Sequential Solutions in Machine Scheduling Games

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Abstract. We consider the classical machine scheduling, where n jobs need to be scheduled on m machines, and where job j scheduled on machine i contributes $p_{i,j} \in \mathbb{R}$ to the load of machine i, with the goal of minimizing the makespan, i.e., the maximum load of any machine in the schedule. We study inefficiency of schedules that are obtained when jobs arrive sequentially one by one, and the jobs choose themselves the machine on which they will be scheduled, aiming at being scheduled on a machine with small load. We measure the inefficiency of a schedule as the ratio of the makespan obtained in the worst-case equilibrium schedule, and of the optimum makespan. This ratio is known as the *sequential price of anarchy (**SPoA**)*. We also introduce two alternative inefficiency measures, which allow for a favorable choice of the order in which the jobs make their decisions. As our first result, we disprove the conjecture of [23] claiming that the sequential price of anarchy for $m = 2$ machines is at most 3. We show that the sequential price of anarchy grows at least linearly with the number n of players, assuming arbitrary tie-breaking rules. That is, we show **SPoA** $\in \Omega(n)$. Complementing this result, we show that **SPoA** $\in O(n)$, reducing previously known exponential bound for 2 machines. Furthermore, we show that there exists an order of the jobs, resulting in makespan that is at most linearly larger than the optimum makespan. To the end, we show that if an authority can change the order of the jobs adaptively to the decisions made by the jobs so far (but cannot influence the decisions of the jobs), then there exists an adaptive ordering in which the jobs end up in an optimum schedule.

Keywords: Machine Scheduling; Price of Anarchy; Price of Stability;

1 Introduction

We consider the classical optimization problem of scheduling n jobs on m *unrelated* machines. In this problem, each job has a (possibly different) processing time on each of the m machines, and a schedule is simply an assignment of jobs to machines. For any such schedule, the load of a machine is the sum of all processing times of the jobs assigned to that machine. In this optimization problem, the objective is to find a schedule minimizing the *makespan*, that is, the maximum load among the machines.

In the *game-theoretic* version of this scheduling problem, also known as the *load balancing game*, jobs correspond to players who *selfishly* choose the machine to which the job is assigned. The cost of a player is the *load* of the machine to which the player assigned its own job. Such a setting models, for example, the situation where the machines correspond to servers, and the communication with a server has a latency that depends on the total traffic to the server.

The decisions of the players lead to some *equilibrium* in which no player has an incentive to deviate, though the resulting schedule may not necessarily be optimal in terms of makespan. Such an equilibrium might have a rather high *social cost*, that is, the makespan of the corresponding schedule[5] is not guaranteed to be the optimal one, as in Example 1 below.

Example 1 (two jobs on two unrelated machines [5]). Consider two jobs and two unrelated machines, where the processing times are given by the following table:

	job 1	job 2
machine 1	1	ℓ
machine 2	ℓ	1

[5] When each player chooses deterministically one machine, this definition is obvious. When equilibria are *mixed* or randomized, each player chooses one machine according to some probability distribution, and the social cost is the expected makespan of the resulting schedule.

The allocation represented by the gray box is a pure Nash equilibrium in the load balancing game (if a job moves to the other machine, its own cost increases from ℓ to $\ell+1$), and has makespan ℓ. The optimal makespan is 1 (swap the allocations). This example shows that the makespan of an equilibrium can be arbitrarily larger than the optimum.

The inefficiency of equilibria is a central concept in algorithmic game theory. Typically, one aims to quantify the *efficiency loss* resulting from a *selfish behavior* of the players, where the loss is measured in terms of the social cost. Arguably, the two most popular measures of inefficiency of equilibria are the *price of anarchy (**PoA**)* [27] and the *price of stability (**PoS**)* [4], which, intuitively, consider the *most pessimistic* and the *most optimistic* scenario:

- The *price of anarchy* is the ratio of the cost of the *worst* equilibrium over the *optimal social cost*;
- The *price of stability* is the ratio of the cost of the *best* equilibrium over the *optimal social cost*.

The price of anarchy corresponds to the situation (anarchy) in which there is no authority, and players converge to some equilibrium by themselves. In the price of stability, one envisions that there are means to suggest the players how to play, and if that is an equilibrium, then they will indeed follow the suggestion, as no unilateral deviation can improve a player's individual cost. Furthermore, the price of stability provides a lower bound on the efficiency loss of an equilibrium outcome, if, for example, no equilibrium is actually a social optimum.

Example 1 thus shows that the **price of anarchy** of load balancing games is **unbounded even for two jobs and two machines**. Interestingly, the price of stability instead is one (**PoS** =1), for any number of jobs and any number of machines. This is because there is always an optimal solution that is also a pure Nash equilibrium [18] (see Section 1.3 for details). In a pure Nash equilibrium, players choose their strategies deterministically, as opposed to *mixed* Nash equilibria. In this work, we will also focus on the case in which players act deterministically, though in a sequential fashion (see below).

As the price of anarchy for unrelated machines is very high (unbounded in general), one may ask whether Nash equilibria are really what happens as an outcome in the game, or whether a central authority, which cannot influence the choices of the players (jobs), may alter some aspects of the scheduling setting, and as a result, improve the performance of the resulting equilibria.

Motivated by these issues, in [28] the authors consider the variant in which players, instead of choosing their strategies simultaneously, play sequentially taking their decisions based on the previous choices and also knowing the order of players that will make play. Formally, this corresponds to an *extensive-form game*, and the corresponding equilibrium concept is called a *subgame-perfect equilibrium*. Players always choose their strategy deterministically. The resulting inefficiency measure is called the *sequential price of anarchy (**SPoA**)*.

There are two main motivations to study a sequential variant of the load balancing game. First, assuming that all players decide simultaneously to choose the machine to process their jobs is a too strong and unnatural modeling assumption in many situations; furthermore, expecting that all players choose the worst-case machine, as was the case in Example 1, is unnatural as well. Second, one may have the power to explicitly ask the players to make sequential decisions, and make this the policy, which the players are aware of, with the view of lowering the loss of efficiency of the resulting equilibrium schedules. In a sense, such an approach of adjusting the way the players access the machines resembles *coordination mechanisms* [11], which are scheduling policies aiming to achieve a small price of anarchy (see Section 1.3 for more details).

1.1 Prior results (SPoA for unrelated machines)

The first bounds on the **SPoA** for unrelated machines have been obtained in [28], showing that

$$n \leq \mathbf{SPoA} \leq m \cdot 2^n.$$

Therefore, **SPoA** is *constant* for a constant number of machines and jobs, while **PoA** is *unbounded* even for two jobs and two machines (recall Example 1). The large gap in the previous bound naturally suggests the question of what happens for *many jobs* and *many machines*. This was addressed by [7] which improved significantly the prior bounds by showing that

$$2^{\Omega(\sqrt{n})} \leq \mathbf{SPoA} \leq 2^n.$$

At this point, one should note that these lower bounds use a *non-constant* number of machines. In other words, it still might be possible that for a *constant number of machines* the **SPoA** is constant. For *two machines*, [23] proved a lower bound **SPoA** ≥ 3, and in the same work the authors made the following conjecture:

Conjecture 1. [23] *For two unrelated machines,* ***SPoA*** $= 3$ *for any number of jobs.*

1.2 Our contributions

In this paper, we disprove Conjecture 1 by showing that in fact, **SPoA** on two machines is *not even constant*. Indeed, it must grow linearly and the conjecture fails already for few jobs:

- For *five jobs* we have **SPoA** ≥ 4 (Theorem 2);
- In general, with arbitrary tie-breaking rules, it holds that **SPoA** $\geq \Omega(n)$ (Theorem 3).

Note that the result of Theorem 3 uses suitable player-specific tie-breaking rules (see Definition 1). We discuss the implications of using tie-breaking rules more in detail at the end of this subsection.

While Theorem 2 settles the conjecture, the result of Theorem 3 says that **SPoA** is non-constant already for two machines (as the number of jobs grows) for generic tie-breaking rules. We actually conjecture that there exist instances for which the **SPoA** is unbounded without having ties. Moreover, it implies a *strong separation* with the case of *identical* machines, where **SPoA** $\leq 2 - \frac{1}{m}$, for any number m of machines [23]. In Theorem 4 we show that **SPoA** is upper bounded by $2(n-1)$, reducing the exponential upper bound obtained in [7] for arbitrarily many machines to linear bound for 2 machines.

The original idea behind the notion of price of stability (**PoS**) is that an authority can suggest to the players how to play:

> *[...] The best Nash equilibrium solution has a natural meaning of stability in this context – it is the optimal solution that can be proposed from which no user will defect. [...] As a result, the global performance of the system may not be as good as in a case where a central authority can simply dictate a solution; rather, we need to understand the quality of solutions that are consistent with self-interested behavior.* [4]

We borrow this idea of an authority suggesting desirable equilibria. Specifically for our setting, the authority suggests the order in which players make their decisions, so to induce a good equilibrium. This can be viewed as the price of stability (**PoS**) for these sequential games. We introduce this notion in two variants (a weaker and a stronger):

- *Sequential Price of Stability (**SPoS**).* The authority can choose the order of the players' moves. This order determines the tree structure of the corresponding game.
- *Adaptive Sequential Price of Stability (**adaptive SPoS**).* The authority decides the order of the players' moves *adaptively* according to the choices made at each step.

The study of these two notions for two unrelated machines is also motivated by our lower bound, and by the lack of any good upper bound on this problem. We prove the following upper bounds for two unrelated machines (Theorems 5 and 6):

$$\mathbf{SPoS} \leq \frac{n}{2} + 1\ , \qquad\qquad \textbf{adaptive SPoS} = 1\ .$$

The next natural question is to consider *three* or more machines. Here we show an impossibility result, namely **adaptive SPoS** $\geq 3/2$ already for three machines (Theorem 7). That is, even with the strongest type of adaptive authority, it is not possible to achieve the optimum. This shows a possible disadvantage of having players capable of complex reasoning, like in extensive-form games. In the classical strategic-games setting, where we consider pure Nash equilibrium, here is an optimum which *is* an equilibrium, that is, **PoS** $= 1$ for any number of machines and jobs. This result follows from [18] (see Section 1.3 for details)

As mentioned above, some of our results rely on the use of a suitable tie-breaking rules. Using tie-breaking rules to prove lower bounds on the **SPoA** is not new: in [16] the authors showed that, in *routing games*, the sequential price of anarchy is *unbounded*. Their proof is based on carefully chosen tie-breaking rules. This way of using tie-breaking rules is not part of the players' strategy interactions. In contrast, some works consider settings where among equivalent choices, each player i can use the one that hurts prior agents who chose a strategy that player i would prefer they had not chosen (see [30]).

1.3 Further related work

The load balancing games considered in this work are one of the most studied models in algorithmic game theory (see, e.g., [27, 26, 20, 2, 15, 19, 17]). In all these works, players correspond to jobs, their cost is the load of the machine they choose, and the social cost is defined as the makespan of the jobs allocation. In particular, the seminal paper [27] which introduced the concept of the price of anarchy, considers the case of *identical* and *related* machines, two simpler versions of unrelated machines (related machines is the setting where each machine has a speed, each job has a certain size, and the processing time equals the job size divided by the machine speed; the case of identical machines is the restriction in which all speeds are the same).

Interestingly, the price of anarchy for *related* or *identical* machines is much better than in the case of unrelated machines (where the price of anarchy is unbounded). Indeed, for related and identical machines, the price of anarchy is *bounded* for any *constant number of machines* [27, 15, 26, 21, 20, 19, 17] (some of these results give bounds also for *mixed* Nash equilibria). Specifically, for pure Nash equilibria, $\mathbf{PoA} = (2 - \frac{2}{m+1})$ for identical machines as implied by the analysis of [21], while $\mathbf{PoA} = O(\frac{\log m}{\log \log m})$ for related machines [15].

As already mentioned above, the **PoS** for *unrelated* machines is 1. This is due to the work [18] which shows that, starting from any schedule, an iterative process of applying unilateral improving-strategy changes of players leads to a pure Nash equilibrium (the same property has been observed earlier in [22] for related machines). This condition implies the existence of a pure Nash equilibrium.

Load balancing games on identical and related machines are a special case of *weighted singleton congestion games*. In a singleton weighted congestion game, there are m resources, and n players, each player i having a weight w_i. Every resource r has a cost function c_r associated with it. In the game, every player i chooses one resource s_i as its strategy, resulting in cost $c_r(\sum_{i:s_i=r} w_i)$ of the resource, which is also the cost of every player i that chooses resource r as its strategy. Obviously, seeing the machines in the load balancing games as resources, seeing the jobs as the players, seeing the job sizes as weights w_i, and setting $c_r(x) = x/\text{speed}_r$, the singleton congestion game models the load balancing games on m related machines, where machine r has speed speed_r, and the processing time of job i on machine r is w_i/speed_r. Load balancing games on unrelated machines have, to the best of our knowledge, no counterpart in congestion games.

Requiring that players make their decisions sequentially, according to a given and known order can be seen as a mean of a central authority that can control access to the resources (machines), but not the choices of the players (jobs). In this sense, changing the access from simultaneous to sequential can be seen as a kind of control mechanism like a *coordination mechanism* [11]. In load balancing games where the cost of a player (job) is the completion time of the job (and not the total load of the machine on which the job is scheduled), a coordination mechanism is a scheduling policy, one for every machine, which determines the order of the jobs in which they will be scheduled on the machine. The scheduling policy needs to be fixed and (publicly) known to the players. For load balancing games in normal form (i.e., where players make simultaneous decisions, as opposed to the sequential decisions, which we consider in this paper), coordination mechanisms have been studied both for the version where the social cost is the makespan (see, e.g., [25, 8, 6, 9] and the references therein), or the total (weighted) completion time (see, e.g., [24, 14, 12, 1, 31] and the references therein).

As already discussed above, the concept of a sequential price of anarchy is not new. In addition to the results for unrelated machines discussed in Section 1.1, the sequential price of anarchy has been studied also for other games. These include congestion games with affine delay functions [16], isolation games [3], and network congestion games [13]. Interestingly, the latter work shows that the sequential price of anarchy for these games is *unbounded*, as opposed to the price of anarchy which was known to be 5/2.

Naturally, there is a huge literature on the classical algorithm-theoretic research on machine scheduling, see, e.g., the textbook [29] and the survey [10] for fundamental results and further references.

2 Preliminaries

In unrelated machine scheduling there are n jobs and m machines, and the processing time of job j on machine i is denoted by p_{ij}. A solution (or schedule) consists of an assignment of each job to one of the machines, that is, a vector $s = (s_1, \ldots, s_n)$ where s_j is the machine to which job j is assigned to. The *load* $l_i(s)$ of a machine i in schedule s is the sum of the processing times of all jobs allocated to it, that is,

$l_i(s) = \sum_{j:s_j=i} p_{ij}$. The social cost of a solution s is the *makespan*, that is, the maximum load among all machines.

Each job j is a *player* who attempts to minimize her own cost $cost_j(s)$, that is, the load of the machine she chooses: $cost_j(s) = l_{s_j}$. Every player j decides s_j, the assignment of job j to a machine. The combination of all players strategies gives a schedule $s = (s_1, \ldots, s_n)$.

In the extensive-form version of these games, players play sequentially; they decide their strategies based on the choices of the previous players and knowing that the remaining players will play rationally. We consider a *full information* game. As players enter the game sequentially, they can compute their optimal moves by the so-called *backward induction*: the last player makes her move greedily, the player before the last makes the move also greedily (taking into account what the last player will do), and so on. Any game of this type can be modeled by a *decision tree*, which is a rooted tree where the non-leaf vertices correspond to the players in certain states, while edges correspond to the strategies available to the players in a given state.

Each leaf corresponds to a solution (schedule), which is simply the strategies on the unique leaf-to-root path. Given the processing times $P = (p_{ij})$, the players can compute the loads on the machines in each of the leaves. In case of ties, all players know the deterministic tie-breaking rules of all the other players. A player can calculate what the final outcome would be for each of her strategies, and choose the strategy that minimizes her cost. This method is called backward induction. Strategies obtained in this way for each internal node constitute what is called the *subgame-perfect equilibrium*: for each subtree, we know what is the outcome achieved by the players in this subtree if they play rationally. We usually represent the strategies (edges) that are chosen by players in the **subgame perfect equilibrium** in **bold**, and the other strategies as *dashed* edges.

It is easy to see that a subgame-perfect equilibrium always exists and it is unique, for given tie-breaking rules. On the other hand, its computation is difficult, as proved in [28]:

Theorem 1. *[28] Computing the outcome of a subgame perfect equilibrium in Unrelated Machine Scheduling is PSPACE-complete.*

Notation and formal definitions. We consider n jobs and m machines, denoted by $J = (J_1, J_2, \ldots, J_n)$ and $M = (M_1, M_2, \ldots, M_m)$ respectively. The processing times are given by a matrix $P = (p_{ij})$, with p_{ij} being the processing time of job J_j on machine M_i. The set of all such nonnegative $n \times m$ matrices is denoted by $\mathcal{P}_{n,m}$ and it represents the possible instances of the game. For any $P \in \mathcal{P}_{n,m}$ as above, we denote by $\mathcal{T}_{n,m}$ the set of all possible depth-n, complete m-ary decision trees where each path from the root to a leaf contains every job (player) exactly once. The whole game (and the resulting subgame perfect equilibrium) is fully specified by P, T, and the *tie-breaking rule* used by the players. The most general – worst case – scenario is that ties are arbitrary (see Definition 1). In the following, we do not specify the dependency on the ties, and simply denote by $SPE(P,T)$ the cost (makespan) of the subgame perfect equilibrium of the game. One type of worst-case analysis is to assume the players' order to be adversarial, and the tree T being chosen accordingly. This is the same as saying that players arrive in a fixed order (say $J_1, J_2, \ldots, J_n$) and their costs P is chosen in an adversarial fashion. In this case, we simply write $SPE(P)$ as the tree structure is fixed. For a fixed order σ (a permutation) of the players, and costs P, we also write $SPE(P,\sigma)$ to denote the quantity $SPE(P,T)$ where T is the tree resulting from this order σ of the players.The optimal social cost (makespan) is denoted by $OPT(P)$.

We next introduce formal definitions to quantify the inefficiency of subgame perfect equilibria in various scenarios (from the most pessimistic to the most optimistic). The *sequential price of anarchy (**SPoA**)* compares the worst subgame perfect equilibrium with the optimal social cost,

$$\mathbf{SPoA} = \sup_{P \in \mathcal{P}_{n,m}} \frac{SPE(P)}{OPT(P)}.$$

In the *sequential price of stability (**SPoS**)*, we can choose the order σ in which players play depending on the instance P. The resulting subgame perfect equilibrium has cost $SPE(P,\sigma)$, which is then compared to the optimum,

$$\mathbf{SPoS} = \sup_{P \in \mathcal{P}_{n,m}} \min_{\sigma \in \mathcal{S}_n} \frac{SPE(P,\sigma)}{OPT(P)},$$

where σ ranges over all permutations $\mathcal{S}_n$ of the n players. In *adaptive sequential price of stability (**adaptive SPoS**)*, we can choose the whole structure of the tree, meaning that for each choice of a player, we can adaptively choose which player will play next. This means that every path from any leaf to the root corresponds to a permutation of the players. The adaptive price of stability is then defined as

$$\textbf{adaptive SPoS} = \sup_{P \in \mathcal{P}_{n,m}} \min_{T \in \mathcal{T}_{n,m}} \frac{SPE(P,T)}{OPT(P)}.$$

Note that by definition **adaptive SPoS** $\leq$ **SPoS** $\leq$ **SPoA**.

3 Linear lower bound for SPoA

In this section, we consider the sequential price of anarchy for *two* unrelated machines. In [23] the authors proved a lower bound **SPoA** ≥ 3 for this case, and they conjectured that this was also a tight bound. We show that unfortunately this is not the case: Already for five jobs, **SPoA** ≥ 4, and with more jobs the lower bound grows linearly, i.e., **SPoA** $= \Omega(n)$.

3.1 A lower bound for $n = 5$ players

Theorem 2. *For two machines and at least five jobs, the* ***SPoA*** *is at least* 4.

The proof is in the appendix.

3.2 Faster linear program formulation

Our first lower bound for $n = 5$ players has been obtained by solving a linear program, suggested in [23]. A crucial achievement for the speedup of the program is the discovery of the property that we describe in the following. This property allows excluding a large number of combinations for the last layer of the tree. As the last layer of the tree represents more than half of the internal nodes in the tree, the number of combinations that have to be generated can be reduced drastically. For example, for $n = 5$, the improvement is from $2 \cdot 10^9$ to $6 \cdot 10^6$.

In this approach of linear programming, the variables are the processing times $\{p_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$, and the approach essentially goes as follows:

- Fix the subgame perfect equilibrium structure, that is, the sequence of players and all the decisions in the internal nodes, this also gives the sequential equilibrium;
- Fix the leaf which is the optimum, and impose that the optimum makespan is at most 1;

For every fixed subgame perfect equilibrium tree structure, we have one constraint for each internal node (decision of a player). The optimum state (leaf) should also be fixed and both numbers have to be assumed to be at most 1 by adding two additional constraints to the linear program. By maximizing the maximum value of loads on the machines in the leaf which corresponds to the sequential equilibrium, we get the worst case example for this particular tree structure. There are $2^n - 1$ internal nodes in the decision tree with n players. Therefore, this approach requires exploring 2^{2^n-1} many possible subgame perfect equilibria tree structures, and for each of them, we have to decide where is the optimum among 2^n leaves and solve a linear program of size $2^n \times O(n)$.

We managed to solve the case $n = 5$ players with the aid of a computer program which explored all possible tree structure leading to subgame perfect equilibria; this has been achieved by understanding the structure and by breaking certain symmetries to reduce the search space, as we explain next. It is clear that the extremely fast growing number of possible tree structures makes the program very time-consuming even for small values. Consequently, we tried to exclude combinations from the computation, i.e. we avoid starting the linear programming solver for certain tree structures. There are some trivial cases that we present for the intuition. The first is that not all leaves of the tree should be tested for the position of the optimum. Both the leftmost and the rightmost leaf nodes can be excluded from the possible optimum position due to the property that **SPoA** is 1 in both cases. For the same reason, all tree structures where the equilibrium is located at those extreme leaves can also be ignored. Additionally, the leaf where the equilibrium is achieved should be avoided for the optimum position. The next idea is that trees that are mirror images of other trees with regard to the vertical axis will lead to the same **SPoA**.

During the experimental investigations of possible outcomes based on the structure of the game tree, we found out that a relatively big part of game tree structures always leads to an infeasible linear program, regardless of the position of the optimum. Consider the simple tree structure for *two* players and *two* machines, depicted in Figure 1. Solid lines represent the best responses in each node. It is obvious that, if the best response of player 2 in the left node is to choose machine 1, then the best response in the right node cannot be to choose machine 2. We generalized this observation to any number of players n and a number of machines m, and apply it to the second lowest level of the tree, i.e., to the best responses of the last player J_n.

Additional notation. We denote the nodes of a $T \in \mathcal{T}_{n,m}$ by H and the nodes in the level i by H_i. Clearly, in the case of adaptive sequences, H_i may contain different players, while in the case of fixed sequences, in H_i all the nodes correspond to exactly one player.

Let $p(h)$ denote the parent node of node h in tree t. We define the child index $c(h) = i \in M$ of node h if h is the i-th child of it parent $p(h)$, this means that the edge from $p(h)$ to h represents that the agent corresponding to $p(h)$ selects M_i. Every node h on the j-th layer of t is defined by the choices of the agents playing before agent J_j. These decisions create a unique path from the root of t to h. The nodes on the path from the root r to h are $P(h) = \{r, ..., p(p(h)), p(h), h\} = \{h' \in H | h \in t_{h'}\}$ where $t_{h'}$ is the subtree of t rooted at h'. These definitions allow us to define the set of agents that selected a certain machine i before node h: $B(i, h) = \{j \in J |\ p(h') \in H_j \wedge h' \in P(h) \wedge c(h') = i\}$. Then the observation is the following:

Observation 1 *For every pair of nodes in the second lowest layer $h, h' \in H_n$ if the best response of player J_n in h is i, then it implies that the best response in h' is also i if $\forall i' \in M \setminus i : B(i', h) \subseteq B(i', h')$ holds.*

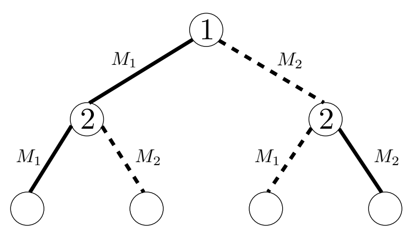


Fig. 1: A tree for the case of two players and two machines. The responses of the players (bold edges) cannot be a subgame perfect equilibrium.

Remark 1. We could not find any example that would give a lower bound on sequential price of anarchy (even locally) better than the one in the proof of Theorem 3. For $n \leq 7$, our computer program searched the whole space and the results obtained above are the best. We believe that the construction from the proof of the theorem gives the best possible lower bound example.

3.3 Linear lower bound

Extending the construction for $n = 5$ players is non-trivial as this seems to require rather involved constants that multiply the ε terms. However, we notice that these terms only help to induce more involved tie-breaking rules of the following form:

Definition 1 (arbitrary tie-breaking rule). *We say that the tie-breaking rule is arbitrary if each player uses a tie-breaking rule between machines which possibly depends on the allocation of all players.*

The following theorem gives our general lower bound:

Theorem 3. *Even for two machines, the* ***SPoA*** *is at least linear in the number n of jobs, in the case of arbitrary tie-breaking rule.*

Proof. We consider the following instance with $n = 3k - 1$ jobs arriving in this order (from left to right),

	J_1	J_2	J_3	J_4	J_5	J_6	$\cdots$	J_{3k-5}	J_{3k-4}	J_{3k-3}	J_{3k-2}	J_{3k-1}
M_1	$k+1$	0	0	k	0	0	$\cdots$	3	0	0	1	2
M_2	0	k	k	0	$k-1$	$k-1$	$\cdots$	0	2	2	1	1

and show that the subgame perfect equilibrium is the gray allocation whose makespan is $k + 2$, while the optimal makespan is 1. This requires players to use the following tie-breaking rules in the first part: if player J_1 chooses machine M_1, then J_2 and J_3 prefer to avoid player J_1, that is, they choose the other machine in case of ties in their final cost.

We prove the claim above by induction on k. The base case is $k = 2$ which follows directly from the example in Equation (10), where we replace $\varepsilon > 0$ with an equivalent tie-breaking rule (and set $\varepsilon = 0$). As for the inductive step, the proof consists of the following steps (which we prove below):

1. If the first three jobs choose their zero-cost machines, then all subsequent jobs implement the subgame perfect equilibrium on the same instance with $k' = k - 1$. The cost of J_1, in this case, is $k' + 2 = k + 1$.
2. If the first job J_1 chooses M_2, then both J_2 and J_3 choose M_1.
3. If the first job J_1 chooses M_1, then all subsequent players will choose the gray allocation (and therefore, the cost of J_1 is $k + 1$ in this scenario as well).

The first two steps above imply that, if J_1 chooses machine M_2, then her cost is $k + 1$. Step 3 says that the same cost occurs if J_1 chooses M_1. We assume the tie-breaking rule for player J_1, in this case, is that she prefers the cost $k+1$ on the first machine M_1. Therefore, by Step 3, J_1 will choose M_1 and all players choose the gray allocation in the subgame perfect equilibrium. Note that the cost on machine M_1 and M_2 is $k + 1$ and $k + 2$, respectively.

Next, we prove the three steps above:

Proof (of Step 1). Note that the sequence starting from J_4 is the same sequence for $k' = k - 1$. Since the first three jobs did not put any cost on the machines, we can apply the inductive hypothesis and assume that all subsequent players play the subgame perfect equilibrium. The resulting cost on machine M_2 will be $k' + 2 = k + 1$, and this is the machine chosen by J_1. □

Proof (of Step 2). Choosing M_2 costs J_2 and J_3 at least k, no matter what the subsequent players do. If they instead choose M_1, by the previous claim, their cost is $k' + 1 = k$ which they both prefer given their tie-breaking rule. □

Proof (of Step 3). In this case, where J_1 is on machine M_1, we assume different tie-breaking rules for the last two players J_{3k-2} and J_{3k-1}, depending on which of the two players J_2 and J_3 choose machine M_2.

Case 1: player J_2 chooses machine M_1. In this case we assume that player J_3 breaks ties in favor of M_2: we will show that choosing M_2 results in the cost of $k + 1$ in the end, instead of some cost on machine M_1, which we already know is at least $k + 1$, because J_1 is already on machine M_1. If job J_3 gets assigned to machine M_2, then by backward induction we can show the following claim:

Claim. No player among $J_4, \ldots, J_{3k-3}$ gets assigned to machine where she has non-zero cost.

Proof (of Claim). Suppose none of them joins the non-zero cost machine. Then the last two players can add at most 3 to the cost $k + 1$ on the first machine and at most cost 2 to the cost k on the second machine. On the other hand, any job among $J_4, \ldots, J_{3k-3}$ adds at least that cost to the non-zero cost machine she joins. By backward induction we can assume that none of them chooses non-zero cost machine. □

Because of the last claim, only the last two players are left to decide. We assume that player J_{3k-2} prefers to choose machine M_1 and pays the cost $k + 2$ instead of choosing machine M_2 and paying the cost $k + 2$ on that machine, while the last player J_{3k-1} prefers to choose machine M_2 and incur the cost $k + 1$. Therefore, job J_2 pays the cost $k+2$, but in this case we can assume that she prefers cost $k+2$ that she has to pay on machine M_2, given this is achievable.

Case 2: job J_2 gets assigned to machine M_2, as in the claimed subgame perfect equilibrium state. Similarly to the previous case, by backward induction we conclude that all players $J_3, \ldots, J_{3k-3}$ get assigned to the machine where they have cost 0, and the last two jobs J_{3k-2} and J_{3k-1} choose machine M_2, here again we assume that player J_{3k-2} prefers to pay $k+2$ on machine M_2 than to pay the same cost on machine M_1. In this way the cost on machine M_1 is $k+1$, while the cost on machine M_2 is $k+2$.

This finishes the proof of Step 3 and of the theorem. □

Note that it is important to have seemingly equivalent jobs J_2 and J_3. They use different tie-breaking rules, which creates the asymmetry between them and increases the **SPoA**.

We solved linear programs with strict inequalities obtained from the subgame perfect equilibria tree structure given in the example from the proof of Theorem 3, by introducing small ε for strict inequalities. We found solutions for $n = 8$ and $n = 11$, that is linear programs are feasible. Therefore, at least for small n's we can drop the assumption about tie-breaking rules. As the solutions replace the ε terms by rather more complicated coefficients, we do not present them here. For the general case, we conjecture that the statement of Theorem 3 holds without the assumption on the tie-breaking rules, and that the latter are merely used to make the analysis easier:

Conjecture 2. For two machines, the **SPoA** is at least linear in n.

4 Linear upper bound for SPoA

Additional notation. To prove the upper bound for **SPoA**, we introduce some additional notation. We define a vector $D = (d_1, d_2)$ of initial load on the machines before the jobs play the game. Consequentially, the load of each machine i becomes

$$l_i(D, s) = d_i + \sum_{j:s_j=i} p_{ij} \,,$$

where $s = (s_1, s_2, \ldots, s_n)$ is the schedule (SPE) achieved by the jobs playing the game with initial load D on the machines; the cost of each job j is

$$cost_j(D, s) = l_{s_j}(D, s) \,.$$

The notation for the makespan is renewed as $SPE_D(P)$ for the SPE with initial load D. Additionally, we define $\Delta SPE(P)$ as the maximum possible increase of the makespan due to the players, with processing time P, for any initial load D:

$$\Delta SPE(P) = \sup_D \{SPE_D(P) - ||D||_\infty\} \,.$$

Moreover, for a given P, we use $P_{[u:v]}$ to represent the processing times only for jobs $(J_u, J_{u+1}, \ldots, J_v)$, that is, $P_{[u:v]} = (p_{ij})$ where $j = u, u+1, \ldots, v$.

We first prove a key lemma showing that each job can only contribute a certain amount (bounded by the total minimum processing time) to the makespan:

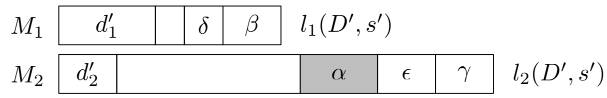
Lemma 2. $\Delta SPE\left(P_{[\ell:n]}\right) - \Delta SPE\left(P_{[\ell+1:n]}\right) \le \sum_{j=\ell}^{n} \min_i p_{ij}$ *for* $\ell = 1, 2, \ldots, n-1$.

Proof. For an arbitrary $\ell \in \{1, 2, \ldots, n-1\}$, giving a processing time $P_{[\ell:n]}$ and an initial load D, we suppose w.l.o.g. that job ℓ chooses machine 1. After job ℓ makes its decision, choosing machine 1, the game consists of the rest players, $\{\ell+1, \ell+2, \ldots, n\}$ starting with a new initial load $D' = (d_1 + p_{1\ell}, d_2)$. Thus,

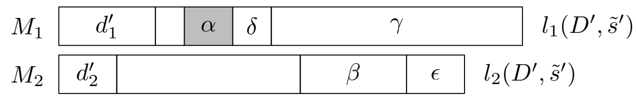
$$SPE_D\left(P_{[\ell:n]}\right) = SPE_{D'}\left(P_{[\ell+1:n]}\right) \,.$$

We first discuss the trivial case when $d_1 + p_{1\ell} < d_2$. In this case, it holds that $||D'||_\infty = ||D||_\infty = d_2$, which indicates that

$$SPE_D\left(P_{[\ell:n]}\right) = SPE_{D'}\left(P_{[\ell+1:n]}\right) \le \Delta SPE\left(P_{[\ell+1:n]}\right) + ||D'||_\infty = \Delta SPE\left(P_{[\ell+1:n]}\right) + ||D||_\infty \,,$$



(a) The original schedule s'



(b) The schedule $\tilde{s}'$ when job α moves to machine 1

Fig. 2: Proof of Claim

that is,

$$SPE_D\left(P_{[\ell:n]}\right) - ||D||_\infty \leq \Delta SPE\left(P_{[\ell+1:n]}\right) . \tag{1}$$

We then consider the other case $d_1 + p_{1\ell} \geq d_2$, that is, $||D'||_\infty = d_1 + p_{1,\ell}$. Let $s' = (s_{\ell+1}, s_{\ell+2}, \ldots, s_n)$ be the schedule of the jobs $\{J_{\ell+1}, J_{\ell+2}, \ldots, J_n\}$ playing with initial load D'. We know that

$$SPE_D\left(P_{[\ell:n]}\right) = SPE_{D'}\left(P_{[\ell+1:n]}\right) = \max\{l_1(D', s'), l_2(D', s')\} . \tag{2}$$

Claim. $\max\{l_1(D', s'), l_2(D', s')\} \leq l_1(D', s') + \sum_{j=\ell+1}^{n} \min_i p_{ij}$.

Proof (of Claim). If $\max\{l_1(D', s'), l_2(D', s')\} = l_1(D', s')$, the claim is obviously true. Thus we only need to prove $l_2(D', s') \leq l_1(D', s') + \sum_{j=\ell+1}^{n} \min_i p_{ij}$.

Let J_α denote the last job who chooses machine 2 and has a longer processing time on machine 2 than on machine 1, i.e., $p_{1\alpha} \leq p_{2\alpha}$. Let $\tilde{s}'$ be the new schedule if job J_α chooses machine 1. In schedule $\tilde{s}'$, the decisions of jobs $\{J_{\alpha+1}, J_{\alpha+2}, \ldots, J_n\}$ may be different from schedule s'. We divide the jobs $\{J_{\alpha+1}, J_{\alpha+2}, \ldots, J_n\}$ into 4 subsets, namely, β, γ, δ and ε, depending on the differences between s' and $\tilde{s}'$ (as shown in Figure 2):

- Jobs in β are on machine 1 in s' but on machine 2 in $\tilde{s}'$;
- Jobs in γ are on machine 2 in s' but on machine 1 in $\tilde{s}'$;
- Jobs in δ are on machine 1 in both s' and $\tilde{s}'$;
- Jobs in ε are on machine 2 in both s' and $\tilde{s}'$.

For simplicity, we define some notations to represent the total processing time of the job sets:

$$\alpha_1 = p_{1\alpha} , \qquad \beta_1 = \sum_{j\in\beta} p_{1j} , \qquad \gamma_1 = \sum_{j\in\gamma} p_{1j} , \qquad \varepsilon_1 = \sum_{j\in\varepsilon} p_{1j} ,$$
$$\alpha_2 = p_{2\alpha} , \qquad \beta_2 = \sum_{j\in\beta} p_{2j} , \qquad \gamma_2 = \sum_{j\in\gamma} p_{2j} , \qquad \varepsilon_2 = \sum_{j\in\varepsilon} p_{2j} .$$

In the following, we will prove $l_2(D', s') \leq l_1(D', s') + \alpha_1 + \gamma_2 + \varepsilon_2$. According to the definition of job J_α, we know the jobs in γ and ε have shorter processing times on machine 2, thus it follows that $\alpha_1 + \gamma_2 + \varepsilon_2 \leq \sum_{j=\alpha}^{n} \min_i p_{ij} \leq \sum_{j=\ell+1}^{n} \min_i p_{ij}$, meaning that the claim is true.

We prove the inequality by contradiction, assuming that

$$l_1(D', s') < l_2(D', s') - \alpha_1 - \gamma_2 - \varepsilon_2 . \tag{3}$$

Intuitively, when job J_α moves to machine 1, if the following jobs $\{J_{\alpha+1}, J_{\alpha+2}, \ldots, J_n\}$ make the same decisions as in schedule s', the cost of job J_α (i.e., $l_1(D', s') + \alpha_1$) is lower than the cost (i.e., $l_2(D', s')$) of job J_α in the original schedule s', since inequality (3) holds. To guarantee that job J_α has no incentive to move to machine 1, the cost of J_α when moving to machine 1 should be higher than $l_2(D', s')$. In other words, when job J_α moves to machine 1, there must be some jobs (i.e., γ) originally on machine 2 also move to machine 1, increasing the load of machine 1 to a value higher than $l_2(D', s')$. Moreover, the incentive of the jobs in γ moving to machine 1 is due to the increase of the load of machine 2 by some jobs (i.e., β) originally on machine 1 moving to machine 2 when J_α moves to machine 1. In the following, we will show that the jobs in β have no incentive to move to machine 2 if inequality (3) holds, which gives a contradiction.

By definition, the loads of machine 1 and 2 in $\tilde{s}'$ are

$$l_1(D', \tilde{s}') = l_1(D', s') + \alpha_1 - \beta_1 + \gamma_1\,, \tag{4}$$

$$l_2(D', \tilde{s}') = l_2(D', s') - \alpha_2 + \beta_2 - \gamma_2\,. \tag{5}$$

Since schedule s' is a equilibrium, it holds that the cost of job J_α in s' is no greater than that in $\tilde{s}'$, that is,

$$l_2(D', s') \le l_1(D', \tilde{s}')\,. \tag{6}$$

First, we know that the job set γ is nonempty, otherwise

$$\begin{aligned} l_1(D', \tilde{s}') &= l_1(D', s') + \alpha_1 - \beta_1 + \gamma_1 && \text{by (4)} \\ &< l_2(D', s') - \alpha_1 - \gamma_2 - \varepsilon_2 + \alpha_1 - \beta_1 + \gamma_1 && \text{by (3)} \\ &\le l_2(D', s')\,, \end{aligned}$$

which contradicts with (6).

Now that γ is nonempty, in schedule $\tilde{s}'$, the cost of jobs in γ is $l_1(D', \tilde{s}')$. The reason why jobs in γ move to machine 1 when job J_α moves to machine 1 is that if any job in γ stays at machine 2, the cost will be higher than $l_1(D', \tilde{s}')$. Thus we have $l_2(D', \tilde{s}') + \gamma_2 \ge l_1(D', \tilde{s}')$. Since $l_1(D', \tilde{s}') \ge l_2(D', s')$ (inequality (6)), it follows that $l_2(D', \tilde{s}') + \gamma_2 \ge l_2(D', s')$. Together with (5) we get $\beta_2 \ge \alpha_2$.

The cost of jobs in β in $\tilde{s}'$ is $l_2(D', \tilde{s}')$. However, we notice that if jobs in β choose machine 1 (after job J_α chooses machine 1), the cost of jobs in β is at most $l_1(D', s') + \alpha_1$ (since jobs in γ will choose machine 2 in this case), and the cost $l_1(D', s') + \alpha_1$ is smaller than $l_2(D', \tilde{s}')$ because

$$\begin{aligned} l_1(D', s') + \alpha_1 &< l_2(D', s') - \gamma_2 - \varepsilon_2 && \text{by (3)} \\ &\le l_2(D', s') - \gamma_2 \\ &\le l_2(D', s') - \alpha_2 + \beta_2 - \gamma_2 && \text{by } \beta_2 \ge \alpha_2 \\ &= l_2(D', \tilde{s}') && \text{by (5)}\,. \end{aligned}$$

Therefore, the jobs in β have no incentive to choose machine 2 in $\tilde{s}'$, since the cost of choosing machine 1 is lower. In other words, schedule $\tilde{s}'$ is not a equilibrium, which is a contradiction. Thus, we conclude that $l_1(D', s') \ge l_2(D', s') - \alpha_1 - \gamma_2 - \varepsilon_2$, which proves this claim. □

From (2) and the above claim, we have

$$\begin{aligned} SPE_D\left(P_{[\ell:n]}\right) &\le l_1(D', s') + \sum_{j=\ell+1}^{n} \min_i p_{ij} \\ &\le \|D'\|_\infty + \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell+1}^{n} \min_i p_{ij}\,. \end{aligned} \tag{7}$$

Moreover, the cost of job J_ℓ, namely $l_1(D', s')$, must be no greater than the cost of choosing machine 2:

$$l_1(D', s') \le l_2(D'', s'')\,,$$

where $D'' = (d_1, d_2 + p_{2\ell})$ is the initial load if job J_ℓ chooses machine 2, and s'' is the schedule of the jobs $\{J_{\ell+1}, J_{\ell+2}, \ldots, J_n\}$ playing with initial load D''. Thus we obtain

$$\begin{aligned} SPE_D\left(P_{[\ell:n]}\right) &\le l_1(D', s') + \sum_{j=\ell+1}^{n} \min_i p_{ij} \\ &\le l_2(D'', s'') + \sum_{j=\ell+1}^{n} \min_i p_{ij} \\ &\le \|D''\|_\infty + \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell+1}^{n} \min_i p_{ij}\,. \end{aligned} \tag{8}$$

According to (7) and (8), it holds that

$$SPE_D\left(P_{[\ell:n]}\right) \leq \min\left\{||D'||_\infty, ||D''||_\infty\right\} + \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell+1}^{n} \min_i p_{ij} \,.$$

As $D' = (d_1 + p_{1\ell}, d_2)$ and $D'' = (d_1, d_2 + p_{2\ell})$, we know

$$\min\left\{||D'||_\infty, ||D''||_\infty\right\} \leq \max\left\{d_1, d_2\right\} + \min\left\{p_{1\ell}, p_{2\ell}\right\} \,.$$

Thus it follows that

$$\begin{aligned} SPE_D\left(P_{[\ell:n]}\right) &\leq \max\left\{d_1, d_2\right\} + \min\left\{p_{1\ell}, p_{2\ell}\right\} + \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell+1}^{n} \min_i p_{ij} \\ &= ||D||_\infty + \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell}^{n} \min_i p_{ij} \,, \end{aligned}$$

that is,

$$SPE_D\left(P_{[\ell:n]}\right) - ||D||_\infty \leq \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell}^{n} \min_i p_{ij} \,. \tag{9}$$

Since inequality (1) holds for the case $d_1 + p_{1\ell} < d_2$, and inequality (9) holds for the case $d_1 + p_{1\ell} \geq d_2$, we obtain that inequality (9) holds for any D, that is, $\Delta SPE\left(P_{[\ell:n]}\right) \leq \Delta SPE\left(P_{[\ell+1:n]}\right) + \sum_{j=\ell}^{n} \min_i p_{ij}$, which concludes the proof of the lemma. □

Theorem 4. *For two machines, the* ***SPoA*** *is at most* $2(n-1)$.

Proof. Applying Lemma 2, we have

$$\begin{aligned} \Delta SPE\left(P_{[1:n]}\right) &\leq \Delta SPE\left(P_{[2:n]}\right) + \sum_{j=1}^{n} \min_i p_{ij} \\ &\leq \Delta SPE\left(P_{[3:n]}\right) + 2\sum_{j=1}^{n} \min_i p_{ij} \\ &\leq \dots \\ &\leq (n-1)\sum_{j=1}^{n} \min_i p_{ij} \,. \end{aligned}$$

Since the optimal cost is at least $OPT \geq \sum_{j=1}^{n} \min_i p_{ij}/2$ (for 2 machines), it follows that

$$\mathbf{SPoA} \leq \frac{\Delta SPE\left(P_{[1:n]}\right)}{OPT} \leq 2(n-1) \,,$$

which completes the proof. □

5 Linear upper bound on the SPoS

In this section, we give a *linear upper bound* on the sequential price of stability for two machines (Theorem 5 below). Unlike in the case of the sequential price of anarchy, here we have the freedom to choose the order of the players. Each player can choose *any* tie-breaking rule. Since we consider a full information setting, the tie-breaking rules are also public knowledge.

Though finding the best order can be difficult, we found that a large set of permutations already gives a linear upper bound on **SPoS**. In particular, it is enough that the authority divides the players into *two groups* and puts players in the first group first, followed by the players from the second group. Inside each group players can form *any order*. The main result of this section is the following theorem:

Theorem 5. *For two machines, the **SPoS** is at most* $\frac{n}{2} + 1$.

The proof is in the appendix. This result cannot be extended to *three* or more machines, because the third machine changes the logic of the proof. In particular, we can no longer assume that the players on the second machine in the optimal assignment can guarantee low costs for themselves by simply staying on that machine. For two machines, we conjecture that actually there is always an order which leads to the optimum:

Conjecture 3. For two machines, the **SPoS** is 1.

Though we are not able to prove this conjecture, in the next section, we introduce a more restricted solution concept, and show that in that case the optimum can be achieved.

6 Achieving the optimum: the adaptive SPoS

In this section, we study the adaptive sequential price of stability. Unlike the previous models, here we assume that there is some authority, which has full control over the order of the players' arrival in the game. It does not only fix the initial complete order, but can also change the order of arrivals depending on the decision that previous players made. On the other hand, the players still have the freedom to choose any action in a given state, each of them aiming at minimizing her own final cost. The players also know the whole decision tree, and thus the way the authority chooses the order. As in the previous section, each player can use *any* tie-breaking rule, and the tie-breaking rules are also known to all players.

This model is the closest instantiation of a general extensive form game compared to the previously studied models in this paper. In this way, the authority has an option to punish players for deviating from the optimal path (path leading to a social optimum) by placing different players after the deviating decisions of the deviating player. As a result, rational players may achieve much better solutions in the end. The following theorem shows that achieving the optimum solution is possible for 2 machines:

Theorem 6. *For two machines, the **adaptive SPoS** is* 1.

The proof is in the appendix. The previous result cannot be extended to more than 2 machines:

Theorem 7. *For three or more machines, the **adaptive SPoS** is at least* $\frac{3}{2}$.

Proof. Consider the following instance with three machines and three jobs, where the optimum is shown as gray boxes:

	J_1	J_2	J_3
M_1	$4-\varepsilon$	2	2
M_2	4	3	3
M_3	6	$6-\varepsilon$	$6-\varepsilon$

We distinguish two cases for the first player to move (the root of the tree), and show that in neither case the players will implement the optimum:

1. *The first to move is* J_1. This player will choose the cheapest machine M_1, because none will join this machine. Indeed, the second player to move will choose M_2 knowing that the last one will then choose M_3.
2. *The first to move is* J_2 *or* J_3. This player will choose M_2 and *not* M_1. Indeed, if the first player to move, say J_2, chooses M_1, then either (I) the other two follow also the optimum (which costs 4 to J_2) or (II) they choose another solution, whose cost is at least $6-\varepsilon$. In the latter case, we have the lower bound. In case (I), we argue that choosing M_2 is better for J_2, because no other player will join: for the following players, being both on machine M_1 is already cheaper than being on M_2 with J_2.

In the first case, given that J_1 is allocated to M_1, the cheapest solution costs $6-\varepsilon$. In the second case, one among J_2 or J_3 is allocated to M_2. The best solution, in this case, costs again $6-\varepsilon$. This completes the proof. □

Remark 2. The following example shows that the analysis of Theorem 6 cannot be extended to 3 machines even in the case of identical machines. Assume that we have $m = 3$ machines, the initial loads on these machines are $(0, 2, 6)$ and there are 3 jobs left to be assigned with processing times $7, 5$ and 5. Note that the constrained optimum here is $(10, 9, 6)$, that is the first job with processing time 7 gets assigned to the second machine M_2, while both jobs with processing times 5 and 5 get assigned to machine M_1. On the other hand, if any of these players chooses different machine their cost is strictly decreasing in the subgame perfect equilibrium solution. We did not find any example showing that **adaptive SPoS** > 1 for more than 2 identical machines, unlike the case of unrelated machines.

7 Conclusions

In this paper, we disprove a conjecture from [23] and give a linear lower bound construction for the sequential price of anarchy. On the other hand, we show linear upper bound. For the best sequence of players, we prove a linear upper bound, that is 4 times lower than the upper bound for sequential price of anarchy. Moreover, we prove the existence of a sequential extensive game which gives an optimum solution. One possible direction for future research is to investigate whether the sequential price of stability is 1 for any number of *identical* machines. In this work, we give some evidence that the case of three (or more) machines is different from the case of two machines (see Theorem 7 and Remark 2).

Our linear lower bound on the sequential price of anarchy (Theorem 3) suggests that subgame perfect equilibria do not guarantee in the worst case a price of anarchy independent of the number of jobs, even for two machines. Though our lower bound is based on a suitable tie-breaking rule, we believe it holds without any tie being involved (Conjecture 2).

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8 Appendix

Proof (of Theorem 2). Consider the following instance with jobs arriving in this order (from left to right),

	J_1	J_2	J_3	J_4	J_5
M_1	$3-11\varepsilon$	ε	ε	$1-2\varepsilon$	$2-8\varepsilon$
M_2	ε	$2-9\varepsilon$	$2-8\varepsilon$	$1-2\varepsilon$	$1-2\varepsilon$

(10)

where the subgame perfect equilibrium is shown as gray boxes. Note that the optimum has cost 1, while the subgame perfect equilibrium costs $4-13\varepsilon$. By letting ε tend to 0 we get the desired result.

For ease of presentation, we assume that *players break ties in favor of machine* M_1 (this assumption can be dropped by using more involved coefficients for the ε terms – see Remark 3). To see why the allocation corresponding to the gray boxes is indeed a subgame perfect equilibrium, we note the following:

1. If the first three jobs follow the optimum, then J_4 prefers to deviate to M_2, which causes J_5 to switch to machine M_1:

	J_1	J_2	J_3
M_1	$3-11\varepsilon$	ε	ε
M_2	ε	$2-9\varepsilon$	$2-8\varepsilon$

$\Rightarrow$

J_4	J_5
$1-2\varepsilon$	$2-8\varepsilon$
$1-2\varepsilon$	$1-2\varepsilon$

Now the cost for J_3 would be $2-6\varepsilon$.

2. Given the previous situation, J_3 prefers to deviate to M_2 because in this way J_4 and J_5 choose M_1, and her cost will be $2-7\varepsilon$:

	J_1	J_2
M_1	$3-11\varepsilon$	ε
M_2	ε	$2-9\varepsilon$

$\Rightarrow$

J_3	J_4	J_5
ε	$1-2\varepsilon$	$2-8\varepsilon$
$2-8\varepsilon$	$1-2\varepsilon$	$1-2\varepsilon$

Now the cost for J_2 would be $3-9\varepsilon$.

3. Given the previous situation, J_2 prefers to deviate to M_2 because in this way J_3 and J_4 choose M_1, J_5 chooses M_2, and the cost for J_2 is $3-10\varepsilon$:

	J_1
M_1	$3-11\varepsilon$
M_2	ε

$\Rightarrow$

J_2	J_3	J_4	J_5
ε	ε	$1-2\varepsilon$	$2-8\varepsilon$
$2-9\varepsilon$	$2-8\varepsilon$	$1-2\varepsilon$	$1-2\varepsilon$

Now the cost for J_1 would be $3-10\varepsilon$.

We have thus shown that, if J_1 chooses M_2 then her cost will be $3-10\varepsilon$. To conclude the proof, observe that if J_1 chooses M_1, then by similar arguments as above job J_2 prefers to choose machine M_2 and all players will choose the allocation in (10), the cost for J_1, in this case, is also $3-10\varepsilon$. Since players break ties in favor of M_1, we conclude that the subgame perfect equilibrium is the one in (10). □

Remark 3 (tie-breaking rule). In the construction above, we have used the fact that players break ties in favor of M_1. This assumption can be removed by using slightly more involved coefficients for the ε terms, so that ties never occur.

Proof (of Theorem 5). Consider an optimal assignment and denote the corresponding makespan by OPT. By renaming jobs and machines, we can assume without loss of generality that in this optimal assignment machine M_1 gets the first $k \leq \frac{n}{2}$ jobs, and machine M_2 gets all the other jobs:

$$\{J_1, J_2, \ldots, J_k\} \to M_1 \ , \qquad \{J_{k+1}, \ldots, J_n\} \to M_2 \ .$$

Take the sequence given by the jobs allocated to M_1 followed by the jobs allocated to M_2,

$$J_1, J_2, \ldots, J_k, J_{k+1}, \ldots, J_n.$$

We prove that for this sequence there is a subgame perfect equilibrium whose makespan is at most $(k+1) \cdot OPT$.

In the proof, we consider the *first player who deviates* from the optimal allocation. We distinguish two cases, corresponding to the next two claims.

Claim. If the first player J_d who deviates is in $\{J_{k+1}, \ldots, J_n\}$, then she does not improve her own cost.

Proof (of Claim). Observe that all players in $\{J_1, J_2, \ldots, J_k\}$, who came before player J_d, did not deviate. Machine M_1 has thus exactly the jobs that it gets in the optimum. If J_d stays on M_2, her cost will be at most OPT (in the worst-case, all subsequent jobs choose to stay on M_2). Moving to M_1 will in the end produce a schedule with fewer jobs on M_2 and more jobs on M_1, compared to the optimum. The cost on M_1 is therefore at least OPT (otherwise the new schedule has smaller makespan than OPT, a contradiction with the optimality), which is the cost of J_d when deviating.

The remaining case is the following one.

Claim. If the first player J_d who deviates is in $\{J_1, \ldots, J_k\}$, then any subgame perfect equilibrium has makespan at most $(t+1) \cdot OPT$ where $t = k + 1 - d$.

Proof (of Claim). The proof is by induction on t. For $t = 1$, the deviating player is the last in $\{J_1, \ldots, J_k\}$, i.e., J_k. Note that, if J_k does not deviate, then by the previous claim, J_k guarantees her cost to be at most OPT. Thus, if J_k deviates to M_2, then in the resulting equilibrium schedule she cannot pay more, i.e., she pays at most OPT. We now argue that if J_k deviates, M_1 will have, in the resulting equilibrium, load at most $2 \cdot OPT$. Clearly, in the resulting equilibrium, the load on M_2 is at most OPT. Some of the jobs among $J_{k+1}, \ldots, J_n$ may be assigned to M_1 in this equilibrium. Moving them all to machine M_2 will result in a load on M_2 being at most $2 \cdot OPT$ (since all these jobs are, in the optimum solution, on M_2). Hence, each job among $J_{k+1}, \ldots, J_n$ that decided to move to M_1 in the resulting equilibrium cannot have a worse cost than that of staying on M_2, which guarantees cost at most $2 \cdot OPT$.

For $t > 0$, the first player who deviates from the optimal assignment is J_{k-t}. We argue similarly that the makespan is at most $(t+1) \cdot OPT$. By induction we can assume that if player J_{k-t} stays on the first machine then she is guaranteed to pay at most $t \cdot OPT$. Since in the subgame perfect equilibrium J_{k-t} chooses the second machine, we know that she is paying at most $t \cdot OPT$ on the second machine. Thus, the cost on the second machine is at most $t \cdot OPT$. We next argue that the cost on the first machine is at most $(t+1) \cdot OPT$. If no player in $\{J_{k+1}, \ldots, J_n\}$ chooses the first machine, then the cost of this machine is at most OPT. Otherwise, if some player J'' in $\{J_{k+1}, \ldots, J_n\}$ chooses the first machine, then we show that she is paying at most $(t+1) \cdot OPT$, thus implying that the cost of the first machine is at most $(t+1) \cdot OPT$. This is because, if J'' would choose the second machine, she would pay at most $(t+1) \cdot OPT$: indeed, the cost on the second machine was at most $t \cdot OPT$ and even if all players after J'' choose M_2, they will contribute at most another additional factor OPT (because the players after J'' are in $\{J_{k+1}, \ldots, J_n\}$).

We have thus shown that the cost on the first machine is at most $(t+1) \cdot OPT$, and therefore this sequence results in a solution which has makespan at most $(k+1) \cdot OPT$, which completes the proof. □

The two claims above imply the theorem. □

Proof (of Theorem 6). [Main Idea] The main idea of the proof is as follows. Each internal node of a tree corresponds to a choice of some player, and the path (edges) to that node correspond to an allocation of a *subset of players* (the nodes on the node-to-root path). We consider the corresponding *constrained optimum*, that is, allocation of all remaining jobs that minimizes the makespan, given the fixed allocation of the previous players. Among these remaining players, we then find a particular one for which the constrained optimum is better than any constrained optimum if she deviates. If this player deviates, we can *punish* such deviation by letting the others implement the more expensive constrained optimum (by adaptively fixing their order).

Proof. For any node h of the tree, let S_h be the subset of players that appeared on the previous nodes (i.e., from the parent of h up to the root), and let A_h be the resulting allocation (described by the path). Let opt_h be the constrained optimum, that is, the allocation of all remaining jobs R_h which, combined with A_h, minimizes the resulting makespan. We now choose a suitable player $J^*(h)$ to put on node h, according to the following:

Claim. There exists a player $J^*(h)$ among the remaining players R_h such that the following holds. If $J^*(h)$ deviates from the constrained optimum opt_h, then the new constrained optimum (if implemented) is not less costly for $J^*(h)$.

Proof (of Claim). Consider the best alternative solution opt'_h, that is, the best allocation for the remaining jobs R_h, given the allocation A_h of jobs in S_h, such that at least one job in R_h is not allocated as in opt_h. Observe the following:

– There must be a job $J' \in R_h$ which opt'_h allocates differently from opt_h and in opt'_h this job is on a machine determining the makespan of opt'_h.

This means that if J' deviates from opt_h, then she is choosing the machine according to opt'_h. Consider the resulting constraints A'_h and R'_h, where we extend A_h by allocating J' as in opt'_h (therefore the remaining players are $R'_h = R_h \setminus \{J'\}$). Since opt'_h is the best solution among all solutions in which at least one job from R_h is not allocated as in opt_h, then opt'_h is a constrained optimum for such A'_h and R'_h. The cost of J' in solution opt'_h is the corresponding makespan, which cannot be smaller than the makespan of opt_h. We can thus choose $J^*(h) = J'$ and observe that deviating from opt_h makes $J^*(h)$ incur at least the same cost as in opt_h. □

At each node h, the chosen player $J^*(h)$ can either follow the constrained optimum, or deviate. backward induction guarantees that in either case, the remaining players implement the resulting constrained optimum. The above claim implies that $J^*(h)$ does not deviate from the constrained optimum opt_h (under a suitable tie-breaking rule). □