Distributed Localization of Wireless Sensor Network Using Communication Wheel

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Abstract

We study the *network localization problem*, i.e., the problem of determining node positions of a wireless sensor network modeled as a unit disk graph. In an arbitrarily deployed network, positions of all nodes of the network may not be uniquely determined. It is known that even if the network corresponds to a unique solution, no polynomial-time algorithm can solve this problem in the worst case, unless RP = NP. So we are interested in algorithms that efficiently localize the network partially. A widely used technique that can efficiently localize a uniquely localizable portion of the network is *trilateration*: starting from three *anchors* (nodes with known positions), nodes having at least three localized neighbors are sequentially localized. However, the performance of trilateration can substantially differ for different choices of the initial three anchors. In this paper, we propose a distributed localization scheme with a theoretical characterization of nodes that are guaranteed to be localized. In particular, our proposed distributed algorithm starts localization from a *strongly interior node* and provided that the subgraph induced by the strongly interior nodes is connected, it localizes all nodes of the network except some *boundary nodes* and *isolated weakly interior nodes*.

1 Introduction

A wireless sensor network (WSN) is a wireless network consisting of a large number of small autonomous sensors spatially distributed in a region to monitor physical or environmental parameters. The sensor nodes are low-cost, low-power, autonomous, multi-functional devices equipped with sensing, processing, and communication capabilities. The knowledge of the physical location of sensor nodes is essential in many applications where the geographical information of the sensed data is important, for example, event detection, environment and habitat monitoring, target tracking, pervasive medical care, etc. The positional information of the nodes also supports many fundamental location-aware protocols, like

geographic routing, topology control, coverage, etc. One method of determining the location of the nodes is by equipping the sensor nodes with Global Positioning System (GPS). However, the installation of GPS on each node of a large scale WSN is expensive and the power consumption of GPS reduces the battery life of the sensor nodes. Moreover, it is not suitable in dense forests, underground or indoor environment where GPS signals are unavailable. Therefore, novel schemes have been proposed to determine the positions of the nodes in a network where only some special nodes called *anchors* are aware of their positions with respect to some global coordinate system (e.g., [1, 4-6, 21, 24]). In these schemes, the nodes can measure the distances to their neighboring nodes and using these distance information they try to determine their positions. This process of computing the positions of the nodes is called *range-based network localization* or simply *network localization*.

The network localization problem can be abstracted as the following: given a weighted graph with edge weights equal to the distances between the respective nodes and coordinates of some nodes, called anchors, with respect to some coordinate system, we have to compute the coordinates of all other nodes in that coordinate system. A network, with the given positions of anchors and distances between adjacent nodes, is said to be uniquely localizable if all nodes of the network have unique positions consistent with the given data, i.e., there is a unique solution. Obviously, if the given instance corresponds to multiple feasible solutions, the actual positions of the nodes can not be determined. The unique localizability of a network is completely determined by certain combinatorial properties of the network graph and the number of anchors. Graph rigidity theory [8, 13, 14] provides the following necessary and sufficient condition for unique localizability [8]: a network is uniquely localizable if and only if it has at least 3 anchors and the network graph is *globally rigid* (See Section 2.3 for definition). However, unless a network is highly dense and regular, it is unlikely that the network is globally rigid. But even if a network is not globally rigid as a whole, a large portion of the network may be globally rigid. For the remaining nodes, there are multiple feasible solutions and hence, their actual positions can not be determined. In the decision version of the problem, also known as GRAPH EMBEDDING or GRAPH REALIZATION problem, given a weighted graph we have to determine whether there is an embedding of the graph in Euclidean plane so that the distances between the adjacent vertices are equal to the edge weights. This problem has been shown to be strongly NP-hard [25]. In [8], it is shown that the problem remains NP-hard even when the graph is globally rigid. However, these results are for general graphs. In a sensor network, only nodes that are within a certain communication range, say r, can measure their relative distances. Therefore, the network can be better modeled as a unit disk graph: two nodes are adjacent if and only if their distance is $\leq r$. In this version of the problem, apart from the coordinates of the anchors and the distances between the adjacent nodes, we have a third type of information: the distances between the non-adjacent nodes are > r. The decision version of this problem, also known as UNIT DISK GRAPH RECONSTRUCTION problem, is that given a weighted graph with weights < r, we have to determine whether there is an embedding of the graph in Euclidean plane so that 1) the distances between the adjacent vertices are equal to the edge weights, and 2) the distance between any pair of non-adjacent nodes is > r. It is shown in [2] that UNIT DISK GRAPH $\operatorname{Reconstruction}$ is NP-hard. Therefore, there is no efficient algorithm that solves the localization problem in the worst case unless P = NP. It is further shown in [2] that a similar result holds even for instances that have unique reconstructions: there is no efficient randomized algorithm that solves the localization problem even for instances that have unique reconstructions unless RP = NP.

Since a real life instance may not have unique solution and even if it has, it is unlikely that there is an efficient algorithm that solves the problem, we are interested in efficient heuristics that partially localize the network. A very popular technique is *trilateration* which efficiently localizes a globally rigid subgraph of the network. It is based on the simple fact that the position of a node can be determined from its distance from three non-collinear nodes with known coordinates. The algorithm starts with at least three anchor nodes and then nodes adjacent to at least three nodes with known coordinates are sequentially localized. It is computationally efficient and very easy to implement in distributed setting, thus widely used in practice. In this paper, we are interested in *anchor-free localization*, i.e., there are no anchor nodes. Since for localization at least three anchor nodes are necessary, in the anchor-free case, some three mutually adjacent nodes of the network fix their coordinates (respecting their mutual

distances) in some virtual coordinate system. These three nodes play the role of anchors. However, in case of trilateration, the performance of the algorithm can drastically differ for different choices of the initial three nodes. In this paper, we address this issue and propose a distributed anchor-free localization scheme with a theoretical characterization of nodes that are guaranteed to be localized. In our approach, a node, based on its local information, can categorize itself as either *strongly interior*, *non-isolated weakly interior*, *isolated weakly interior* or *boundary*. Provided that the *strong interior*, i.e., the subgraph induced by the set of strongly interior nodes, is connected, one strongly interior node is chosen by a leader election protocol. Our sequential localization algorithm starts from that strongly interior node, and it is theoretically guaranteed to localize all nodes except some boundary and isolated weakly interior nodes. Due to the space restrictions, it is not possible to present a comprehensive survey of the large number of works on localization (e.g., [1,3-6,9-12,15,17,21,22,24,26-28,32] etc.) in the literature. The readers are instead referred to the surveys [7,18,20,30] and the references therein.

2 Preliminaries

2.1 Basic Model and Assumptions

The mathematical model of wireless sensor network considered in this work is described in the following:

- A set of n sensors is arbitrarily deployed in \mathbb{R}^2 . Each sensor node has computation and wireless communication capabilities.
- There is a constant r > 0, called the *communication range*, such that any two sensor nodes can directly communicate with each other if and only if the distance between them is ≤ r. This implies that the corresponding communication network can be modeled as a *unit disk graph (UDG)*: two nodes are adjacent if and only if they are at most r distance apart. We assume that this graph is connected. Note that if the graph is not connected, then it is impossible to localize the entire network consistently.
- The euclidean distance between a pair of sensors can be measured directly and accurately if and only if they are at most *r* distance apart. Hence, if a sensor node can directly communicate with another node, then it also knows the distance between them.
- The sensor nodes are assumed to be in general positions, i.e., no three points are collinear. This is not a major assumption, as the nodes of a randomly deployed network are almost always in general positions.

2.2 Definitions and Notations

Let \mathcal{V} be the set of n sensors at positions in \mathbb{R}^2 . The corresponding wireless sensor network can be modeled as an undirected edge-weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$, where

- $\mathcal{V} = \{v_1, \dots, v_n\}$ is the set of sensors,
- $(v_i, v_j) \in \mathcal{E}$, i.e., v_i is adjacent to v_j if and only if $d(v_i, v_j) \leq r$, where r is the communication range of the sensors,
- the edge-weight $w : \mathcal{E} \longrightarrow \mathbb{R}$ is given by $w(v_i, v_j) = d(v_i, v_j)$.

We call G the *underlying network graph* of the wireless sensor network. As mentioned previously, we assume that the graph G is connected.

A sensor node $v \in \mathcal{V}$ is called an *interior node* if for every point $z \in \partial(\mathcal{Z}(v))$, where $\partial(\mathcal{Z}(v))$ is the boundary of $\mathcal{Z}(v)$, we have $z \in \mathcal{Z}(v')$ for some $v' \in \mathcal{V} \setminus \{v\}$. If $v \in \mathcal{V}$ is not an interior node, then it is called a *boundary node*. An interior node $v \in \mathcal{V}$ is said to be a strongly interior node if every node

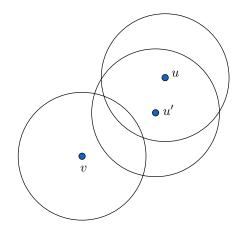


Figure 1: u is not a maximal neighbor of v as $u \preceq_v u'$.

in $\mathcal{N}(v)$ is an interior node. An interior node $v \in \mathcal{V}$ is said to be *a weakly interior node* if at least one node in $\mathcal{N}(v)$ is a boundary node. A weakly interior node is said to be *isolated* if it is not adjacent to any strongly interior node. The subgraph of \mathcal{G} induced by the set of all interior nodes is called the *interior* of \mathcal{G} . Similarly, the subgraph of \mathcal{G} induced by the set of all strongly interior nodes is called the *strong interior* of \mathcal{G} .

If $v, v' \in \mathcal{V}$ are adjacent to each other, then we shall refer to the intersections of $\partial(\mathcal{Z}(v))$ and $\partial(\mathcal{Z}(v'))$ as their *boundary intersections*. We shall denote these boundary intersections as CW(v, v') and CCW(v, v') according to the following rule: if one traverses from CCW(v, v') to CW(v, v') along $\partial(\mathcal{Z}(v))$ in clockwise direction, it sweeps an angle $< \pi$ about the center v.

Given a node v, we define a partial order relation \leq_v on $\mathcal{N}(v)$ as following: for $u, u' \in \mathcal{N}(v)$, $u \leq_v u'$ if and only if $\mathcal{Z}(u) \cap \partial(\mathcal{Z}(v)) \subseteq \mathcal{Z}(u') \cap \partial(\mathcal{Z}(v))$. See Fig. 1. A node $u \in \mathcal{N}(v)$ is said to be a *maximal* neighbor of v if it is a maximal element in $\mathcal{N}(v)$ with respect to \leq_v , i.e., there is no $u' \in \mathcal{N}(v) \setminus \{u\}$, such that $u \leq_v u'$.

2.3 Some Results from Graph Rigidity Theory

In this section, we present some basic definitions and results in graph rigidity. For a detailed exposition on graph rigidity, the readers are referred to [14].

A d-dimensional framework is a pair (G, ρ) , where G = (V, E) is a connected simple graph and the realization ρ is a map $\rho : V \longrightarrow \mathbb{R}^d$. Two frameworks (G, ρ_1) and (G, ρ_2) are said to be equivalent if $d(\rho_1(u), \rho_1(v)) = d(\rho_2(u), \rho_2(v))$, for all $(u, v) \in E$. Frameworks (G, ρ_1) and (G, ρ_2) are said to be congruent if $d(\rho_1(u), \rho_1(v)) = d(\rho_2(u), \rho_2(v))$, for all $u, v \in V$. In other words, two frameworks are said to be congruent if one can be obtained from another by an isometry of \mathbb{R}^d . A realization is generic if the vertex coordinates are algebraically independent over rationals. The framework (G, ρ) is rigid if \exists an $\varepsilon > 0$ such that if (G, ρ') is equivalent to (G, ρ) and $d(\rho(u), \rho'(u)) < \varepsilon$ for all $u \in V$, then (G, ρ') is congruent to (G, ρ) . Intuitively, it means that the framework can not be continuously deformed. (G, ρ) is said to be globally rigid if every framework which is equivalent to (G, ρ) depends only on the graph G, if (G, ρ) is generic. The set of generic realizations is dense in the realization space and thus almost all realizations of a graph are generic. So, we say that a graph G is rigid in \mathbb{R}^2 if every generic realization of G in \mathbb{R}^2 is rigid.

Theorem 1. [14] A graph G is globally rigid in \mathbb{R}^2 if and only if either G is a complete graph on at most three vertices or G is 3-connected, rigid and remains rigid even after deleting an edge.

Theorem 2. [8] If a network has at least 3 anchors and the underlying network graph is globally rigid, then it is uniquely localizable.

The condition of having at least 3 anchors is also necessary for unique localizability in order to rule out the trivial transformations. Since we are considering anchor-free localization, some three mutually adjacent nodes of the network will play the role anchors by fixing their coordinates (respecting their mutual distances) in some virtual coordinate system. The remaining nodes of the network have to find their position according to this coordinate system. It should be noted here that for networks that do not satisfy the condition that two nodes are adjacent if and only if they are within some fixed distance, the condition of having globally rigid underlying network graph is also necessary. In our model, where two nodes are adjacent if and only if the distance between them is at most r, the network can be uniquely localizable even if its underlying network graph is not globally rigid.

3 Construction of a Globally Rigid Subgraph Using Communication Wheels

In this section, we shall show that if the strong interior is connected, then the network has a globally rigid subgraph containing all strongly interior nodes, and all non-isolated weakly interior nodes. The proof is constructive and will lead to our localization algorithm presented in section 4.

We first present some results that will be frequently used in the paper. Lemmas 2-5 follow from elementary geometric arguments.

Lemma 1. Let v_1 be an interior node and $v_2 \in \mathcal{N}(v_1)$. Then

- 1. $CCW(v_1, v_2) \in \mathcal{Z}(v_3)$ for some $v_3 \in \mathcal{N}(v_1) \setminus \{v_2\}$, such that $CCW(v_1, v_3) \notin \mathcal{Z}(v_2)$,
- 2. $CW(v_1, v_2) \in \mathcal{Z}(v_4)$ for some $v_4 \in \mathcal{N}(v_1) \setminus \{v_2\}$, such that $CW(v_1, v_4) \notin \mathcal{Z}(v_2)$

Proof. It is sufficient to prove only the first part. We shall prove by contradiction. So, assume that there is no such node in $\mathcal{N}(v_1) \setminus \{v_2\}$. Let $P = CCW(v_1, v_2)$. Let us partition the set of neighbors of v_1 into two sets as: $A = \{v \in \mathcal{N}(v_1) \mid P \in \mathcal{Z}(v)\}$ and $B = \{v \in \mathcal{N}(v_1) \mid P \notin \mathcal{Z}(v)\}$. $A \neq \emptyset$, since $v_2 \in A$. $B \neq \emptyset$, because the diametrically opposite point of P on $\partial(\mathcal{Z}(v_1))$ must be covered by some node which does not cover P.

Fix the ray $v_1 P$ as a reference axis. Now for each $v \in \mathcal{N}(v_1)$, shoot rays from v_1 passing through $CCW(v_1, v)$ for $v \in A$ and $CW(v_1, v)$ for $v \in B$. For each $v \in \mathcal{N}(v_1)$, let θ_v be the angle formed by the corresponding ray measured counterclockwise from the reference axis $v_1 P$. Let $\theta = min\{\theta_v \mid v \in B\}$. We must have $\theta > 0$, since for any $v \in B$, $\theta_v > 0$. Also, it implies from our hypothesis that $max\{\theta_v \mid v \in A\} = 0$. Then clearly any point on $\partial(\mathcal{Z}(v_1))$ making an angle in between $(0, \theta)$ with the ray $v_1 P$ is not covered by any neighbor of v_1 (See Fig. 2). This contradicts the fact that v_1 is an interior node.

Lemma 2. If u and u' are two distinct neighbors of $v \in \mathcal{V}$ such that $u \leq_v u'$, then d(u, v) > d(u', v).

Lemma 3. For distinct $v, u, u' \in \mathcal{V}$, $u \preceq_v u' \Leftrightarrow v \preceq_u u'$.

Lemma 4. For distinct $v, u \in V$, u is a maximal neighbor of v if and only if v is a maximal neighbor of u.

Lemma 5. For distinct $v, u, u' \in \mathcal{V}$, $u \preceq_v u' \Rightarrow u \not\preceq_{u'} v$.

A wheel graph [29] of order n or simply an n-wheel, $n \ge 3$, is a simple graph which consists of cycle of order n and another vertex called the *hub* such that every vertex of the cycle is connected to the hub. The vertices on the cycle are called the *rim vertices*. An edge joining a rim vertex and the hub is

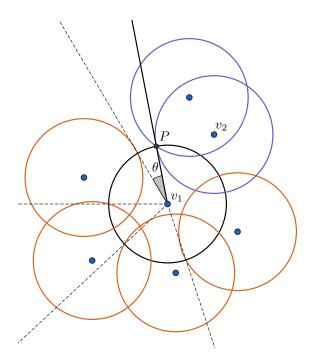


Figure 2: Illustration of the constructions in the proof of Lemma 1. The purple and the orange circles correspond to the boundaries of the communication zones of the nodes in the set A and B respectively.

called a *spoke*, and an edge joining two consecutive rim vertices is called a *rim edge*. By Theorem 1, it follows that a wheel is globally rigid.

The most crucial part of our algorithm is the construction of a special structure called the *communication wheel*. The definition of communication wheel closely resembles to that of *sensing wheel* used in [23], where the authors devised a wheel based centralized sequential localization algorithm for a restricted class of sensing covered networks over a convex region.

Communication wheel: For any interior node $v \in V$, we define a *communication wheel* of v as a subgraph W of \mathcal{G} such that

- 1. W is a wheel graph with v as the hub and the rim nodes $\{v_1,\ldots,v_m\}$ being maximal neighbors of v
- 2. $CCW(v, v_i) \in \mathcal{Z}(v_{i+1})$ and $CW(v, v_i) \in \mathcal{Z}(v_{i-1})$, for $i = 1, \ldots, m$, where v_{m+1} means v_1 and v_0 means v_m .

For a rim node v' of a communication wheel W of v, we can denote the two neighboring rim nodes of v' as $CCW_W(v')$ and $CW_W(v')$ so that $CCW(v, v') \in \mathcal{Z}(CCW_W(v'))$ and $CW(v, v') \in \mathcal{Z}(CW_W(v'))$.

Lemma 6. If W is a communication wheel of v, then $\partial(\mathcal{Z}(v)) \subset \bigcup_{u \in \mathcal{V}(W) \setminus \{v\}} \mathcal{Z}(u)$.

Proof. Follows immediately from the definition of communication wheel.

Theorem 3. If $v \in V$ is an interior node and v_1 a maximal neighbor of v, then v has a communication wheel W having v_1 as a rim node.

Proof. First observe that for any maximal neighbor v' of v, $|\partial(\mathcal{Z}(v)) \cap \partial(\mathcal{Z}(v'))| = 2$, i.e., CW(v, v') and CCW(v, v') are distinct points. If not, then suppose that v' is a maximal neighbor of v such that $\partial(\mathcal{Z}(v))$ and $\partial(\mathcal{Z}(v'))$ intersect at a single point, say P. Then by Lemma 1, there is another neighbor

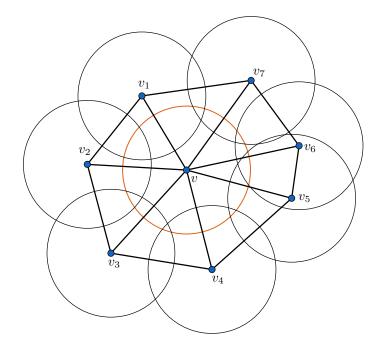


Figure 3: A communication wheel of v with rim nodes $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7 .

of v, say v'', such that $P \in \mathcal{Z}(v'')$. Hence we have $v'' \neq v'$, such that $\partial(\mathcal{Z}(v)) \cap \mathcal{Z}(v') = \{P\} \subset \partial(\mathcal{Z}(v)) \cap \mathcal{Z}(v'')$. This contradicts the fact that v' is a maximal neighbor of v.

Now take any maximal neighbor v_1 of v. By Lemma 1, choose a maximal $v_2 \in \mathcal{N}(v) \setminus \{v_1\}$, such that $CCW(v, v_1) \in \mathcal{Z}(v_2)$ and $CCW(v, v_2) \notin \mathcal{Z}(v_1)$. Notice that $CW(v, v_1) \notin \mathcal{Z}(v_2)$, because otherwise $v_1 \preceq_v v_2$. Since $CCW(v, v_2) \notin \mathcal{Z}(v_1)$, by again invoking Lemma 1, we can choose a maximal $v_3 \in \mathcal{N}(v) \setminus \{v_1, v_2\}$, such that $CCW(v, v_2) \in \mathcal{Z}(v_3)$ and $CCW(v, v_3) \notin \mathcal{Z}(v_2)$. Continuing in this manner, after some m steps we shall find $v_m \in \mathcal{N}(v) \setminus \{v_1, \ldots, v_{m-1}\}$, such that $CCW(v, v_{m-1}) \in \mathcal{Z}(v_m)$ and $CW(v, v_1) \in \mathcal{Z}(v_m)$. It is easy to see that a communication wheel of v can be formed with $\{v_1, \ldots, v_m\}$ as rim nodes.

Corollary 1. $v \in V$ is an interior node if and only if it has a communication wheel.

Lemma 7. Let $v \in V$ be an interior node and W be a communication wheel of v. If $u \in V$ is a neighbor of v, then u is either a rim node of W or adjacent to some rim node of W.

Proof. Easy to see.

Lemma 8. Let $v \in V$ be an interior node and W be a communication wheel of v. If $u \in V$ is a neighbor of v, which is adjacent to exactly one rim node of W, say u', then $u \preceq_v u'$.

Proof. Easy to see.

Lemma 9. Let $v \in V$ be a strongly interior node and u a neighbor of v. If W_1 is a communication wheel of v, then there is a globally rigid subgraph of \mathcal{G} containing v, u and W_1 .

Proof. If u is a rim node of W_1 , then we are done, since a wheel graph is globally rigid. So, suppose that u is not a rim node of W_1 .

Then by Lemma 7, u is adjacent to a rim node of W_1 , say v_i . If u is adjacent to another rim node, then u can be added to W_1 to form a globally rigid graph. Hence, we assume that u is adjacent to only one rim node of W_1 , i.e., v_i . Then by Lemma 8, we have $u \leq_v v_i$.

Since v is a strongly interior node and v_i is a neighbor of v, v_i must be an interior node. Also, since v_i is a maximal neighbor of v, v is also a maximal neighbor of v_i , by Lemma 4. Hence, by Theorem 3, v_i has a communication wheel W_2 having v as a rim node.

See Fig. 5a. Let $v_{i-1} = CW_{W_1}(v_i)$ and $v_{i+1} = CCW_{W_1}(v_i)$. Also, let $v'_{i+1} = CW_{W_2}(v)$ and $v'_{i-1} = CCW_{W_2}(v)$. Now let $A = CCW(v, v_i) = CW(v_i, v)$ and $B = CW(v, v_i) = CCW(v_i, v)$. So we must have $A \in \mathcal{Z}(v_{i+1}) \cap \mathcal{Z}(v'_{i+1})$ and $B \in \mathcal{Z}(v_{i-1}) \cap \mathcal{Z}(v'_{i-1})$. This implies that v_{i-1}, v'_{i-1} and v_{i+1}, v'_{i+1} are adjacent. So v_{i-1} and v_{i+1} can be added to the list of rim nodes of W_2 and construct a wheel W_3 of v_i having v_{i-1}, v, v_{i+1} as rim nodes. Then the two globally rigid graphs W_1 and W_3 have three nodes common, namely v_{i-1}, v, v_{i+1} . Hence $W_1 \cup W_3$ is globally rigid. See Fig. 5b.

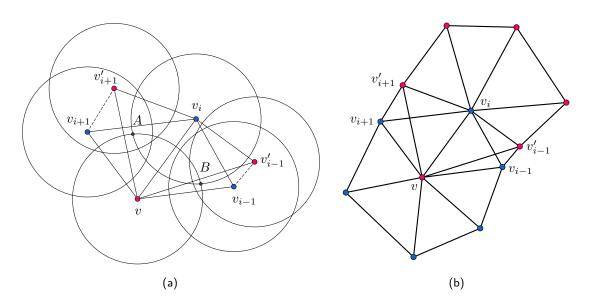


Figure 4: Illustrations supporting the proof of Lemma 9.

Now it is sufficient to prove that u is adjacent to at least three nodes of $W_1 \cup W_3$. Let $\mathcal{R}im(W_3) = \mathcal{V}(W_3) \setminus \{v_i\}$ be the set of rim nodes of W_3 . We have $\partial(\mathcal{Z}(v_i)) \subset \bigcup_{w \in \mathcal{R}im(W_3)} \mathcal{Z}(w)$. Since $u \preceq_v v_i$, we have $u \not\preceq_{v_i} v$, by Lemma 5. In other words, $\partial(\mathcal{Z}(v_i)) \cap \mathcal{Z}(u) \not\subseteq \partial(\mathcal{Z}(v_i)) \cap \mathcal{Z}(v)$. Therefore, we have $\partial(\mathcal{Z}(v_i)) \cap \mathcal{Z}(u) \subset \partial(\mathcal{Z}(v_i)) \subset \bigcup_{w \in \mathcal{R}im(W_3)} \mathcal{Z}(w) \Rightarrow (\partial(\mathcal{Z}(v_i)) \cap \mathcal{Z}(u)) \cap (\bigcup_{w \in \mathcal{R}im(W_3) \setminus \{v\}} \mathcal{Z}(w)) \neq \emptyset$. This implies that u is adjacent to some rim node of W_3 other than v. We already have assumed

 $\neq v$. This implies that u is adjacent to some rim node of W_3 other than v. We already have assumed that u is adjacent to v and v_i . Hence, u is adjacent to at least three nodes of $W_1 \cup W_3$.

Theorem 4. If $v \in V$ is a strongly interior node, then there is a subgraph \mathcal{H}_v of \mathcal{G} containing v such that 1) \mathcal{H}_v contains all neighbors of v, 2) \mathcal{H}_v is globally rigid.

Proof. Let W_1 be a communication wheel of v. Then by Lemma 9, for each neighbor of v not in W_1 , we obtain a globally rigid subgraph of \mathcal{G} containing the neighbor, v and W_1 . So any two of these globally rigid subgraphs have at least three nodes in common. Hence, these graphs constitute to form the desired globally rigid graph.

Theorem 5. If the strong interior of \mathcal{G} is connected, then \mathcal{G} has a globally rigid subgraph \mathcal{R} which contains 1) all strongly interior nodes, 2) all non-isolated weakly interior nodes.

Proof. Choose any strongly interior node v. Denote the subgraph of \mathcal{G} consisting of only the node v as \mathcal{R}_0 . Since the strong interior of \mathcal{G} is connected, every strongly interior node of \mathcal{G} is connected to v by a path consisting strongly interior nodes. The distance of a strongly interior node from v is defined as

the smallest length of such a path. Let m be the maximum distance of a strongly interior node from v. We shall prove the theorem by inductively constructing globally rigid subgraphs $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_m$ where \mathcal{R}_j contains all strongly interior nodes at a distance at most j from v. \mathcal{R}_0 is globally rigid as it is only a singleton node. \mathcal{R}_1 is constructed using Theorem 4. Suppose that $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_j$, $1 \leq j < m$, are already constructed. Now consider a strongly interior node v' at a distance j+1from v. In a smallest path from v to v', let v' be adjacent to v''. Clearly v'' is in \mathcal{R}_i . Since \mathcal{R}_i is globally rigid, v'' is adjacent to at least three nodes v_1, v_2, v_3 in \mathcal{R}_j (\mathcal{R}_j has at least four nodes as it contains the communication wheel of v). By Theorem 4, there is a globally rigid graph containing v''and its neighbors v', v_1, v_2, v_3 . Union of this graph and \mathcal{R}_i is globally rigid as there are at least three nodes in common, namely v'', v_1, v_2, v_3 , etc. Similarly for each strongly interior node at a distance j+1 from v, we extend the subgraph \mathcal{R}_j preserving global rigidity to eventually obtain a globally rigid graph \mathcal{R}_{j+1} containing all the strongly interior nodes at a distance at most j+1 from v. The inductive argument leads to the globally rigid subgraph \mathcal{R}_m which contains all strongly interior nodes in $\mathcal{G}.$ Each non-isolated weakly interior node v''' is adjacent to some strongly interior node in \mathcal{R}_m . For each such v''', again by the same construction, we can extend \mathcal{R}_m preserving global rigidity to include v''', if it is not already in \mathcal{R}_m . The resulting graph is the desired globally rigid subgraph \mathcal{R} .

4 The Localization Algorithm

In the beginning, each node messages its neighbor list along their distances form itself to all its neighbors. Therefore, every node $u \in \mathcal{V}$ knows the neighbors of all its neighbors and also if v is a neighbor of u and w is a neighbor of v, then u knows d(v, w) as well. The three main stages of our algorithm are 1) construction of communication wheel, 2) leader election, and 3) propagation. They are discussed in detail in the following subsections.

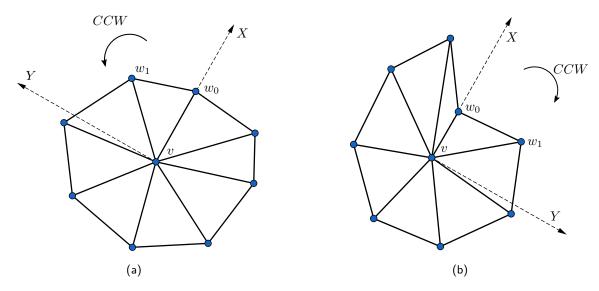
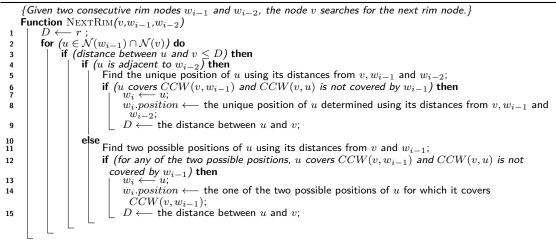


Figure 5: A node v executing Algorithm 1 sets its local coordinate system in such a way that w_1 gets positive Y-coordinate.

Algorithm 1: COMMUNICATION WHEEL	
{	The node v constructs a communication wheel. If it successfully constructs a communication wheel, it declares
	itself as an interior node, or otherwise a boundary node.}
Procedure COMMUNICATIONWHEEL(v)	
1	$w = \lfloor \ \rfloor$;
2	$v.position \leftarrow origin;$
3	$w_0 \leftarrow$ closest neighbor of v ;
4	w_0 position \leftarrow on the X-axis according to the distance between v and w_0 ;
5	$w_1 \leftarrow$ the common neighbor of v and w_0 closest to v that covers a boundary-intersection of v and w_0 , such
	that $w_1 eq v w_0$;
6	if (no such w_1 is found) then
7	$v.type \leftarrow boundary;$
8	break;
9 10	else $w_{1.position} \leftarrow$ one of the two possible positions preserving the distances from v and w_0 such that w_1
10	has positive Y-coordinate:
11	$CCW(v, w_0) \leftarrow$ the boundary-intersection of v and w_0 covered w_1
12	i = 1:
13	do
14	i + +;
15	NEXTRIM (v, w_{i-1}, w_{i-2}) ;
16	while $(w_i \neq null \& w_i \text{ does not cover } CW(v, w_0));$
17	if $(w_i = null)$ then
18	$v.type \leftarrow boundary;$
19	else $v.type \leftarrow interior;$
20	

Algorithm 2: NEXTRIM



4.1 Construction of Communication Wheel

Each sensor node v starts off computations by executing the COMMUNICATIONWHEEL algorithm. The algorithm finds if the node is interior or boundary, and also constructs a communication wheel if it is interior. A pseudocode description of the procedure is presented in Algorithm 1. The algorithm COMMUNICATIONWHEEL is similar to the constructions used in the proof of the Theorem 3. To construct a communication wheel of a node v, if it exists, we first need to find a maximal neighbor. In view of Lemma 2, the closest neighbor of a node is guaranteed to be a maximal neighbor. After finding the closest neighbor, call it w_0 , v assigns its position on the X-axis and itself at the origin. Then it searches for a common neighbor of v and w_0 that covers a boundary-intersection of v and w_0 . If no such node is found, then v is a boundary node. If more than one of such nodes are found, the one closest to v is to be taken. Let us call this node w_1 . Now the distance of w_1 from v and w_0 is known. From this data, there are two possible coordinates for w_1 so that its Y-coordinate is positive. In other words, v sets its local coordinate system in such a way that w_1 gets positive Y-coordinate. Also

set the boundary-intersection of v and w_0 that is covered by w_1 as $CCW(v, w_0)$. In other words v sets 'counterclockwise' to be the direction in which if one rotates a ray, from the origin towards the positive direction of the X-axis, by $\frac{\pi}{2}$, it coincides with the positive direction of the Y-axis. While discussing Algorithm 1, 'counterclockwise' and 'clockwise' will always be with respect to the local coordinate system of node executing the algorithm. Note that since w_0 is a maximal neighbor of v, $CW(v, w_0)$ is not covered by w_1 . After fixing the positions of w_0 and w_1 , the subroutine NEXTRIM is recursively called to find the subsequent rim nodes of the communication wheel. Given two consecutive rim nodes w_{i-1} and w_{i-2} , having positions fixed, NEXTRIM(v, w_{i-1}, w_{i-2}) finds the next rim node w_i . The program terminates when either NEXTRIM reports a failure or returns a node that covers $CW(v, w_0)$. A pseudocode description of the NEXTRIM function is presented in Algorithm 2.

Theorem 6. The algorithm COMMUNICATIONWHEEL is correct, i.e., if v is an interior node then COMMUNICATIONWHEEL(v) constructs a communication wheel of v and declares it as an interior node; and otherwise declares it as a boundary node.

Proof. The algorithm replicates the proof of Theorem 3. The algorithm starts off with fixing a maximal neighbor of v as the first rim node w_0 . Then it recursively finds the rim nodes w_i such that w_i covers $CCW(v, w_{i-1})$ and $CCW(v, w_i)$ is not covered by w_{i-1} . The algorithm terminates when there is no such w_i or when w_i covers $CW(v, w_0)$. Thus in view of the proof of Theorem 3, we only need to show that these steps are correctly executed.

Initialization: The closest neighbor of v is set as w_0 . Hence, w_0 is a maximal neighbor of v by Lemma 2. Note that in order to compute the communication wheel of v, if it exists, first we need to fix the positions of (i.e., assign virtual coordinates to) at least three nodes of the communication wheel, preserving their mutual distances. So, first v is assigned with virtual coordinates (0, 0). If the distance between v and w_0 is d, then the coordinates of w_0 are set as (d, 0). Now we have to check if there is a common neighbor w of v and w_0 that covers a boundary intersection of v and w_0 , and such that $w \not\leq_v w_0$. For any common neighbor w of v and w_0 , this can be easily checked from $d(v, w_0), d(v, w)$ and $d(w, w_0)$. If no such w is found, then v is obviously a boundary node. Otherwise one such node that is closest to v is set as the next rim node w_1 . From $d(v, w_1)$ and $d(w_0, w_1)$, two possible coordinates of w_1 can be found, one with positive Y-coordinate and one with negative Y-coordinate. Then v sets its local coordinate system in such a way that w_1 gets positive Y-coordinate, and hence w_1 covers $CCW(v, w_0)$. Recall that here counterclockwise and clockwise is defined with respect to the local coordinate system of v as described in Section 4.1.

Recursion: After w_0 and w_1 are fixed, the algorithm will recursively call NEXTRIM (v, w_{i-1}, w_{i-2}) to find the next rim node w_i , if it exists. In the for loop (line 2 in Algorithm 2), the common neighbors of v and w_{i-1} are scanned through to find nodes $u \in \mathcal{N}(w_{i-1}) \cap \mathcal{N}(v)$ such that u covers $CCW(v, w_{i-1})$ and CCW(v, u) is not covered by w_{i-1} . Among these nodes, the one closest to v is set as the next rim node w_i . If no such node is found, then v is a boundary node. Notice that in order to check whether u covers $CCW(v, w_{i-1})$ or not, the exact position of u needs to be known. As we scan through $\mathcal{N}(w_{i-1}) \cap \mathcal{N}(v)$, there are two cases to consider:

Case 1. Suppose that u is adjacent to w_{i-2} . Now the positions of v, w_{i-1} and w_{i-2} are known. Hence the position of u can ascertained from its distances from v, w_{i-1} and w_{i-2} . Once the position of u is found, it can be checked whether it covers $CCW(v, w_{i-1})$ and also whether w_{i-1} covers CCW(v, u).

Case 2. Suppose that u is not adjacent to w_{i-2} . Two possible positions of u can be found from its distance from v and w_{i-1} . Call these two possible positions U_1 and U_2 . U_1 and U_2 are mirror images of each other with respect to the line joining v and w_{i-1} . $CCW(v, w_{i-1})$ and $CW(v, w_{i-1})$ are also mirror images of each other with respect to the line joining v and w_{i-1} . Hence if a node at U_1 covers $CCW(v, w_{i-1})$, then a node at U_2 covers $CW(v, w_{i-1})$ as well. Similarly if a node at U_1 covers neither $CCW(v, w_{i-1})$ nor $CW(v, w_{i-1})$, then the same is true for a node at U_2 . So consider the following three possibilities:

Case 2a. If for both positions U_1 and U_2 , no boundary intersection between v and w_{i-1} is covered, then u does not meet the desired criteria that it is to cover $CCW(v, w_{i-1})$.

Case 2b. Suppose that for both positions U_1 and U_2 , both of the boundary intersections between v and w_{i-1} are covered. This can not happen as this would imply that $w_{i-i} \leq v u$ and hence is adjacent to w_{i-2} , contradicting our assumption.

Case 2c. Suppose that u, if situated at U_1 , covers $CCW(v, w_{i-1})$ (and not $CW(v, w_{i-1})$). Hence u, if it is at U_2 , would cover $CW(v, w_{i-1})$ and would not cover $CCW(v, w_{i-1})$. In this case, the algorithm determines the position of u to be U_1 . We shall prove that U_1 is indeed the correct position of u. Suppose on the contrary that the actual position of u is U_2 . First observe that $CW(v, w_{i-1}) \in \mathcal{Z}(w_{i-2}) \cap \partial(\mathcal{Z}(v))$. Otherwise, it implies that $w_{i-2} \preceq_v w_{i-1}$. We argue that this is impossible. For i = 2, it is obvious since $w_0 \not\leq_v w_1$. Recall that w_0 is the closest neighbor, and hence is a maximal neighbor, of v. For i > 2, we assume as induction hypothesis that rim nodes w_{i-1},\ldots,w_1,w_0 are successfully found by the algorithm. Also each w_i , for $j=1,\ldots,i-1$, must satisfy the two criteria: 1) w_i covers $CCW(v, w_{i-1})$ and 2) $CCW(v, w_i)$ is not covered by w_{i-1} . In fact, as mentioned earlier, the algorithm chooses as w_i the closest among the nodes that satisfy these two criteria. So in particular, w_{i-2} is the closest neighbor of v such that 1) w_{i-2} covers $CCW(v, w_{i-3})$ and 2) $CCW(v, w_{i-2})$ is not covered by w_{i-3} . Clearly if $w_{i-2} \preceq_v w_{i-1}$, w_{i-1} also satisfies the two aforesaid conditions. But by Lemma 2, w_{i-1} is closer to v than w_{i-2} . This contradicts the fact that w_{i-2} is the closest node satisfying the two aforesaid conditions. So we have $CW(v, w_{i-1}) \in \mathcal{Z}(w_{i-2})$. Also $CW(v,w_{i-1})\in\mathcal{Z}(u)$ as u is assumed to be at U_2 . Hence $\mathcal{Z}(w_{i-2})\cap\mathcal{Z}(u)
eq\emptyset$. This is a contradiction as u and w_{i-2} are not adjacent.

Termination: It is easy to see that the algorithm terminates.

4.2 Leader Election

Once a node identifies itself as interior or boundary, it announces the result to all its neighbors. Hence, every node can determine if it is a strongly interior node or not. Since the strong interior is connected and the nodes have unique id's, the strongly interior nodes can elect a leader among themselves by executing a leader election protocol [19].

4.3 Propagation

Starting from the leader, different nodes will gradually get localized via message passing. The correctness of the process will follow from the discussions in this subsection and the proofs of Theorem 4 and 5. There are five types of messages that a sensor node can send to another node:

- 1. "I am at"
- 2. "You are at"
- 3. "Construct wheel with me at \ldots and v at \ldots "
- 4. "Construct wheel with me at ..., you at ..., v at ... and find u"
- 5. "*u* is at ...".

The nodes of the network will be localized in the local coordinate system of the leader v_l set during its execution of Algorithm 1. Henceforth, this coordinate system will be referred to as the *global coordinate system*. So the leader first localizes itself by setting its coordinates to (0,0). Any nonleader node u is localized by either receiving a "You are at ..." message or receiving at least three "I am at ..." messages. In the first case, some node has calculated the coordinates of u and has sent it to u. In the second case, u receives the coordinates of at least three neighbors and therefore, can calculate its own coordinates. When a node is localized, it announces its coordinates to all its

neighbors. After setting its coordinates to (0,0) and v_l initiates the localization of \mathcal{H}_{v_l} (See Theorem 4). It first announces its coordinates to all its neighbors via the message "I am at (0,0)". During the construction of its communication wheel, v_l had assigned coordinates to the rim nodes. So v_l sends these coordinates to the corresponding rim nodes via the message "You are at ...". Let us denote the communication wheel of v_l as $\mathcal{W}(v_l)$ and the set of all rim nodes as $\mathcal{R}im(v_l)$. When a rim node receives this message, it sets its coordinates accordingly and announces it to all its neighbors via the message "I am at ...". Notice that a rim node does not need to send this message to v_l . There are multiple such modifications that can be made to reduce the number of messages used in the algorithm. But we do not mention them for simplicity of the description. Now if a neighbor of v_l is adjacent to at least two nodes of $\mathcal{R}im(v_l)$, then it can localize itself, since it will receive "I am at ..." messages from at least three nodes, i.e., one from v_l and at least two from $\mathcal{R}im(v_l)$. But if a neighbor of v_l is adjacent to only one vertex from $\mathcal{R}im(v_l)$, then it may not be localized. To resolve this, v_l computes $|\mathcal{N}(u) \cap \mathcal{R}im(v_l)|$ for all $u \in \mathcal{N}(v_l)$. If it finds a $u \in \mathcal{N}(v_l)$ with $\mathcal{N}(u) \cap \mathcal{R}im(v_l) = \{v_i\}$, it sends the message "Construct wheel with me at ... and v_{i+1} at ..." to v_i , where v_{i+1} is a neighboring rim node of v_i in $\mathcal{W}(v_l)$. When v_i receives this message from v_l , it does the following. Since v_l is a strongly interior node, v_i must be an interior node. Therefore, v_i has already computed the communication wheel $\mathcal{W}(v_i)$ and coordinates of each of its nodes with respect to its local coordinate system. Since v_l is a maximal neighbor of v_i (by Lemma 4), it is adjacent to at least two nodes of $\mathcal{R}im(v_i)$ (by Lemma 8). Hence, v_i can compute the coordinates of v_l with respect to its local coordinate system. Let \mathcal{W}' be the globally rigid graph $v_l \cup \mathcal{W}(v_i)$. Now, from the proof of Theorem 4, it is known that v_{i+1} is adjacent to at least three nodes of \mathcal{W}' . Hence, v_i can also compute the coordinates of v_{i+1} with respect to its local coordinate system. So, v_i has the coordinates of all nodes of $\mathcal{W}'' = v_{i+1} \cup \mathcal{W}'$ with respect to its local coordinate system. Now, v_i will compute the positions of all nodes of \mathcal{W}'' with respect to the global coordinate system set by v_l . Let us call them the *true positions* of the nodes. Note that v_i knows the true positions of at least three nodes of \mathcal{W}'' , namely, itself, v_l , and v_{i+1} . With this information, v_i can determine the formula that transforms its local coordinate system to the global coordinate system. Hence, v_i computes the true positions of all nodes in \mathcal{W}'' and informs them via "You are at ..." messages. Hence, all nodes in \mathcal{W}' will be localized and will announce their locations to all their neighbors. Since u is adjacent to at least three nodes in \mathcal{W}'' (from the proof of Theorem 4), it will also get localized. Therefore, we see that every neighbor of v_l eventually gets localized.

The localization propagates as each strongly interior node localizes its neighbors. However, a strongly interior node v can compute the positions of its neighbors only with respect to its local coordinate system. Hence, in order to compute the true positions (i.e., to perform coordinate transformation), it needs to know its true position and that of at least two neighbors. Hence, when a localized strongly interior node v receives at least two "I am at ..." messages, it starts to localize its neighbors in the following way. Let u be a neighbor of v. If u is adjacent at least two nodes of $\mathcal{R}im(v)$, then v can compute the position of u in terms of its local coordinate system. Otherwise, if $\mathcal{N}(u) \cap \mathcal{R}im(v) = \{v_i\}$, then v sends the message "Construct wheel with me at ..., you at ..., v_{i+1} at ... and find u' to v_i , where v_{i+1} is a neighboring rim node to v_i in $\mathcal{W}(v)$, and positions mentioned in the message are given in local coordinates of v. Again, as v_i is a maximal neighbor of v, by Lemma 4 and 8, v is either in $\mathcal{W}(v_i)$ or adjacent to at least three nodes in $\mathcal{W}(v_i)$. Also, v_{i+1} is either in $\mathcal{W}' = \mathcal{W}(v_i) \cup v$ or adjacent to at least three nodes in \mathcal{W}' (from the proof of Theorem 4). Hence, from the data received from v, v_i can compute the positions of all the nodes in $\mathcal{W}'' = \mathcal{W}' \cup v_{i+1}$ in terms of the local coordinates of v. From the proof of Theorem 4, u is either in \mathcal{W}'' or adjacent to at least three nodes in \mathcal{W}'' . So v_i can compute the position of u in terms of the local coordinate system of v, and then sends the information back to v via the message "u is at ...". Hence, v computes the positions of all its neighbors in its local coordinate system. So, v now knows the positions of three nodes (namely, itself and the two nodes from which it has received " $I \; am \; at \; \dots$ " message) with respect to both its local coordinate system and the global coordinate system set by the leader. Hence, v can find the formula that transformations its local coordinate system to the global coordinate system. Hence, v computes the true positions of all its neighbors and then informs them via "You are at ..." messages. However, it still remains to prove that every non-leader localized strongly interior node v always receives at least two "I am at ..."

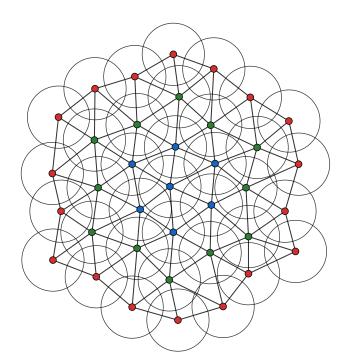


Figure 6: The red, green and blue nodes are respectively boundary, weakly interior and strongly interior nodes. It is easy to see that trilateration does not progress beyond the base step for any choice of the initial triangle. However, our algorithm always localizes all the nodes of the network from any initial strongly interior node. This structure can be extended arbitrarily. Hence for any $n \in \mathbb{N}$, we have a network of size $\geq n$, such that 1) it is always entirely localized by our algorithm, 2) but trilateration fails to localize more than 3 nodes for any choice of the initial triangle.

messages, that triggers the propagation. Since v is localized, either it has received three " $I am at \ldots$ " messages or one " $You \ are \ at \ \ldots$ " message. If it is the first case, then we are done. In the later case, v receives the " $You \ are \ at \ \ldots$ " message from a localized interior node, say v'. Then by Lemma 7, v is either in the communication wheel of v', or adjacent to at least one of its rim nodes. Observe that v' is localized and has also localized all its rim nodes. So, v will get least two " $I \ am \ at \ \ldots$ " messages, as all localized nodes announce their positions. Therefore, all strongly interior and non-isolated weakly interior nodes get localized, while some some boundary nodes and isolated weakly interior nodes may not get localized. Therefore, all strongly interior and non-isolated weakly interior nodes get localized. Also, if a localized weakly interior node receives " $I \ am \ at \ \ldots$ " messages from at least two nodes, it can localize all the rim nodes and also neighbors that are adjacent to at least two rim nodes.

5 Concluding Remarks

Our algorithm works under the condition that the strong interior of the network is connected. Relaxing this condition, it would be interesting to characterize the conditions under which localization starting from different components of the strong interior can be stitched together. It would be also interesting to study impact noisy distance measurement on our algorithm. Our algorithm also works under the strong assumption of uniform communication range. An important direction of future research would be to see if our approach can be extend to networks with sensors having irregular communication range, e.g., quasi unit disk networks [16]. Another problem is to compare the class of networks that are fully localized by our algorithm to those that are fully localized by trilateration. The example in Fig. 6 shows

a class of network in which trilateration does not progress beyond the base step for any choice of the initial triangle, but our algorithm always localizes all the nodes from any initial strongly interior node.

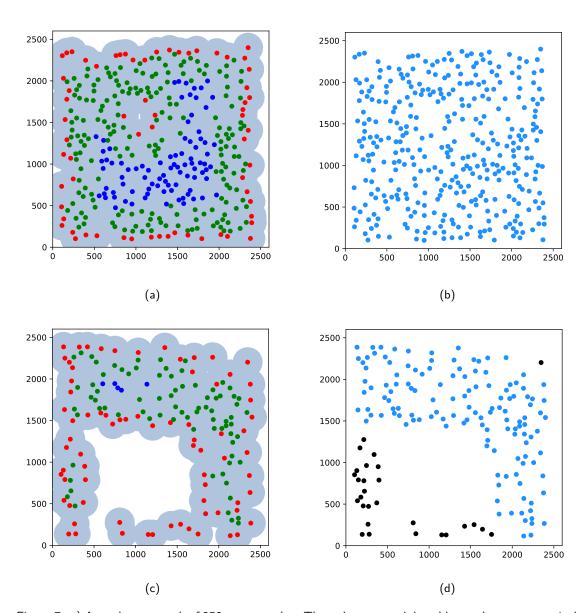
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References

- J. Albowicz, Alvin Chen, and Lixia Zhang. Recursive position estimation in sensor networks. In 9th International Conference on Network Protocols (ICNP 2001), 11-14 November 2001, Riverside, CA, USA, pages 35–43. IEEE Computer Society, 2001. doi: 10.1109/ICNP.2001.992758.
- [2] James Aspnes, David Kiyoshi Goldenberg, and Yang Richard Yang. On the computational complexity of sensor network localization. In Algorithmic Aspects of Wireless Sensor Networks: First International Workshop, ALGOSENSORS 2004, Turku, Finland, July 16, 2004. Proceedings, volume 3121 of Lecture Notes in Computer Science, pages 32–44. Springer, 2004. doi: 10.1007/978-3-540-27820-7_5.
- [3] Aline Baggio and Koen Langendoen. Monte carlo localization for mobile wireless sensor networks. *Ad Hoc Networks*, 6(5):718–733, 2008. doi: 10.1016/j.adhoc.2007.06.004.
- [4] Pratik Biswas, Tzu-Chen Liang, Ta-Chung Wang, and Yinyu Ye. Semidefinite programming based algorithms for sensor network localization. ACM Trans. Sens. Networks, 2(2):188–220, 2006. doi: 10.1145/1149283.1149286.
- [5] Pratik Biswas, Kim-Chuan Toh, and Yinyu Ye. A distributed SDP approach for large-scale noisy anchor-free graph realization with applications to molecular conformation. *SIAM J. Sci. Comput.*, 30(3):1251–1277, 2008. doi: 10.1137/05062754X.
- [6] Nirupama Bulusu, John S. Heidemann, and Deborah Estrin. Gps-less low-cost outdoor localization for very small devices. *IEEE Wirel. Commun.*, 7(5):28–34, 2000. doi: 10.1109/98.878533.
- [7] Tashnim J. S. Chowdhury, Colin Elkin, Vijay Devabhaktuni, Danda B. Rawat, and Jared Oluoch. Advances on localization techniques for wireless sensor networks: A survey. *Comput. Networks*, 110:284–305, 2016. doi: 10.1016/j.comnet.2016.10.006.
- [8] Tolga Eren, David Kiyoshi Goldenberg, Walter Whiteley, Yang Richard Yang, A. Stephen Morse, Brian D. O. Anderson, and Peter N. Belhumeur. Rigidity, computation, and randomization in network localization. In *Proceedings IEEE INFOCOM 2004, The 23rd Annual Joint Conference of the IEEE Computer and Communications Societies, Hong Kong, China, March 7-11, 2004*, pages 2673–2684. IEEE, 2004. doi: 10.1109/INFCOM.2004.1354686.
- [9] Jia Fang, Ming Cao, A. Stephen Morse, and Brian D. O. Anderson. Sequential localization of sensor networks. SIAM J. Control. Optim., 48(1):321–350, 2009. doi: 10.1137/070679144.
- [10] David Kiyoshi Goldenberg, Pascal Bihler, Yang Richard Yang, Ming Cao, Jia Fang, A. Stephen Morse, and Brian D. O. Anderson. Localization in sparse networks using sweeps. In Mario Gerla, Chiara Petrioli, and Ramachandran Ramjee, editors, *Proceedings of the 12th Annual International Conference on Mobile Computing and Networking, MOBICOM 2006, Los Angeles, CA, USA, September 23-29, 2006*, pages 110–121. ACM, 2006. doi: 10.1145/1161089.1161103.

- [11] David Kiyoshi Goldenberg, Arvind Krishnamurthy, Wesley C. Maness, Yang Richard Yang, Anthony Young, A. Stephen Morse, Andreas Savvides, and Brian D. O. Anderson. Network localization in partially localizable networks. In *INFOCOM 2005. 24th Annual Joint Conference of the IEEE Computer and Communications Societies*, 13-17 March 2005, Miami, FL, USA, pages 313–326. IEEE, 2005. doi: 10.1109/INFCOM.2005.1497902.
- [12] Tian He, Chengdu Huang, Brian M. Blum, John A. Stankovic, and Tarek F. Abdelzaher. Range-free localization schemes for large scale sensor networks. In David B. Johnson, Anthony D. Joseph, and Nitin H. Vaidya, editors, *Proceedings of the Ninth Annual International Conference on Mobile Computing and Networking, MOBICOM 2003, 2003, San Diego, CA, USA, September 14-19, 2003*, pages 81–95. ACM, 2003. doi: 10.1145/938985.938995.
- Bruce Hendrickson. Conditions for unique graph realizations. SIAM J. Comput., 21(1):65–84, 1992. doi: 10.1137/0221008.
- [14] Bill Jackson and Tibor Jordán. Connected rigidity matroids and unique realizations of graphs. J. Comb. Theory, Ser. B, 94(1):1–29, 2005. doi: 10.1016/j.jctb.2004.11.002.
- [15] Xiang Ji. Sensor positioning in wireless ad-hoc sensor networks with multidimensional scaling. In Proceedings IEEE INFOCOM 2004, The 23rd Annual Joint Conference of the IEEE Computer and Communications Societies, Hong Kong, China, March 7-11, 2004, pages 2652–2661. IEEE, 2004. doi: 10.1109/INFCOM.2004.1354684.
- [16] Fabian Kuhn, Roger Wattenhofer, and Aaron Zollinger. Ad hoc networks beyond unit disk graphs. Wirel. Networks, 14(5):715–729, 2008. doi: 10.1007/s11276-007-0045-6.
- [17] Sol Lederer, Yue Wang, and Jie Gao. Connectivity-based localization of large-scale sensor networks with complex shape. ACM Trans. Sens. Networks, 5(4):31:1–31:32, 2009. doi: 10.1145/1614379. 1614383.
- [18] Yunhao Liu, Zheng Yang, Xiaoping Wang, and Lirong Jian. Location, localization, and localizability. J. Comput. Sci. Technol., 25(2):274–297, 2010. doi: 10.1007/s11390-010-9324-2.
- [19] Nancy A Lynch. Distributed algorithms. Elsevier, 1996.
- [20] Guoqiang Mao, Baris Fidan, and Brian D. O. Anderson. Wireless sensor network localization techniques. *Comput. Networks*, 51(10):2529–2553, 2007. doi: 10.1016/j.comnet.2006.11.018.
- [21] David C. Moore, John J. Leonard, Daniela Rus, and Seth J. Teller. Robust distributed network localization with noisy range measurements. In John A. Stankovic, Anish Arora, and Ramesh Govindan, editors, Proceedings of the 2nd International Conference on Embedded Networked Sensor Systems, SenSys 2004, Baltimore, MD, USA, November 3-5, 2004, pages 50–61. ACM, 2004. doi: 10.1145/1031495.1031502.
- [22] Rong Peng and Mihail L. Sichitiu. Angle of arrival localization for wireless sensor networks. In Proceedings of the Third Annual IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks, SECON 2006, September 25-28, 2006, Reston, VA, USA, pages 374–382. IEEE, 2006. doi: 10.1109/SAHCN.2006.288442.
- [23] B. Sau and K. Mukhopadhyaya. Length-based anchor-free localization in a fully covered sensor network. In 2009 First International Communication Systems and Networks and Workshops, pages 1–10, Jan 2009. doi: 10.1109/COMSNETS.2009.4808851.
- [24] Andreas Savvides, Chih-Chieh Han, and Mani B. Srivastava. Dynamic fine-grained localization in ad-hoc networks of sensors. In Christopher Rose, editor, MOBICOM 2001, Proceedings of the seventh annual international conference on Mobile computing and networking, Rome, Italy, July 16-21, 2001, pages 166–179. ACM, 2001. doi: 10.1145/381677.381693.

- [25] James B Saxe. Embeddability of weighted graphs in k-space is strongly np-hard. In Proc. of 17th Allerton Conference in Communications, Control and Computing, Monticello, IL, pages 480–489, 1979.
- [26] Yi Shang and Wheeler Ruml. Improved mds-based localization. In Proceedings IEEE INFOCOM 2004, The 23rd Annual Joint Conference of the IEEE Computer and Communications Societies, Hong Kong, China, March 7-11, 2004, pages 2640–2651. IEEE, 2004. doi: 10.1109/INFCOM. 2004.1354683.
- [27] Yi Shang, Wheeler Ruml, Ying Zhang, and Markus P. J. Fromherz. Localization from mere connectivity. In *Proceedings of the 4th ACM Interational Symposium on Mobile Ad Hoc Networking and Computing, MobiHoc 2003, Annapolis, Maryland, USA, June 1-3, 2003*, pages 201–212. ACM, 2003. doi: 10.1145/778415.778439.
- [28] Francesco Betti Sorbelli, Sajal K. Das, Cristina M. Pinotti, and Simone Silvestri. Range based algorithms for precise localization of terrestrial objects using a drone. *Pervasive Mob. Comput.*, 48:20–42, 2018. doi: 10.1016/j.pmcj.2018.05.007.
- [29] W.T. Tutte. Graph Theory. Cambridge Mathematical Library. Cambridge University Press, 2001.
- [30] Jing Wang, Ratan K Ghosh, and Sajal K Das. A survey on sensor localization. Journal of Control Theory and Applications, 8(1):2–11, 2010. doi: 10.1007/s11768-010-9187-7.
- [31] Walter Whiteley. Some matroids from discrete applied geometry. *Contemporary Mathematics*, 197:171–312, 1996.
- [32] Zheng Yang, Yunhao Liu, and Xiang-Yang Li. Beyond trilateration: On the localizability of wireless ad hoc networks. *IEEE/ACM Trans. Netw.*, 18(6):1806–1814, 2010. doi: 10.1109/TNET.2010. 2049578.



A Appendix: Simulation Results

Figure 7: a) A random network of 350 sensor nodes. The red, green and deep blue nodes are respectively boundary, weakly interior and strongly interior nodes. Disks of radii $\frac{r}{2}$ around each node is shown in pale blue. b) The network being dense, all 350 nodes are successfully localized by our algorithm. c) A relatively sparse and irregular network of 160 sensor nodes. d) Despite having only 5 strongly interior nodes, our algorithm successfully localizes 132 nodes, and thus achieves the maximum number of nodes localized by trilateration starting from any possible triangle (See Fig.8a). The blue nodes are successfully localized, and the black nodes are not localized.

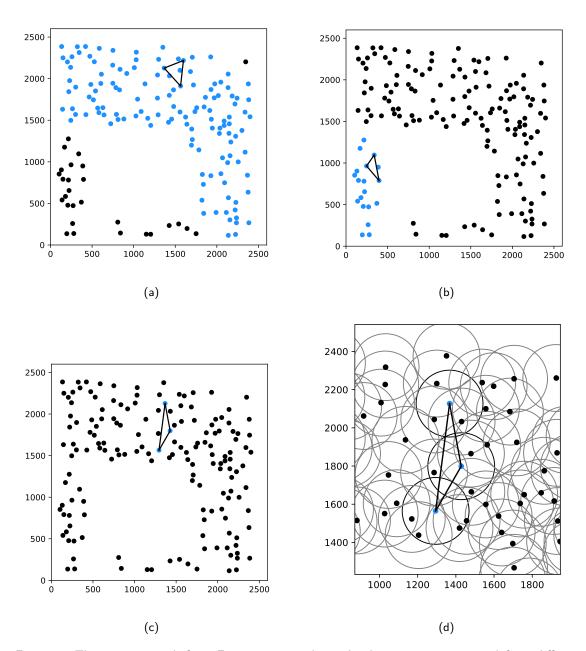


Figure 8: The same network from Fig. 7c is consider and trilateration is attempted from different initial triangles. a) In the overall best result, 132 nodes are localized b) In this case, only 19 nodes are localized. c) In the worst case, trilateration does not progress beyond the initial triangle. d) A closer view of Fig. 8c.