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Clique-width: Harnessing the power of atoms

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Abstract

Many NP-complete graph problems are polynomial-time solvable on graph classes of bounded clique-width. Several of these problems are polynomial-time solvable on a hereditary graph class $\mathcal{G}$ if they are so on the atoms (graphs with no clique cut-set) of $\mathcal{G}$. Hence, we initiate a systematic study into boundedness of clique-width of atoms of hereditary graph classes. A graph G is *H-free* if H is not an induced subgraph of G, and it is *(H_1, H_2)-free* if it is both H_1-free and H_2-free. A class of H-free graphs has bounded clique-width if and only if its atoms have this property. This is no longer true for (H_1, H_2)-free graphs, as evidenced by one known example. We prove the existence of another such pair (H_1, H_2) and classify the boundedness of clique-width on (H_1, H_2)-free atoms for all but 18 cases.

KEYWORDS
atom, clique-width, hereditary graph class

1 | INTRODUCTION

Many hard graph problems become tractable when restricting the input to some graph class. The two central questions are "for which graph classes does a graph problem become tractable" and "for which graph classes does it stay computationally hard?" Ideally, we wish to answer these questions for a large set of problems simultaneously instead of considering individual problems one by one.

Graph width parameters [31, 45, 47, 51, 65] help make such results possible. A graph class has *bounded* width if there is a constant c such that the width of all its members is at most c. As we discuss below, there are several meta-theorems that provide sufficient conditions for a problem to be tractable on a graph class of bounded width.

Two popular width parameters are treewidth (tw) and clique-width (cw). For every graph G the inequality $\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)-1}$ holds [23]. Hence, every problem that is polynomial-time solvable on graphs of bounded clique-width is also polynomial-time solvable on graphs of bounded treewidth. However, the converse statement does not hold: there exist graph problems, such as List Colouring, which are polynomial-time solvable on graphs of bounded treewidth [50], but NP-complete on graphs of bounded clique-width [27]. Thus, the trade-off between treewidth and clique-width is that the former can be used to solve more problems, but the latter is *more powerful* in the sense that it can be used to solve problems for larger graph classes.

Courcelle [24] proved that every graph problem definable in monadic second-order logic (MSO_2) is linear-time solvable on graphs of bounded treewidth. Courcelle, Makowsky, and Rotics [26] showed that every graph problem definable in the more restricted logic MSO_1 is polynomial-time solvable even on graphs of bounded clique-width (see [25] for details on MSO_1 and MSO_2). Since then, several clique-width meta-theorems for graph problems not definable in MSO_1 have been developed [38, 42, 52, 61].

All of the above meta-theorems require a constant-width decomposition of the graph. We can compute such a decomposition in polynomial time for treewidth [4] and clique-width [60], but for other width parameters, such as mim-width, which is even more powerful than clique-width [65], it is not known whether this is possible and this problem may turn out to be harder. For instance, unless $\mathsf{NP} = \mathsf{ZPP}$, there is no constant-factor approximation algorithm for mim-width that runs in polynomial time [62]. Meta-theorems for mim-width [2, 18] currently require an appropriate constant-width decomposition as part of the input (which may still be found in polynomial time for some graph classes).

Our focus. In our paper we concentrate on *clique-width*[1] in an attempt to find *larger* graph classes for which certain NP-complete graph problems become tractable without the requirement of an appropriate decomposition as part of the input. The type of graph classes we consider all have the natural property that they are closed under vertex deletion. Such graph classes are said to be *hereditary* and there is a long-standing study on the boundedness of clique-width for hereditary graph classes (see e.g. [3, 6, 8, 9, 12–15, 28, 30, 32, 33, 35, 36, 45, 51, 57]).

Besides capturing many well-known classes, the framework of hereditary graph classes also enables us to perform a *systematic* study of a width parameter or graph problem. This is because every hereditary graph class $\mathcal{G}$ is readily seen to be uniquely characterised by a minimal (but not necessarily finite) set $\mathcal{F}_\mathcal{G}$ of forbidden induced subgraphs. If $|\mathcal{F}_\mathcal{G}| = 1$ or $|\mathcal{F}_\mathcal{G}| = 2$, then $\mathcal{G}$ is said to be *monogenic* or *bigenic*, respectively. Monogenic and bigenic graph classes already have

[1]See Section 2 for a definition of clique-width and other terminology used in Section 1.

a rich structure, and studying their properties has led to deep insights into the complexity of bounding graph parameters and solving graph problems. This is evidenced, for example, by extensive studies on the classes of bull-free graphs [20] or claw-free graphs [21, 46], and surveys for graph problems or parameters specifically restricted to bigenic graph classes [31, 43].

It is well known (see e.g. [36]) that a monogenic class of graphs has bounded clique-width if and only if it is a subclass of the class $\mathcal{G}$ with $\mathcal{F}_{\mathcal{G}} = \{P_4\}$. The survey [31] gives a state-of-the-art theorem on the boundedness and unboundedness of clique-width of bigenic graph classes. Unlike treewidth, for which a complete dichotomy is known [5], and mim-width, for which there is an infinite number of open cases [17], this state-of-the-art theorem shows that there are still five open cases (up to an equivalence relation); see also Section 4. From the same theorem we observe that many graph classes are of unbounded clique-width. However, if a graph class has unbounded clique-width, then this does not mean that a graph problem must be NP-hard on this class. For example, COLOURING is polynomial-time solvable on the (bigenic) class of (C_4, P_6)-free graphs [41], which contains the class of split graphs and thus has unbounded clique-width [57]. In this case it turns out that the *atoms* (graphs with no clique cut-set) in the class of (C_4, P_6)-free graphs *do* have bounded clique-width. This immediately gives us an algorithm for the whole class of (C_4, P_6)-free graphs due to Tarjan's decomposition theorem [64].

In fact, Tarjan's result holds not only for COLOURING, but also for many other graph problems. For instance, several other classical graph problems, such as MINIMUM FILL-IN, MAXIMUM CLIQUE, MAXIMUM WEIGHTED INDEPENDENT SET [64] (see [1] for the unweighted variant) and MAXIMUM INDUCED MATCHING [16] are polynomial-time solvable on a hereditary graph class $\mathcal{G}$ if and only if this is the case on the atoms of $\mathcal{G}$. Hence, we aim to investigate, in a systematic way, the following natural research question:

Which hereditary graph classes of unbounded *clique-width have the property that their atoms have* bounded *clique-width?*

Known results. For monogenic graph classes, the restriction to atoms does not yield any algorithmic advantages, as shown by Gaspers et al. [41].

> **Theorem 1** (Gaspers et al. [41]). *Let H be a graph. The class of H-free atoms has bounded clique-width if and only if the class of H-free graphs has bounded clique-width* (*so, if and only if H is an induced subgraph of P_4*).

The result for (C_4, P_6)-free graphs [41] shows that the situation is different for bigenic classes. We are aware of other hereditary graph classes $\mathcal{G}$ with this property, but in those cases $|\mathcal{F}_{\mathcal{G}}| > 2$. Chordal graphs, or equivalently, $(C_4, C_5, \ldots)$-free graphs have unbounded clique-width [57], but chordal atoms are complete graphs [37] and have clique-width at most 2. The same holds for any subclass of chordal graphs of unbounded clique-width, such as the class of split graphs [57], or equivalently, $(C_4, C_5, 2P_2)$-free graphs. Moreover, Cameron et al. [19] proved that (cap,C_4)-free odd-signable atoms have clique-width at most 48, whereas the class of all (cap,C_4)-free odd-signable graphs contains the class of split graphs and thus has unbounded clique-width. We refer to [39, 40] for some examples of polynomial-time algorithms for COLOURING on hereditary graph classes that exploit the fact that atoms of subclasses of these graph classes have bounded clique-width.

Our results. Due to Theorem 1, and motivated by the aforementioned algorithmic applications, we focus on the atoms of bigenic graph classes. Recall that the class of (C_4, P_6)-free

FIGURE 1 The two forbidden induced subgraphs from Theorem 2.

graphs has unbounded clique-width but its atoms have bounded clique-width [41]. This also holds, for instance, for its subclass of $(C_4, 2P_2)$-free graphs and thus for (C_4, P_5)-free graphs and $(C_4, P_2 + P_3)$-free graphs. We determine a new, incomparable case where we forbid $2P_2$ and $\overline{P_2 + P_3}$ (also known as the *paraglider* [49]); see Figure 1 for illustrations of these forbidden induced subgraphs.

Theorem 2. *The class of* $(2P_2, \overline{P_2 + P_3})$*-free atoms has bounded clique-width* (*whereas the class of* $(2P_2, \overline{P_2 + P_3})$*-free graphs has unbounded clique-width*).

We prove Theorem 2 in Section 3 after first giving an outline. Our approach shares some similarities with the approach Malyshev and Lobanova [58] used to show that (WEIGHTED) COLOURING is polynomial-time solvable on $(P_5, \overline{P_2 + P_3})$-free graphs. We explain the differences between both approaches and the new ingredients of our proof in detail in Section 3. Here, we only discuss a complication that makes proving the boundedness of clique-width of atoms more difficult in general. Namely, when working with atoms, we need to be careful with performing complementation operations. In particular, a class of (H_1, H_2)-free graphs has bounded clique-width if only if the class of $(\overline{H_1}, \overline{H_2})$-free graphs has bounded clique-width. However, this equivalence relation no longer holds for classes of (H_1, H_2)-free atoms. For example, (C_4, P_5)-free (and even (C_4, P_6)-free) atoms have bounded clique-width [41], but we will prove that $(\overline{C_4}, \overline{P_5})$-free atoms have unbounded clique-width.

We also identify a number of new bigenic graph classes whose atoms already have unbounded clique-width. We prove this by giving two general techniques for constructing atoms of unbounded clique-width (see Lemmas 13 and 14) and by modifying existing graph constructions for proving the unbounded clique-width of the whole class; see Section 5. Combining the constructions from Section 5 with Theorem 2 and the state-of-the-art theorem on clique-width from [31] yields the following summary; see Section 2 for definitions of the notation used.

Theorem 3. *For graphs* H_1 *and* H_2*, let* $\mathcal{G}$ *be the class of* (H_1, H_2)*-free graphs.*

1. *The class of atoms in* $\mathcal{G}$ *has bounded clique-width if*
   (i) H_1 *or* $H_2 \subseteq_i P_4$,
   (ii) $H_1 =$ paw *or* K_s *and* $H_2 = P_1 + P_3$ *or* tP_1 *for some* $s, t \geq 1$,
   (iii) $H_1 \subseteq_i$ paw *and* $H_2 \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + P_2 + P_3, P_1 + P_5, P_1 + S_{1,1,2}, P_2 + P_4, P_6, S_{1,1,3}$ *or* $S_{1,2,2}$,
   (iv) $H_1 \subseteq_i P_1 + P_3$ *and* $H_2 \subseteq_i \overline{K_{1,3} + 3P_1}, \overline{K_{1,3} + P_2}, \overline{P_1 + P_2 + P_3}, \overline{P_1 + P_5}, \overline{P_1 + S_{1,1,2}}, \overline{P_2 + P_4}, \overline{P_6}, \overline{S_{1,1,3}}$ *or* $\overline{S_{1,2,2}}$,
   (v) $H_1 \subseteq_i$ diamond *and* $H_2 \subseteq_i P_1 + 2P_2, 3P_1 + P_2$ *or* $P_2 + P_3$,

(vi) $H_1 \subseteq_i 2P_1 + P_2$ *and* $H_2 \subseteq_i \overline{P_1 + 2P_2}$, $\overline{3P_1 + P_2}$ *or* $\overline{P_2 + P_3}$,
(vii) $H_1 \subseteq_i$ gem *and* $H_2 \subseteq_i P_1 + P_4$ *or* P_5,
(viii) $H_1 \subseteq_i P_1 + P_4$ *and* $H_2 \subseteq_i \overline{P_5}$,
(ix) $H_1 \subseteq_i K_3 + P_1$ *and* $H_2 \subseteq_i K_{1,3}$,
(x) $H_1 \subseteq_i \overline{2P_1 + P_3}$ *and* $H_2 \subseteq_i 2P_1 + P_3$,
(xi) $H_1 \subseteq_i P_6$ *and* $H_2 \subseteq_i C_4$ *or*
(xii) $H_1 \subseteq_i 2P_2$ *and* $H_2 \subseteq_i \overline{P_2 + P_3}$.

2. *The class of atoms in* $\mathcal{G}$ *has unbounded clique-width if*
(i) $H_1 \notin \mathcal{S}$ *and* $H_2 \notin \mathcal{S}$,
(ii) $H_1 \notin \overline{\mathcal{S}}$ *and* $H_2 \notin \overline{\mathcal{S}}$,
(iii) $H_1 \supseteq_i K_3 + P_1$ *and* $H_2 \supseteq_i 4P_1$ *or* $2P_2$,
(iv) $H_1 \supseteq_i K_{1,3}$ *and* $H_2 \supseteq_i K_4$ *or* C_4,
(v) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i K_{1,3}$, $5P_1$, $P_2 + P_4$ *or* $P_1 + P_6$,
(vi) $H_1 \supseteq_i 2P_1 + P_2$ *and* $H_2 \supseteq_i K_3 + P_1$, K_5, $\overline{P_2 + P_4}$ *or* $\overline{P_6}$,
(vii) $H_1 \supseteq_i K_3$ *and* $H_2 \supseteq_i 2P_1 + 2P_2$, $2P_1 + P_4$, $4P_1 + P_2$, $3P_2$ *or* $2P_3$,
(viii) $H_1 \supseteq_i 3P_1$ *and* $H_2 \supseteq_i \overline{2P_1 + 2P_2}$, $\overline{2P_1 + P_4}$, $\overline{4P_1 + P_2}$, $\overline{3P_2}$ *or* $\overline{2P_3}$,
(ix) $H_1 \supseteq_i K_4$ *and* $H_2 \supseteq_i P_1 + P_4$, $3P_1 + P_2$ *or* $2P_2$,
(x) $H_1 \supseteq_i 4P_1$ *and* $H_2 \supseteq_i$ gem, $\overline{3P_1 + P_2}$ *or* C_4,
(xi) $H_1 \supseteq_i$ gem, $\overline{P_1 + 2P_2}$ *or* $\overline{P_2 + P_3}$ *and* $H_2 \supseteq_i P_1 + 2P_2$ *or* P_6,
(xii) $H_1 \supseteq_i P_1 + P_4$ *and* $H_2 \supseteq_i \overline{P_1 + 2P_2}$ *or*
(xiii) $H_1 \supseteq_i 2P_2$ *and* $H_2 \supseteq_i \overline{P_2 + P_4}$, $\overline{3P_2}$ *or* $\overline{P_5}$.

We prove Theorem 3 in Section 6. Due to this theorem, we are left with 18 open cases, which we list in Section 6 (see Open Problem 2). In Section 7 we discuss directions for future work.

2 | PRELIMINARIES

We first give some general graph terminology and notation, followed by some terminology related to clique-width.

2.1 | General graph terminology

Let G be a graph. For a subset $S \subseteq V(G)$, the subgraph of G *induced by* S is the graph $G[S]$, which has vertex set S and edge set $\{uv \mid uv \in E(G), u, v \in S\}$. If $S = \{s_1, ..., s_r\}$, we may write $G[s_1, ..., s_r]$ instead of $G[\{s_1, ..., s_r\}]$. We write $F \subseteq_i G$ to indicate that F is an induced subgraph of G. For a subset $S \subseteq V(G)$, we let $G \setminus S = G[V(G) \setminus S]$. A *(connected) component* of G is a maximal connected subgraph of G. The *neighbourhood* of a vertex $u \in V(G)$ is the set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. Two vertices in G are *false twins* if they have the same neighbourhood; note that such vertices must be non-adjacent.

An *independent set* in a graph G is a subset of $V(G)$ that consists of pairwise non-adjacent vertices; a *clique* in G is a subset of pairwise adjacent vertices. A clique $K \subseteq V(G)$ is a *clique cut-set* of G if $G \setminus K$ is disconnected. A graph with no clique cut-sets is an *atom*. Note that if a graph is disconnected, then the empty set is a clique cut-set, so atoms are connected by definition.

Let H be a graph. A graph G is *H-free* if G does not contain H as an induced subgraph. Let $\{H_1, ..., H_p\}$ be a set of graphs. Then G is *$(H_1, ..., H_p)$-free* if it is H_i-free for all $i \in \{1, ..., p\}$.

Let X and Y be two disjoint vertex subsets of a graph G. The edges between X and Y form a *matching* if every vertex in X is adjacent to at most one vertex in Y and vice versa. A vertex $x \in V(G) \setminus Y$ is *(anti)-complete* to Y if it is (non)-adjacent to every vertex in Y. Similarly, X is *complete* to Y if every vertex of X is complete to Y and *anti-complete* to Y if every vertex of X is anti-complete to Y. A vertex $u \in V(G)$ is *dominating* if it is complete to $V(G) \setminus \{u\}$. For $k \geq 1$, a *k-subdivision* of G is the operation of replacing each edge uv of G with a $(k+1)$-edge path, whose end-vertices are identified with u and v, respectively. The *complement* $\overline{G}$ of G has vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{uv \mid u, v \in V(G), u \neq v, uv \notin E(G)\}$. The *line graph* of G is the graph with vertex set $E(G)$ and an edge between two vertices e_1 and e_2 if and only if e_1 and e_2 share a common end-vertex in G.

A graph is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets. A graph is *complete multi-partite* if its vertex can be partitioned into r independent sets $V_1, ..., V_r$ for some integer $r \geq 1$ such that V_i is complete to V_j for every pair i, j with $1 \leq i < j \leq r$; if $r = 2$, we say that the graph is *complete bipartite*.

The graph $G_1 + G_2$ is the *disjoint union* of two vertex-disjoint graphs G_1 and G_2 and has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graph rG is the disjoint union of r copies of a graph G.

The graphs C_t, K_t and P_t denote the cycle, complete graph and path on t vertices, respectively. The graph $K_{s,t}$ denotes the complete bipartite graph whose two partition classes contain s and t vertices, respectively. The paw is the graph $\overline{P_1 + P_3}$, the diamond is the graph $\overline{2P_1 + P_2}$, and the gem is the graph $\overline{P_1 + P_4}$. The claw is the graph with vertices x, y_1, y_2, y_3 and edges xy_i for $i \in \{1, 2, 3\}$. The *subdivided claw* $S_{h,i,j}$, for $1 \leq h \leq i \leq j$ is the tree with one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. We let $\mathcal{S}$ denote the class of graphs every connected component of which is either a subdivided claw or a path on at least one vertex. Note that $S_{1,1,1}$ is isomorphic to the claw $K_{1,3}$.

A graph is *chordal* if it has no induced cycles on more than four vertices, that is, if it is $(C_4, C_5, ...)$-free. A graph is *co-chordal* if its complement is chordal. A graph is *split* if its vertex set can be partitioned into a clique K and an independent set I. A graph is *bipartite chain* if it is bipartite, say with bipartition classes X and Y, such that the vertices of X can be ordered $x_1, ..., x_p$ with the property that $N(x_1) \subseteq N(x_2) \subseteq ... \subseteq N(x_p)$. The following observation is well known and easy to see.

Lemma 4. *A graph is bipartite chain if and only if it is bipartite and $2P_2$-free.*

2.2 | Clique-width

The *clique-width* of a graph G, denoted by $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G using the following four operations:

1. create a new graph consisting of a single vertex v with label i;
2. take the disjoint union of two labelled graphs G_1 and G_2;
3. add an edge between every vertex with label i and every vertex with label j $(i \neq j)$;
4. relabel every vertex with label i to have label j.

A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that $\mathrm{cw}(G) \leq c$ for every $G \in \mathcal{G}$; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

For an induced subgraph G' of a graph G, the *subgraph complementation* acting on G with respect to G' replaces every edge of G' by a non-edge, and vice versa. Hence, the resulting graph has vertex set $V(G)$ and edge set $(E(G)\setminus E(G')) \cup E(\overline{G'})$. For two disjoint vertex subsets S and T in G, the *bipartite complementation* acting on G with respect to S and T replaces every edge with one end-vertex in S and the other in T by a non-edge and vice versa.

For a constant $k \geq 0$ and a graph operation γ, a graph class $\mathcal{G}'$ is (k, γ)-*obtained* from a graph class $\mathcal{G}$ if

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$, there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

Then γ *preserves* boundedness of clique-width if for every constant k and every graph class $\mathcal{G}$, every graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [54].
Fact 2. Subgraph complementation preserves boundedness of clique-width [51].
Fact 3. Bipartite complementation preserves boundedness of clique-width [51].

We finish this section with making two further observations that we will need later on. First, we make the following well-known observation on bipartite chain graphs, which is readily seen.

Lemma 5. *Bipartite chain graphs have clique-width at most* 3.

Let $G = (K \cup I, E)$ be a split graph with clique K and independent set I. If there is a vertex $v \in I$ with $N(v) \subsetneq K$, then $N(v)$ is a clique cut-set of G. Furthermore, if $|I| > 1$ then K is a clique cut-set. It follows that split atoms are complete graphs. It is readily seen that complete graphs have clique-width at most 2. Hence, we can make the following observation (which also follows from a theorem of Dirac [37] and the fact that every split graph is chordal).

Lemma 6. *Split atoms are complete graphs and have clique-width at most* 2.

3 | PROOF OF THEOREM 2

Here, we prove Theorem 2, namely, that the class of $(2P_2, \overline{P_2 + P_3})$-free atoms has bounded clique-width. Our approach is based on the following three claims:

(i) $(2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_5 have bounded clique-width.
(ii) $(2P_2, \overline{P_2 + P_3})$-free atoms with an induced C_4 have bounded clique-width.
(iii) $(C_4, C_5, 2P_2, \overline{P_2 + P_3})$-free atoms have bounded clique-width.

We prove Claims (i) and (ii) in Lemmas 7 and 8, respectively, whereas Claim (iii) follows from the fact that $(C_4, C_5, 2P_2)$-free graphs are split graphs and so, by Lemma 6, the atoms in this

class are complete graphs and therefore have clique-width at most 2. We partition the vertex set of an arbitrary $(2P_2, \overline{P_2 + P_3})$-free atom G into a number of different subsets according to their neighbourhoods in an induced C_5 in Lemma 7 or an induced C_4 in Lemma 8. We then analyse the properties of these different subsets of $V(G)$ and how they are connected to each other, and use this knowledge to apply a number of appropriate vertex deletions, subgraph complementations and bipartite complementations. These operations will modify G into a graph G' that is a disjoint union of a number of smaller "easy" graphs known to have "small" clique-width. We then use Facts 1–3 to conclude that G also has small clique-width.

This approach works, as we will:

- apply the vertex deletions, subgraph complementations and bipartite complementations only a constant number of times;
- not use the properties of being an atom or being $(2P_2, \overline{P_2 + P_3})$-free once we "leave the graph class" due to applying the above graph operations.

Our approach is similar to the approach used by Malyshev and Lobanova [58] for showing that Colouring is polynomial-time solvable on the superclass of $(P_5, \overline{P_2 + P_3})$-free graphs. However, we note the following two techniques that can be used in the design of algorithms for Colouring on hereditary graph classes, but cannot be used for proving boundedness of clique-width. Both these techniques were used in [58].

1 Prime atoms restriction: A set $X \subseteq V(G)$ is a *module* if all vertices in X have the same set of neighbours in $V(G) \setminus X$. A module X in a graph G is *trivial* if it contains either all or at most one vertex of G. A graph G is *prime* if it has no non-trivial modules. To solve Colouring in polynomial time on some hereditary graph class $\mathcal{G}$, one may restrict to prime atoms in $\mathcal{G}$ [48]. Malyshev and Lobanova proved that $(P_5, \overline{P_2 + P_3})$-free prime atoms with an induced C_5 are $3P_1$-free or have a bounded number of vertices. In both cases, Colouring can be solved in polynomial time. We cannot make the pre-assumption that our atoms are prime. To see this, let G be a split graph that is not complete. Add two new non-adjacent vertices u and v to G and make them complete to the rest of $V(G)$. Let $\mathcal{G}$ be the (hereditary) graph class that consists of all these "enhanced" split graphs and their induced subgraphs. These enhanced split graphs are atoms, which have unbounded clique-width due to Fact 1 and the fact that split graphs have unbounded clique-width [57]. However, the prime atoms in $\mathcal{G}$ are P_1 and P_2,[2] which have clique-width 1 and 2, respectively.
2 Perfect graphs restriction: Malyshev and Lobanova observed that $(P_5, \overline{P_2 + P_3}, C_5)$-free graphs are perfect. Hence, Colouring can be solved in polynomial time on such graphs [44]. However, being perfect does not imply boundedness of clique-width. For instance, split graphs are perfect graphs with unbounded clique-width [57].

Lemma 7. *The class of $(2P_2, \overline{P_2 + P_3})$-free atoms that contain an induced C_5 has bounded clique-width.*

[2]Let D be a prime atom in $\mathcal{G}$. As D is prime, D cannot contain both u and v. This means that D is a split graph. By Lemma 6, as D is an atom, it must be a complete graph. As D is prime, this implies that $|V(D)| \leq 2$.

Proof. Suppose G is a $(2P_2, \overline{P_2 + P_3})$-free atom containing an induced cycle C on five vertices, say $v_1, ..., v_5$ in that order. For $S \subseteq \{1, ..., 5\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i | i \in S\}$.

To simplify notation, in the following claims, subscripts on vertices and vertex sets should be interpreted modulo 5 and whenever possible we will write V_i instead of $V_{\{i\}}$, write $V_{i,j}$ instead of $V_{\{i,j\}}$, and so on.

Claim 7.1. For $i \in \{1, ..., 5\}$, $V_i \cup V_{i,i+1} \cup V_{i-1,i,i+1}$ are empty.

Proof of Claim. Suppose, for contradiction, that $x \in V_2 \cup V_{2,3} \cup V_{1,2,3}$. Then $G[x, v_2, v_4, v_5]$ is a $2P_2$, a contradiction. The claim follows by symmetry. ◇

By Claim 7.1, the only non-empty sets V_S are those of the form $V_\emptyset$, $V_{i,i+2}$, $V_{i,i+1,i+3}$, $V_{i,i+1,i+2,i+3}$ and $V_{1,2,3,4,5}$. We now prove a sequence of claims.

Claim 7.2. For $i \in \{1, ..., 5\}$, $V_\emptyset \cup V_{i,i+2}$ is independent.

Proof of Claim. Suppose, for contradiction, that $x, y \in V_\emptyset \cup V_{1,3}$ are adjacent. Then $G[v_4, v_5, x, y]$ is a $2P_2$, a contradiction. The claim follows by symmetry. ◇

Claim 7.3. For $i \in \{1, ..., 5\}$, $|V_{i,i+1,i+3} \cup V_{i,i+1,i+2,i+3}| \leq 1$.

Proof of Claim. Suppose, for contradiction that there are distinct vertices $x, y \in V_{1,2,4} \cup V_{1,2,3,4}$. Then $G[v_1, v_4, x, v_5, y]$ or $G[x, y, v_1, v_4, v_2]$ is a $\overline{P_2 + P_3}$ if x is adjacent or non-adjacent to y, respectively, a contradiction. The claim follows by symmetry. ◇

Claim 7.4. For $i \in \{1, ..., 5\}$, there is at most one edge between $V_{i,i+2}$ and $V_{i,i-2}$.

Proof of Claim. Suppose, for contradiction, that a vertex $x \in V_{1,3}$ has two neighbours $y, y' \in V_{1,4}$. By Claim 7.2, the sets $V_{1,3}$ and $V_{1,4}$ are independent. In particular, this means that y is non-adjacent to y'. Therefore $G[y, y', x, v_4, v_1]$ is a $\overline{P_2 + P_3}$, a contradiction. It follows that every vertex in $V_{1,3}$ has at most one neighbour in $V_{1,4}$. By symmetry, every vertex in $V_{1,4}$ has at most one neighbour in $V_{1,3}$ and so the edges between $V_{1,3}$ and $V_{1,4}$ form a matching. Since G is $2P_2$-free, it follows that there is at most one edge between $V_{1,3}$ and $V_{1,4}$. The claim follows by symmetry. ◇

Claim 7.5. For $i \in \{1, ..., 5\}$, $V_{i,i+2}$ is complete to $V_{i-1,i+1} \cup V_{i+1,i+3}$.

Proof of Claim. Suppose, for contradiction, that $x \in V_{1,3}$ is non-adjacent to $y \in V_{2,4}$. Then $G[x, v_1, y, v_4]$ is a $2P_2$, a contradiction. The claim follows by symmetry. ◇

Claim 7.6. If $x \in V_{1,2,3,4,5}$, then x is complete to $V(G) \setminus \{x\}$. In particular, this implies that $V_{1,2,3,4,5}$ is a clique.

Proof of Claim. Let $x \in V_{1,2,3,4,5}$ and suppose, for contradiction, that $y \in V(G) \setminus \{x\}$ is non-adjacent to x. Clearly $y \notin V(C)$. If $y \in V_{1,2,3,4,5}$, then $G[x, y, v_1, v_4, v_2]$ is a $\overline{P_2 + P_3}$, a

contradiction. If y is adjacent to v_i and v_{i+2}, but not to v_{i+1} for some $i \in \{1, ..., 5\}$, then $G[v_i, v_{i+2}, v_{i+1}, y, x]$ is a $\overline{P_2 + P_3}$, a contradiction. By Claim 7.1, it follows that $y \in V_\emptyset$. Note that this implies that every vertex of $V_{1,2,3,4,5}$ is adjacent to every other vertex in $V(G) \setminus V_\emptyset$.

Since G is an atom, $N(y)$ cannot be a clique, and so it must contain two non-adjacent vertices, say u and v. By Claim 7.2, $u, v \notin V_\emptyset$ and for all $i \in \{1, ..., 5\}$, $u, v \notin V_{i,i+2}$. Since every vertex of $V_{1,2,3,4,5}$ is adjacent to every other vertex in $V(G) \setminus V_\emptyset$, neither u nor v is equal to x and, furthermore, x is adjacent to both u and v. By Claim 7.1, it follows that u and v must each have at least three neighbours in C. Therefore u and v must have a common neighbour in C; let v_i be such a common neighbour. Now $G[u, v, x, y, v_i]$ is a $\overline{P_2 + P_3}$, a contradiction. This completes the proof of the claim. ◇

We now show how to use a bounded number of vertex deletions, complementations and bipartite complementations to change G into an edgeless graph. First, by Claim 7.3, we can make $V_{i,i+1,i+3}$ and $V_{i,i+1,i+2,i+3}$ empty for all $i \in \{1, ..., 5\}$ by deleting at most five vertices. See Figure 2 for an illustration of the resulting graph. Next, by Claim 7.6 we can apply a bipartite complementation between $V_{1,2,3,4,5}$ and the rest of the graph to disconnect $G[V_{1,2,3,4,5}]$ from it. Next, by Claim 7.6 we can apply a complementation to $V_{1,2,3,4,5}$, which turns it into an independent set. Now, by Claim 7.1, the only other vertices remaining are those in C, those in $V_\emptyset$ and those in $V_{i,i+2}$ for $i \in \{1, ..., 5\}$. Next, by Claim 7.4, we can make $V_{i,i+2}$ anti-complete to $V_{i,i-2}$ for all $i \in \{1, ..., 5\}$ by deleting at most five vertices. By Claim 7.2, the only remaining edges are those between $V_{i-1,i+1} \cup \{v_i\}$ and $V_{i,i+2} \cup \{v_{i+1}\}$ for $i \in \{1, ..., 5\}$. By Claim 7.5 combined with the definition of $V_{i,i+2}$, we can apply a bipartite complementation between each of these pairs to remove all remaining edges of the graph. Thus, applying at most ten vertex deletions, six bipartite complementations and one complementation operation to G, we obtain an edgeless

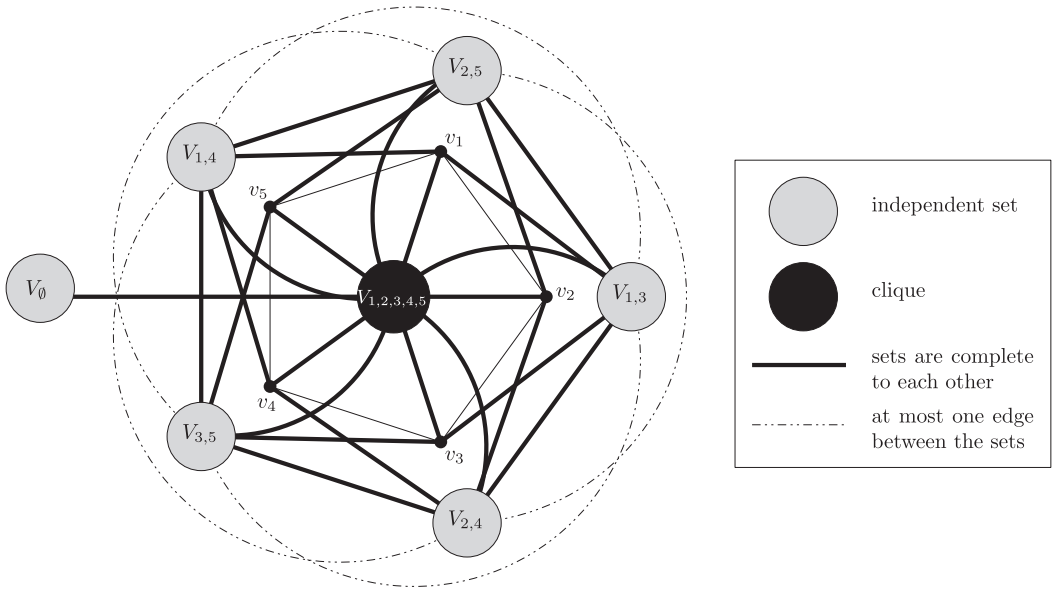

FIGURE 2 The configuration of the sets in the proof of Lemma 7 obtained after deleting the vertex sets $V_{i,i+1,i+3}$ and $V_{i,i+1,i+2,i+3}$ for $i \in \{1, 2, 3, 4, 5\}$. Note that vertices in $V_\emptyset$ now have no neighbours outside $V_{1,2,3,4,5}$.

graph, which has clique-width 1. By Facts 1, 2 and 3 it follows that G has bounded clique-width. □

Lemma 8. *The class of $(2P_2, \overline{P_2 + P_3})$-free atoms that contain an induced C_4 has bounded clique-width.*

Proof. Suppose G is a $(2P_2, \overline{P_2 + P_3})$-free atom containing an induced cycle C on four vertices, say $v_1, ..., v_4$ in that order. By Lemma 7, we may assume that G is C_5-free. For $S \subseteq \{1, ..., 4\}$, let V_S be the set of vertices $x \in V(G) \setminus V(C)$ such that $N(x) \cap V(C) = \{v_i | i \in S\}$.

To simplify notation, in the following claims, subscripts on vertices and vertex sets should be interpreted modulo 4 and whenever possible we will write V_i instead of $V_{\{i\}}$, write $V_{i,j}$ instead of $V_{\{i,j\}}$, and so on.

Claim 8.1. For $i \in \{1, ..., 4\}$, $V_{i,i+1,i+2}$ is empty.

Proof of Claim. Suppose, for contradiction, that $x \in V_{1,2,3}$. Then $G[v_1, v_3, v_2, v_4, x]$ is a $\overline{P_2 + P_3}$, a contradiction. The claim follows by symmetry. ◇

See Figure 3 for an illustration of the remaining sets V_S that can be non-empty.

Claim 8.2. For $i \in \{1, ..., 4\}$, $V_\emptyset \cup V_i \cup V_{i+1} \cup V_{i,i+1}$ is an independent set.

Proof of Claim. Suppose, for contradiction, that $x, y \in V_\emptyset \cup V_1 \cup V_2 \cup V_{1,2}$ are adjacent. Then $G[x, y, v_3, v_4]$ is a $2P_2$, a contradiction. The claim follows by symmetry. ◇

Claim 8.3. For $i \in \{1, ..., 4\}$, $V_{i,i+1} \cup V_{i,i+2}$ and $V_{i,i+1} \cup V_{i+1,i+3}$ are independent sets.

Proof of Claim. Suppose, for contradiction, that $x, y \in V_{1,2} \cup V_{1,3}$ are adjacent. By Claim 8.2, x and y cannot both be in $V_{1,2}$, so assume without loss of generality that $x \in V_{1,3}$. Now $G[x, v_2, v_1, v_3, y]$ is a $\overline{P_2 + P_3}$ (regardless of whether $y \in V_{1,2}$ or $y \in V_{1,3}$), a contradiction. The claim follows by symmetry. ◇

Claim 8.4. $G[V_{1,2,3,4}]$ is $(P_1 + P_2)$-free and so it has bounded clique-width.

Proof of Claim. Suppose, for contradiction, that $x, y, y' \in V_{1,2,3,4}$ induce a $P_1 + P_2$ in G. Then $G[v_1, v_3, y, x, y']$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore $G[V_{1,2,3,4}]$ is $(P_1 + P_2)$-free and so P_4-free, so it has bounded clique-width by Theorem 1. ◇

Claim 8.5. For $i \in \{1, 2\}$, $V_{i,i+2}$ is complete to $V_{1,2,3,4}$.

Proof of Claim. Suppose, for contradiction, that $x \in V_{1,3}$ is non-adjacent to $y \in V_{1,2,3,4}$. Then $G[v_1, v_3, v_2, x, y]$ is a $\overline{P_2 + P_3}$, a contradiction. The claim follows by symmetry. ◇

Claim 8.6. For $i \in \{1, 2, 3, 4\}$ either $V_{i-1} \cup V_{i-1,i}$ or $V_{i,i+1} \cup V_{i+1}$ is empty.

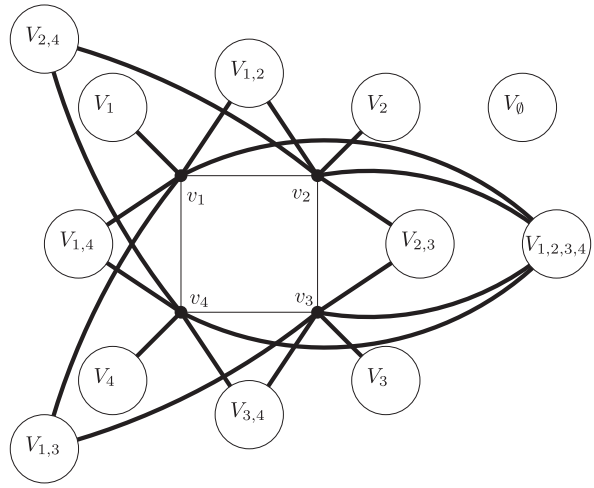

FIGURE 3 The possible non-empty sets V_S in the initial situation in the proof of Lemma 8. Edges between the different sets V_S are not drawn.

Proof of Claim. Suppose, for contradiction, that $x \in V_1 \cup V_{1,2}$ and $y \in V_{2,3} \cup V_3$. Then $G[v_1, x, y, v_3, v_4]$ is a C_5 or $G[x, v_1, y, v_3]$ is a $2P_2$ if x is adjacent or non-adjacent to y, respectively, a contradiction. The claim follows by symmetry. ◇

Claim 8.7. If $x \in V_\emptyset$, then x has at least two neighbours in one of $V_{1,3}$ and $V_{2,4}$ and is anti-complete to the other. Furthermore, in this case x is complete to $V_{1,2,3,4}$.

Proof of Claim. Suppose $x \in V_\emptyset$. Since G is not an atom, $N(x)$ cannot be a clique, and so must contain two non-adjacent vertices y, y'. By Claims 8.1 and 8.2, and the definition of $V_\emptyset$, it follows that $y, y' \in V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$. If $y, y' \in V_{1,2,3,4}$, then $G[y, y', v_1, x, v_2]$ is a $\overline{P_2 + P_3}$, a contradiction. By Claim 8.5, $V_{1,2,3,4}$ is complete to $V_{1,3} \cup V_{2,4}$, so it follows that $y, y' \in V_{1,3} \cup V_{2,4}$. If $y \in V_{1,3}$ and $y' \in V_{2,4}$, then $G[v_1, v_2, y', x, y]$ is a C_5, a contradiction. It follows that $y, y' \in V_{1,3}$ or $y, y' \in V_{2,4}$.

Suppose $y, y' \in V_{1,3}$. If $z \in V_{2,4}$ is a neighbour of x, then z must be adjacent to y and y' (since, as shown above, x cannot have a pair of non-adjacent neighbours one of which is in $V_{1,3}$ and the other of which is in $V_{2,4}$), in which case $G[y, y', x, v_1, z]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore x cannot have a neighbour in $V_{2,4}$. If $z \in V_{1,2,3,4}$ is a non-neighbour of x, then z must be adjacent to y and y' by Claim 8.5, so $G[y, y', v_1, x, z]$ is a $\overline{P_2 + P_3}$, a contradiction. Therefore x is complete to $V_{1,2,3,4}$. The claim follows by symmetry. ◇

Claim 8.8. For $i \in \{1, 2\}$, $|V_{i,i+1} \cup V_{i+2,i+3}| \leq 2$.

Proof of Claim. Suppose, for contradiction, that $|V_{1,2} \cup V_{3,4}| \geq 3$. First note that if $x \in V_{1,2}$, $y \in V_{3,4}$ are non-adjacent, then $G[v_1, x, v_3, y]$ is a $2P_2$, a contradiction. Therefore $V_{1,2}$ is complete to $V_{3,4}$. By Claim 8.2, both $V_{1,2}$ and $V_{3,4}$ are independent sets. If $x \in V_{1,2}$ and $y, y' \in V_{3,4}$, then $G[y, y', v_3, x, v_4]$ is a $\overline{P_2 + P_3}$, a contradiction. By symmetry, we conclude that either $V_{1,2}$ or $V_{3,4}$ is empty.

Suppose $V_{3,4}$ is empty, so $V_{1,2}$ contains at least three vertices and let $x \in V_{1,2}$ be such a vertex. Since G is an atom, $N(x)$ cannot be a clique, so x must have two neighbours y, y' that are non-adjacent. By Claims 8.1, 8.2, 8.3 and 8.6, and the definition of $V_{1,2}$, every neighbour of $x \in V_{1,2}$ lies in $\{v_1, v_2\} \cup V_{1,2,3,4}$. Since v_1 is complete to $\{v_2\} \cup V_{1,2,3,4}$ and v_2 is complete to $\{v_1\} \cup V_{1,2,3,4}$, it follows that $y, y' \in V_{1,2,3,4}$. Now $G[y, y', v_1, v_3, x]$ is a $\overline{P_2 + P_3}$, a contradiction. The claim follows by symmetry. ◇

Claim 8.9. For $i \in \{1, 2, 3, 4\}$, V_i is complete to $V_{1,2,3,4}$ and at most one vertex of $V_{i,i+2}$ has neighbours in V_i.

Proof of Claim. Suppose $x \in V_1$. Since G is an atom, x must have two neighbours y, y' that are non-adjacent. By Claims 8.1, 8.2 and 8.6, and the definition of V_1, every neighbour of x lies in $\{v_1\} \cup V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$. If $y, y' \in V_{1,3} \cup V_{1,2,3,4}$, then $G[y, y', v_1, v_3, x]$ is a $\overline{P_2 + P_3}$, a contradiction. The vertex v_1 is complete to $V_{1,3} \cup V_{1,2,3,4}$. Therefore without loss of generality, we may assume $y \in V_{2,4}$. Furthermore, note that $V_{1,3}$ is an independent set by Claim 8.3, so x has at most one neighbour in $V_{1,3}$. Since V_1 is an independent set by Claim 8.2, it follows that $G[V_1 \cup V_{1,3}]$ is a bipartite graph with parts V_1 and $V_{1,3}$. Since G is $2P_2$-free, it follows that no two vertices in V_1 can have different neighbours in $V_{1,3}$. Therefore at most one vertex of $V_{1,3}$ has a neighbour in V_1. Now if $z \in V_{1,2,3,4}$, then z is adjacent to y by Claim 8.5. If x is non-adjacent to z, then $G[v_1, y, v_2, x, z]$ is a $\overline{P_2 + P_3}$, a contradiction. We conclude that V_1 is complete to $V_{1,2,3,4}$. The claim follows by symmetry. ◇

We now proceed as follows. By Claim 8.1, the set $V_{1,2,3} \cup V_{2,3,4} \cup V_{1,3,4} \cup V_{1,2,4}$ is empty. By Claims 8.6 and 8.8, there are at most two vertices in $V_{1,2} \cup V_{2,3} \cup V_{3,4} \cup V_{1,4}$, so after doing at most two vertex deletions, we may assume these sets are empty (note that the resulting graph may no longer be an atom). Applying four further vertex deletions, we can remove the cycle C from G. By Claim 8.9, at most one vertex of $V_{1,3}$ (resp. $V_{2,4}$) has a neighbour in V_1 (resp. V_2). Therefore, applying at most two further vertex deletions, we may assume that $V_{1,3}$ is anti-complete to V_1 and $V_{2,4}$ is anti-complete to V_2. By Claim 8.6, we may assume without loss of generality that V_3 and V_4 are empty (see Figure 4 for an illustration of the resulting graph).

The remaining vertices of G all lie in $V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4} \cup V_{1,2,3,4}$ and by Fact 1, it suffices to show that this modified graph has bounded clique-width. By Claims 8.5, 8.7 and 8.9, $V_{1,2,3,4}$ is complete to $V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4}$, and so applying a bipartite complementation between these two sets disconnects $G[V_{1,2,3,4}]$ from the rest of the graph. By Claim 8.4, $G[V_{1,2,3,4}]$ has bounded clique-width, so by Fact 3, we may assume $V_{1,2,3,4}$ is empty.

By Claim 8.7, we can partition $V_\emptyset$ into the set $V_\emptyset^{1,3}$ of vertices that have neighbours in $V_{1,3}$ and the set $V_\emptyset^{2,4}$ of vertices that have neighbours in $V_{2,4}$. Now Claims 8.2 and 8.3 imply that $V_\emptyset^{2,4} \cup V_1 \cup V_{1,3}$ and $V_\emptyset^{1,3} \cup V_2 \cup V_{2,4}$ are independent sets (recall that $V_{1,3}$ is now anti-complete to V_1 and $V_{2,4}$ is now anti-complete to V_2), and so $G[V_\emptyset \cup V_1 \cup V_2 \cup V_{1,3} \cup V_{2,4}]$ is a $2P_2$-free bipartite graph, so it is a bipartite chain graph by Lemma 4 and thus has bounded clique-width by Lemma 5. By Fact 1, this completes the proof. □

The class of split graphs is the class of $(C_4, C_5, 2P_2)$-free graphs. Since split graphs therefore form a subclass of the class of $(2P_2, \overline{P_2 + P_3})$-free graphs, and split graphs have unbounded clique-width [57], it follows that $(2P_2, \overline{P_2 + P_3})$-free graphs also have unbounded clique-width. Recall that by Lemma 6, split atoms are complete graphs and therefore have clique-width at

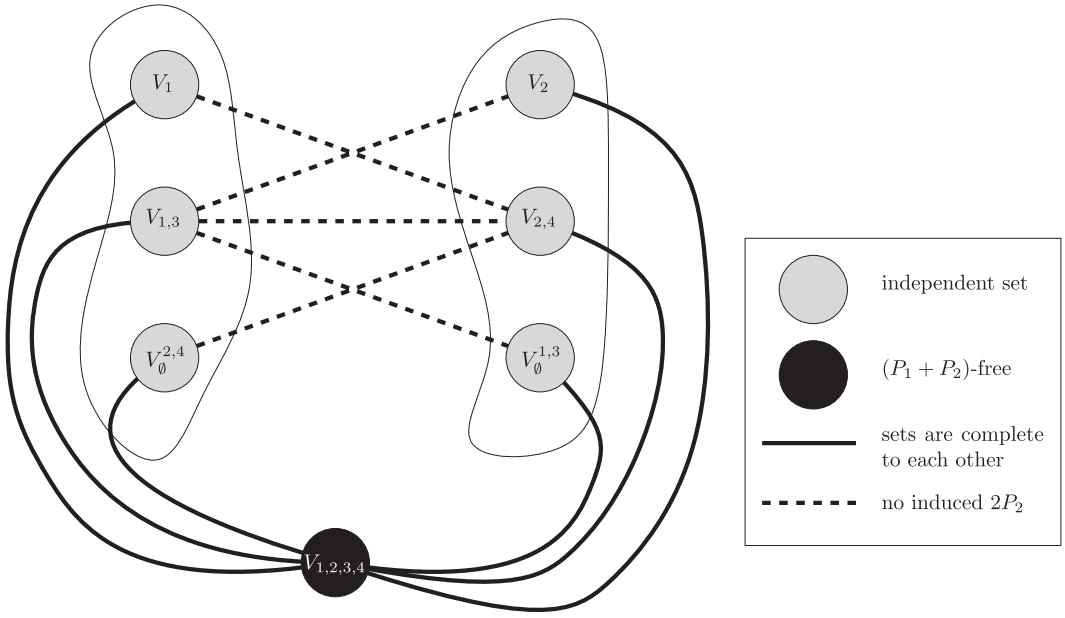

FIGURE 4 The configuration in the proof of Lemma 8 after deleting the at most six vertices in $V_{1,2} \cup V_{2,3} \cup V_{3,4} \cup V_{1,4} \cup V(C)$ along with at most one vertex in each of $V_{1,3}$ and $V_{2,4}$. The sets $V_\emptyset^{2,4} \cup V_1 \cup V_{1,3}$ and $V_\emptyset^{1,3} \cup V_2 \cup V_{2,4}$ are independent.

most 2. The $(2P_2, \overline{P_2 + P_3})$-free atoms that are not split must therefore contain an induced C_4 or C_5. Applying Lemmas 7 and 8, we obtain Theorem 2, which we restate below.

Theorem 2 (restated). *The class of* $(2P_2, \overline{P_2 + P_3})$*-free atoms has bounded clique-width* (*whereas the class of* $(2P_2, \overline{P_2 + P_3})$*-free graphs has unbounded clique-width*).

4 | CLIQUE-WIDTH SUMMARY FOR GENERAL BIGENIC CLASSES

In this section we present the state-of-the-art for boundedness of clique-width of general bigenic classes. We will use these results in Section 5, where we prove our results on unboundedness of clique-width of atoms in bigenic classes.

Let H_1, H_2, H_3, H_4 be four graphs. Then the classes of (H_1, H_2)-free graphs and (H_3, H_4)-free graphs are said to be *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations: (i) complementing both graphs in the pair, and (ii) if one of the graphs in the pair is $3P_1$, replacing it with $P_1 + P_3$ or vice versa. If two classes are equivalent, then one of them has bounded clique-width if and only if the other one does [36].

Recall that the subdivided claw $S_{h,i,j}$, for $1 \leq h \leq i \leq j$ is the tree with one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Also recall that $\mathcal{S}$ denotes the class of graphs every connected component of which is either a subdivided claw or a path. Moreover, recall that the paw is the graph $\overline{P_1 + P_3}$, the diamond is the graph $\overline{2P_1 + P_2}$ and the gem is the graph $\overline{P_1 + P_4}$.

Theorem 9 (Dabrowski et al. [31]). *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

1. *$\mathcal{G}$ has bounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   (i) *H_1 or $H_2 \subseteq_i P_4$,*
   (ii) *$H_1 = K_s$ and $H_2 = tP_1$ for some $s, t \geq 1$,*
   (iii) *$H_1 \subseteq_i$ paw and $H_2 \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + P_2 + P_3, P_1 + P_5, P_1 + S_{1,1,2}, P_2 + P_4, P_6, S_{1,1,3}$ or $S_{1,2,2}$,*
   (iv) *$H_1 \subseteq_i$ diamond and $H_2 \subseteq_i P_1 + 2P_2, 3P_1 + P_2$ or $P_2 + P_3$,*
   (v) *$H_1 \subseteq_i$ gem and $H_2 \subseteq_i P_1 + P_4$ or P_5,*
   (vi) *$H_1 \subseteq_i K_3 + P_1$ and $H_2 \subseteq_i K_{1,3}$ or*
   (vii) *$H_1 \subseteq_i \overline{2P_1 + P_3}$ and $H_2 \subseteq_i 2P_1 + P_3$.*
2. *$\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   (i) *$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$,*
   (ii) *$H_1 \notin \overline{\mathcal{S}}$ and $H_2 \notin \overline{\mathcal{S}}$,*
   (iii) *$H_1 \supseteq_i K_3 + P_1$ or C_4 and $H_2 \supseteq_i 4P_1$ or $2P_2$,*
   (iv) *$H_1 \supseteq_i$ diamond and $H_2 \supseteq_i K_{1,3}, 5P_1, P_2 + P_4$ or P_6,*
   (v) *$H_1 \supseteq_i K_3$ and $H_2 \supseteq_i 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2$ or $2P_3$,*
   (vi) *$H_1 \supseteq_i K_4$ and $H_2 \supseteq_i P_1 + P_4$ or $3P_1 + P_2$ or*
   (vii) *$H_1 \supseteq_i$ gem and $H_2 \supseteq_i P_1 + 2P_2$.*

As mentioned in Section 1, Theorem 9 does not cover five (non-equivalent) cases (see also Open Problem 2, where these open cases are marked with a *).

Open Problem 1. *Does the class of (H_1, H_2)-free graphs have bounded or unbounded clique-width when:*

(i) *$H_1 = K_3$ and $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$,*
(ii) *$H_1 =$ diamond and $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$,*
(iii) *$H_1 =$ gem and $H_2 = P_2 + P_3$.*

5 | ATOMS OF UNBOUNDED CLIQUE-WIDTH

In this section we show our results for pairs (H_1, H_2), for which the class of (H_1, H_2)-free atoms has unbounded clique-width. We start by giving a number of known and new lemmas, some of which have wider applicability.

Lemma 10 (Dabrowski and Paulusma [36]). *For $m \geq 0$ and $n > m + 1$ the clique-width of a graph G is at least $\left\lfloor \frac{n-1}{m+1} \right\rfloor + 1$ if $V(G)$ has a partition into sets $V_{i,j}$ $(i, j \in \{0, ..., n\})$ with the following properties:*

1. *$|V_{i,0}| \leq 1$ for all $i \in \{1, ..., n\}$,*
2. *$|V_{0,j}| \leq 1$ for all $j \in \{1, ..., n\}$,*

3. $|V_{i,j}| \geq 1$ *for all* $i, j \in \{1, ..., n\}$,
4. $G[\cup_{j=0}^{n} V_{i,j}]$ *is connected for all* $i \in \{1, ..., n\}$,
5. $G[\cup_{i=0}^{n} V_{i,j}]$ *is connected for all* $j \in \{1, ..., n\}$,
6. *for* $i, j, k \in \{1, ..., n\}$, *if a vertex of* $V_{k,0}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $i \leq k$,
7. *for* $i, j, k \in \{1, ..., n\}$, *if a vertex of* $V_{0,k}$ *is adjacent to a vertex of* $V_{i,j}$ *then* $j \leq k$ *and*
8. *for* $i, j, k, \ell \in \{1, ..., n\}$, *if a vertex of* $V_{i,j}$ *is adjacent to a vertex of* $V_{k,\ell}$ *then* $|k - i| \leq m$ *and* $|\ell - j| \leq m$.

The next lemma concerns walls. We do not formally define the wall, but instead we refer to Figure 5, in which three examples of walls of different heights are depicted; see, for example, [22] for a formal definition.

Lemma 11 (Lozin and Rautenbach [55]). *For any constant* $k \geq 0$, *the class of* k*-subdivided walls has unbounded clique-width.*

Lemma 12. *Let* H_1, H_2 *be graphs. If* $H_1, H_2 \notin \mathcal{S}$ *or* $\overline{H_1}, \overline{H_2} \notin \mathcal{S}$ *then the class of* (H_1, H_2)*-free atoms has unbounded clique-width.*

Proof. Let $k = \max(|V(H_1)|, |V(H_2)|)$. Let H be a k-subdivided wall of height at least 2 (see Figure 5). Note that the clique-width of k-subdivided walls is unbounded by Lemma 11. By Fact 2, it follows that the clique-width of graphs of the form $\overline{H}$ is also unbounded.

We claim that if $H_1, H_2 \notin \mathcal{S}$ then H is (H_1, H_2)-free. Let $i \in \{1, 2\}$. It is easy to verify that if H_i contains a cycle, then H is H_i-free (due to the choice of k). Similarly, if H_i contains an induced tree with two vertices of degree at least 3 or a vertex of degree at least 4, then H is H_i-free. Therefore, if H_i is an induced subgraph of H, then H_i is a forest and every component of H_i must be a tree in which at most one vertex has degree 3 and all other vertices have degree at most 2. In other words, if H_i is an induced subgraph of H, then $H_i \in \mathcal{S}$. We conclude that if $H_1, H_2 \notin \mathcal{S}$, then H is (H_1, H_2)-free. This also implies that if $\overline{H_1}, \overline{H_2} \notin \mathcal{S}$, then $\overline{H}$ is (H_1, H_2)-free.

It remains to show that H and $\overline{H}$ are atoms. Indeed, H is a bipartite graph, so every clique cut-set consists of at most two vertices; it is easy to verify that there is no vertex whose removal disconnects H and no edge such that removing both of its end-vertices disconnects H. Therefore H is indeed an atom.

Now, $\overline{H}$ is a co-bipartite graph, so it can be partitioned into two cliques A and B. Note that $|A|, |B| > 12$ by construction. Suppose, for contradiction, that X is a clique cut-set in $\overline{H}$. Let $Y = V(\overline{H}) \setminus X$ and note that $\overline{H}[Y]$ is disconnected, so it contains two vertices a, b that are non-adjacent. Since A is a clique and B is a clique, we may assume $a \in A$ and $b \in B$. Now Y cannot contain vertices $a' \in A$, $b' \in B$ that are adjacent in $\overline{H}$, as in that

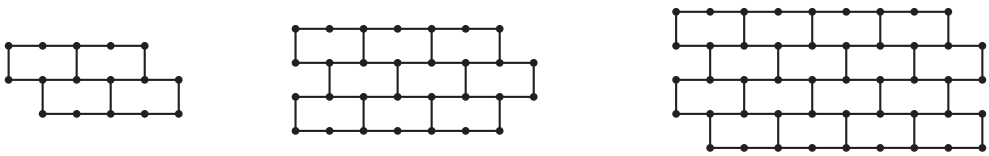

FIGURE 5 Walls of height 2, 3 and 4, respectively.

case $\{a', b'\}$ would dominate $\overline{H}$, contradicting the assumption that $\overline{H}[Y]$ is disconnected. In H every vertex has either two or three neighbours, so in $\overline{H}$ every vertex has either two or three non-neighbours. Since $a \in A \cap Y$, there can be at most three vertices in $B \cap Y$ and similarly, there can be at most three vertices in $A \cap Y$. Since every vertex in $B \cap Y$ has at most three non-neighbours in A, it follows that at most nine vertices of A have non-neighbours in $B \cap Y$. Since $|A| > 12 \geq 9 + |A \cap Y|$, there must be a vertex in $a' \in A \setminus Y = A \cap X$ that has no non-neighbours in $B \cap Y$ and therefore has a non-neighbour $b' \in B \setminus Y = B \cup X$. This contradicts the fact that X is a clique in $\overline{H}$. Therefore $\overline{H}$ is indeed an atom. □

Lemma 13. *Let $\mathcal{H}$ be a set of graphs such that no graph in $\mathcal{H}$ contains a pair of vertices that are false twins. Then the class of $\mathcal{H}$-free atoms has bounded clique-width if and only if the class of $\mathcal{H}$-free graphs does.*

Proof. Clearly, if the class of $\mathcal{H}$-free graphs has bounded clique-width, then the class of $\mathcal{H}$-free atoms does. Now suppose that the class of $\mathcal{H}$-free graphs has unbounded clique-width. Let $\mathcal{F}$ be the class of connected $\mathcal{H}$-free graphs on at least two vertices. Since the clique-width of a graph is equal to the maximum of the clique-widths of its components, it follows that $\mathcal{F}$ has unbounded clique-width. For every graph $F \in \mathcal{F}$, we construct the graph F', which has vertex set $V(F') = \{v, v' | v \in V(F)\}$ and edge set $E(F') = \{uv, uv', u'v, u'v' | uv \in E(F)\}$. So, for every $v \in V(F)$ we have introduced a new vertex v' such that v and v' are false twins in F'. Let $\mathcal{F}'$ be the set of such graphs F'. Since for every $F \in \mathcal{F}$, the graph F' contains F as an induced subgraph, it follows that $\mathcal{F}'$ has unbounded clique-width.

We claim that every graph in $\mathcal{F}'$ is an atom. Indeed, suppose, for contradiction, that X is a clique cut-set of a graph $F' \in \mathcal{F}'$. Since for every $v \in V(F)$, v is non-adjacent to v' in F', it follows that at most one of v and v' is in X. Since v and v' are false twins in F' we may replace all vertices $v \in X \cap V(F)$ by their false twins v' and the resulting set X' will still be a clique cut-set. By construction, the graph F is connected and every vertex in $V(F') \setminus X'$ has a neighbour in $V(F)$ in the graph $F' \setminus X'$. Therefore $F' \setminus X'$ is connected, a contradiction. It follows that every graph in $\mathcal{F}'$ is indeed an atom.

It remains to show that the graphs in $\mathcal{F}'$ are $\mathcal{H}$-free. Indeed, suppose, for contradiction, that $H \in \mathcal{H}$ is an induced subgraph of $F' \in \mathcal{F}'$. Since for every $v \in V(F)$, the vertices v and v' are false twins in F', and H does not have a pair of false twins, it follows that at most one of v and v' is in the induced copy of H found in F'. Furthermore, if v' is in this induced copy, then we can replace it by v. Thus we find that there is an induced copy of H in F' all of whose vertices lie in $V(F)$. Therefore H is an induced subgraph of F. This is a contradiction as $F \in \mathcal{F}$ and the graphs in $\mathcal{F}$ are $\mathcal{H}$-free. We have therefore shown that the graphs in $\mathcal{F}'$ are $\mathcal{H}$-free atoms and that $\mathcal{F}'$ has unbounded clique-width. This completes the proof. □

Observe that the condition in the following lemma holds if and only if for every graph $H \in \mathcal{H}$, the graph $\overline{H}$ does not have a component isomorphic to P_1 or P_2.

Lemma 14. *Let $\mathcal{H}$ be a set of graphs such that no graph in $\mathcal{H}$ contains a dominating vertex and no graph in $\mathcal{H}$ contains a pair of non-adjacent vertices that are complete to the*

remainder of the graph. Then the class of $\mathcal{H}$-free atoms has bounded clique-width if and only if the class of $\mathcal{H}$-free graphs does.

Proof. Clearly, if the class of $\mathcal{H}$-free graphs has bounded clique-width, then the class of $\mathcal{H}$-free atoms does. Now suppose that the class of $\mathcal{H}$-free graphs has unbounded clique-width. Let $\mathcal{F}$ be the class of $\mathcal{H}$-free graphs that contain at least one non-edge. Since complete graphs have clique-width at most 2 and the class of $\mathcal{H}$-free graphs has unbounded clique-width, it follows that $\mathcal{F}$ has unbounded clique-width. For every graph $F \in \mathcal{F}$, we construct the graph F' by adding two new vertices x, x' and adding edges to make $\{x, x'\}$ complete to the remainder of the graph (note that x is non-adjacent to x' in F'). Let $\mathcal{F}'$ be the set of such graphs F'. Since for every $F \in \mathcal{F}$, the graph F' contains F as an induced subgraph, it follows that $\mathcal{F}'$ has unbounded clique-width.

We claim that every graph in $\mathcal{F}'$ is an atom. Suppose, for contradiction, that $F' \in \mathcal{F}'$ has a clique cut-set X. Since x and x' are non-adjacent, it follows that either x or x' are not in X; since x and x' are false twins, we may assume $x \notin X$. Since F is not a complete graph, there must be a vertex $y \in V(F) \setminus X$. Since x is complete to $V(F) \setminus X$ in $F' \setminus X$, every vertex of $V(F) \setminus X$ is in the same component of $F' \setminus X$ as x. Since y is complete to $\{x, x'\} \setminus X$ in $F' \setminus X$, every vertex of $\{x, x'\} \setminus X$ is in the same component of $F' \setminus X$ as y. Therefore $F' \setminus X$ is connected, a contradiction. It follows that every graph in $\mathcal{F}'$ is indeed an atom.

It remains to show that the graphs in $\mathcal{F}'$ are $\mathcal{H}$-free. Indeed, suppose, for contradiction, that $H \in \mathcal{H}$ is an induced subgraph of $F' \in \mathcal{F}'$. Since H does not contain a pair of non-adjacent vertices that are complete to the rest of the graph, this induced copy of H in F' cannot contain both x and x'. Since H does not have a dominating vertex, the induced copy of H in F' cannot contain exactly one of x and x'. Therefore the induced copy of H in F' must consist of only vertices in $V(F)$. Therefore H is an induced subgraph of F. This is a contradiction as $F \in \mathcal{F}$ and the graphs in $\mathcal{F}$ are $\mathcal{H}$-free. We have therefore shown that the graphs in $\mathcal{F}'$ are $\mathcal{H}$-free atoms and that $\mathcal{F}'$ has unbounded clique-width. This completes the proof. □

Lemma 15. *The class of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free atoms and the class of $(2P_2, K_3 + P_1, 4P_1, 2P_1 + P_2)$-free atoms have unbounded clique-width (see Figure 6 for illustrations of the forbidden induced subgraphs).*

Proof. Brandstädt et al. [10, Theorem 10(ii)] constructed a family of graphs H_n that have unbounded clique-width and are $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free. The graph H_n is constructed from the 1-subdivided $n \times n$ grid by adding new edges incident to the vertices added by the subdivision as follows: in each cell of the subdivided grid, the left vertex added by the subdivision is made adjacent to the top one, and the bottom vertex added by the subdivision is made adjacent to the right one (see also Figure 7 or see [10, Section 6.2] for a formal definition). However, the graph H_n has clique cut-sets, so it is not an atom. On the other hand, since the class of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free graphs has unbounded clique-width, Fact 2 implies that the class of $(2P_2, K_3 + P_1, 4P_1, 2P_1 + P_2)$-free graphs has unbounded clique-width. We observe that every graph in $\{2P_2, K_3 + P_1, 4P_1, 2P_1 + P_2\}$ has no dominating vertex and no two non-adjacent vertices that are complete to the remainder of the graph. Therefore, by Lemma 14, the class of $(2P_2, K_3 + P_1, 4P_1, 2P_1 + P_2)$-free atoms has unbounded clique-width.

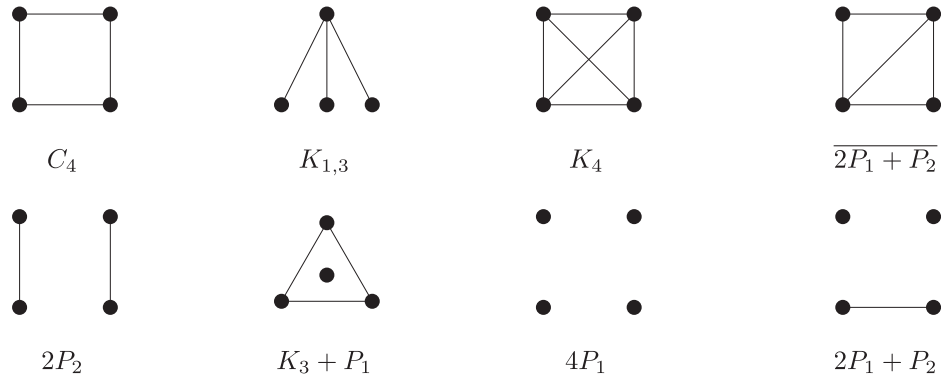

FIGURE 6 The forbidden induced subgraphs for the classes of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free graphs and $(2P_2, K_3 + P_1, 4P_1, 2P_1 + P_2)$-free graphs mentioned in Lemma 15.

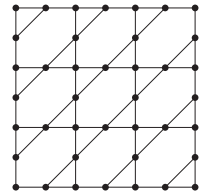

FIGURE 7 The graph H_n from the proof of Lemma 15 ($n = 4$ shown).

We now prove that the class of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free atoms has unbounded clique-width. Consider a wall of height $k \geq 2$ and let J_k be its line graph. It is easy to verify that for every k, the graph H_n contains J_k as an induced subgraph if n is sufficiently large. Similarly, for every n, the graph J_k contains H_n as an induced subgraph if k is sufficiently large. Therefore, by [10, Theorem 10(ii)], the graph J_k is $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free and this family of graphs has unbounded clique-width (the former can also be seen by inspection and latter can also be seen by using Lemma 10). Every clique in J_k contains at most three vertices and it is easy to verify that J_k does not contain a clique cut-set on at most three vertices, so J_k is an atom. This completes the proof. □

Lemma 16. *The class of* $(\overline{2P_1 + P_2}, 5P_1)$*-free atoms and the class of* $(2P_1 + P_2, K_5)$*-free atoms has unbounded clique-width* (*see Figure 8 for illustrations of the forbidden induced subgraphs*).

Proof. We use the construction from [29], which was used to show that $(\overline{2P_1 + P_2}, 5P_1)$-free graphs have unbounded clique-width. Consider a wall of height $2n + 1$ for some $n \geq 2$. Colour the vertices on the top row with colours 1, 2, 3, 4, 1, 2, 3, 4, ... and on the next row with colours 3, 4, 1, 2, 3, 4, 1, 2, ..., then alternate these colourings on the following rows, so that no vertex has two neighbours that have the same colour (see also Figure 9). Add edges to make each colour class into a clique and let G_n be the resulting graph. Now G_n is $(\overline{2P_1 + P_2}, 5P_1)$-free and the family of such graphs had unbounded clique-width [29] (the former can also be seen by inspection and the latter follows from

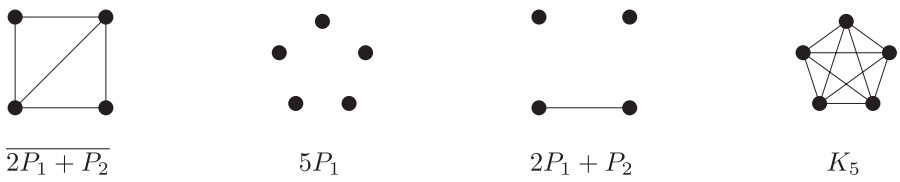

FIGURE 8 The forbidden induced subgraphs for the classes of $(\overline{2P_1 + P_2}, 5P_1)$-free graphs and $(2P_1 + P_2, K_5)$-free graphs mentioned in Lemma 16.

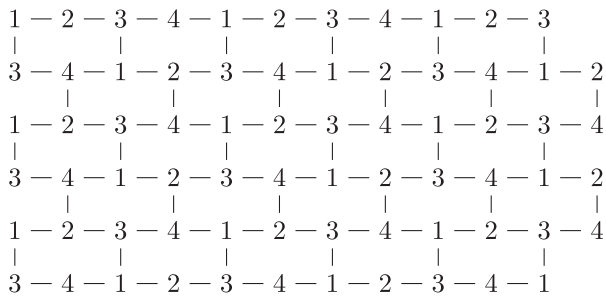

FIGURE 9 The graph G_n from the proof of Lemma 16 ($n = 2$ shown).

combining Lemma 10 with Fact 2). By Fact 2, the family of graphs $\overline{G_n}$ also has unbounded clique-width.

It remains to show that G_n and $\overline{G_n}$ are atoms. Let V_i be the set of vertices with colour i. Suppose, for contradiction, that G_n has a clique cut-set X. If $X \subseteq V_i$ for some $i \in \{1, 2, 3, 4\}$, then all vertices of $G_n \setminus V_i$ are in the same component of $G_n \setminus X$. Since every vertex in V_i has at least one neighbour outside of V_i, it follows that every vertex of $G_n \setminus X$ is in the same component of $G_n \setminus X$ in this case, a contradiction. We may therefore assume that X contains vertices in at least two sets V_i. By construction, each vertex in a set V_i has at most one neighbour in each V_j for $j \in \{1, 2, 3, 4\} \setminus \{i\}$. Therefore X has at most one vertex in each V_i. Therefore, there must be a vertex in $V_1 \setminus X$ that has a neighbour in each of $V_2 \setminus X$, $V_3 \setminus X$ and $V_4 \setminus X$. Since each set V_i is a clique, it follows that $G_n \setminus X$ is connected. This contradiction implies that G_n is indeed an atom. Now suppose, for contradiction, that $\overline{G_n}$ has a clique cut-set X. Since $V_1, ..., V_4$ are independent sets in $\overline{G_n}$, X contains at most one vertex of any V_i. Since in $\overline{G_n}$ every vertex of V_i has at most one non-neighbour in each V_j for $j \in \{1, 2, 3, 4\} \setminus \{i\}$, it follows that $\overline{G_n} \setminus X$ must be connected. This contradiction implies that $\overline{G_n}$ is indeed an atom. □

Lemma 17. *The class of $(\overline{2P_1 + P_2}, P_2 + P_4, P_1 + P_6)$-free and the class of $(2P_1 + P_2, \overline{P_2 + P_4}, \overline{P_1 + P_6})$-free atoms have unbounded clique-width (see Figure 10 for illustrations of the forbidden induced subgraphs).*

Proof. We modify the construction of the graph G_n, which was used in [30] to prove that $(\overline{2P_1 + P_2}, P_2 + P_4)$-free graphs have unbounded clique-width. Consider a wall of height $n \geq 2$. A wall is a bipartite graph; let A and C be the two sets in its bipartition. Consider a

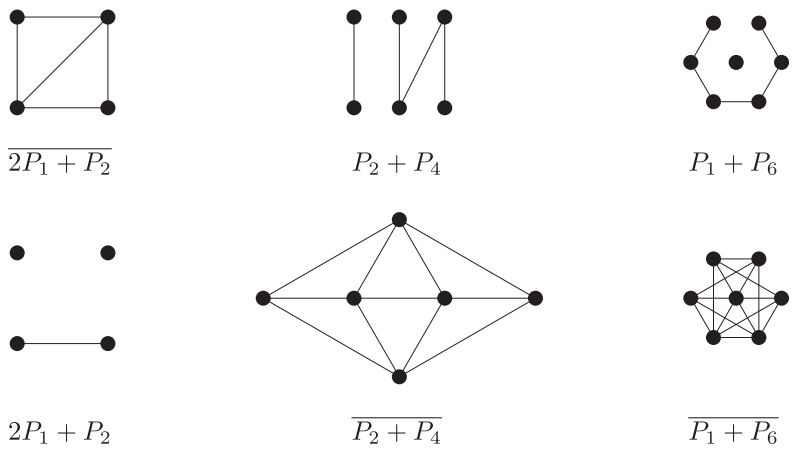

FIGURE 10 The forbidden induced subgraphs for the classes of $(\overline{2P_1 + P_2}, P_2 + P_4, P_1 + P_6)$-free graphs and $(2P_1 + P_2, \overline{P_2 + P_4}, \overline{P_1 + P_6})$-free graphs mentioned in Lemma 17.

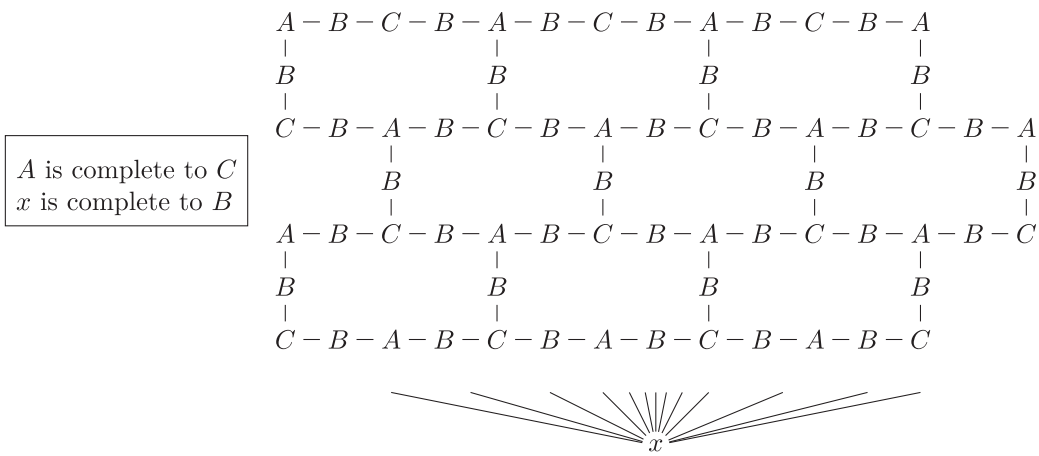

FIGURE 11 The graph H_n from the proof of Lemma 17 ($n = 3$ shown). Vertices are denoted A, B or C if they are in the corresponding set.

1-subdivision of the wall and let B be the set of vertices introduced by the subdivision. Finally, we add edges to make A complete to C. Let G_n be the resulting graph. Then G_n is $(\overline{2P_1 + P_2}, P_2 + P_4)$-free and the family of such graphs G_n has unbounded clique-width [30] (the former also follows by inspection and the latter follows from combining Lemma 11 with Fact 3). Let H_n be the graph obtained from G_n by adding a vertex x complete to B, see Figure 11. Since H_n contains G_n as an induced subgraph, the family of graphs H_n has unbounded clique-width.

Now G_n is $(\overline{2P_1 + P_2}, P_2 + P_4)$-free, so if H_n contains an induced copy of $\overline{2P_1 + P_2}$ or $P_2 + P_4$, then one of its vertices must be x. The neighbourhood of x in H_n is B, which is an independent set. Every vertex of $\overline{2P_1 + P_2}$ has two neighbours that are adjacent to each other, so H_n is $\overline{2P_1 + P_2}$-free. Suppose, for contradiction, that H_n contains an induced $P_2 + P_4$, say with vertex set Y. As observed above, $x \in Y$. Now x has either one or two neighbours in $H_n[Y]$. If x has one neighbour in $H_n[Y]$, then this neighbour must be in B,

and then there can be no other vertices in $B \cap Y$, so $|(A \cup C) \cap Y| = 4$, but $H_n[A \cup C]$ is a complete bipartite graph, so $H_n[(A \cup C) \cap Y]$ is isomorphic to $C_4, K_{1,3}$ or $4P_1$, contradicting the fact that $P_2 + P_4$ is $(C_4, K_{1,3}, 4P_1)$-free. Therefore x has two neighbours in $H_n[Y]$, so it is in the P_4 component of $H_n[Y]$. These two neighbours of x must be in B, so the P_4 component containing x must contain a vertex of A or C and the remaining P_2 component of $H_n[Y]$ must lie in $A \cup C$. Since $H_n[A \cup C]$ is a complete bipartite graph, it follows that there is an edge between the P_2 component and the P_4 component. This contradiction implies that H_n is indeed $(P_2 + P_4)$-free.

Suppose, for contradiction, that H_n contains an induced $P_1 + P_6$, say with vertex set $\{v\} \cup Y$ where v is the vertex in the P_1 component and Y is the vertex set of the P_6 component. If there are three vertices in $B \cap Y$ and $x \in Y$, then $H_n[Y]$ contains an induced $K_{1,3}$, a contradiction. Note that P_6 has two vertices of degree 1 and four vertices of degree 2, but every vertex in B has only two neighbours apart from x: one in each of A and C. Therefore, if there are three vertices in $B \cap Y$, then one of these vertices b must have neighbours $a \in A \cap Y$ and $c \in C \cap Y$, in which case $H_n[a, b, c]$ is a K_3, a contradiction. We conclude that there are at most two vertices in $B \cap Y$. If $x \notin Y$ then there are at least four vertices in $(A \cup C) \cap Y$, contradicting the fact that P_6 is $(C_4, K_{1,3}, 4P_1)$-free. Therefore, $x \in Y$, and so $v \in A \cup C$ (say A) because x is complete to B. Since $\{x\} \cup A$ is independent and P_6 is $4P_1$-free, it follows that $|A \cap Y| \leq 2$, and so there is at least one vertex $c \in C \cap Y$. But c is complete to A, so it is adjacent to v, a contradiction. This contradiction implies that H_n is indeed $(P_1 + P_6)$-free.

It remains to show that H_n and $\overline{H_n}$ are atoms. Suppose, for contradiction, that H_n contains a clique cut-set X. If $x \in X$ then X contains at most one additional vertex, which must lie in B; it is easy to verify that $H_n \setminus X$ is connected in this case. We may therefore assume that $x \notin X$. Since A, B and C are independent sets, X contains at most one vertex in each of these sets. Since $x \notin X$, and x is complete to B, all vertices of $B \setminus X$ are in the same component of $H_n \setminus X$. Since every vertex of B has a neighbour in A and C, there must be a vertex in $B \setminus X$ that has neighbours in both $A \setminus X$ and $C \setminus X$. Since A is complete to C, it follows that every vertex in $V(H_n) \setminus X$ is in the same component of $H_n \setminus X$. This contradiction implies that H_n is indeed an atom. Now suppose, for contradiction, that $\overline{H_n}$ contains a clique cut-set X. Since A is anti-complete to C in $\overline{H_n}$, X cannot contain vertices in both A and C; by symmetry we may assume that X does not contain any vertices of C. Now C is a clique and, since every vertex of B has a neighbour in C, every vertex in $(B \cup C) \setminus X$ is in the same component of $\overline{H_n} \setminus X$. If $x \notin X$, then every vertex in $A \setminus X$ is adjacent to x, which is complete to C, so every vertex in $V(\overline{H_n}) \setminus X$ is in the same component of $\overline{H_n} \setminus X$, a contradiction. We may therefore assume that $x \in X$. Then no vertex of B is in X, so $X \subseteq A \cup \{x\}$. Since every vertex of A has a neighbour in B, it follows that every vertex of A has a neighbour in $B \setminus X = B$. Therefore every vertex of $V(\overline{H_n}) \setminus X$ is in the same component of $\overline{H_n} \setminus X$. This contradiction implies that $\overline{H_n}$ is indeed an atom. □

Lemma 18. *The class of* $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2)$*-free bipartite atoms and the class of* $(\overline{2P_1 + 2P_2}, \overline{2P_1 + P_4}, \overline{4P_1 + P_2}, \overline{3P_2})$*-free co-bipartite atoms have unbounded clique-width* (*see Figure 12 for illustrations of the forbidden induced subgraphs*).

Proof. Let H_n be a 1-subdivided wall of height $n \geq 2$ and note that the class of such graphs has unbounded clique-width by Lemma 11. Note that H_n is connected and bipartite, say with parts V_1 and V_2. Let G_n be the graph obtained from H_n by applying a

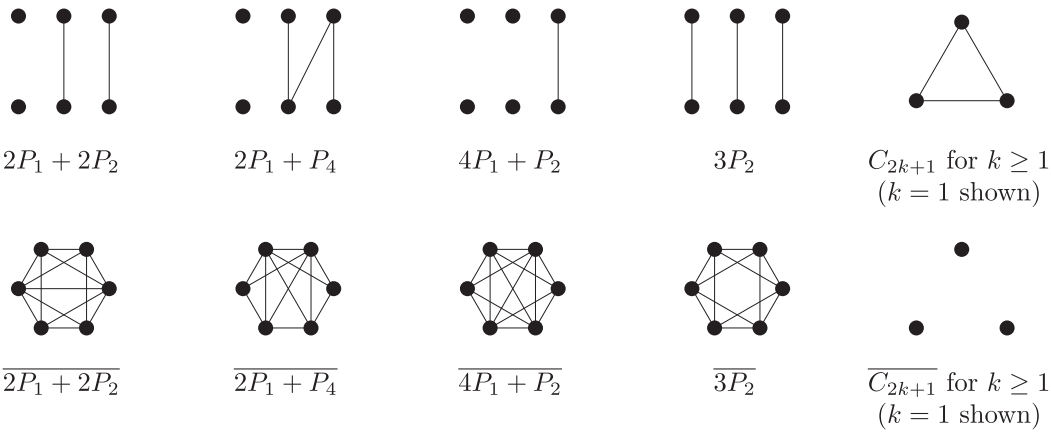

FIGURE 12 The forbidden induced subgraphs for the classes of $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2)$-free bipartite graphs and $(\overline{2P_1 + 2P_2}, \overline{2P_1 + P_4}, \overline{4P_1 + P_2}, \overline{3P_2})$-free co-bipartite graphs mentioned in Lemma 18.

bipartite complementation between V_1 and V_2. By Fact 3, the family of such graphs G_n also has unbounded clique-width and by Fact 2 the family of graphs $\overline{G_n}$ also has unbounded clique-width (see also Figure 13). Now G_n is a $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2)$-free bipartite graph (by inspection, or see e.g. [35, 56]).

It remains to show that G_n and $\overline{G_n}$ are atoms. Suppose, for contradiction, that X is a clique cut-set of G_n. Since V_1 and V_2 are independent, X contains at most one vertex from each of these sets. Since every vertex of V_1 has at most three non-neighbours in V_2 and vice versa, it follows that $G_n \setminus X$ is connected. This contradiction shows that G_n is indeed an atom. Now suppose, for contradiction, that X is a clique cut-set of $\overline{G_n}$. If X is a subset of either V_1 or V_2, say V_1, then every vertex of V_2 lies outside X. Since every vertex of V_1 has a neighbour in V_2, it follows that $\overline{G_n} \setminus X$ is connected in this case. Therefore X must contain at least one vertex of V_1 and at least one vertex of V_2. In $\overline{G_n}$, every vertex in V_1 has at most three neighbours in V_2 and vice versa, so X contains at most three vertices from V_1 and at most three vertices from V_2. In $\overline{G_n}$, every vertex in V_1 has at most three neighbours in V_2, and every vertex in V_2 has at least one neighbour in V_1, and $|V_2| > 12 = 9 + 3$. Hence, there must be a vertex in $V_2 \setminus X$ with a neighbour in $V_1 \setminus X$. Since V_1 and V_2 are cliques, it follows that $\overline{G_n} \setminus X$ is connected. This completes the proof. □

Lemma 19. *The class of $2P_3$-free bipartite atoms and the class of $\overline{2P_3}$-free co-bipartite atoms have unbounded clique-width (see Figure 14 for illustrations of the forbidden induced subgraphs).*

Proof. We adapt the construction of the graph G_n, which was used by Lozin and Volz [56] to show that the class of $2P_3$-free bipartite graphs has unbounded clique-width. For $n \geq 3$, construct the graph G_n as follows. Let the vertex set of G_n be $\{v_{i,j} \mid i \in \{0, ..., n\}, j \in \{1, ..., n\}\} \cup \{w_{i,j} \mid i \in \{1, ..., n\}, j \in \{0, ..., n\}\}$. For $i, j, k \in \{1, ..., n\}$, add an edge between $v_{i,j}$ and $w_{k,0}$ if $k \geq i$ and add an edge between $w_{i,j}$ and $v_{0,k}$ if $k \geq j$. For each $i, j \in \{1, ..., n\}$, add an edge between $v_{i,j}$ and $w_{i,j}$ and an edge between $v_{0,j}$ and $w_{i,0}$. Let G_n be the resulting graph. Then G_n is a $2P_3$-free bipartite graph and the family of such graphs has unbounded clique-width [56] (the former can also be seen by inspection

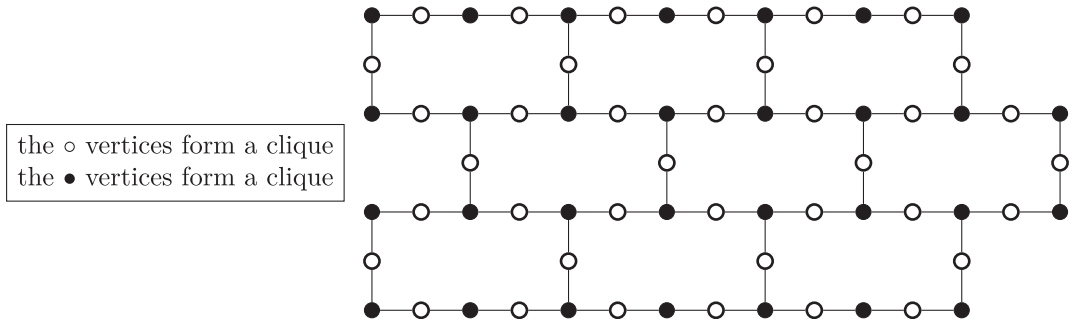

FIGURE 13 The graph $\overline{G_n}$ from the proof of Lemma 18 ($n = 3$ shown).

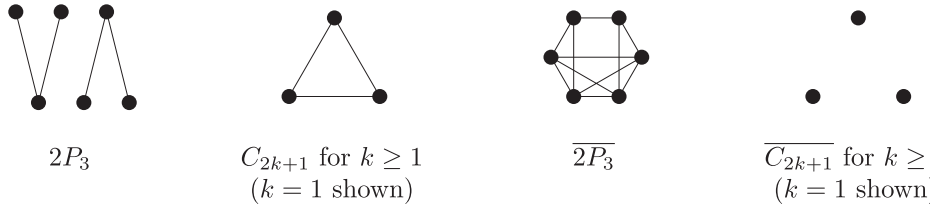

FIGURE 14 The forbidden induced subgraphs for the classes of $2P_3$-free bipartite graphs and $\overline{2P_3}$-free co-bipartite graphs mentioned in Lemma 19.

and the latter follows from Lemma 10). Therefore the class of $2P_3$-free bipartite graphs has unbounded clique-width, and so Fact 2 implies that the class of $\overline{2P_3}$-free co-bipartite graphs has unbounded clique-width.

We observe that every graph in $\{\overline{2P_3}, \overline{C_3}, \overline{C_5}, \overline{C_7}, ...\}$ has no dominating vertex and no two non-adjacent vertices that are complete to the remainder of the graph. Therefore, by Lemma 14, it follows that the class of $\overline{2P_3}$-free co-bipartite atoms has unbounded clique-width. Now let H_n be the graph obtained from G_n by deleting $v_{n,n}$ and $w_{n,n}$ (see Figure 15) and note that H_n is a $2P_3$-free bipartite graph. By Fact 1, the family of graphs H_n has unbounded clique-width.

We now prove that the class of $2P_3$-free bipartite atoms has unbounded clique-width. Let V be the set of vertices in the $2P_3$-free bipartite graph H_n of the form $v_{i,j}$ and let W be the set of vertices in H_n of the form $w_{i,j}$ and note that V and W are independent sets. Suppose, for contradiction, that H_n has a clique cut-set X. Since H_n is bipartite, every clique cut-set in H_n contains at most one vertex from each part, so $|X| \leq 2$. If X does not contain $v_{0,n}$, then every vertex in $W \setminus X$ is in the same component of $H_n \setminus X$. Since every vertex in V has at least two neighbours in W, and at most one vertex of W is in X, it follows that every vertex of $V \setminus X$ is in the same component of $H_n \setminus X$, and so $H_n \setminus X$ is connected. This contradiction implies that $v_{0,n} \in X$. By symmetry, $w_{n,0} \in X$. By construction, deleting $v_{0,n}$ and $w_{n,0}$ does not disconnect H_n, so H_n is indeed an atom. This completes the proof. □

Lemma 20. *The class of* $(K_4, P_1 + P_4)$*-free atoms and the class of* $(4P_1, \overline{P_1 + P_4})$*-free atoms have unbounded clique-width* (*see Figure 16 for illustrations of the forbidden induced subgraphs*).

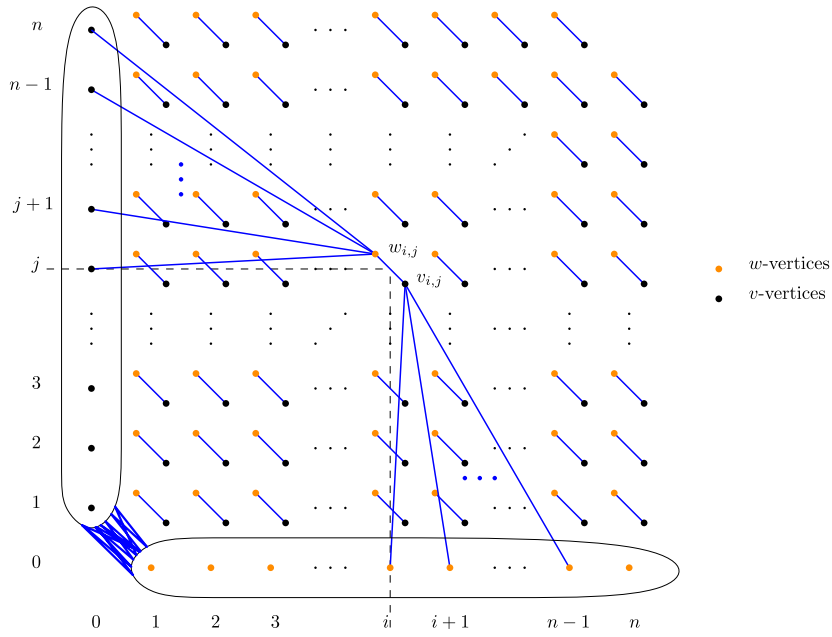

FIGURE 15 The graph H_n from the proof of Lemma 19. For clarity, the edges incident to $v_{i,j}$ and $w_{i,j}$ when $i, j \in \{1, ..., n\}$ are depicted for only one such pair of vertices. [Color figure can be viewed at wileyonlinelibrary.com]

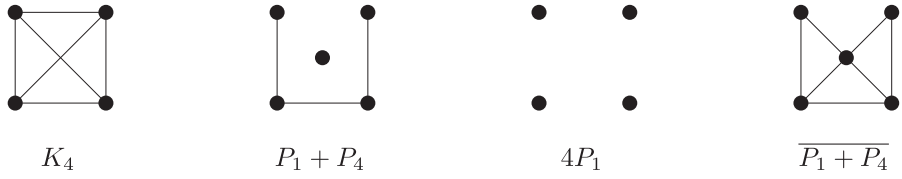

FIGURE 16 The forbidden induced subgraphs for the classes of $(K_4, P_1 + P_4)$-free graphs and $(4P_1, \overline{P_1 + P_4})$-free graphs mentioned in Lemma 20.

Proof. For this proof we use a construction that is implicit in the proof in [53, Theorem 3] that GRAPH ISOMORPHISM is GI-complete on the class of $(K_4, P_1 + P_4)$-free graphs; we give an explicit construction. Consider a 1-subdivided wall of height $n \geq 2$. This graph is bipartite; let P and Q be the two parts of its bipartition with the vertices in Q being the vertices added by the subdivision. Consider a 3-subdivision of this 1-subdivided wall (so the resulting graph is a 7-subdivided wall). Partition the vertices introduced by this 3-subdivision as follows: let A be the set of vertices that are adjacent to vertices of P, let C be the set of vertices that are adjacent to vertices of Q, and let B be the set of remaining vertices introduced by the 3-subdivision (which have a neighbour in both A and C. Apply complementations to $P \cup C$, $Q \cup A$, and B (these sets will become cliques). Let H_n be the resulting graph (see also Figure 17) and note that $\overline{H_n}$ is $(K_4, P_1 + P_4)$-free and that the family of such graphs has unbounded clique-width [63] (the former statement can be seen by inspection and the latter can be seen by combining Lemma 11 and Fact 2). Therefore the class of $(K_4, P_1 + P_4)$-free graphs has unbounded clique-width.

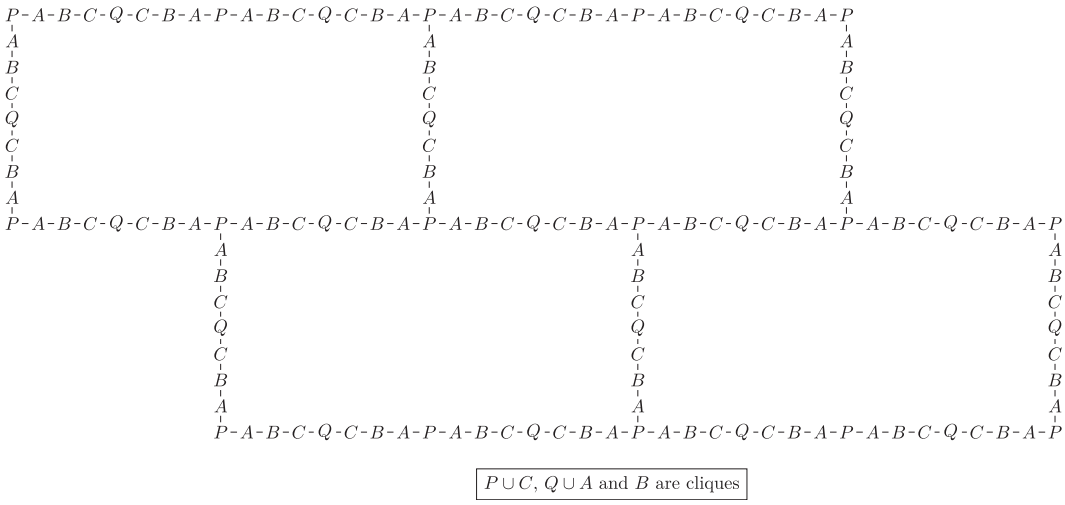

FIGURE 17 The graph H_n from the proof of Lemma 20 ($n = 2$ shown).

We observe that neither K_4 nor $P_1 + P_4$ contains a pair of false twins. Therefore, by Lemma 13, the class of $(K_4, P_1 + P_4)$-free atoms has unbounded clique-width.

We now prove that the class of $(4P_1, \overline{P_1 + P_4})$-free atoms has unbounded clique-width. As $\overline{H_n}$ is $(K_4, P_1 + P_4)$-free, it follows that H_n is $(4P_1, \overline{P_1 + P_4})$-free. By Fact 2, it follows that the family of graphs H_n also has unbounded clique-width. It remains to show that H_n is an atom. Suppose, for contradiction, that H_n has a clique cut-set X. Recall that $P \cup C$, $Q \cup A$, and B are cliques. As A and C are anti-complete to each other, it follows that X contains vertices from at most one of A or C. Similarly, since P, Q and B are pairwise anti-complete, it follows that X contains vertices from at most one of P, Q or B. Note that every vertex from $P \cup Q \cup B$ has a neighbour in A and in C. Since A and C are cliques and X contains vertices in at most one of these sets, it follows that the vertices in $(P \cup Q \cup B) \setminus X$ all lie in the same component of $H_n \setminus X$. Similarly, every vertex of $A \cup C$ has a neighbour in both P and Q. Since X contains vertices from at most one of P or Q, it follows that the vertices in $(A \cup C) \setminus X$ are in the same component of $H_n \setminus X$ as the vertices of $(P \cup Q \cup B) \setminus X$. Therefore $H_n \setminus X$ is connected. This contradiction implies that H_n is indeed an atom. This completes the proof. □

Lemma 21. *The class of* $(4P_1, \overline{3P_1 + P_2})$*-free atoms and the class of* $(\overline{4P_1}, 3P_1 + P_2)$*-free atoms have unbounded clique-width* (*see Figure* 18 *for illustrations of the forbidden induced subgraphs*).

Proof. We use the construction of [29] for proving that the class of $(4P_1, \overline{3P_1 + P_2})$-free graphs has unbounded clique-width. Let $n \geq 7$ and consider an $n \times n$ grid H_n and for $i, j \in \{0, ..., n - 1\}$, let $v_{i,j}$ be the vertex of H_n with x-coordinate i and y-coordinate j. For $k \in \{0, 1, 2\}$, let $V_k = \{v_{i,j} \mid i + j \equiv k \bmod 3\}$ (see also Figure 19 for a depiction of this 3-colouring). Apply a complementation to each V_k. Let G_n be the resulting graph. The resulting graph G_n is $(4P_1, \overline{3P_1 + P_2})$-free and the family of graphs G_n has unbounded clique-width [29] (the first of these statements can also be seen by inspection and the

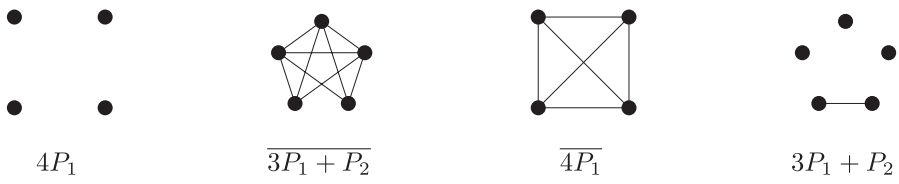

FIGURE 18 The forbidden induced subgraphs for the classes of $(4P_1, \overline{3P_1 + P_2})$-free graphs and $(\overline{4P_1}, 3P_1 + P_2)$-free graphs mentioned in Lemma 21.

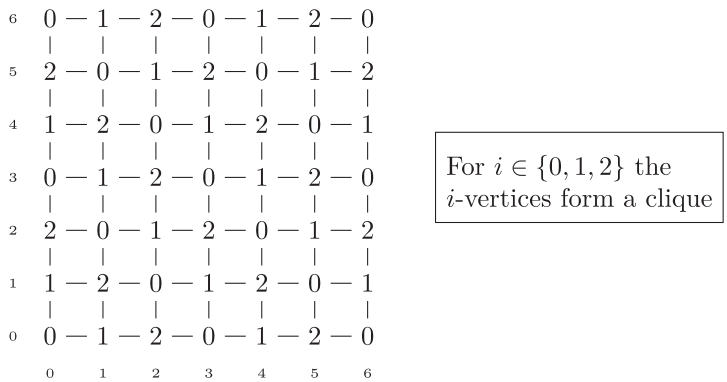

FIGURE 19 The graph G_n from the proof of Lemma 21 ($n = 7$ shown).

latter follows from combining Lemma 10 and Fact 2). By Fact 2, it follows that the family of graphs $\overline{G_n}$ also has unbounded clique-width.

It remains to show that G_n and $\overline{G_n}$ are atoms. Suppose, for contradiction, that G_n has a clique cut-set X. If $X \subseteq V_i$ for some $i \in \{0, 1, 2\}$, then all vertices of $G_n \setminus V_i$ are in the same component of $G_n \setminus X$. Since every vertex in V_i has at least one neighbour outside of V_i, it follows that every vertex of $G_n \setminus X$ is in the same component of $G_n \setminus X$ in this case, a contradiction. We may therefore assume that X contains vertices in at least two sets V_i. By construction, each vertex in a set V_i has at most two neighbours in each V_j for $j \in \{0, 1, 2\} \setminus \{i\}$. Therefore X has at most two vertices in each V_i. Since $n \geq 7$, there must be at least 15 vertices in V_0 that have neighbours in both V_1 and V_2 (see also Figure 19). Since very vertex in $V_1 \cup V_2$ has at most two neighbours in V_0, there must be a vertex in $V_0 \setminus X$ that has a neighbour in both $V_1 \setminus X$ and $V_2 \setminus X$. Since each set V_i is a clique, it follows that $G_n \setminus X$ is connected. This contradiction implies that G_n is indeed an atom. Now suppose, for contradiction, that $\overline{G_n}$ has a clique cut-set X. Since V_0, V_1 and V_2 are independent sets in $\overline{G_n}$, X contains at most one vertex of any V_i. Since every vertex of V_i has at most two non-neighbours in each V_j for $j \in \{0, 1, 2\} \setminus \{i\}$, it follows that $\overline{G_n} \setminus X$ must be connected. This contradiction implies that $\overline{G_n}$ is indeed an atom. □

Lemma 22. *The class of K_4-free co-chordal atoms and the class of $(C_4, 4P_1)$-free atoms have unbounded clique-width (see Figure 20 for illustrations of the forbidden induced subgraphs).*

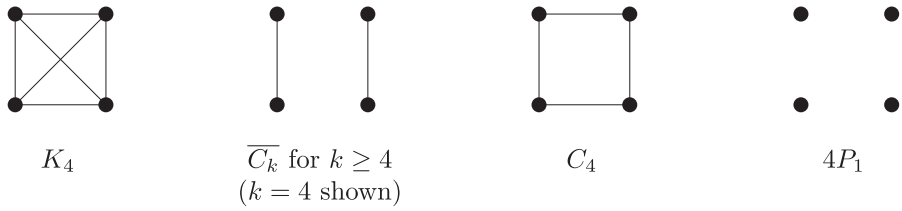

FIGURE 20 The forbidden induced subgraphs for the classes of K_4-free co-chordal graphs and $(C_4, 4P_1)$-free graphs mentioned in Lemma 22.

Proof. In [10, Theorem 11], Brandstädt et al. constructed a family of graphs G_n that are K_4-free co-chordal and have unbounded clique-width. The construction of G_n for $n \geq 3$ is as follows. Let the vertex set of G_n be $\{v_{i,j} \mid i, j \in \{0, ..., n\}, (i, j) \neq (0, 0)\}$. For $i, j, k \in \{1, ..., n\}$, add an edge between $v_{i,j}$ and $v_{k,0}$ if $k \geq i$ and add an edge between $v_{i,j}$ and $v_{0,k}$ if $k \geq j$. For each $i, j \in \{1, ..., n\}$, add an edge between $v_{i,0}$ and $v_{0,j}$. As shown in the proof of [10, Theorem 11], G_n is a K_4-free co-chordal graph and the family of such graphs has unbounded clique-width (the former property can also be seen by inspection and the latter follows from Lemma 10). Therefore the class of K_4-free co-chordal graphs had unbounded clique-width. We observe that neither K_4 nor $\overline{C_r}$ for any $r \geq 4$ contains a pair of false twins. Therefore, by Lemma 13, the class of K_4-free co-chordal atoms has unbounded clique-width.

We now prove that the class of $(C_4, 4P_1)$-free atoms has unbounded clique-width. Observe that the family of graphs of the form $\overline{G_n}$ is $4P_1$-free and chordal, and by Fact 2, it has unbounded clique-width. Now $\overline{G_n}$ is not an atom, since the set of vertices $\{v_{i,j} \mid i, j \in \{1, ..., n\}\}$ is a clique cut-set in $\overline{G_n}$. We construct a graph J_n from $\overline{G_{n+1}}$ as follows (see also Figure 21). Delete the vertices in the set $\{v_{1,i}, v_{i,1} | i \in \{1, ..., n + 1\}$ and add the edge $v_{0,n+1}v_{n+1,0}$. Let J_n be the resulting graph. Now J_n contains $\overline{G_{n-1}}$ as an induced subgraph, so the family of graphs of the form J_n has unbounded clique-width. We claim that J_n is $(C_4, 4P_1)$-free. Note that $J_n \setminus \{v_{0,n+1}\}$ and $J_n \setminus \{v_{n+1,0}\}$ are induced subgraphs of $\overline{G_{n+1}}$, which is $(C_4, 4P_1)$-free. Therefore we only need to verify that there is no induced $4P_1$ or C_4 in J_n that contains both $v_{0,n+1}$ and $v_{n+1,0}$. Since $v_{0,n+1}$ is adjacent to $v_{n+1,0}$, there cannot be an induced $4P_1$ in J_n that contains both these vertices. Now $N(v_{0,n+1}) = \{v_{n+1,0}\} \cup \{v_{0,1}, ..., v_{0,n}\}$ and $N(v_{n+1,0}) = \{v_{0,n+1}\} \cup \{v_{1,0}, ..., v_{n,0}\}$. Since no vertex in $\{v_{0,1}, ..., v_{0,n}\}$ has a neighbour in $\{v_{1,0}, ..., v_{n,0}\}$ in J_n, it follows that J_n is indeed C_4-free.

It remains to show that J_n is an atom. Suppose, for contradiction, that X is a clique cut-set of J_n. First, suppose that $v_{0,n+1}$ is in X. Then $v_{1,0} \notin X$ and $v_{n+1,n+1} \notin X$ as $v_{0,n+1}$ is non-adjacent to these vertices. As $v_{1,0}$ and $v_{n+1,n+1}$ are adjacent and every vertex in $J_n \setminus \{v_{0,n+1}\}$ is adjacent to at least one of these vertices, we find that X is not a clique cut-set. We may therefore assume that $v_{0,n+1} \notin X$ and by symmetry, that $v_{n+1,0} \notin X$. We partition the vertices $v_{i,j}$ in J_n into three sets A, B and C, if $j = 0, i = 0$ or $i, j \neq 0$, respectively, and note that each of these sets is a clique. Since A and B are cliques and the vertices $v_{0,n+1}$ and $v_{n+1,0}$ are adjacent, it follows that all vertices in $(A \cup B) \setminus X$ are in the same component of $J_n \setminus X$. Note that every vertex from C has at least one neighbour in both A and B. However, X cannot contain vertices from both A and B since $A \setminus \{v_{0,n+1}\}$ and $B \setminus \{v_{n+1,0}\}$ are anti-complete. Therefore $J_n \setminus X$ is connected. This contradiction implies that J_n is an atom. □

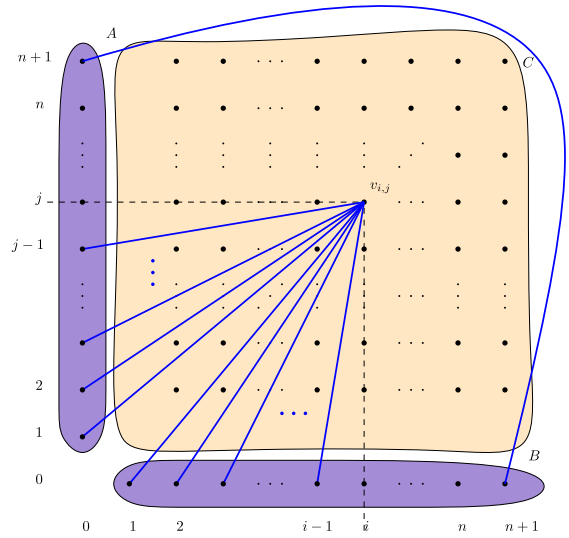

FIGURE 21 The graph J_n from the proof of Lemma 22. The sets A, B and C are cliques. For clarity, the edges between $A \cup B$ and C are depicted for only one vertex in C. [Color figure can be viewed at wileyonlinelibrary.com]

Lemma 23. *The class of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free atoms and the class of $(P_1 + P_4, \overline{P_1 + 2P_2})$-free atoms have unbounded clique-width (see Figure 22 for illustrations of the forbidden induced subgraphs).*

Proof. We use the construction from [7], which was used to show that $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs have unbounded clique-width. We copy this construction below. Let $t \geq 2$ and let G be the $t \times t$ square grid. Let $v_1^G, ..., v_n^G$ be the vertices of G and let $e_1^G, ..., e_m^G$ be the edges of G. We construct a graph $q(G)$ from G as follows (see also Figure 23):

1. Create a complete multi-partite graph with a partition $(A_1^G, ..., A_n^G)$, where $|A_i^G| = d_G(v_i^G)$ for $i \in \{1, ..., n\}$ and let $A^G = \bigcup A_i^G$.
2. Create a complete multi-partite graph with partition $(B_1^G, ..., B_m^G)$, where $|B_i^G| = 2$ for $i \in \{1, ..., m\}$ and let $B^G = \bigcup B_i^G$.
3. Take the disjoint union of the two graphs above, then for each edge $e_i^G = v_{i_1}^G v_{i_2}^G$ in G in turn, add an edge from one vertex of B_i^G to a vertex of $A_{i_1}^G$ and an edge from the other vertex of B_i^G to a vertex of $A_{i_2}^G$. Do this in such a way that the edges added between A^G and B^G form a perfect matching.

In [7] it was shown that the graph $q(G)$ is $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free and that the clique-width of such graphs is unbounded. Therefore the class of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs has unbounded clique-width. We observe that neither $\overline{P_1 + P_4}$ nor $P_1 + 2P_2$ contains a pair of false twins. Therefore, by Lemma 13, it follows that the class of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free atoms has unbounded clique-width.

We now prove that the class of $(P_1 + P_4, \overline{P_1 + 2P_2})$-free atoms has unbounded clique-width. By Fact 2, the class of $(P_1 + P_4, \overline{P_1 + 2P_2})$-free graphs of the form $\overline{q(G)}$ also has

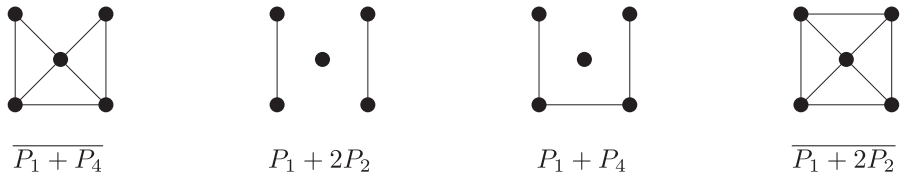

FIGURE 22 The forbidden induced subgraphs for the classes of $(\overline{P_1 + P_4}, P_1 + 2P_2)$-free graphs and $(P_1 + P_4, \overline{P_1 + 2P_2})$-free graphs mentioned in Lemma 23.

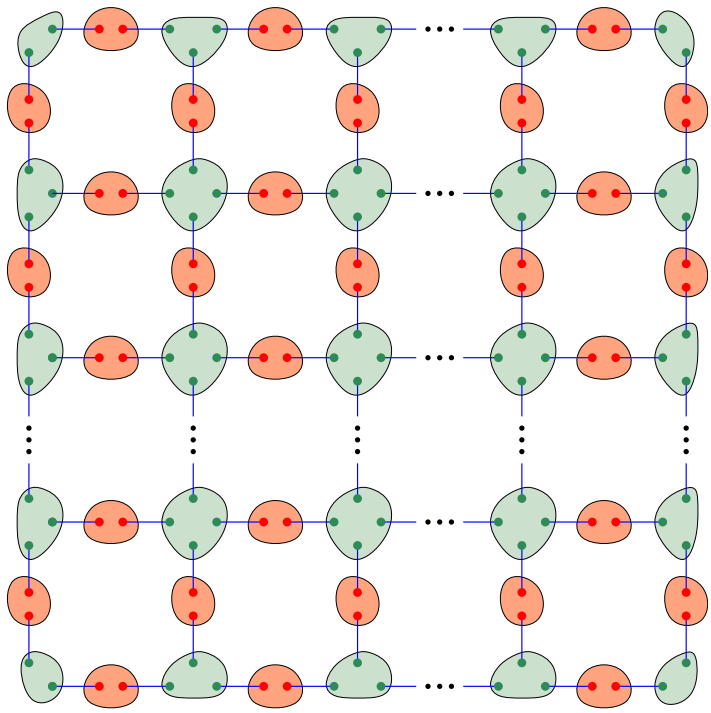

FIGURE 23 The graph $q[G]$ in Lemma 23. Edges between partition classes A_i^G and edges between partition classes B_i^G are not shown. [Color figure can be viewed at wileyonlinelibrary.com]

unbounded clique-width. It therefore suffices to show that $\overline{q(G)}$ is an atom. Suppose, for contradiction, that X is a clique cut-set in $\overline{q(G)}$. In $\overline{q(G)}$, A^G induces a disjoint union of cliques of the form A_i^G, and B^G induces a disjoint union of cliques of the form B_i^G. Therefore $X \in A_i^G \cup B_j^G$ for some i, j. Let $k \neq i$ and $\ell \neq j$. Then in $\overline{q(G)}$ every vertex in B^G has a neighbour in A_k^G and every vertex in A^G has a neighbour in B_ℓ^G. Since A_k^G and B_ℓ^G are cliques, it follows that $\overline{q(G)} \setminus X$ is connected, a contradiction. Therefore $\overline{q(G)}$ is indeed an atom. □

Lemma 24. *The class of $(2P_1 + P_2, \overline{P_6})$-free atoms has unbounded clique-width (see Figure 24 for illustrations of the forbidden induced subgraphs).*

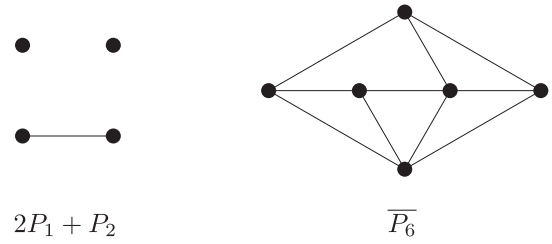

FIGURE 24 The forbidden induced subgraphs for the class of $(2P_1 + P_2, \overline{P_6})$-free graphs mentioned in Lemma 24.

FIGURE 25 The forbidden induced subgraphs for the class of $(\overline{P_1 + P_4}, P_6)$-free graphs mentioned in Lemma 25.

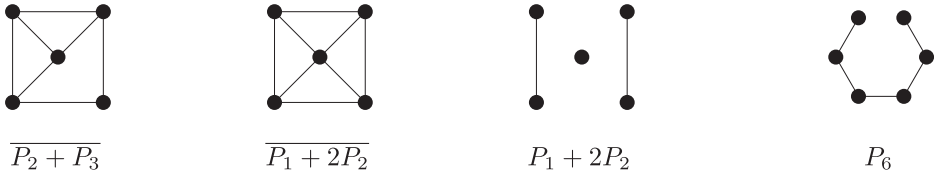

FIGURE 26 The forbidden induced subgraphs for the class of $(\overline{P_2 + P_3}, \overline{P_1 + 2P_2}, P_1 + 2P_2, P_6)$-free graphs mentioned in Lemma 26.

Proof. By Theorem 9.2(iv), the class of $(2P_1 + P_2, \overline{P_6})$-free graphs has unbounded clique-width. We observe that neither $2P_1 + P_2$ nor $\overline{P_6}$ has a dominating vertex or a pair of non-adjacent vertices that are complete to the remainder of the graph. Therefore, by Lemma 14, it follows that the class of $(2P_1 + P_2, \overline{P_6})$-free atoms has unbounded clique-width. □

Lemma 25. *The class of $(\overline{P_1 + P_4}, P_6)$-free atoms has unbounded clique-width (see Figure 25 for illustrations of the forbidden induced subgraphs).*

Proof. By Theorem 9.2(iv), the class of $(\overline{P_1 + P_4}, P_6)$-free graphs has unbounded clique-width. We observe that neither $\overline{P_1 + P_4}$ nor P_6 contains a pair of false twins. Therefore, by Lemma 13, it follows that the class of $(\overline{P_1 + P_4}, P_6)$-free atoms has unbounded clique-width. □

Lemma 26. *The class of $(\overline{P_2 + P_3}, \overline{P_1 + 2P_2}, P_1 + 2P_2, P_6)$-free atoms has unbounded clique-width (see Figure 26 for illustrations of the forbidden induced subgraphs).*

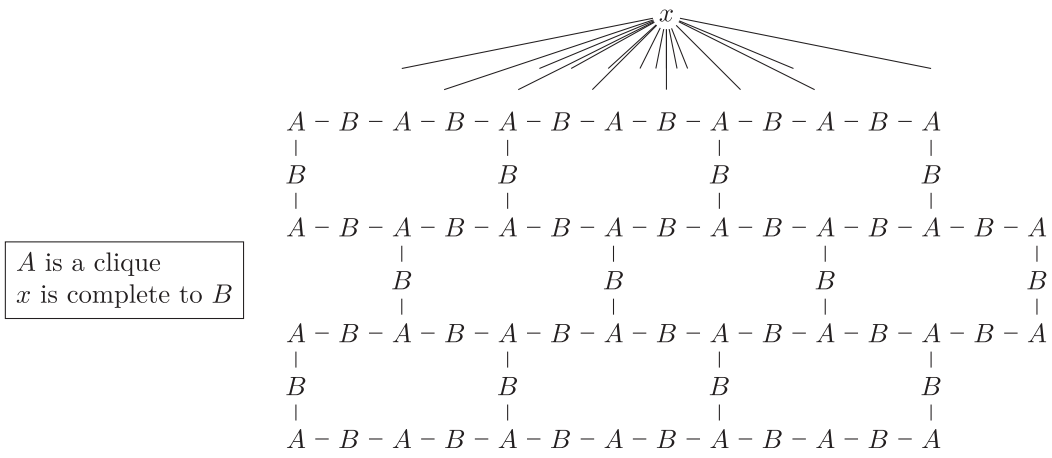

FIGURE 27 The graph H_n from the proof of Lemma 26 ($n = 3$ shown). Vertices are denoted A or B if they are in the corresponding set.

Proof. Consider a 1-subdivision of a wall of height $n \geq 2$. Let A be the set of original vertices of the wall and let B be the set of vertices introduced by the subdivision. Apply a complementation to A and add a vertex x complete to B. Let H_n be the resulting graph (see Figure 27). By Lemma 11, combined with Facts 1 and 2, the family of such graphs has unbounded clique-width. Note that x is complete to the independent set B and anti-complete to the clique A. Every vertex in B has exactly two neighbours in A and every vertex in A has either two or three neighbours in B. Furthermore, no two vertices in B have the same pair of neighbours in A.

We prove that H_n is an atom. Suppose, for contradiction, that X is a clique cut-set of H_n. If $x \in X$, then X can contain at most one other vertex (which must be in the independent set B). Since every vertex in B has neighbours in the clique A, it follows that $H_n \setminus X$ is connected. We may therefore assume that $x \notin X$. Since B is an independent set, $|B \cap X| \leq 1$, so every vertex of A has a neighbour in $B \setminus X$. Since x is complete to B, it follows that every vertex of $B \setminus X$ is in the same component of $B \setminus X$ as x, and so $H_n \setminus X$ is connected, a contradiction. We conclude that H_n is indeed an atom.

It remains to show that H_n is $(\overline{P_2 + P_3}, \overline{P_1 + 2P_2}, P_1 + 2P_2, P_6)$-free. Note that $H_n[A \cup B]$ is a split graph, so it is $(2P_2, \overline{2P_2})$-free. Therefore every induced $2P_2$ or $\overline{2P_2}$ in H_n contains the vertex x. Suppose, for contradiction, that H_n contains an induced $\overline{P_2 + P_3}$ or $\overline{P_1 + 2P_2}$, say on vertex set Y. Since $\overline{P_2 + P_3}$ and $\overline{P_1 + 2P_2}$ each contain an induced $\overline{2P_2}$, it follows that $x \in Y$. Since x has two neighbours and one non-neighbour in this $\overline{2P_2}$, this $\overline{2P_2}$ consists of the vertex x, two vertices in B and one vertex in A. Now Y contains one more vertex y, which is adjacent to either three or four of the remaining vertices of Y. Now y cannot be in B, since B is an independent set and there are two vertices in $(B \cap Y) \setminus \{y\}$, so $y \in A$. Therefore $A \cap Y$ contains two vertices with two common neighbours in B, contradicting the fact that no two vertices of B have the same two neighbours in A. We conclude that H_n is indeed $(\overline{P_2 + P_3}, \overline{P_1 + 2P_2})$-free. Now suppose, for contradiction, that H_n contains an induced $P_1 + 2P_2$ or an induced P_6, say on vertex set Y. Since $P_1 + 2P_2$ and P_6 each contain an induced $2P_2$, we find that $x \in Y$. Every vertex in $P_1 + 2P_2$ and P_6 has at least three non-neighbours. Since x is complete to B, it follows that $|A \cap Y| \geq 3$. But A is a clique and so $H_n[A \cap Y]$ contains a K_3, which is a

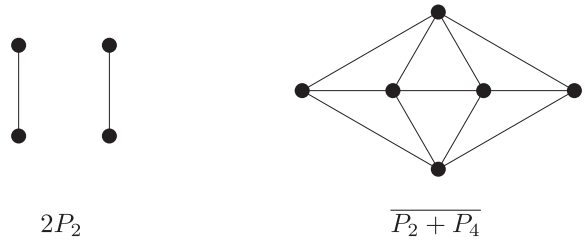

FIGURE 28 The forbidden induced subgraphs for the class of $(2P_2, \overline{P_2 + P_4})$-free graphs mentioned in Lemma 27.

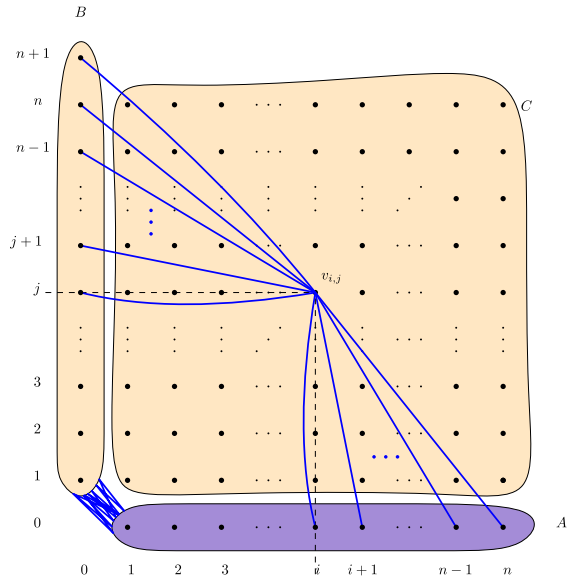

FIGURE 29 The graph G_n constructed in the proof of Lemma 27. The set A is a clique. For clarity, the edges between $A \cup B$ and C are depicted for only one vertex in C. [Color figure can be viewed at wileyonlinelibrary.com]

contradiction, since $P_1 + 2P_2$ and P_6 are K_3-free. We conclude that H_n is $(P_1 + 2P_2, P_6)$-free. Hence, H_n is $(\overline{P_2 + P_3}, \overline{P_1 + 2P_2}, P_1 + 2P_2, P_6)$-free. □

Lemma 27. *The class of $(2P_2, \overline{P_2 + P_4})$-free atoms has unbounded clique-width (see Figure 28 for illustrations of the forbidden induced subgraphs).*

Proof. Let $n \geq 2$ and construct the graph G_n as follows (see also Figure 29). Let the vertex set of G_n be $\{v_{i,j} \mid i, j \in \{0, ..., n\}, (i, j) \neq (0, 0)\} \cup \{v_{0,n+1}\}$. For $i, j, k \in \{1, ..., n\}$, add an edge between $v_{i,j}$ and $v_{k,0}$ if $k \geq i$, add an edge between $v_{i,j}$ and $v_{0,k}$ if $k \geq j$, and add an edge between $v_{i,j}$ and $v_{0,n+1}$. Let $A = \{v_{i,0} \mid i \in \{1, ..., n\}\}$, $B = \{v_{0,j} \mid j \in \{1, ..., n + 1\}\}$, and $C = \{v_{i,j} \mid i, j \in \{1, ..., n\}\}$. Apply a bipartite complementation between A and B and

apply a complementation to A. By Lemma 10 combined with Facts 1–3, the family of graphs G_n has unbounded clique-width.

We claim that G_n is an atom. Suppose, for contradiction, that X is a clique cut-set of G_n. Since $v_{0,n}$ and $v_{0,n+1}$ are non-adjacent, but complete to C, it follows that every vertex of C is in the same component of $G_n \setminus X$. Since C is independent, at most one vertex of C is in X. Since every vertex in $A \cup B$ has at least two neighbours in C, it follows that every vertex of $G_n \setminus X$ is in the same component of $G_n \setminus X$, a contradiction. Therefore G_n is indeed an atom.

Now suppose, for contradiction, that G_n contains an induced subgraph isomorphic to $2P_2$, say on vertex set Y. Since $G_n[A \cup C]$ is a split graph, it is $2P_2$-free, so Y must contain at least one vertex of B. Since $G_n[B \cup C]$ is a bipartite chain graph, it is $2P_2$-free, so Y contains at least one vertex of A. Since A is complete to B, it follows that Y contains exactly one vertex of A and exactly one vertex of B, and these two vertices are adjacent. Therefore $Y \cap C$ contains two adjacent vertices, contradicting the fact that C is independent. We conclude that G_n is $2P_2$-free.

Next suppose, for contradiction, that G_n contains an induced subgraph isomorphic to $\overline{P_2 + P_4}$, say on vertex set Y. Since $\overline{P_2 + P_4}$ is $3P_1$-free, and B and C are independent, it follows that $|B \cap Y| \leq 2$ and $|C \cap Y| \leq 2$. Therefore $|A \cap Y| \geq 6 - 2 - 2 = 2$. Now $G_n[A \cup B]$ and $G_n[A \cup C]$ are split graphs and therefore C_4-free. Since $\overline{P_2 + P_4}$ contains an induced $\overline{2P_2} = C_4$, it follows that $|C \cap Y| \geq 1$ and $|B \cap Y| \geq 1$. Since A is a clique that is complete to B and $\overline{P_2 + P_4}$ is K_4-free, it follows that $|A \cap Y| \leq 2$. Therefore $|A \cap Y| = |B \cap Y| = |C \cap Y| = 2$. Since B and C are cliques in $\overline{G_n}$, we observe that all vertices in $B \cap Y$ are in the same component of $\overline{G_n}[Y]$, and all vertices in $C \cap Y$ are in the same component of $\overline{G_n}[Y]$. Furthermore, since in $\overline{G_n}$ the set A is independent and anti-complete to B, the vertices in $(A \cup C) \cap Y$ form the P_4-component of $\overline{G_n}[Y]$, and vertices in $B \cap Y$ form the P_2-component of $\overline{G_n}[Y]$. Therefore $\overline{G_n}[(A \cup C) \cap Y]$ is isomorphic to P_4, which means that in $\overline{G_n}$ there must be two vertices in the clique C that have private neighbours in the independent set A. By the construction of $\overline{G_n}$, the vertices in C can be linearly ordered according to their neighbourhood in A, a contradiction. Therefore G_n is indeed $\overline{P_2 + P_4}$-free. □

Lemma 28. *The class of* $(2P_2, \overline{P_5}, \overline{3P_2})$*-free atoms has unbounded clique-width* (*see Figure 30 for illustrations of the forbidden induced subgraphs*).

Proof. Consider a wall of height $k \geq 2$ and note that it is a bipartite graph, say with parts A and B. Apply a complementation to A and add two vertices x and y that are complete to $A \cup B$. Let H_k be the resulting graph (see Figure 31). By Lemma 11, combined with Facts 1 and 2, the class of such graphs has unbounded clique-width.

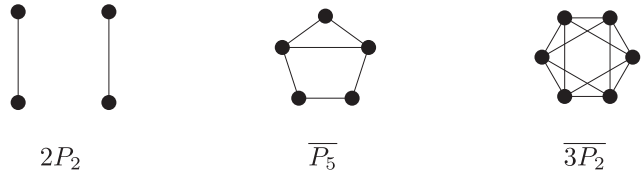

FIGURE 30 The forbidden induced subgraphs for the class of $(2P_2, \overline{P_5}, \overline{3P_2})$-free graphs mentioned in Lemma 28.

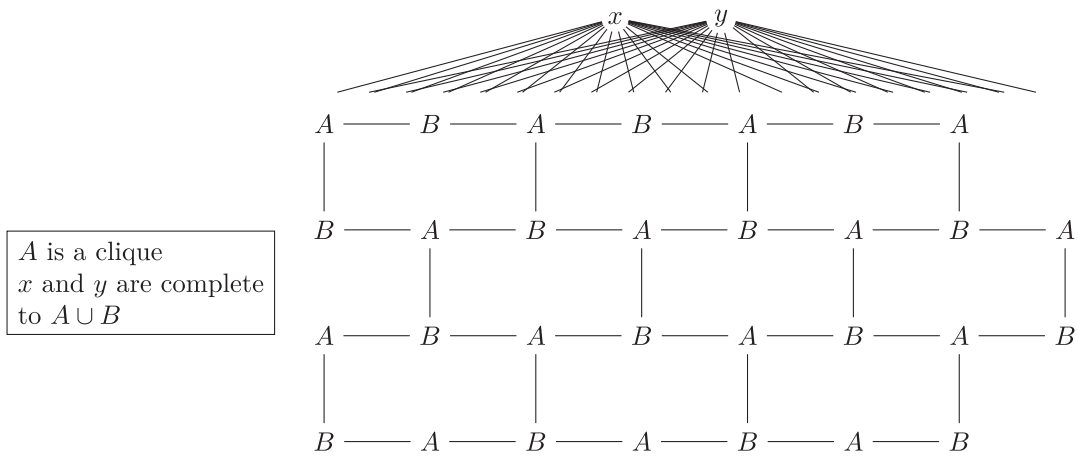

FIGURE 31 The graph H_k from the proof of Lemma 28 ($k = 3$ shown). Vertices are denoted A or B if they are in the corresponding set.

We claim that H_k is an atom. Suppose, for contradiction, that X is a clique cut-set of H_k. Since x is non-adjacent to y, at most one of x and y is in X. Without loss of generality, we may assume that $y \notin X$. Now y is adjacent to every vertex of $A \cup B$, so $H_k[\{y\} \cup (A \cup B) \setminus X]$ is connected. Since $A \cup B$ is not a clique, there must be at least one vertex in $(A \cup B) \setminus X$. Therefore, if $x \notin X$ then x is in the same component of $H_k \setminus X$ as y is. It follows that $H_k \setminus X$ is connected, a contradiction. Therefore H_k is indeed an atom.

It remains to show that H_k is $(2P_2, \overline{P_5}, \overline{3P_2})$-free. First, note that $H_k \setminus \{x\}$ and $H_k \setminus \{y\}$ are split graphs, so they are $(2P_2, \overline{2P_2})$-free, and therefore $(2P_2, \overline{P_5}, \overline{P_1 + 2P_2})$-free and note that this also implies that H_k is $\overline{3P_2}$-free. Therefore, if H_k contains an induced $2P_2$ or $\overline{P_5}$, then this induced copy must contain both x and y. Since x and y are false twins in H_k, but $2P_2$ and $\overline{P_5}$ do not contain two vertices that are false twins, it follows that H_k is $(2P_2, \overline{P_5})$-free. This completes the proof. □

It is not known whether the clique-width of $(K_3, S_{1,2,3})$-free graphs is bounded or unbounded. Recall that this case is equivalent to the open case for $(3P_1, \overline{S_{1,2,3}})$-free graphs; see also Open Problem 1. As a final result of this section, we observe that the boundedness for these cases also matches their atom counterparts.

Lemma 29. *The class of $(K_3, S_{1,2,3})$-free atoms has bounded clique-width if and only if the class of $(K_3, S_{1,2,3})$-free graphs has bounded clique-width if and only if the class of $(3P_1, \overline{S_{1,2,3}})$-free atoms has bounded clique-width (see Figure 32 for illustrations of the forbidden induced subgraphs).*

Proof. We first observe that neither K_3 nor $S_{1,2,3}$ contains a pair of false twins. Therefore, by Lemma 13, the class of $(K_3, S_{1,2,3})$-free atoms has bounded clique-width if and only if the class of $(K_3, S_{1,2,3})$-free graphs has bounded clique-width. By Fact 2, the class of $(K_3, S_{1,2,3})$-free graphs has bounded clique-width if and only if the class of $(3P_1, \overline{S_{1,2,3}})$-free graphs has bounded clique-width.

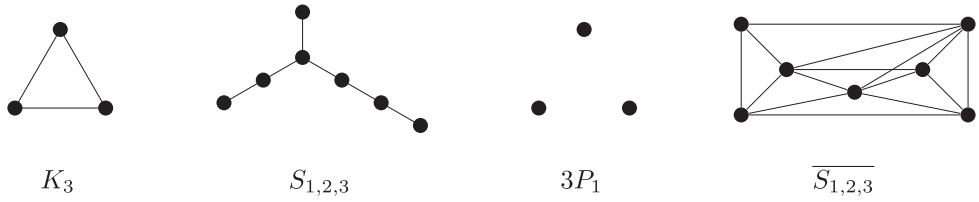

FIGURE 32 The forbidden induced subgraphs for the classes of $(K_3, S_{1,2,3})$-free graphs and $(3P_1, \overline{S_{1,2,3}})$-free graphs mentioned in Lemma 29.

We observe that neither $3P_1$ nor $\overline{S_{1,2,3}}$ has a dominating vertex or a pair of non-adjacent vertices that are complete to the remainder of the graph. Therefore, by Lemma 14, the class of $(3P_1, \overline{S_{1,2,3}})$-free graphs has bounded clique-width if and only if the class of $(3P_1, \overline{S_{1,2,3}})$-free atoms has bounded clique-width. □

6 | PROOF OF THEOREM 3

Recall the definition of equivalent bigenic classes given at the start of Section 4. To make Theorem 3 easier to compare to Theorem 9, in this section we will use the following reformulation of it, where we group classes together if they are equivalent, and we will prove this reformulated version of the theorem instead (it is easy to verify that Theorems 3 and 30 cover the same graph classes).

Theorem 30. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs.*

1. *The class of atoms in $\mathcal{G}$ has bounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   (i) H_1 *or* $H_2 \subseteq_i P_4$,
   (ii) $H_1 = K_s$ *and* $H_2 = tP_1$ *for some* $s, t \geq 1$,
   (iii) $H_1 \subseteq_i$ paw *and* $H_2 \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + P_2 + P_3, P_1 + P_5, P_1 + S_{1,1,2}, P_2 + P_4, P_6, S_{1,1,3}$ *or* $S_{1,2,2}$,
   (iv) $H_1 \subseteq_i$ diamond *and* $H_2 \subseteq_i P_1 + 2P_2, 3P_1 + P_2$ *or* $P_2 + P_3$,
   (v) $H_1 \subseteq_i$ gem *and* $H_2 \subseteq_i P_1 + P_4$ *or* P_5,
   (vi) $H_1 \subseteq_i K_3 + P_1$ *and* $H_2 \subseteq_i K_{1,3}$ *or*
   (vii) $H_1 \subseteq_i \overline{2P_1 + P_3}$ *and* $H_2 \subseteq_i 2P_1 + P_3$.
2. *The class of atoms in $\mathcal{G}$ has bounded clique-width if $\mathcal{G}$ is a subclass of the class of:*
   (i) $(P_6, \overline{2P_2})$*-free graphs or*
   (ii) $(2P_2, \overline{P_2 + P_3})$*-free graphs.*
3. *The class of atoms in $\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:*
   (i) $H_1 \notin \mathcal{S}$ *and* $H_2 \notin \mathcal{S}$,
   (ii) $H_1 \notin \overline{\mathcal{S}}$ *and* $H_2 \notin \overline{\mathcal{S}}$,
   (iii) $H_1 \supseteq_i K_3 + P_1$ *and* $H_2 \supseteq_i 4P_1$ *or* $2P_2$,
   (iv) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i K_{1,3}, 5P_1$ *or* $P_2 + P_4$,
   (v) $H_1 \supseteq_i K_3$ *and* $H_2 \supseteq_i 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2$ *or* $2P_3$,
   (vi) $H_1 \supseteq_i K_4$ *and* $H_2 \supseteq_i P_1 + P_4, 3P_1 + P_2$ *or* $2P_2$ *or*

(vii) $H_1 \supseteq_i$ gem *and* $H_2 \supseteq_i P_1 + 2P_2$.

4. *The class of atoms in* $\mathcal{G}$ *has unbounded clique-width if it contains the class of* (H_1, H_2)*-free graphs such that one of the following holds:*
   (i) $H_1 \supseteq_i$ diamond *and* $H_2 \supseteq_i P_1 + P_6$,
   (ii) $H_1 \supseteq_i 2P_1 + P_2$ *and* $H_2 \supseteq_i \overline{P_6}$,
   (iii) $H_1 \supseteq_i$ gem *and* $H_2 \supseteq_i P_6$,
   (iv) $H_1 \supseteq_i P_1 + 2P_2$ *or* P_6 *and* $H_2 \supseteq_i \overline{P_1 + 2P_2}$ *or* $\overline{P_2 + P_3}$ *or*
   (v) $H_1 \supseteq_i 2P_2$ *and* $H_2 \supseteq_i \overline{P_2 + P_4}, \overline{3P_2}$ *or* $\overline{P_5}$.

Proof. We start by considering the bounded cases. Theorem 30.1 follows immediately from Theorem 9.1. Theorem 30.2(i) follows from the fact that $(P_6, \overline{2P_2})$-free atoms have bounded clique-width [41]. Theorem 30.2(ii) follows from the fact that $(2P_2, \overline{P_2 + P_3})$-free atoms have bounded clique-width (Theorem 2). Next, we consider the unbounded cases. Theorem 30.3(i) and Theorem 30.3(ii) follow from Lemma 12. Theorem 30.3(iii) follows from Lemma 15. Theorem 30.3(iv) follows from Lemmas 15, 16 and 17. Theorem 30.3(v) follows from Lemmas 18 and 19. Theorem 30.3(vi) follows from Lemmas 20, 21 and 22. Theorem 30.3(vii) follows from Lemma 23. Theorem 30.4(i) follows from Lemma 17. Theorem 30.4(ii) follows from Lemma 24. Theorem 30.4(iii) follows from Lemma 25. Theorem 30.4(iv) follows from Lemma 26. Theorem 30.4(v) follows from Lemmas 27 and 28. □

In the open problem below, the cases marked with a * are those for which even the boundedness of clique-width of the whole class of (H_1, H_2)-free graphs is unknown (see also Open Problem 1 in Section 4).

Open Problem 2. *Does the class of* (H_1, H_2)*-free atoms have bounded clique-width if*

(i) $H_1 =$ diamond *and* $H_2 = P_6$,
(ii) $H_1 = C_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_4, 3P_2\}$,
(iii) $H_1 = \overline{P_1 + 2P_2}$ *and* $H_2 \in \{2P_2, P_2 + P_3, P_5\}$,
(iv) $H_1 = \overline{P_2 + P_3}$ *and* $H_2 \in \{P_2 + P_3, P_5\}$,
*(v) $H_1 = K_3$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$,
*(vi) $H_1 = 3P_1$ *and* $H_2 = \overline{P_1 + S_{1,1,3}}$,
*(vii) $H_1 =$ diamond *and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$,
*(viii) $H_1 = 2P_1 + P_2$ *and* $H_2 \in \{\overline{P_1 + P_2 + P_3}, \overline{P_1 + P_5}\}$,
*(ix) $H_1 =$ gem *and* $H_2 = P_2 + P_3$ *or*
*(x) $H_1 = P_1 + P_4$ *and* $H_2 = \overline{P_2 + P_3}$.

Olariu [59] proved that every connected $\overline{P_1 + P_3}$-free graph is either K_3-free or complete multi-partite. Since complete multi-partite graphs and their complements have bounded clique-width, when looking at boundedness of clique-width of a hereditary class, forbidding $\overline{P_1 + P_3}$ as an induced subgraph is equivalent to forbidding K_3 and forbidding $P_1 + P_3$ is equivalent to forbidding $3P_1$. Thus, when studying boundedness of clique-width we may assume that we never explicitly forbid $\overline{P_1 + P_3}$ or $P_1 + P_3$. Furthermore, by Lemma 29, the class of $(K_3, S_{1,2,3})$-free atoms has bounded clique-width if and only if the class of $(3P_1, \overline{S_{1,2,3}})$-free atoms has bounded clique-width, so we may assume $\{H_1, H_2\} \neq \{3P_1, \overline{S_{1,2,3}}\}$. We now state the following theorem.

Theorem 31. *Let H_1 and H_2 be graphs (which are not isomorphic to $\overline{P_1 + P_3}$ or $P_1 + P_3$) with $\{H_1, H_2\} \neq \{3P_1, \overline{S_{1,2,3}}\}$ and let $\mathcal{G}$ be the class of (H_1, H_2)-free graphs. Then (un)boundedness of clique-width for atoms in $\mathcal{G}$ does not follow from Theorem 30 if and only if this class is listed in Open Problem 2.*

Proof. First, note that Theorem 30 does not specify the (un)boundedness of clique-width for atoms in any of the classes listed in Open Problem 2.

Consider the classes listed in Open Problem 1. For all bigenic classes $\mathcal{G}$ for which the (un)boundedness of clique-width of general graphs is not listed in Theorem 9, an equivalent class is listed in Open Problem 1 (see [31, 36]). Since the results in Theorem 30.2 do not solve these cases when restricted to atoms, these classes (and their complements, apart from the $H_1 = 3P_1$, $H_2 = \overline{S_{1,2,3}}$) appear in Open Problem 2(v)–(x). The only other classes we need to consider are those for which Theorem 9.2 states that the class $\mathcal{G}$ has unbounded clique-width, but the class of atoms in $\mathcal{G}$ might not have unbounded clique-width.

There are two classes listed in Theorem 9.2 that are not listed in Theorem 30.3, namely, the class of $(\overline{2P_2}, 2P_2)$-free graphs and the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs. The class of $(\overline{2P_2}, 2P_2)$-free graphs is only equivalent to itself. The class of $(\overline{2P_1 + P_2}, P_6)$-free graphs equivalent to only one other class, namely, the class of $(2P_1 + P_2, \overline{P_6})$-free graphs. However, the class of $(2P_1 + P_2, \overline{P_6})$-free atoms has unbounded clique-width by Theorem 30.4(ii). We therefore only need to consider the class of $(\overline{2P_2}, 2P_2)$-free graphs and the class of $(2P_1 + P_2, \overline{P_6})$-free, together with any bigenic classes $\mathcal{G}'$ that are extensions of these classes such that Theorem 30 does not specify that the atoms of $\mathcal{G}'$ have unbounded clique-width.

We start by considering extensions of the classes of $(\overline{2P_1 + P_2}, P_6)$-free graphs. Consider graphs H_1, H_2 with $\overline{2P_1 + P_2} \subseteq_i H_1$ and $P_6 \subseteq_i H_2$ such that the class of (H_1, H_2)-free atoms has bounded clique-width, but Theorem 30 does not state that (H_1, H_2)-free atoms have unbounded clique-width. By Theorem 30.3(ii), it follows that $\overline{H_1} \in \mathcal{S}$. By Theorem 30.3(iii), it follows that $\overline{H_1}$ is $K_{1,3}$-free, so it is a linear forest. By Theorem 30.3(vi), it follows that $\overline{H_1}$ is $4P_1$-free. By Theorem 30.4(iii), $\overline{H_1}$ must be $(P_1 + P_4)$-free. By Theorem 30.4(iv), $\overline{H_1}$ must be $(P_1 + 2P_2, P_2 + P_3)$-free. The 1-vertex extensions of $2P_1 + P_2$ in $\mathcal{S}$ are $3P_1 + P_2$, $P_1 + 2P_2$, $2P_1 + P_3$, $P_1 + P_4$, $P_2 + P_3$ and $S_{1,1,2}$, none of which are $(K_{1,3}, 4P_1, P_1 + P_4, P_1 + 2P_2, P_2 + P_3)$-free. We conclude that $\overline{H_1} = 2P_1 + P_2$. By Theorem 30.3(i), it follows that $H_2 \in \mathcal{S}$. By Theorem 30.3(iv), it follows that H_2 is $(K_{1,3}, P_2 + P_4)$-free. By Theorem 30.4(i), it follows that H_2 is $(P_1 + P_6)$-free. The 1-vertex extensions of P_6 that are in $\mathcal{S}$ are $P_1 + P_6$, P_7, $S_{1,1,4}$ and $S_{1,2,3}$, none of which are $(K_{1,3}, P_2 + P_4, P_1 + P_6)$-free. We conclude that $H_2 = P_6$. Therefore, we do not need to consider any extensions of $(\overline{2P_1 + P_2}, P_6)$-free graphs, apart from the class of $(\overline{2P_1 + P_2}, P_6)$-free graphs itself, and this is listed in Open Problem 2(i).

Now consider graphs H_1, H_2 with $2P_2 \subseteq_i H_1, \overline{H_2}$ such that the class of (H_1, H_2)-free atoms has bounded clique-width, but Theorem 30 does not state that (H_1, H_2)-free atoms have unbounded clique-width. By Theorems 30.3(i) and 3(ii), respectively, H_1 and $\overline{H_2}$ must both be in $\mathcal{S}$. By Theorem 30.3(iii), it follows that H_1 and $\overline{H_2}$ are $K_{1,3}$-free, so they are both linear forests. By Theorem 30.3(vi), H_1 and $\overline{H_2}$ are $4P_1$-free, and because they are bipartite, this means they each contain at most six vertices. Since H_1 and $\overline{H_2}$ are linear forests on at most six vertices containing an induced $2P_2$, it follows that $H_1, \overline{H_2} \in \{2P_2, P_1 + 2P_2, P_2 + P_3, P_5, 2P_1 + 2P_2, 3P_2, P_1 + P_2 + P_3, P_2 + P_4, P_1 + P_5, 2P_3, P_6\}$. Since H_1

and $\overline{H_2}$ are $4P_1$-free, it follows that $H_1, \overline{H_2} \in \{2P_2, P_1 + 2P_2, P_2 + P_3, P_5, 3P_2, P_2 + P_4, P_6\}$. By Theorem 30.4(v), $\overline{H_2}$ is $(P_2 + P_4, 3P_2, P_5)$-free, and so $\overline{H_2} \in \{2P_2, P_1 + 2P_2, P_2 + P_3\}$. Now if $\overline{H_2} = 2P_2$, then by Theorem 30.2(i), we may assume that H_1 is not an induced subgraph of P_6, so $H_1 \in \{P_1 + 2P_2, 3P_2, P_2 + P_4\}$ and these cases are listed in Open Problem 2(ii). Otherwise, $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$. In this case by Theorem 30.4(iv) H_1 is $(P_1 + 2P_2, P_6)$-free, so $H_1 \in \{2P_2, P_2 + P_3, P_5\}$. If $\overline{H_2} = P_1 + 2P_2$ then $H_1 \in \{2P_2, P_2 + P_3, P_5\}$ and these cases are listed in Open Problem 2(iii). If $\overline{H_2} = P_2 + P_3$ then by Theorem 30.2(ii), H_1 is not an induced subgraph of $2P_2$, so $H_1 \in \{P_2 + P_3, P_5\}$ and these cases are listed in Open Problem 2(iv). □

7 | CONCLUSIONS

Motivated by algorithmic applications, we determined a new class of (H_1, H_2)-free graphs of unbounded clique-width whose atoms have *bounded* clique-width, namely, when $(H_1, H_2) = (2P_2, \overline{P_2 + P_3})$ (in fact, our proof for $(2P_2, \overline{P_2 + P_3})$-free atoms also works for *linear* clique-width). We also identified a number of classes of (H_1, H_2)-free graphs of unbounded clique-width whose atoms still have *unbounded* clique-width. In particular, our results show that boundedness of clique-width of (H_1, H_2)-free atoms does not necessarily imply boundedness of clique-width of $(\overline{H_1}, \overline{H_2})$-free atoms. For example, (C_4, P_5)-free atoms have bounded clique-width [41], but we proved that $(\overline{C_4}, \overline{P_5})$-free atoms have unbounded clique-width (Lemma 28).

We also presented a summary theorem (Theorem 3), from which we deduced a list of *18* remaining cases of pairs (H_1, H_2) for which we do not know whether the clique-width of (H_1, H_2)-free atoms is bounded; see also Open Problem 2 and Theorem 31. In particular, we ask whether boundedness of clique-width of $(2P_2, \overline{P_2 + P_3})$-free atoms can be extended to $(P_5, \overline{P_2 + P_3})$-free atoms. Is boundedness of clique-width the underlying reason why COLOURING is polynomial-time solvable on $(P_5, \overline{P_2 + P_3})$-free graphs [58]? Brandstädt and Hoàng [11] showed that $(P_5, \overline{P_2 + P_3})$-free atoms with no dominating vertices and no vertex pairs $\{x, y\}$ with $N(x) \subseteq N(y)$ are either isomorphic to some specific graph G^* or all their induced C_5s are dominating. Recently, Huang and Karthick [49] proved a more refined decomposition. However, it is not clear how to use these results to prove boundedness of clique-width of $(P_5, \overline{P_2 + P_3})$-free atoms, and additional insights are needed.

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REFERENCES

1. V. E. Alekseev, *On easy and hard hereditary classes of graphs with respect to the independent set problem*, Discrete Appl. Math. **132** (2003), no. 1–3, 17–26. https://doi.org/10.1016/S0166-218X(03)00387-1
2. R. Belmonte and M. Vatshelle, *Graph classes with structured neighborhoods and algorithmic applications*, Theoret. Comput. Sci. **511** (2013), 54–65. https://doi.org/10.1016/j.tcs.2013.01.011
3. A. Blanché, K. K. Dabrowski, M. Johnson, V. V. Lozin, D. Paulusma, and V. Zamaraev, *Clique-width for graph classes closed under complementation*, SIAM J. Discrete Math. **34** (2020), no. 2, 1107–1147. https://doi.org/10.1137/18M1235016
4. H. L. Bodlaender, *A linear-time algorithm for finding tree-decompositions of small treewidth*, SIAM J. Comput. **25** (1996), no. 6, 1305–1317. https://doi.org/10.1137/S0097539793251219
5. H. L. Bodlaender, N. Brettell, M. Johnson, G. Paesani, D. Paulusma, and E. J. van Leeuwen, *Steiner trees for hereditary graph classes: A treewidth perspective*, Theoret. Comput. Sci. **867** (2021), 30–39. https://doi.org/10.1016/j.tcs.2021.03.012
6. R. Boliac and V. V. Lozin, *On the clique-width of graphs in hereditary classes*, Proc. ISAAC 2002 LNCS. **2518** (2002), 44–54. https://doi.org/10.1007/3-540-36136-7_5
7. M. Bonamy, N. Bousquet, K. K. Dabrowski, M. Johnson, D. Paulusma, and T. Pierron, *Graph isomorphism for (H_1, H_2)-free graphs: An almost complete dichotomy*, Algorithmica. **83** (2021), no. 3, 822–852. https://doi.org/10.1007/s00453-020-00747-x
8. A. Brandstädt, K. K. Dabrowski, S. Huang, and D. Paulusma, *Bounding the clique-width of H-free split graphs*, Discrete Appl. Math. **211** (2016), 30–39. https://doi.org/10.1016/j.dam.2016.04.003
9. A. Brandstädt, K. K. Dabrowski, S. Huang, and D. Paulusma, *Bounding the clique-width of H-free chordal graphs*, J. Graph Theory. **86** (2017), no. 1, 42–77. https://doi.org/10.1002/jgt.22111
10. A. Brandstädt, J. Engelfriet, H.-O. Le, and V. V. Lozin, *Clique-width for 4-vertex forbidden subgraphs*, Theory Comput. Syst. **39** (2006), no. 4, 561–590. https://doi.org/10.1007/s00224-005-1199-1
11. A. Brandstädt and C. T. Hoàng, *On clique separators, nearly chordal graphs, and the maximum weight stable set problem*, Theoret. Comput. Sci. **389** (2007), no. 1–2, 295–306. https://doi.org/10.1016/j.tcs.2007.09.031
12. A. Brandstädt, T. Klembt, and S. Mahfud, *P_6- and triangle-free graphs revisited: Structure and bounded clique-width*, Discrete Math. Theoret. Comput. Sci. **8** (2006), no. 1, 173–188. https://dmtcs.episciences.org/372
13. A. Brandstädt, H.-O. Le, and R. Mosca, *Gem- and co-gem-free graphs have bounded clique-width*, Int. J. Found. Comput. Sci. **15** (2004), no. 1, 163–185. https://doi.org/10.1142/S0129054104002364
14. A. Brandstädt, H.-O. Le, and R. Mosca, *Chordal co-gem-free and (P_5, gem)-free graphs have bounded clique-width*, Discrete Appl. Math. **145** (2005), no. 2, 232–241. https://doi.org/10.1016/j.dam.2004.01.014
15. A. Brandstädt and S. Mahfud, *Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time*, Inform. Process. Lett. **84** (2002), no. 5, 251–259. https://doi.org/10.1016/S0020-0190(02)00291-0
16. A. Brandstädt and R. Mosca, *On distance-3 matchings and induced matchings*, Discrete Appl. Math. **159** (2011), no. 7, 509–520. https://doi.org/10.1016/j.dam.2010.05.022
17. N. Brettell, J. Horsfield, A. Munaro, G. Paesani, and D. Paulusma, *Bounding the mim-width of hereditary graph classes*, J. Graph Theory. **99** (2022), no. 1, 117–151. https://doi.org/10.1002/jgt.22730
18. B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle, *Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems*, Theoret. Comput. Sci. **511** (2013), 66–76. https://doi.org/10.1016/j.tcs.2013.01.009
19. K. Cameron, M. V. G. da Silva, S. Huang, and K. Vušković, *Structure and algorithms for (cap, even hole)-free graphs*, Discrete Math. **341** (2018), no. 2, 463–473. https://doi.org/10.1016/j.disc.2017.09.013
20. M. Chudnovsky, *The structure of bull-free graphs II and III—A summary*, J. Combin. Theory Ser. B. **102** (2012), no. 1, 252–282. https://doi.org/10.1016/j.jctb.2011.07.002
21. M. Chudnovsky and P. D. Seymour, *The structure of claw-free graphs*, Lond. Math. Soc. Lect. Note Ser. **327** (2005), 153–171. https://doi.org/10.1017/CBO9780511734885.008

22. J. Chuzhoy, *Improved bounds for the flat wall theorem*, P. Indyk (ed). Proc. SODA 2015, SIAM, Philadelphia, Pennsylvania, United States, 2015, pp. 256–275.
23. D. G. Corneil and U. Rotics, *On the relationship between clique-width and treewidth*, SIAM J. Comput. **34** (2005), 825–847. https://doi.org/10.1137/S0097539701385351
24. B. Courcelle, *The monadic second-order logic of graphs. I. Recognizable sets of finite graphs*, Inform. and Comput. **85** (1990), no. 1, 12–75. https://doi.org/10.1016/0890-5401(90)90043-H
25. B. Courcelle and J. Engelfriet, *Graph structure and monadic second-order logic: A language–theoretic approach*, Encyclopedia of Mathematics and its Applications, vol. **138**, Cambridge University Press, Cambridge, UK, 2012. https://doi.org/10.1017/CBO9780511977619
26. B. Courcelle, J. A. Makowsky, and U. Rotics, *Linear time solvable optimization problems on graphs of bounded clique-width*, Theory Comput. Syst. **33** (2000), no. 2, 125–150. https://doi.org/10.1007/s002249910009
27. B. Courcelle and S. Olariu, *Upper bounds to the clique width of graphs*, Discrete Appl. Math. **101** (2000), no. 1–3, 77–114. https://doi.org/10.1016/S0166-218X(99)00184-5
28. K. K. Dabrowski, F. Dross, and D. Paulusma, *Colouring diamond-free graphs*, J. Comput. System Sci. **89** (2017), 410–431. https://doi.org/10.1016/j.jcss.2017.06.005
29. K. K. Dabrowski, P. A. Golovach, and D. Paulusma, *Colouring of graphs with Ramsey-type forbidden subgraphs*, Theoret. Comput. Sci. **522** (2014), 34–43. https://doi.org/10.1016/j.tcs.2013.12.004
30. K. K. Dabrowski, S. Huang, and D. Paulusma, *Bounding clique-width via perfect graphs*, J. Comput. System Sci. **104** (2019), 202–215. https://doi.org/10.1016/j.jcss.2016.06.007
31. K. K. Dabrowski, M. Johnson, and D. Paulusma, *Clique-width for hereditary graph classes*, Lond. Math. Soc. Lect. Note Ser. **456** (2019), 1–56. https://doi.org/10.1017/9781108649094.002
32. K. K. Dabrowski, V. V. Lozin, and D. Paulusma, *Clique-width and well-quasi-ordering of triangle-free graph classes*, J. Comput. System Sci. **108** (2020), 64–91. https://doi.org/10.1016/j.jcss.2019.09.001
33. K. K. Dabrowski, V. V. Lozin, R. Raman, and B. Ries, *Colouring vertices of triangle-free graphs without forests*, Discrete Math. **312** (2012), no. 7, 1372–1385. https://doi.org/10.1016/j.disc.2011.12.012
34. K. K. Dabrowski, T. Masařík, J. Novotná, D. Paulusma, and P. Rzążewski, *Clique-width: Harnessing the power of atoms*, Proc. WG 2020 LNCS. **12301** (2020), 119–133. https://doi.org/10.1007/978-3-030-60440-0_10
35. K. K. Dabrowski and D. Paulusma, *Classifying the clique-width of H-free bipartite graphs*, Discrete Appl. Math. **200** (2016), 43–51. https://doi.org/10.1016/j.dam.2015.06.030
36. K. K. Dabrowski and D. Paulusma, *Clique-width of graph classes defined by two forbidden induced subgraphs*, Computer J. **59** (2016), no. 5, 650–666. https://doi.org/10.1093/comjnl/bxv096
37. G. A. Dirac, *On rigid circuit graphs*, Abh. Math. Semin. Univ. Hambg. **25** (1961), no. 1–2, 71–76. https://doi.org/10.1007/BF02992776
38. W. Espelage, F. Gurski, and E. Wanke, *How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time*, Proc. WG 2001 LNCS. **2204** (2001), 117–128. https://doi.org/10.1007/3-540-45477-2_12
39. A. M. Foley, D. J. Fraser, C. T. Hoàng, K. Holmes, and T. P. LaMantia, *The intersection of two vertex coloring problems*, Graphs Combin. **36** (2020), no. 1, 125–138. https://doi.org/10.1007/s00373-019-02123-1
40. D. J. Fraser, A. M. Hamel, C. T. Hoàng, K. Holmes, and T. P. LaMantia, *Characterizations of* $(4K_1, C_4, C_5)$*-free graphs*, Discrete Appl. Math. **231** (2017), 166–174. https://doi.org/10.1016/j.dam.2016.08.016
41. S. Gaspers, S. Huang, and D. Paulusma, *Colouring square-free graphs without long induced paths*, J. Comput. System Sci. **106** (2019), 60–79. https://doi.org/10.1016/j.jcss.2019.06.002
42. M. U. Gerber and D. Kobler, *Algorithms for vertex-partitioning problems on graphs with fixed clique-width*, Theoret. Comput. Sci. **299** (2003), no. 1, 719–734. https://doi.org/10.1016/S0304-3975(02)00725-9
43. P. A. Golovach, M. Johnson, D. Paulusma, and J. Song, *A survey on the computational complexity of colouring graphs with forbidden subgraphs*, J. Graph Theory. **84** (2017), no. 4, 331–363. https://doi.org/10.1002/jgt.22028
44. M. Grötschel, L. Lovász, and A. Schrijver, *Polynomial algorithms for perfect graphs*, Ann. Discrete Math. **21** (1984), 325–356. https://doi.org/10.1016/S0304-0208(08)72943-8
45. F. Gurski, *The behavior of clique-width under graph operations and graph transformations*, Theory Comput. Syst. **60** (2017), no. 2, 346–376. https://doi.org/10.1007/s00224-016-9685-1
46. D. Hermelin, M. Mnich, E. J. van Leeuwen, and G. J. Woeginger, *Domination when the stars are out*, ACM Trans. Algorith. **15** (2019), no. 2, 25:1–25:90. https://doi.org/10.1145/3301445

47. P. Hliněný, S.-i. Oum, D. Seese, and G. Gottlob, *Width parameters beyond tree-width and their applications*, Computer J. **51** (2008), no. 3, 326–362. https://doi.org/10.1093/comjnl/bxm052
48. C. T. Hoàng and D. A. Lazzarato, *Polynomial-time algorithms for minimum weighted colorings of* $(P_5, \overline{P_5})$*-free graphs and similar graph classes*, Discrete Appl. Math. **186** (2015), 106–111. https://doi.org/10.1016/j.dam.2015.01.022
49. S. Huang and T. Karthick, *On graphs with no induced five-vertex path or paraglider*, J. Graph Theory. **97** (2021), no. 2, 305–323. https://doi.org/10.1002/jgt.22656
50. K. Jansen and P. Scheffler, *Generalized coloring for tree-like graphs*, Discrete Appl. Math. **75** (1997), no. 2, 135–155. https://doi.org/10.1016/S0166-218X(96)00085-6
51. M. Kamiński, V. V. Lozin, and M. Milanič, *Recent developments on graphs of bounded clique-width*, Discrete Appl. Math. **157** (2009), no. 12, 2747–2761. https://doi.org/10.1016/j.dam.2008.08.022
52. D. Kobler and U. Rotics, *Edge dominating set and colorings on graphs with fixed clique-width*, Discrete Appl. Math. **126** (2003), no. 2–3, 197–221. https://doi.org/10.1016/S0166-218X(02)00198-1
53. S. Kratsch and P. Schweitzer, *Graph isomorphism for graph classes characterized by two forbidden induced subgraphs*, Discrete Appl. Math. **216** (2017), Part 1, 240–253. https://doi.org/10.1016/j.dam.2014.10.026
54. V. V. Lozin and D. Rautenbach, *On the band-, tree-, and clique-width of graphs with bounded vertex degree*, SIAM J. Discrete Math. **18** (2004), no. 1, 195–206. https://doi.org/10.1137/S0895480102419755
55. V. V. Lozin and D. Rautenbach, *The tree- and clique-width of bipartite graphs in special classes*, Australas. J. Combin. **34** (2006), 57–67. https://ajc.maths.uq.edu.au/pdf/34/ajc_v34_p057.pdf
56. V. V. Lozin and J. Volz, *The clique-width of bipartite graphs in monogenic classes*, Int. J. Found. Comput. Sci. **19** (2008), no. 02, 477–494. https://doi.org/10.1142/S0129054108005772
57. J. A. Makowsky and U. Rotics, *On the clique-width of graphs with few* P_4*'s*, Int. J. Found. Comput. Sci. **10** (1999), no. 3, 329–348. https://doi.org/10.1142/S0129054199000241
58. D. S. Malyshev and O. O. Lobanova, *Two complexity results for the vertex coloring problem*, Discrete Appl. Math. **219** (2017), 158–166. https://doi.org/10.1016/j.dam.2016.10.025
59. S. Olariu, *Paw-free graphs*, Inform. Process. Lett. **28** (1988), no. 1, 53–54. https://doi.org/10.1016/0020-0190(88)90143-3
60. S.-i. Oum and P. D. Seymour, *Approximating clique-width and branch-width*, J. Combin. Theory Ser. B. **96** (2006), no. 4, 514–528. https://doi.org/10.1016/j.jctb.2005.10.006
61. M. Rao, *MSOL partitioning problems on graphs of bounded treewidth and clique-width*, Theoret. Comput. Sci. **377** (2007), no. 1–3, 260–267. https://doi.org/10.1016/j.tcs.2007.03.043
62. S. H. Sæther, and M. Vatshelle, *Hardness of computing width parameters based on branch decompositions over the vertex set*, Theoret. Comput. Sci. **615** (2016), 120–125. https://doi.org/10.1016/j.tcs.2015.11.039
63. P. Schweitzer, *Towards an isomorphism dichotomy for hereditary graph classes*, Theory Comput. Syst. **61** (2017), no. 4, 1084–1127. https://doi.org/10.1007/s00224-017-9775-8
64. R. E. Tarjan, *Decomposition by clique separators*, Discrete Math. **55** (1985), no. 2, 221–232. https://doi.org/10.1016/0012-365X(85)90051-2
65. M. Vatshelle, *New width parameters of graphs*, Ph.D. thesis, University of Bergen, 2012.

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