Quantum-over-classical Advantage in Solving Multiplayer Games

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Abstract. We study the applicability of quantum algorithms in computational game theory and generalize some results related to Subtraction games, which are sometimes referred to as one-heap Nim games. In quantum game theory, a subset of Subtraction games became the first explicitly defined class of zero-sum combinatorial games with provable separation between quantum and classical complexity of solving them. For a narrower subset of Subtraction games, an exact quantum sublinear algorithm is known that surpasses all deterministic algorithms for finding solutions with probability 1.

Typically, both Nim and Subtraction games are defined for only two players. We extend some known results to games for three or more players, while maintaining the same classical and quantum complexities: $\Theta(n^2)$ and $\tilde{O}(n^{1.5})$ respectively.

Keywords: quantum game theory, quantum combinatorial games, quantum multiplayer games, quantum algorithm, Nim, subtraction game

1 Introduction

Quantum game theory traditionally is being studied in the context of nonlocal properties of quantum particles, and usually stays apart from quantum computing and quantum algorithms. In contrast, *Quantum combinatorial game theory* is an approximately yearling branch of game theory, where quantum algorithms are applied for solving classical combinatorial games.

First explicit examples of combinatorial games with quantum-better-thanclassical solving algorithms are some subsets of *Subtraction games*. [11] identify a specific subset of size $const^{n^2}$ of Subtraction games which are solvable by a quantum algorithm in time $O(n^{1.5} \log n)$ in bounded error setting. [9] identify a smaller set of size $const^{\sqrt{n} \log n}$ of restricted Subtraction games which are solvable by an exact quantum algorithm in time $O(n^{1.5})$. Deterministic algorithms for both classes of games require $\Omega(n^2)$ steps for solving. (Hereafter n + 1 stands for the number of positions in a game, regardless which of the players has to make the next move.)

Latvia, Riga, Latvia

A Subtraction game is similar to a canonical Nim game [6] in several senses. Nim is a notable game in game theory because it traditionally serves as a "base case" for Sprague-Grundy theorem [13,8], which establishes a deterministic upper bound for solving many combinatorial games. Similarly, Subtraction games seem to become a good candidate for being a "base sample" for game-solving Quantum Dynamic Programming. And like many games are known to be reducible to some Nim games, also many games on graphs can be reduced to the corresponding Subtraction games. Finally, the rules of these games have very similar definitions.

The difference between these two games is that in a Subtraction game, the players deal with just one heap of stones, though with certain limitations imposed on the number of stones they can take from the heap. The most common limitation for Subtraction games is defining a maximum for the number of stones to be taken away, and this kind of Subtraction game has very fast deterministic solutions in $\tilde{O}(1)$. Here we study a much more general class of such limitations and thus a broader class of Subtraction games.

We investigate algorithms for *solving* Subtraction games, that is determining the payoffs of all the players, assuming each of them to play optimally. We exploit techniques similar to ones in [11] and [9] to establish upper bounds for the quantum complexity and asymptotes for the classical complexity.

The paper is organized in the following way. Section 2 contains basic definitions. In Section 3, we present evaluations of classical complexity and quantum algorithms for solving a special class of games which we call balanced Subtraction games. Finally, in Section 4 we analyze the complexity of solving the so called restricted Subtraction games.

2 Definitions

2.1 Subtraction Games

In a play of a Subtraction game players $1, \ldots, k$ sequentially remove some positive amounts of stones from a heap, with player l being followed by player $(l \mod k + 1)$.

Let n be the initial number of stones in the heap, and Γ be a lower-triangular binary matrix of size $n \times n$, with rows numbered from 1 to n and columns numbered from 0 to n - 1. A player which has to make the next move, can remove j - i stones ($0 \le i < j \le n$) from the heap with exactly j stones left iff $\Gamma_{ji} = 1$. In simple terms, Γ_{ji} indicates the possibility for a player to receive position "j stones" from the predecessor and pass position "i stones" to the follower on the next turn.

If in some position a player, say player l_0 , cannot make a legal move, then the play ends, and each player l receives their payoff $(l - l_0 + k) \mod k$. That is, player l_0 is the loser, the previous player is the major winner with payoff (k - 1), the previous-to-previous player gets (k - 2) and so on. In order to become the major winner, a player has to take all the remaining stones, or to leave a number of stones j such that no allowed moves would remain: $\sum_{i=0}^{j-1} \Gamma_{ji} = 0$.

Obviously, the rules of a Subtraction game are fully determined by such matrix Γ , so hereafter we use letter Γ to denote a corresponding game. We also reserve the name n to denote the initial number of stones. This number n also corresponds to the dimension of the matrix Γ . Note that there are only n(n+1)/2 meaningful bits in the matrix Γ , as a player cannot increase the number of stones in the heap or leave it as is: $\Gamma_{ji} = 0$ for all $j \leq i$.

Finally, we note that the selected payoff function is not a must. It may be arbitrary, provided that each player has strict preferences over the set of all k possible endings. Otherwise, if some preferences are not strict, the concept of optimal behavior will not be well-defined, and the required assumption of optimal players will fail.

2.2 Winning Function

Let \mathcal{G} be a set of lower-triangular binary matrices of size $n \times n$, with rows numbered from 1 to n and columns numbered from 0 to n - 1.

We define a winning function WIN : $\mathcal{G} \otimes \mathcal{N}_n \to \{\$w\}_{0 \leq w < k}$, such that WIN $(\Gamma, j) = \$w$ iff a player gets payoff \$w given position "j stones" in game Γ , under assumption of optimal players:

$$\operatorname{WIN}\left(\Gamma,j\right) = \begin{cases} \$0 & \text{if } j = 0, \\ \$0 & \text{if } \sum_{i} \Gamma_{ji} = 0, \\ \$ \left(\max_{i:\Gamma_{ji}=1} \operatorname{WIN}\left(\Gamma,i\right) - 1\right) \bmod k & \text{otherwise.} \end{cases}$$

We also use notation WIN $(\Gamma) = WIN(\Gamma, n)$ for the value of game Γ .

2.3 Properties of Subtraction Games

In this work we stick to the conventional terminology of [11, Section 2.3] and [9, Section 2.1], and use the following definitions.

We call a game Γ losing if the first player loses it assuming other players are optimal:

$$WIN\left(\Gamma\right) = \$0. \tag{1}$$

We call a game Γ balanced if the values of WIN (Γ, j) are uniformly distributed over the set $\{\$w\}_{0 \le w \le k}$:

$$\forall w : \left| \# \left\{ j : \operatorname{WIN}\left(\Gamma, j\right) = \$w \right\}_{1 \le j \le n} - \frac{n}{k} \right| \le o\left(\frac{n}{k}\right).$$

$$(2)$$

When considering a random balanced game we hereafter implicitly bear in mind the following procedure of picking a game:

- 1. Assign each position "j stones" one of the values from $\{\$0, \$1, \ldots, \$(k-1)\}$ in the uniform fashion.
- 2. Assign position "0 stones" value \$0.
- 3. Initially assign $\Gamma = [0]_{ii}$.
- 4. For each position "j stones" with value w put $\Gamma_{ji} = 1$ with probability 1/2 whenever i < j and position "i stones" is assigned one of the values $w = (w+1) \mod k$, $w = (w-1), \ldots, w$.
- 5. Additionally, for each position "j stones" with value w put $\Gamma_{ji} = 1$ for one i such that position "i stones" is assigned value $(w + 1) \mod k$, whenever it was not already done in the previous step.
- 6. If the previous step failed because for some position "j stones" it is not possible to find a position "i stones" with the appropriate value, then start everything from the beginning.

Should one feel that discarding in the last step essentially destroys the uniformity of WIN (Γ, j) , they can at the second step assign each position "w stones", $0 \le w < k$, value $(k - w) \mod k$. This will make the last step obsolete, as no failure can occur, and will preserve the perfect uniformity. Our further observations are valid for either kind of picking a random balanced Subtraction game.

Finally, we call a game \varGamma restricted if in each position at most one move is possible:

$$\forall j : \sum_{i} \Gamma_{ji} \le 1. \tag{3}$$

2.4 Computational Model

To evaluate the complexity of a quantum algorithm, we use the standard form of the quantum query model. It is a generalization of the decision tree model of classical computation that is commonly used to lower bound the amount of time required by a computation.

Let $f: D \to \{0, 1\}, D \subseteq \{0, 1\}^n$ be an n variable function we wish to compute on an input $x \in D$. We have an oracle access to the input x — it is realized by a specific unitary transformation usually defined as $|i\rangle|z\rangle|w\rangle \to |i\rangle|z \oplus x_i\rangle|w\rangle$ where the $|i\rangle$ register indicates the index of the variable we are querying, $|z\rangle$ is the output register, and $|w\rangle$ is some auxiliary work-space. An algorithm in the query model consists of alternating applications of arbitrary unitaries independent of the input and the query unitary, and a measurement in the end. The smallest number of queries for an algorithm that outputs f(x) with probability $\geq \frac{2}{3}$ on all x is called the quantum query complexity of the function f and is denoted by Q(f). In this paper, as running time of an algorithm, we mean a number of queries to oracle.

More information on quantum computation and query model can be found in [2,1].

To distinguish ordinary deterministic and randomized complexities from the quantum complexity, they are traditionally called by one term *classical complexity*.

3 Balanced Subtraction Games

3.1 Classical Query Complexity

In this subsection we limit our considerations with the number of players k = 3, for the sake of simplicity:

- \star a player who cannot make a move at the end of a play gets \$0;
- \star the previous player who managed to make the last move gets \$2;
- \star the previous-to-previous player gets \$1.

All similar results also hold for arbitrary k but require larger formulations, which in our mind are redundant for understanding.

Lemma 1. Let Γ be a losing balanced Subtraction game Γ picked uniformly at random. Let game Γ' differ from Γ in exactly one random bit of their binary representations: HAMMINGDISTANCE $(\Gamma, \Gamma') = 1$.

Then $\Pr\left[\operatorname{Win}\left(\Gamma'\right) \neq \operatorname{Win}\left(\Gamma\right)\right] \geq 1/18.$

Proof. First, we mention the following three facts about balanced losing games.

★ Balancedness (2) of Γ implies:

$$\Pr_{j}\left[\operatorname{WIN}\left(\Gamma,j\right) = \$1\right] = \Pr_{j}\left[\operatorname{WIN}\left(\Gamma,j\right) = \$0\right] = \frac{1}{3}.$$
(4)

* Losingness (1) of Γ implies for each j, 0 < j < n:

either
$$\Gamma_{nj} = 0$$
 or WIN $(\Gamma, j) =$ \$1, or both; (5)

otherwise the first player would be able to take n - j stones in the first turn and thus have a positive payoff.

 \star Assuming Γ to be picked at random, losingness (1) implies

$$\Pr\left[\Gamma_{nj} = 1 \mid \operatorname{Win}\left(\Gamma, j\right) = \$1\right] = \frac{1}{2},\tag{6}$$

since possibility or impossibility of a (worst possible) move which leads to position "j stones" with WIN $(\Gamma, j) =$ 1 does not affect the value of any position.

Now let $\Gamma'_{ji} \neq \Gamma_{ji}$ for some pair of indices j, i picked at random, s.t. $0 \leq i < j \leq n$. Then we are interested in the value of

$$\Pr_{j,i} \left[\operatorname{Win} \left(\Gamma' \right) \neq \operatorname{Win} \left(\Gamma \right) \right].$$

Let us consider four possible cases for j and i to evaluate this probability. Readers who only care about large values of n, are welcome to skip all but the last case, as the former ones are highly unlikely to happen for random j and i. 1. j = n, and i = 0.

Inversion of bit Γ_{n0} changes the value of game Γ with certainty, since Γ is losing (1), so and $\Gamma_{n0} = 0$, but Γ' with $\Gamma'_{n0} = 1$ can be won by taking all n stones in the first turn:

$$\Pr\left[\operatorname{Win}\left(\Gamma'\right)\neq\operatorname{Win}\left(\Gamma\right)\right]=1$$

2. j = n, and i > 0.

(4) \wedge (5) implies that $\Pr\left[\text{WIN}\left(\Gamma,i\right) \neq \$1 \wedge \Gamma_{ni} = 0\right] = 1 - \frac{1}{3} = \frac{2}{3}$. Under this condition, inversion of bit Γ_{ni} changes the value of game Γ from \$0 to \$ (WIN $(\Gamma, i) - 1$) mod 3 > \$0:

$$\Pr\left[\operatorname{Win}\left(\Gamma'\right)\neq\operatorname{Win}\left(\Gamma\right)\right]=\frac{2}{3}$$

3. j < n, and i = 0.

(4) \wedge (5) \wedge (6) implies that $\Pr[\text{WIN}(\Gamma, j) = \$1 \wedge \Gamma_{nj} = 1] = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ Here WIN $(\Gamma, j) = \$1$ means that $\Gamma_{j0} = 0$, and inversion of this bit changes the value of WIN (Γ, j) from \$1 to \$2. And $\Gamma_{nj} = 1$ means that the value of WIN (Γ) is at least (WIN $(\Gamma, j) - \$1$) mod k. These two facts together mean that inversion of bit Γ_{j0} changes the value of WIN (Γ, n) from \$0 to \$1. Therefore:

$$\Pr\left[\operatorname{Win}\left(\Gamma'\right)\neq\operatorname{Win}\left(\Gamma\right)\right]=\frac{1}{6}$$

4. 0 < i < j < n.

 $\begin{array}{l} (4) \wedge (5) \wedge (6) \text{ implies that } \Pr \left[\operatorname{WIN} \left({\varGamma,j} \right) = \$1 \wedge \operatorname{WIN} \left({\varGamma,i} \right) = \$0 \wedge {\varGamma_{nj}} = 1 \right] = \\ \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{18}. \text{ Here } \operatorname{WIN} \left({\varGamma,j} \right) = \$1 \wedge \operatorname{WIN} \left({\varGamma,i} \right) = \$0 \text{ means that } {\varGamma_{ji}} = 0, \\ \text{and inversion of this bit changes the value of } \operatorname{WIN} \left({\varGamma,j} \right) \text{ from } \$1 \text{ to } \$2. \text{ And } \\ {\varGamma_{nj}} = 1 \text{ means that the value of } \operatorname{WIN} \left({\varGamma,j} \right) \text{ is at least } \left(\operatorname{WIN} \left({\varGamma,j} \right) - \$1 \right) \text{ mod } k. \\ \text{These two facts together mean that inversion of bit } {\varGamma_{ji}} \text{ changes the value of } \\ \operatorname{WIN} \left({\varGamma,n} \right) \text{ from } \$0 \text{ to } \$1. \text{ Therefore:} \end{array}$

$$\Pr\left[\operatorname{Win}\left(\Gamma'\right)\neq\operatorname{Win}\left(\Gamma\right)\right]=\frac{1}{18}$$

In either case we have that $\Pr\left[\operatorname{Win}\left(\Gamma'\right) \neq \operatorname{Win}\left(\Gamma\right)\right] \geq \frac{1}{18}$.

Theorem 1. There is no deterministic or randomized algorithm for solving function WIN faster than in $\Omega(n^2)$ steps.

The proof of this theorem is next to the obvious implication of Lemma 1 and is identical to [11, Theorem 1]. We refer to the aforementioned work for the details and for the remarks on the eligibility of this proof, which are also relevant here.

3.2 Quantum Algorithm

In this subsection we suggest a quantum algorithm for solving an arbitrary k-player Subtraction game. We assume the reader to be familiar with the basics of quantum computing and, in particular, with Grover Search algorithm [7,3]. Among other problems, this algorithm is also applicable for searching in directed acyclic graphs (DAGs) [10,4]. In this paper we apply it to Subtraction games which essentially are games on DAGs: if a game-representing binary Γ is treated as an adjacency matrix of DAG G = (V, E), then its set V corresponds to n positions of the game, and its set E corresponds to all the legal moves.

The algorithm determines an optimal move in each possible position, thus providing a strong solution of a game. But as we are interested in the value of WIN (Γ) only, we provide a solution which only returns the value of a game. The algorithm searches for the maximum among directly accessible vertices ADJ $[j] \stackrel{\text{def}}{=} \{i : \Gamma_{ji}\}$. This algorithm has two important properties:

- * its expected running time is $O\left(\sqrt{\deg j}\right)$, where $\deg j \stackrel{\text{def}}{=} |\operatorname{ADJ}[j]|$ is the number of vertices directly accessible from the vertex j;
- \star it returns a vertex *i'* with the maximal value of WIN with a constant probability (say 0.5) if there exist one or more vertices with maximal values.

In Algorithm 1, we use Dürr-Høyer Algorithm [5] for minimum search in the form of GROVER_MAX subroutine which returns maximum value among its arguments. We store the search results in the array w and reuse them in all the subsequent searches.

Algorithm 1 Quantum algorithm for solving a k -player Subtraction game Γ	
$w_0 \leftarrow \$0$	
for $j = 1 \dots n$ do	$\triangleright O(n)$
$w_j \leftarrow \$0$	
for $z = 1 \dots 2 \cdot \log_2 n$ do	$\triangleright O\left(\log n\right)$
$w_j \leftarrow \max(w_j, \text{GROVER_MAX} \{\$(w_i - 1) \mod k \mid i \in \text{Adj}[j]\})$	$\triangleright O\left(\sqrt{\deg j}\right)$
end for	
end for	
return w_n	

Theorem 2. Algorithm 1 computes WIN (Γ) in expected running time $O\left(\sqrt{n|E|}\log n\right)$ and with error probability $\epsilon \lesssim 1/n$.

Proof. The correctness of the algorithm is obvious: each of the variables w_j (for j running from 1 to n) is assigned a value WIN (Γ, j) according to the definition of the function WIN (Γ, j) .

The time complexity follows from Cauchy-Bunyakovsky-Schwarz inequality:

$$\sum_{j=1}^{n} \sqrt{\deg j} \le \sum_{j=1}^{n} \sqrt{\mathbb{E}_j \left[\deg j\right]} = \sum_{j=1}^{n} \sqrt{|E|/n} = \sqrt{n |E|}.$$

The probability of error in evaluating one particular w_j is $2^{-2\log_2 n} = 1/n^2$, so the probability of no error at all among evaluations of w_1, \ldots, w_n is $(1-1/n^2)^n \gtrsim 1-1/n$.

We note that, for a random Subtraction game, the expected number of edges $\mathbb{E}[|E|] = \Theta(n^2)$, and then we conclude that, while the best classical algorithms require time $\Theta(n^2)$ to solve a Subtraction game, there exists a polynomially faster quantum algorithm which runs in time $O(n^{3/2} \log n)$.

The exact-time algorithm for a small number of players. If k is a small constant, one can apply a quantum algorithm that works in exact time $O(\sqrt{n|E|}\log n)$ (in contrast to Algorithm 1 which has the same evaluation for the expected running time). Algorithm 2 runs Grover's Search k - 1 times instead of running one search for the maximum. At the t-th step the value t is to be searched for among the values from the adjacent vertices, for t running from (k - 1) down to 0. Obviously, the first found value is equal to the maximal payoff available in the considered position "j stones". Subroutine GROVER_t in Algorithm 2 searches for the value t among its arguments and returns True with probability 0.5 when there is such value, and False otherwise. It has to be run $2 \cdot \log_2 k \cdot \log_2 n$ times to amplify the probability of success in case if the arguments contain value t.

Algorithm 2 Quantum algorithm for solving a k-player Subtraction game Γ , where k is a small constant

 $w_0 \leftarrow \$0$ for $j = 1 \dots n$ do $\triangleright O(n)$ $w_j \leftarrow \$0$ $t \leftarrow \$ (k-1)$ while $t > \$0 \land w_j = \0 do for $z = 1 \dots 2 \cdot \log_2 n \cdot \log_2 k$ do $\triangleright O(\log n)$ if $\operatorname{GROVER}_t \{ \$ (w_i - 1) \mod k \mid i \in \operatorname{ADJ}[j] \}$ then $\triangleright O\left(\sqrt{\deg j}\right)$ $w_j \leftarrow t$ end if end for $t \leftarrow t - \$1$ end while end for return w_n

Theorem 3. If k is a small constant, Algorithm 2 computes WIN (Γ) in exact running time $O\left(\sqrt{n |E|} \log n\right)$ and with error probability $\epsilon \leq 1/n$.

4 Restricted Subtraction Games

4.1 Concept of Solution

Restricted Subtraction games are, in some sense, degenerated, as players essentially have no choice in either position. Nevertheless, these games are of natural interest in terms of the computational complexity of Boolean functions. Namely, they became the first functions formulated in terms of a game, that demonstrate polynomial separation between exact quantum query complexity and classical query complexity.

We note that the adaptive deterministic query complexity of WIN (Γ) for a restricted Subtraction game Γ is n. The upper bound n follows from a very simple analysis of Algorithm 3, the lower bound n also is obvious.

Algorithm 3 Computing WIN (Γ) for a restricted Subtraction game Γ

```
w_n \leftarrow \$0

j \leftarrow n

for i = n - 1, \dots, 0 do

if \Gamma_{ji} = 1 then

j \leftarrow i

w_n \leftarrow \$ (w_n - 1) \mod k

end if

end for

return w_n
```

Instead of computing WIN (Γ) for a restricted Subtraction game Γ , [9] aims for a more ambitious problem of solving all positions of a game, i.e. of finding vector $W = [WIN(\Gamma, j)]_{j}$.

4.2 Classical Query Complexity

Theorem 4. Classical query complexity of computing $[WIN(\Gamma, j)]_j$ for a kplayer restricted Subtraction game Γ is $\Theta(n^2)$.

The proof is identical to one of [9, Theorem 1], which was formulated for the two-player games, but is valid also for the multiplayer games.

4.3 Quantum Algorithm

To solve a restricted Subtraction game, we modify Algorithm 1 according to the idea from [9]. We run *exact* Grover's search for a non-zero element in each row. Exact Grover's search [12] is a modification of Grover's algorithm, which returns the position of the non-zero element in a binary string with Hamming weight 1, or *False* if its Hamming weight is 0. It cannot handle strings with a bigger

Hamming weight, but for the promised input, it works in exact time $O(\sqrt{n})$ for an *n*-bit binary string, and with no errors. Subroutine EXACT_GROVER of Algorithm 4 refers to the exact Grover's search. In contrast to Algorithms 1 and 2, this subroutine has to be called just once as its result is exact and does not need to be amplified.

Algorithm 4 Quantum algorithm for solving a restricted	ed k-player game Γ
$w_0 \leftarrow \$0$	
for $j = 1 \dots n$ do	$\triangleright O(n)$
$t \leftarrow \text{Exact}_\text{Grover} \{ \text{Adj} [j] \}$	$\triangleright O\left(\sqrt{\deg j}\right)$
$w_j \leftarrow \$ (w_t - 1) \mod k$	
end for	
return w	

Theorem 5. Algorithm 4 computes WIN (Γ) in exact running time $O(n^{1.5})$ and with no error.

The theorem follows from the properties of exact Grover's search and the evaluation of $O(\sqrt{\deg j})$ as in the proof of Theorem 2.

5 Conclusion

Recent results in quantum game theory have stepped into the field of combinatorial games. In this work we generalized several of these results for solving multiplayer combinatorial games. In particular, we established several upper bounds for quantum query complexity, which generally correspond to the running time of a quantum algorithm. We also derived several classical lower bounds for these problems. We did not focus on the classical upper bounds, but they obviously coincide with the lower bounds, which can be shown just by describing the straightforward dynamic programming approach.

The polynomial separation between quantum and classical complexities was shown using different kinds of games and different concepts of solution, but all of them engage Subtraction games for the demonstration of the power of quantum algorithms in combinatorial game theory. Perhaps, one should expect better and more general bounds to emerge for the quantum complexity of solving Subtraction games. Of course, we hope also for detecting other examples of games with quantum-smaller-than-classical complexity.

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