i}\left(g_{1} x_{1}^{-1}, \ldots, g_{n} x_{n}^{-1}\right)=0\right),
\end{aligned}
$$

which is in turn equivalent to

$$
\begin{aligned}
& \widehat{\iota} \doteq \bigwedge_{k=1}^{m} \forall g_{1} \ldots \forall g_{n} \forall x_{1} \ldots \forall x_{n}\left(\bigwedge_{i=1}^{m} f_{i}\left(g_{1}, \ldots, g_{n}\right)\right.=0 \wedge \bigwedge_{i=1}^{m} f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right)=0 \\
&\left.\longrightarrow x^{d_{k}} f_{k}\left(g_{1} x_{1}^{-1}, \ldots, g_{n} x_{n}^{-1}\right) \prod_{j=1}^{n} g_{j} x_{j}=0\right)
\end{aligned}
$$

Hence, by Hilbert's Nullstellensatz, 1 holds in $\overline{\mathbb{K}}$ if and only if

$$
\begin{equation*}
x^{d_{k}} f_{k}\left(g_{1} x_{1}^{-1}, \ldots, g_{n} x_{n}^{-1}\right) \prod_{j=1}^{n} g_{j} x_{j} \in R_{10} \quad \text { for all } \quad k \in M, \tag{7}
\end{equation*}
$$

where $R_{\text {臬 }}=\sqrt{\left\langle f_{i}\left(g_{1}, \ldots, g_{n}\right), f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right) \mid i \in M\right\rangle}$.
Similarly, our $\mathcal{L}$-sentence $\mu$ for condition (2) translates into

$$
\begin{aligned}
& \widehat{\mu} \doteq \bigwedge_{k=1}^{m} \forall g_{1} \ldots \forall g_{n} \forall x_{1} \ldots \forall x_{n} \forall y_{1} \ldots \forall y_{n} \\
& \\
& \left(\begin{array}{l}
\left(\bigwedge_{i=1}^{m} f_{i}\left(g_{1}, \ldots, g_{n}\right)=0 \wedge \bigwedge_{i=1}^{m} f_{i}\left(g_{1} x_{1}, \ldots, g_{n} x_{n}\right)=0 \wedge \bigwedge_{i=1}^{m} f_{i}\left(g_{1} y_{1}, \ldots, g_{n} y_{n}\right)=0\right. \\
\end{array} \quad \longrightarrow f_{k}\left(g_{1} x_{1} y_{1}, \ldots, g_{n} x_{n} y_{n}\right) \prod_{j=1}^{n} g_{j} x_{j} y_{j}=0\right)
\end{aligned}
$$

Again, by Hilbert's Nullstellensatz, 2 holds in $\overline{\mathbb{K}}$ if and only if

$$
\begin{equation*}
f_{k}\left(g_{1} x_{1} y_{1}, \ldots, g_{n} x_{n} y_{n}\right) \prod_{j=1}^{n} g_{j} x_{j} y_{j} \in R_{2} \quad \text { for all } \quad k \in M, \tag{8}
\end{equation*}
$$

where $R_{2}=\sqrt{\left\langle f_{i}(g), f_{i}(g x), f_{i}(g y) \mid i \in M\right\rangle}$.
Next, our $\mathcal{L}$-sentence $\eta$ is is equivalent to

$$
\widehat{\eta} \doteq \forall x_{1} \ldots \forall x_{n}\left(\bigwedge_{i=1}^{m} f_{i}=0 \longrightarrow \prod_{j=1}^{n} x_{j}=0\right)
$$

Using once more Hilbert's Nullstellensatz, $\overline{\mathbb{K}} \models \widehat{\eta}$ if and only if

$$
\begin{equation*}
\prod_{j=1}^{n} x_{j} \in R_{23} \tag{9}
\end{equation*}
$$

where $R_{3}=\sqrt{\left\langle f_{1}, \ldots, f_{m}\right\rangle}$. Hence our non-emptiness condition (3) holds in $\overline{\mathbb{K}}$ if and only if

$$
\begin{equation*}
\prod_{j=1}^{n} x_{j} \notin R_{\text {豚 }} \tag{10}
\end{equation*}
$$

Finally, our $\mathcal{L}$-sentence $\gamma$ for condition (4) is equivalent to

$$
\widehat{\gamma} \doteq \bigwedge_{k=1}^{m}\left(0=0 \longrightarrow f_{k}(1, \ldots, 1)=0\right)
$$

Here Hilbert's Nullstellensatz tells us that condition (4) holds in $\overline{\mathbb{K}}$ if and only if

$$
\begin{equation*}
f_{k}(1, \ldots, 1) \in R_{4} \quad \text { for all } \quad k \in M, \tag{11}
\end{equation*}
$$

where $R_{4}=\sqrt{\langle 0\rangle}=\langle 0\rangle$. Notice that the radical membership test quite naturally reduces to the obvious test with plugging in.

## 5 Complexity

Let us briefly discuss asymptotic complexity bounds around problems and methods addressed here. We do so very roughly, in terms of the input word length. The cited literature provides more precise bounds in terms of several complexity parameters, such as numbers of quantifiers, or degrees.

The decision problem for algebraically closed fields is double exponential [30 in general, but only single exponential when the number of quantifier alternations is bounded [25], which covers in particular our universal formulas. The decision problem for real closed fields is double exponential as well [12], even for linear problems [51]; again it becomes single exponential when bounding the number of quantifier alternations [26].

Ideal membership tests are at least double exponential [39], and it was widely believed that this would impose a corresponding lower bound also for any algorithm for Hilbert's Nullstellensatz. Quite surprisingly, it turned out that there are indeed single exponential such algorithms [7, 37].

On these grounds it is clear that our coset tests addressed in the previous sections can be solved in single exponential time for algebraically closed fields as well as for real closed fields. Recall that our considering those tests is actually motivated by our interest in shifted toricity, which requires, in addition, the irreducibility of the considered variety over the considered domain. Recently it has been shown that testing shifted toricity, including irreducibility, is also only single exponential over algebraically closed fields as well as real closed fields 27].

Most asymptotically fast algorithms mentioned above are not implemented and it is not clear that they would be efficient in practice.

## 6 Computational Experiments

We have studied 129 models from the BioModels ${ }^{2}$ database 9 . Technically, we took our input from ODEbas $\epsilon^{3}$ which provides preprocessed versions for symbolic computation. Our 129 models establish the complete set currently provided by ODEbase for wich the relevant systems of ordinary differential equations have polynomial vector fields.

We limited ourselves to characteristic 0 and applied the tests (7), (8), (9), (11) derived in Section 4 using Hilbert's Nullstellensatz. Recall that those tests correspond to $\iota, \mu, \eta, \gamma$ from Section 3 respectively, and that one needs $\iota \wedge \mu \wedge \neg \eta$ or $\iota \wedge \mu \wedge \gamma$ for cosets or groups, respectively. From a symbolic computation point of view, we used exclusively polynomial arithmetic and radical membership test. The complete Maple code for computing a single model is displayed in Figure 1 it is surprisingly simple.

We conducted our computations on a 2.40 GHz Intel Xeon E5-4640 with 512 GB RAM and 32 physical cores providing 64 CPUs via hyper-threading. For parallelization of the jobs for the individual models we used GNU Parallel [47. Results and timings are collected in Table 1 With a time limit of one hour CPU time per model we succeeded on 78 models, corresponding to $60 \%$, the largest of which, no. 559, has 90 polynomials in 71 dimensions. The median of the overall computation times for the successful models is 1.419 s . We would like to emphasize that our focus here is illustrating and evaluating our overall approach, rather than obtaining new insights into the models. Therefore our code in Figure 1 is very straightforward without any optimizations. In particular, computation continues even when one relevant subtest has already failed. More comprehensive results on our dataset can be found in [27].

Among our 78 successfully computed models, we detected 20 coset cases, corresponding to $26 \%$. Two out of those 20 are even group cases. Among the 58 other cases, 46 , corresponding

[^1]```
```

ToricHilbert := proc(F:: list(polynom))

```
```

ToricHilbert := proc(F:: list(polynom))
uses PolynomialIdeals;
uses PolynomialIdeals;
local Iota := proc()::truefalse;
local Iota := proc()::truefalse;
local R1, s, prod, f;
local R1, s, prod, f;
R1 :=<op(subs(zip('=', xl, gl), F ) ), op(\operatorname{subs}(zip((x, g) -> x= g*x, xl, gl), F)) >;
R1 :=<op(subs(zip('=', xl, gl), F ) ), op(\operatorname{subs}(zip((x, g) -> x= g*x, xl, gl), F)) >;
s := zip ((x, g) -> x = g/x, xl, gl);
s := zip ((x, g) -> x = g/x, xl, gl);
prod := g * x;
prod := g * x;
for f in subs(s, F) do
for f in subs(s, F) do
if not RadicalMembership(numer(f) * prod, R1) then
if not RadicalMembership(numer(f) * prod, R1) then
return false
return false
end if
end if
end do;
end do;
return true
return true
end proc;
end proc;
local Mu := proc()::truefalse;
local Mu := proc()::truefalse;
local R2, s, prod, f;

```
```

local R2, s, prod, f;

```
```






```
```

    s := zip('=', xl, xl, zip('`*', gl, zip(''*', gl, xl, yl)), f));
    ```
```

    s := zip('=', xl, xl, zip('`*', gl, zip(''*', gl, xl, yl)), f));
    prod := g*x * y;
    prod := g*x * y;
    for f in subs(s, F) do
    for f in subs(s, F) do
    if not RadicalMembership(f * prod, R2) then
if not RadicalMembership(f * prod, R2) then
return false
return false
end if
end if
end do;
end do;
return true
return true
end proc;
end proc;
local Eta := proc():: truefalse;
local Eta := proc():: truefalse;
local R3, prod;
local R3, prod;
R3:=<op(F)>;
R3:=<op(F)>;
prod := foldl('*'`, 1, op(xl));     prod := foldl('*'`, 1, op(xl));
prod := foldl('*'', 1, op(xl));
prod := foldl('*'', 1, op(xl));
end proc;
end proc;
local Gamma := proc()::truefalse;
local Gamma := proc()::truefalse;
local R4, s, f;
local R4, s, f;
cal R4, s, f;
cal R4, s, f;
s := map(x - x m=1, xl );
s := map(x - x m=1, xl );
for f in subs(s, F) do
for f in subs(s, F) do
if not RadicalMembership(f, R4) then
if not RadicalMembership(f, R4) then
return false
return false
end if
end if
end do;
end do;
return true
return true
end proc;
end proc;
local Rename := proc(base::name, l::list(name))::list (name);
local Rename := proc(base::name, l::list(name))::list (name);
uses StringTools;
uses StringTools;
return map(x m cat(base, Select(IsDigit, x)), l)
return map(x m cat(base, Select(IsDigit, x)), l)
end proc;
end proc;
local X, xl, gl, yl, g, x, y, iota, t__iota, mu, t_mu, eta, t_eta, gamma_, t_gamma, coset, group, t;
local X, xl, gl, yl, g, x, y, iota, t__iota, mu, t_mu, eta, t_eta, gamma_, t_gamma, coset, group, t;
t := time();
t := time();
xl := convert(indets(F), list);
xl := convert(indets(F), list);
x := foldl('*',},1,op(xl))
x := foldl('*',},1,op(xl))
gl := Rename('g',
gl := Rename('g',
g := foldl('*', 1, op(gl));
g := foldl('*', 1, op(gl));
yl := Rename('y', xl);
yl := Rename('y', xl);
y := foldl('**', 1, op(yl));
y := foldl('**', 1, op(yl));
t_iota }:=\mathrm{ time(); iota }:=\operatorname{Iota}(); t__iota := time() - t__iota
t_iota }:=\mathrm{ time(); iota }:=\operatorname{Iota}(); t__iota := time() - t__iota
t_mu := time(); mu := Mu(); t_mu := time() - t_mu;
t_mu := time(); mu := Mu(); t_mu := time() - t_mu;
t__eta }:= time(); eta := Eta(); t__eta := time() - t__eta
t__eta }:= time(); eta := Eta(); t__eta := time() - t__eta
t_gamma }:= time(); gamm := Gamma(); t_gamma := time() - t_gamma
t_gamma }:= time(); gamm := Gamma(); t_gamma := time() - t_gamma
coset := iota and mu and not eta;
coset := iota and mu and not eta;
coset := iota and mu and not eta
coset := iota and mu and not eta
t := time() - t;
t := time() - t;
return nops(F), nops(xl), iota, t__iota, mu, t_mu, eta, t__eta, gamm, t_gamma, coset, group, t
return nops(F), nops(xl), iota, t__iota, mu, t_mu, eta, t__eta, gamm, t_gamma, coset, group, t
end proc;

```
end proc;
```149
10
1111121313
14151616
1817
1818
1919
20
```22
23
2424
2526
2727
28
```

32

```33
34
35
3635
363839
```

40

```414242
434545
4647484949
50
5151525353
55
```

5

```
5
5
5
7
7
2.}
2.}
(f) prod,R1) then 10
(f) prod,R1) then 10
            if
            if32
3336
37
```

Figure 1: Maple code for computing one row of Table 1

Table 1: Results and computation times (in seconds) of our computations on models from the BioModels database 9

| model | $m$ | $n$ | $\iota$ | $t_{\iota}$ | $\mu$ | $t_{\mu}$ | $\eta$ | $t_{\eta}$ | $\gamma$ | $t_{\gamma}$ | coset | group | $t_{\Sigma}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 001 | 12 | 12 | true | 7.826 | true | 7.86 | false | 4.267 | false | 0.053 | true | false | 20.007 |
| 040 | 5 | 3 | false | 1.415 | false | 0.173 | false | 0.114 | false | 0.043 | false | false | 1.746 |
| 050 | 14 | 9 | true | 1.051 | true | 2.458 | true | 0.113 | false | 0.05 | false | false | 3.673 |
| 052 | 11 | 6 | true | 3.605 | true | 1.635 | true | 0.096 | false | 0.059 | false | false | 5.396 |
| 057 | 6 | 6 | true | 0.271 | true | 0.263 | false | 0.858 | false | 0.045 | true | false | 1.438 |
| 072 | 7 | 7 | true | 0.763 | true | 0.496 | true | 0.08 | false | 0.06 | false | false | 1.4 |
| 077 | 8 | 7 | true | 0.296 | true | 0.356 | false | 0.097 | false | 0.051 | true | false | 0.801 |
| 080 | 10 | 10 | true | 0.714 | true | 1.341 | true | 0.103 | false | 0.06 | false | false | 2.219 |
| 082 | 10 | 10 | true | 0.384 | true | 0.39 | true | 0.086 | false | 0.041 | false | false | 0.902 |
| 091 | 16 | 14 | true | 0.031 | true | 0.045 | true | 0.003 | false | 0.062 | false | false | 0.142 |
| 092 | 4 | 3 | true | 0.293 | true | 0.244 | false | 0.104 | false | 1.03 | true | false | 1.671 |
| 099 | 7 | 7 | true | 0.298 | true | 0.698 | false | 0.087 | false | 0.036 | true | false | 1.119 |
| 101 | 6 | 6 | false | 4.028 | false | 10.343 | false | 0.917 | false | 0.073 | false | false | 15.361 |
| 104 | 6 | 4 | true | 0.667 | true | 0.146 | true | 0.084 | false | 0.039 | false | false | 0.937 |
| 105 | 39 | 26 | true | 0.455 | true | 0.367 | true | 0.043 | false | 0.038 | false | false | 0.905 |
| 125 | 5 | 5 | false | 0.193 | false | 0.098 | false | 0.078 | false | 0.038 | false | false | 0.408 |
| 150 | 4 | 4 | true | 0.173 | true | 0.153 | false | 0.094 | false | 0.043 | true | false | 0.464 |
| 156 | 3 | 3 | true | 2.638 | true | 0.248 | false | 0.86 | false | 0.052 | true | false | 3.8 |
| 158 | 3 | 3 | false | 0.148 | false | 0.149 | false | 0.16 | false | 0.045 | false | false | 0.503 |
| 159 | 3 | 3 | true | 0.959 | true | 0.175 | false | 0.083 | false | 0.04 | true | false | 1.257 |
| 178 | 6 | 4 | true | 0.52 | true | 1.71 | true | 0.877 | false | 1.201 | false | false | 4.308 |
| 186 | 11 | 10 | true | 31.785 | true | 1026.464 | true | 1.956 | false | 0.095 | false | false | 1060.301 |
| 187 | 11 | 10 | true | 27.734 | true | 1023.648 | true | 0.103 | false | 0.062 | false | false | 1051.548 |
| 188 | 20 | 10 | true | 0.075 | true | 0.079 | true | 0.04 | false | 0.047 | false | false | 0.242 |
| 189 | 18 | 7 | true | 0.035 | true | 0.02 | true | 0.002 | false | 0.062 | false | false | 0.12 |
| 194 | 5 | 5 | false | 2.338 | false | 1.922 | false | 0.612 | false | 0.05 | false | false | 4.922 |
| 197 | 7 | 5 | false | 7.562 | false | 71.864 | false | 0.485 | false | 0.05 | false | false | 79.962 |
| 198 | 12 | 9 | true | 0.397 | true | 0.793 | true | 0.077 | false | 0.042 | false | false | 1.31 |
| 199 | 15 | 8 | true | 1.404 | true | 1.531 | false | 0.215 | false | 0.054 | true | false | 3.205 |
| 220 | 58 | 56 | true | 146.146 | true | 534.832 | true | 6.921 | false | 0.964 | false | false | 688.866 |
| 227 | 60 | 39 | true | 0.273 | true | 0.485 | true | 0.01 | false | 0.077 | false | false | 0.847 |
| 229 | 7 | 7 | true | 1.917 | true | 3.348 | false | 0.131 | false | 0.062 | true | false | 5.458 |
| 233 | 4 | 2 | false | 0.16 | false | 0.44 | false | 0.17 | false | 0.557 | false | false | 1.328 |
| 243 | 23 | 19 | true | 8.598 | true | 1171.687 | true | 2.512 | false | 0.171 | false | false | 1182.97 |
| 259 | 17 | 16 | true | 1.334 | true | 1.913 | true | 0.092 | false | 0.045 | false | false | 3.385 |
| 260 | 17 | 16 | true | 2.182 | true | 0.748 | true | 0.079 | false | 0.047 | false | false | 3.057 |
| 261 | 17 | 16 | true | 3.359 | true | 2.872 | true | 0.113 | false | 0.095 | false | false | 6.44 |
| 262 | 11 | 9 | true | 0.402 | true | 0.41 | true | 0.091 | false | 0.071 | false | false | 0.975 |
| 263 | 11 | 9 | true | 0.379 | true | 0.403 | true | 0.085 | false | 0.066 | false | false | 0.934 |
| 264 | 14 | 11 | true | 1.031 | true | 2.036 | true | 0.136 | false | 0.063 | false | false | 3.268 |
| 267 | 4 | 3 | true | 1.084 | true | 0.246 | true | 0.095 | false | 0.049 | false | false | 1.475 |
| 271 | 6 | 4 | true | 0.286 | true | 0.283 | true | 0.746 | false | 0.045 | false | false | 1.361 |
| 272 | 6 | 4 | true | 0.361 | true | 0.323 | true | 0.086 | false | 0.055 | false | false | 0.826 |
| 281 | 32 | 32 | true | 20.987 | true | 29.791 | true | 0.602 | false | 0.055 | false | false | 51.437 |
| 282 | 6 | 3 | true | 0.205 | true | 0.19 | true | 0.087 | false | 0.046 | false | false | 0.528 |
| 283 | 4 | 3 | true | 0.294 | true | 0.211 | true | 0.087 | false | 0.412 | false | false | 1.005 |
| 289 | 5 | 4 | false | 2.291 | false | 1.118 | false | 0.165 | false | 0.044 | false | false | 3.619 |
| 292 | 6 | 2 | true | 0.06 | true | 0.048 | true | 0.063 | false | 0.046 | false | false | 0.218 |
| 306 | 5 | 2 | true | 0.149 | true | 0.121 | false | 0.079 | false | 0.041 | true | false | 0.391 |
| 307 | 5 | 2 | true | 0.129 | true | 0.121 | true | 0.043 | false | 0.148 | false | false | 0.441 |
| 310 | 4 | 1 | true | 0.053 | true | 0.369 | true | 0.047 | false | 0.04 | false | false | 0.509 |
| 311 | 4 | 1 | true | 0.076 | true | 0.048 | true | 0.224 | false | 0.048 | false | false | 0.397 |
| 312 | 3 | 2 | true | 0.098 | true | 0.512 | true | 0.043 | false | 0.043 | false | false | 0.697 |
| 314 | 12 | 10 | true | 0.515 | true | 1.789 | true | 0.1 | false | 0.059 | false | false | 2.464 |
| 321 | 3 | 3 | true | 0.163 | true | 0.148 | true | 0.042 | false | 0.039 | false | false | 0.393 |
| 357 | 9 | 8 | true | 0.353 | true | 1.517 | true | 0.07 | false | 0.045 | false | false | 1.986 |
| 359 | 9 | 8 | true | 1.677 | true | 3.605 | true | 0.11 | false | 0.055 | false | false | 5.448 |
| 360 | 9 | 8 | true | 0.479 | true | 0.47 | true | 0.096 | false | 0.05 | false | false | 1.096 |
| 361 | 8 | 8 | true | 1.069 | true | 2.746 | true | 0.156 | false | 0.045 | false | false | 4.017 |
| 363 | 4 | 3 | true | 0.244 | true | 0.199 | true | 0.077 | false | 0.041 | false | false | 0.561 |
| 364 | 14 | 12 | true | 2.483 | true | 7.296 | true | 0.55 | false | 0.064 | false | false | 10.394 |
| 413 | 5 | 5 | false | 1.55 | false | 22.323 | false | 0.117 | false | 0.053 | false | false | 24.044 |
| 459 | 4 | 3 | true | 0.542 | true | 0.224 | false | 0.18 | false | 0.068 | true | false | 1.014 |
| 460 | 4 | 3 | false | 1.025 | false | 0.936 | false | 0.143 | false | 0.216 | false | false | 2.321 |
| 475 | 23 | 22 | true | 97.876 | true | 3377.021 | true | 0.231 | false | 0.062 | false | false | 3475.192 |
| 484 | 2 | 1 | true | 0.384 | true | 0.143 | false | 0.099 | false | 0.048 | true | false | 0.674 |
| 485 | 2 | 1 | false | 0.564 | false | 0.354 | false | 0.209 | false | 0.042 | false | false | 1.169 |
| 486 | 2 | 2 | true | 0.119 | true | 0.106 | false | 0.073 | false | 0.041 | true | false | 0.339 |
| 487 | 6 | 6 | true | 0.475 | true | 1.008 | false | 0.099 | false | 0.045 | true | false | 1.628 |
| 491 | 57 | 57 | true | 123.138 | true | 536.865 | false | 2.067 | true | 0.007 | true | true | 662.08 |
| 492 | 52 | 52 | true | 85.606 | true | 284.753 | false | 1.123 | true | 0.003 | true | true | 371.489 |
| 519 | 3 | 3 | true | 1.357 | true | 2.367 | false | 5.142 | false | 0.097 | true | false | 8.964 |
| 546 | 7 | 3 | true | 0.327 | true | 0.338 | true | 0.109 | false | 0.042 | false | false | 0.817 |
| 559 | 90 | 71 | true | 4.742 | true | 7.525 | true | 0.19 | false | 0.053 | false | false | 12.515 |
| 584 | 35 | 9 | true | 0.4 | true | 0.655 | false | 0.095 | false | 0.043 | true | false | 1.194 |
| 619 | 10 | 8 | true | 0.411 | true | 0.443 | true | 0.087 | false | 0.052 | false | false | 0.994 |
| 629 | 5 | 5 | true | 0.209 | true | 0.197 | false | 0.079 | false | 0.046 | true | false | 0.532 |
| 647 | 11 | 11 | false | 0.854 | false | 16.436 | false | 0.165 | false | 0.051 | false | false | 17.507 |

to $78 \%$, fail only due to their emptiness $\eta$; we know from [27] that many such cases exhibit in fact coset structure when considered in suitable lower-dimensional spaces, possibly after prime decomposition. Finally notice that our example reaction from the Introduction, no. 92, is among the smallest ones with a coset structure.

## 7 Conclusions and Future Work

We have used Hilbert's Nullstellensatz to derive important information about the varieties of biological models with a polynomial vector field $F$. The key technical idea was generalizing from pure algebra to more general first-order logic. Recall from Section 3 that except for nonemptiness of $V(F)^{*}$ the information we obtained is valid in all fields of characteristic 0 . Wherever we discovered non-emptiness, this holds at least in all algebraically closed fields of characteristic 0 . For transferring our obtained results to real closed fields, e.g., subtropical methods [46, 21, 32] provide fast heuristic tests for the non-emptiness of $V(F)^{*}$ there.

Technically, we only used polynomial arithmetic and polynomial radical membership tests. This means that on the software side there are many off-the-shelf computer algebra systems and libraries available where our ideas could be implemented, robustly and with little effort. This in turn makes it attractive for the integration with software from systems biology, which could open exciting new perspectives for symbolic computation with applications ranging from the fundamental research in the life sciences to state-of-the-art applied research in medicine and pharmacology.

We had motivated our use of Hilbert's Nullstellensatz by viewing it as a decision procedure for the universal fragment of first-order logic in algebraically closed fields, which is sufficient for our purposes. Our focus on algebraically closed fields here is in accordance with the majority of existing literature on toricity. However, it is generally accepted that from a scientific point of view, real closed fields are the appropriate domain to consider.

We have seen in Section 5 that the theoretical complexities for general decision procedures in algebraically closed fields vs. real closed fields strongly resemble each other. What could now take the place of Hilbert's Nullstellensatz over the reals with respect to practical computations on model sizes as in Table 1 or even larger? A factor of 10 could put us in the realm of models currently used in the development of drugs for diabetes or cancer. One possible answer is satisfiability modulo theories solving (SMT) [41] $]^{4}$ SMT is incomplete in the sense that it often proves or disproves validity, but it can yield "unknown" for specific input problems. When successful, it is typically significantly faster than traditional algebraic decision procedures. For coping with incompleteness one can still fall back into real quantifier elimination. Interest in collaboration between the SMT and the symbolic computation communities exists on both sides (1) 2].

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[^2]
## References

[1] Erika Ábrahám, John Abbott, Bernd Becker, Anna M. Bigatti, Martin Brain, Bruno Buchberger, Alessandro Cimatti, James H. Davenport, Matthew England, Pascal Fontaine, Stephen Forrest, Alberto Griggio, Daniel Kroening, Werner M. Seiler, and Thomas Sturm. Satisfiability checking and symbolic computation. ACM Communications in Computer Algebra, 50(4):145-147, 2016.
[2] Erika Ábrahám, John Abbott, Bernd Becker, Anna M. Bigatti, Martin Brain, Bruno Buchberger, Alessandro Cimatti, James H. Davenport, Matthew England, Pascal Fontaine, Stephen Forrest, Alberto Griggio, Daniel Kroening, Werner M. Seiler, and Thomas Sturm. $\mathrm{SC}^{2}$ : Satisfiability checking meets symbolic computation. In CICM 2016, volume 9791 of LNCS, pages 28-43. Springer, 2016.
[3] Thomas Becker, Volker Weispfenning, and Heinz Kredel. Gröbner Bases, a Computational Approach to Commutative Algebra, volume 141 of Graduate Texts in Mathematics. Springer, 1993.
[4] Ludwig Boltzmann. Lectures on Gas Theory. University of California Press, Berkeley and Los Angeles, CA, 1964.
[5] François Boulier, François Fages, Ovidiu Radulescu, Satya Samal, Andreas Schuppert, Werner Seiler, Thomas Sturm, Sebastian Walcher, and Andreas Weber. The SYMBIONT project: Symbolic methods for biological networks. ACM Communications in Computer Algebra, 52(3):67-70, 2018.
[6] François Boulier, François Fages, Ovidiu Radulescu, Satya Samal, Andreas Schuppert, Werner Seiler, Thomas Sturm, Sebastian Walcher, and Andreas Weber. The SYMBIONT project: Symbolic methods for biological networks. F1000Research, 7(1341), 2018.
[7] W. Dale Brownawell. Bounds for the degrees in the Nullstellensatz. Annals of Mathematics, 126(3):577-591, 1987.
[8] Bruno Buchberger. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal. Doctoral dissertation, Mathematical Institute, University of Innsbruck, Austria, 1965.
[9] Vijayalakshmi Chelliah, Nick Juty, Ishan Ajmera, Raza Ali, Marine Dumousseau, Mihai Glont, Michael Hucka, Gaël Jalowicki, Sarah Keating, Vincent Knight-Schrijver, Audald Lloret-Villas, Kedar Nath Natarajan, Jean-Baptiste Pettit, Nicolas Rodriguez, Michael Schubert, Sarala M. Wimalaratne, Yangyang Zhao, Henning Hermjakob, Nicolas Le Novère, and Camille Laibe. BioModels: Ten-year anniversary. Nucl. Acids Res., 2015.
[10] Carsten Conradi and Thomas Kahle. Detecting binomiality. Advances in Applied Mathematics, 71:52-67, 2015.
[11] Gheorghe Craciun, Alicia Dickenstein, Anne Shiu, and Bernd Sturmfels. Toric dynamical systems. J. Symb. Comput., 44(11):1551-1565, 2009.
[12] James H. Davenport and Joos Heintz. Real quantifier elimination is doubly exponential. $J$. Symb. Comput., 5(1-2):29-35, 1988.
[13] Andreas Dolzmann and Thomas Sturm. Redlog: Computer algebra meets computer logic. ACM SIGSAM Bulletin, 31(2):2-9, 1997.
[14] Andreas Dolzmann and Thomas Sturm. Simplification of quantifier-free formulae over ordered fields. J. Symb. Comput., 24(2):209-231, 1997.
[15] Albert Einstein. Strahlungs-emission und -absorption nach der Quantentheorie. Verh. Dtsch. Phys. Ges., 18:318-323, 1916.
[16] David Eisenbud and Bernd Sturmfels. Binomial ideals. Duke Math. J., 84(1):1-45, 1996.
[17] Jean-Charles Faugère. A new efficient algorithm for computing Gröbner bases (F4). J. Pure Appl. Algebra, 139(1):61-88, 1999.
[18] Martin Feinberg. Complex balancing in general kinetic systems. Arch. Ration. Mech. An., 49(3):187-194, 1972.
[19] Martin Feinberg. Stability of complex isothermal reactors-I. The deficiency zero and deficiency one theorems. Chem. Eng. Sci., 42(10):2229-2268, 1987.
[20] Martin Feinberg. Foundations of Chemical Reaction Network Theory, volume 202 of Applied Mathematical Sciences. Springer, 2019.
[21] Pascal Fontaine, Mizuhito Ogawa, Thomas Sturm, and Xuan Tung Vu. Subtropical satisfiability. In FROCOS 2017, volume 10483 of $L N C S$, pages 189-206. Springer, 2017.
[22] Matilde Esther Fuentes, Ramón Varón, Manuela García-Moreno, and Edelmira Valero. Kinetics of intra- and intermolecular zymogen activation with formation of an en-zyme-zymogen complex. The FEBS Journal, 272(1):85-96, 2005.
[23] William Fulton. Introduction to Toric Varieties, volume 131 of Annals of Mathematics Studies. Princeton University Press, 1993.
[24] Karin Gatermann and Matthias Wolfrum. Bernstein's second theorem and Viro's method for sparse polynomial systems in chemistry. Adv. Appl. Math., 34(2):252-294, 2005.
[25] D. Yu. Grigor'ev. The complexity of the decision problem for the first-order theory of algebraically closed fields. Math. USSR Izv., 29(2):459-475, 1987.
[26] D. Yu. Grigor'ev. Complexity of deciding Tarski algebra. J. Symb. Comput., 5(1-2):65-108, 1988.
[27] Dima Grigoriev, Alexandru Iosif, Hamid Rahkooy, Thomas Sturm, and Andreas Weber. Efficiently and effectively recognizing toricity of steady state varieties. CoRR, abs/1910.04100, 2019.
[28] Dima Grigoriev and Pierre D. Milman. Nash resolution for binomial varieties as Euclidean division. A priori termination bound, polynomial complexity in essential dimension 2. Adv. Math., 231(6):3389-3428, 2012.
[29] Dima Grigoriev and Andreas Weber. Complexity of solving systems with few independent monomials and applications to mass-action kinetics. In CASC 2012, volume 7442 of $L N C S$, pages 143-154. Springer, 2012.
[30] Joos Heintz. Definability and fast quantifier eliminarion in algebraically closed fields. Theor. Comput. Sci., 24:239-277, 1983.
[31] David Hilbert. Über die vollen Invariantensysteme. Math. Ann., 42:313-373, 1893.
[32] Hoon Hong and Thomas Sturm. Positive solutions of systems of signed parametric polynomial inequalities. In CASC 2018, volume 11077 of $L N C S$. Springer, 2018.
[33] F. Horn and R. Jackson. General mass action kinetics. Arch. Ration. Mech. An., 47(2):81116, 1972.
[34] Thomas Kahle. Decompositions of binomial ideals. Ann. I. Stat. Math., 62(4):727-745, 2010.
[35] Thomas Kahle. Decompositions of binomial ideals. Journal of Software for Algebra and Geometry, 4(1):1-5, 2012.
[36] Kiran S. Kedlaya. Finite automata and algebraic extensions of function fields. Journal de Théorie des Nombres de Bordeaux, 18(2):379-420, 2006.
[37] Janos Kollar. Sharp effective Nullstellensatz. J. Am. Math. Soc., 1(4):963-975, 1988.
[38] Marek Košta. New Concepts for Real Quantifier Elimination by Virtual Substitution. Doctoral dissertation, Saarland University, Germany, 2016.
[39] Ernst W. Mayr and Albert R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. Adv. Math., 46(3):305-329, 1982.
[40] Stefan Müller, Elisenda Feliu, Georg Regensburger, Carsten Conradi, Anne Shiu, and Alicia Dickenstein. Sign conditions for injectivity of generalized polynomial maps with applications to chemical reaction networks and real algebraic geometry. Found. Comput. Math., 16(1):6997, 2016.
[41] Robert Nieuwenhuis, Albert Oliveras, and Cesare Tinelli. Solving SAT and SAT modulo theories: From an abstract Davis-Putnam-Logemann-Loveland procedure to DPLL(T). J. ACM, 53(6):937-977, 2006.
[42] Lars Onsager. Reciprocal relations in irreversible processes. I. Phys. Rev., 37(4):405, 1931.
[43] Mercedes Pérez Millán and Alicia Dickenstein. The structure of MESSI biological systems. SIAM J. Appl. Dyn. Syst.,, 17(2):1650-1682, 2018.
[44] Mercedes Pérez Millán, Alicia Dickenstein, Anne Shiu, and Carsten Conradi. Chemical reaction systems with toric steady states. Bull. Math. Biol., 74(5):1027-1065, 2012.
[45] Andreas Seidl. Cylindrical Decomposition Under Application-Oriented Paradigms. Doctoral dissertation, University of Passau, Germany, 2006.
[46] Thomas Sturm. Subtropical real root finding. In ISSAC 2015, pages 347-354. ACM, 2015.
[47] Ole Tange. GNU Parallel: The command-line power tool. login: The USENIX Magazine, 36(1):42-47, 2011.
[48] Alfred Tarski. A decision method for elementary algebra and geometry. Prepared for publication by J. C. C. McKinsey. RAND Report R109, August 1, 1948, Revised May 1951, Second Edition, RAND, Santa Monica, CA, 1957.
[49] Shu Wang, Jia-Ren Lin, Eduardo D. Sontag, and Peter K. Sorger. Inferring reaction network structure from single-cell, multiplex data, using toric systems theory. bioRxiv, page 731018, 2019.
[50] Rudolf Wegscheider. Über simultane Gleichgewichte und die Beziehungen zwischen Thermodynamik und Reactionskinetik homogener Systeme. Monatsh. Chem. Verw. Tl., 22(8):849 906, 1901.
[51] Volker Weispfenning. The complexity of linear problems in fields. J. Symb. Comput., 5(1-2):3-27, 1988.
[52] Volker Weispfenning. Quantifier elimination for real algebra-The quadratic case and beyond. Appl. Algebr. Eng. Comm., 8(2):85-101, 1997.


[^0]:    ${ }^{1}$ https://www.ebi.ac.uk/compneur-srv/biomodels-main/publ-model.do?mid=BIOMD0000000092

[^1]:    ${ }^{2}$ https://www.ebi.ac.uk/biomodels/
    3 http://odebase.cs.uni-bonn.de/

[^2]:    ${ }^{4}$ SMT technically aims at the existential fragment, which in our context is equivalent to the universal fragment via logical negation.

