

Multidimensional Stable Roommates with Master List[‡]

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Abstract

Since the early days of research in algorithms and complexity, the computation of stable matchings is a core topic. While in the classic setting the goal is to match up two agents (either from different “gender” (this is STABLE MARRIAGE) or “unrestricted” (this is STABLE ROOMMATES)), Knuth [1976] triggered the study of three- or multidimensional cases. Here, we focus on the study of MULTIDIMENSIONAL STABLE ROOMMATES, known to be NP-complete since the early 1990’s. Many NP-completeness results, however, rely on general input instances that do not occur in at least some of the specific application scenarios. With the quest for identifying islands of tractability for MULTIDIMENSIONAL STABLE ROOMMATES, we study the case of master lists. Here, as natural in applications where agents express their preferences based on “objective” scores, one roughly speaking assumes that all agent preferences are “derived from” a central master list, implying that the individual agent preferences shall be similar. Master lists have been frequently studied in the two-dimensional (classic) stable matching case, but seemingly almost never for the multidimensional case. This work, also relying on methods from parameterized algorithm design and complexity analysis, performs a first systematic study of MULTIDIMENSIONAL STABLE ROOMMATES under the assumption of master lists.

Keywords. Stable matching, partially ordered sets, NP-hardness, parameterized complexity, distance-from-triviality parameterization

1 Introduction

Computing stable matchings is a core topic in the intersection of algorithm design, algorithmic game theory, and computational social choice. It has numerous applications such as higher education admission in several countries [3, 6], kidney exchange [44], assignment of dormitories [40], P2P-networks [18], wireless three-sided networks [9], and spatial crowdsourcing [33]. The research started in the 1960’s with the seminal work of Gale and Shapley [19], introducing the STABLE MARRIAGE problem: given two different types of agents, called “men” and “women”, each agent of one gender has preferences (i.e., strict orders aka rankings) over the agents of the

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†An extended abstract of this work appears in the *Proceedings of the 16th International Conference on Web and Internet Economics* [7]. This full version now contains full proofs of all results.

opposite gender. Then, the task is to find a matching which is stable. Informally, a matching is *stable* if no pair of agents can improve by breaking up with their currently assigned partners and instead matching to each other.

Many variations of this problem have been studied; STABLE ROOMMATES, with only one type of agents, is among the most prominent ones. Knuth [30] asked for generalizing STABLE MARRIAGE to dimension three, i.e., having three types of agents and having to match the agents to groups of size three, where any such group contains exactly one agent of each type. Here, a matching is called *stable* if there is no group of three agents which would improve by being matched together. We focus on the MULTIDIMENSIONAL STABLE ROOMMATES problem. Here, there is only one type of agents, now having preferences over $(d - 1)$ -sets (that is, sets of size $d - 1$) of (the other) agents.

Example 1. Consider the following instance of 3-DIMENSIONAL STABLE ROOMMATES with six agents a, b, c, d, e , and f .

$$\begin{aligned}
a &: \{b, d\} \succ \{b, c\} \succ \{b, e\} \succ \{b, f\} \succ \{c, d\} \succ \{c, e\} \succ \{c, f\} \succ \{d, e\} \succ \{d, f\} \succ \{e, f\} \\
b &: \{a, d\} \succ \{a, c\} \succ \{a, e\} \succ \{a, f\} \succ \{c, d\} \succ \{c, e\} \succ \{c, f\} \succ \{d, e\} \succ \{d, f\} \succ \{e, f\} \\
c &: \{a, b\} \succ \{a, d\} \succ \{a, e\} \succ \{b, d\} \succ \{a, f\} \succ \{b, e\} \succ \{b, f\} \succ \{d, e\} \succ \{d, f\} \succ \{e, f\} \\
d &: \{a, b\} \succ \{a, c\} \succ \{a, e\} \succ \{a, f\} \succ \{b, c\} \succ \{b, e\} \succ \{b, f\} \succ \{c, e\} \succ \{c, f\} \succ \{e, f\} \\
e &: \{a, b\} \succ \{a, c\} \succ \{a, d\} \succ \{a, f\} \succ \{b, c\} \succ \{b, d\} \succ \{b, f\} \succ \{c, d\} \succ \{c, f\} \succ \{d, f\} \\
f &: \{a, b\} \succ \{a, c\} \succ \{a, d\} \succ \{a, e\} \succ \{b, c\} \succ \{b, d\} \succ \{b, e\} \succ \{c, d\} \succ \{c, e\} \succ \{d, e\}
\end{aligned}$$

Matching $M_1 := \{\{a, b, c\}, \{d, e, f\}\}$ is not stable, as $\{a, b, d\}$ is blocking because a prefers $\{b, d\}$ to $\{b, c\}$, agent b prefers $\{a, d\}$ to $\{a, c\}$, and d prefers $\{a, b\}$ to $\{e, f\}$. However, matching $M_2 := \{\{a, b, d\}, \{c, e, f\}\}$ is stable.

As this problem is NP-complete in general [36], we focus on the case where the preferences of all agents are derived from an ordered master list. For instance, master lists naturally arise when the agent preferences are based on scores, e.g., when assigning junior doctors to medical posts in the UK [26] or when allocating students to dormitories [40]. Master lists have been frequently used in the context of (two-dimensional) stable matchings [4, 26, 38, 40] or the related POPULAR MATCHING problem [29]. We generalize master lists to the multidimensional setting in two natural ways. First, following the above spirit of preference orders, we assume that the master list consists of sets of size $d - 1$. Each agent then derives its preferences from the master list by just deleting all $(d - 1)$ -sets containing the agent itself. For example, in Example 1 agents d, e , and f derive their preferences from the list $\{a, b\} \succ \{a, c\} \succ \{a, d\} \succ \{a, e\} \succ \{a, f\} \succ \{b, c\} \succ \{b, d\} \succ \{b, e\} \succ \{b, f\} \succ \{c, d\} \succ \{c, e\} \succ \{c, f\} \succ \{d, e\} \succ \{d, f\} \succ \{e, f\}$, while a, b , and c do not. In the second way we study, the master list is a poset over the set of agents. In this case, any agent a shall prefer a $(d - 1)$ -set t to a $(d - 1)$ -set t' if t is “better” than t' according to the master list, where “better” means that a does not prefer the k -th best agent of t' to the k -th best agent from t (according to the master list). For any tuples t, t' for which neither t is “better” than t' nor t' is “better” than t , an agent may prefer t to t' or t' to t independently of the other agents. More formally, we require that any agent prefers a set of $d - 1$ agents t to any set of $d - 1$ agents t' dominated by t , where we say that $t = \{a_1, \dots, a_{d-1}\}$ dominates $t' = \{b_1, \dots, b_{d-1}\}$ if $a_i = b_i$ or the master list prefers a_i to b_i for all $i \in [d - 1]$ and $a_i \neq b_i$ for some $i \in [d - 1]$. Then for any two sets $\{a_1, \dots, a_{d-1}\}$ and $\{b_1, \dots, b_{d-1}\}$ of $d - 1$ agents with $\{a_1, \dots, a_{d-1}\}$ dominating $\{b_1, \dots, b_{d-1}\}$, the preferences of any agent must fulfill that the set $\{a_1, \dots, a_{d-1}\}$ is before $\{b_1, \dots, b_{d-1}\}$. In Example 1, the preferences of c, d, e , and f are derived from the list of agents $a \succ b \succ c \succ d \succ e \succ f$. For such master lists, we also relax the condition that the master list is a strict order (that is, for every two different agents

a and b , either a is better than b or b is better than a in the master list) by the condition that the master list is a partially ordered set (poset), and consider the parameterized complexity with respect to parameters measuring the similarity to a strict order. Preferences where such a parameter is small might arise if there are few similar rankings, and each agent derives its ranking from these orders, or if the objective score consists of several attributes and each agent weights these attributes slightly differently. From these rankings of each agent, a master poset arises by saying that agent a is better than agent b if and only if all agents (except for a and b) agree on this.

1.1 Related work

STABLE ROOMMATES can be solved in linear time [25]. If the preferences are incomplete (that is, two agents may prefer being unmatched to being matched together) and derived from a strict master list, then both STABLE MARRIAGE and STABLE ROOMMATES admit a unique stable matching [26].¹

If the preferences are complete but contain ties, then there are three different generalizations of stability studied in the literature. *Weak stability* considers a pair to be blocking if both agents in this pair prefer each other to their assigned partner in the matching. *Strong stability* considers a pair to be blocking if one agent prefers the pair to the agent assigned to it, and the other agent does not prefer the pair to the agent assigned to it. *Super-stability* considers a pair to be blocking if both agents in this pair do not prefer their assigned partner to each other. Finding a weakly stable matching in a STABLE ROOMMATES instance is NP-complete [43]. However, if the preferences are complete, derived from a master list, and contain ties, then one can decide whether a given pair of agents in a STABLE MARRIAGE instance is matched together in some weakly stable matching in linear time [26] (which is NP-complete for general complete preferences [34]), and a weakly stable matching in a STABLE ROOMMATES instance always exists and can be found in linear time. For incomplete preferences derived from a master list with ties, an $O(\sqrt{nm})$ -time algorithm for finding a strongly stable matching is known [38] (where n is the number of agents and m is the number of acceptable pairs), while for general preferences, only an $O(mn)$ -time algorithm is known [31]. Finding a weakly stable matching in a STABLE ROOMMATES instance, however, is NP-complete if the preferences contain ties, are incomplete, and are derived from a master list [26]. Further examples of STABLE MARRIAGE problems becoming easier for complete preferences derived from a master list are given by Scott [46, Chapter 8]. There is quite some work for 3-DIMENSIONAL STABLE MARRIAGE [11, 39, 48, 49], but less so for 3-DIMENSIONAL STABLE ROOMMATES.

While master lists are a standard setting for finding 2-dimensional stable matchings [4, 26, 28, 38, 40], we are only aware of few works combining multidimensional stable matchings with master lists. Escamocher and O’Sullivan [16] gave a recursive formula for the number of 3-dimensional stable matchings for cyclic preferences (i.e., the agents are partitioned into three sets A_0 , A_1 , and A_2 , and each agent from A_i only cares about the agent from A_{i+1} (modulo 3) it is matched to) derived from master lists. Cui and Jia [9] showed that if the preferences are cyclic and the preferences of the agents from A_1 are derived from a master list, while each agent from A_3 is indifferent between all agents from A_1 , then a stable matching always exists and can be found in polynomial time, but it is NP-complete to find a maximum-cardinality stable matching. There is some work on d -dimensional stable matchings and cyclic preferences (without master lists) [23, 32].

Deineko and Woeginger [13] showed that 3-DIMENSIONAL STABLE ROOMMATES is NP-

¹Actually, Irving et al. [26] only state this for STABLE MARRIAGE, but the generalization to STABLE ROOMMATES is trivial (actually, the whole statement is straightforward).

complete for preferences derived from a metric space. For the special case of the Euclidean plane, Arkin et al. [1] showed that a stable matching does not always exist, but left the complexity of deciding existence open.

Huang [24] showed that 3-DIMENSIONAL STABLE ROOMMATES is NP-complete even if the preferences are *consistent*, i.e., for each agent a , there exists a strictly ordered preference list \succ_a over all other agents such that for any two pairs $\{b, c\}$ and $\{d, e\}$ of agents with $b \succ_a d$ and $c \succ_a e$ it holds that a prefers $\{b, c\}$ to $\{d, e\}$. Note that in a 3-DIMENSIONAL STABLE ROOMMATES instance, the preferences of all agents are derived from a strict order \succ as master poset if and only if the preferences of every agent are consistent, and this can be witnessed by the strict order \succ for every agent.

Iwama et al. [27] introduced the NP-complete STABLE ROOMMATES WITH TRIPLE ROOMS, where each agent has preferences over all other agents, and prefers a 2-set p of agents to a 2-set p' if it prefers the best-ranked agent of p to the best-ranked agent of p' , and the second-best agent of p to the second-best agent of p' . They showed that this problem is NP-complete.

Our scenario of MULTIDIMENSIONAL STABLE ROOMMATES can be seen as a special case of finding core-stable outcomes for hedonic games where each agent prefers size- d coalitions over singleton-coalitions which are then preferred over all other coalitions [45, 47]. Notably, there are fixed-parameter tractability results for hedonic games (without fixed “coalition” size as we request) with respect to treewidth (MSO-based) [22, 41]. Other research considers hedonic games with fixed coalition size [8], but aims for Pareto optimal outcomes instead of core stability which we consider.

To the best of our knowledge, the parameterized complexity of multidimensional stable matching problems has not yet been investigated.

1.2 Our contributions

Our results are surveyed in Table 1. To our surprise, even if the preferences are derived from a master list of 2-sets of agents (this is the special case of dimension $d = 3$), a stable matching is not guaranteed to exist (Section 3.1). We use such an instance not admitting a stable matching to show that THREE-DIMENSIONAL STABLE ROOMMATES is NP-complete also when restricted to preferences derived from a master list of 2-sets (Theorem 3.8).

If the preferences are derived from a strict master list of agents, then a unique stable matching always exists and can be found by a straightforward algorithm (Proposition 4.1). When relaxing the condition that the master list is strictly ordered to being a poset (i.e., the master list may also declare two agents “incomparable” instead of stating that one is better than the other, but if agent a is better than b and b is better than c , then also a is better than c), then the problem clearly is NP-complete, as a master list which ties all agents does not impose any condition on the preferences of the agents, and THREE-DIMENSIONAL STABLE ROOMMATES is NP-complete. Consequently, in the spirit of “distance from triviality”-parameterization [21, 37], we investigate the parameterized complexity with respect to several parameters measuring the distance of the poset to a strict order. Note that our algorithm can also solve the corresponding search problem. For the parameter maximum number of agents incomparable to a single agent, we show that MULTIDIMENSIONAL STABLE ROOMMATES is fixed-parameter tractable (FPT) (even when d is part of the input) (Theorem 4.6). If this parameter is bounded, then this results in a special case of 3-dimensional stable matching problems which can be solved by an “efficient” nontrivial algorithm. Considering the stronger parameter width of the master poset, we show THREE-DIMENSIONAL STABLE ROOMMATES to be W[1]-hard, and this is true also for the orthogonal parameter deletion (of agents) distance to a strictly ordered master poset (Theorem 4.27). We also show that THREE-DIMENSIONAL STABLE ROOMMATES is NP-complete

Table 1: Results overview: six variations of MULTIDIMENSIONAL STABLE ROOMMATES. All three studied parameters measure the similarity of the master poset to a strict order. Note that the parameter “max. number κ of incomparable agents” is weaker than the parameter “Width of master poset”, and both parameters are incomparable to the parameter “Deletion distance to strictly ordered master poset”.

Setting/Parameter	Complexity
Master list of 2-sets	NP-complete for $d = 3$ (Theorem 3.8)
Master poset of agents:	
Linear master poset of agents	linear time (Proposition 4.1)
max. number κ of incomparable agents	$O(n^2) + (\kappa^2 2^{12\kappa})^{O(\kappa^2 2^{12\kappa})} n$ (Theorem 4.6)
Width of master poset	W[1]-hard for $d = 3$ (Theorem 4.15)
Incomplete preferences, strictly ordered master poset	NP-complete for $d \geq 3$ (Theorem 4.34)
Deletion distance to strictly ordered master poset	W[1]-hard for $d = 3$ (Theorem 4.27)

even with a strict order of the agents as a master poset if each agent is allowed to declare an arbitrary set of 2-sets unacceptable (Theorem 4.34), contrasting the polynomial-time solvability when every agent accepts every $(d - 1)$ -set of other agents (Proposition 4.1).

1.3 Structure of the paper

After introducing basic notation in Section 2, we consider 3-DIMENSIONAL STABLE ROOMMATES with master list of $(d - 1)$ -sets of agents in Section 3 and show its NP-completeness. Then, we turn to master posets of agents. We show in Section 4.1 that 3-DIMENSIONAL STABLE ROOMMATES is easy if the master list is strictly ordered. Moreover, we consider the case that preferences are incomplete, or that the master list is a poset and investigate parameters measuring the similarity to a strict order in Sections 4.2 and 4.3. In Section 4.4, we show that 3-DIMENSIONAL STABLE ROOMMATES is NP-complete if the master list is strict, but every agent may declare an arbitrary subset of $(d - 1)$ -sets to be not acceptable. Finally, we conclude in Section 5.

2 Preliminaries

Let $[n] := \{1, 2, 3, \dots, n\}$ and $[n, m] := \{n, n+1, \dots, m\}$. A set of cardinality d will also be called *d-set*. For a set X and an integer d , we denote by $\binom{X}{d}$ the set of size- d subsets of X . A *preference list* \succ over a set X is a strict order of X . We call a set of pairwise disjoint d -subsets of a set A of agents a *d-dimensional matching*. Usually, d is clear from the context; if so, then we may only write “matching”. Given a d -dimensional matching M and an agent a , we denote by $M(a)$ the $(d - 1)$ set t such that $t \cup \{a\} \in M$; if for a no such $(d - 1)$ -set exists, then $M(a) := \emptyset$. We say that an agent a *prefers* a $(d - 1)$ -set t to a $(d - 1)$ -set t' if $t \succ_a t'$ where \succ_a is the preference list of a . Any agent prefers any $(d - 1)$ -set not containing itself to being unmatched. A *blocking d-set* for a d -dimensional matching M is a set of d agents $\{a_1, a_2, \dots, a_d\}$ such that, for all $i \in [d]$, either a_i is unmatched in M or $\{a_1, a_2, \dots, a_d\} \setminus \{a_i\} \succ_{a_i} \{b_1^i, b_2^i, \dots, b_{d-1}^i\}$, where $\{b_j^i : j \in [d - 1]\} \cup \{a_i\} \in M$. A matching is called *stable* if it does not admit a blocking d -set. Now, we are ready to define our central problem.

MULTIDIMENSIONAL STABLE ROOMMATES (MDSR)

- Input:* An integer d , a set A of agents together with a preference list \succ_a over $\binom{A \setminus \{a\}}{d-1}$ for each agent $a \in A$.
- Task:* Decide whether a stable matching exists.

Note that we require each agent to list each size- $(d - 1)$ set of other agents. We denote by ℓ -DSR the restriction of MDSR to instances with $d = \ell$. It is known that a 3-dimensional stable matching does not always exist, and 3-DSR is NP-complete [36].

A *master list* ML is a preference list over $\binom{V}{d-1}$. A preference list \succ_v for an agent a is *derived* from a master list ML by deleting all $(d - 1)$ -sets containing a .

Example 2. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of agents, $d = 3$, and let $\{a_1, a_2\} \succ \{a_2, a_4\} \succ \{a_1, a_3\} \succ \{a_3, a_4\} \succ \{a_2, a_3\} \succ \{a_1, a_4\}$ be the master list.

Then the preferences of a_1 are $\{a_2, a_4\} \succ_{a_1} \{a_3, a_4\} \succ_{a_1} \{a_2, a_3\}$, the preferences of a_2 are $\{a_1, a_3\} \succ_{a_2} \{a_3, a_4\} \succ_{a_2} \{a_1, a_4\}$, the preferences of a_3 are $\{a_1, a_2\} \succ_{a_3} \{a_2, a_4\} \succ_{a_3} \{a_1, a_4\}$, and the preferences of a_4 are $\{a_1, a_2\} \succ_{a_4} \{a_1, a_3\} \succ_{a_4} \{a_2, a_3\}$.

Next, we define the MULTIDIMENSIONAL STABLE ROOMMATES WITH MASTER LIST OF $(d - 1)$ -SETS problem (MDSR-ML-SETS).

MDSR-ML-SETS

- Input:* An integer d , a set A of agents, and a master list \succ_{ML} over $\binom{A}{d-1}$, from which the preference list of each agent is derived.
- Task:* Decide whether a stable matching exists.

Again, we denote by ℓ -DSR-ML-SETS the problem MDSR-ML-SETS restricted to instances with $d = \ell$.

We now turn to the case that the master list orders single agents instead of $(d - 1)$ -sets of agents. We first need the definition of a partially ordered set.

A *partially ordered set (poset)* is a pair (V, \succeq) , where \succeq is a binary relation over the set V such that (i) $v \succeq v$ for all $v \in V$, (ii) $v \succeq w$ and $w \succeq v$ if and only if $v = w$, and (iii) if $u \succeq v$ and $v \succeq w$, then $u \succeq w$.

If $v \succeq w$ and $v \neq w$, then we write $v \succ w$. If neither $v \succeq w$ nor $w \succeq v$, then we say that v and w are *incomparable*, and write $v \perp w$. Instead of $v \succeq w$ or $v \succ w$, we may also write $w \preceq v$ or $w \prec v$. A *weak order* is a poset such that for every a, b , and c with $a \perp b$ and $b \perp c$ also $a \perp c$ hold.

A *chain* of (V, \succeq) is a subset $X = \{x_1, x_2, \dots, x_k\} \subseteq V$ such that $x_i \succ x_{i+1}$ for all $i \in [k - 1]$. An *antichain* is a subset $X \subseteq V$ such that for all $v, w \in X$ with $v \neq w$, we have $v \perp w$. The *width* of a poset is the size of a maximum antichain.

For a poset \succ over a set V , $\kappa_{\succ}(v) := |\{w \in V : v \perp w\}|$ be the number of elements incomparable with v . We define $\kappa(\succ) := \max_{v \in V} \kappa_{\succ}(v)$. As an example, consider the poset $(\{v_1, v_2, v_3, v_4\}, \succ)$ with $v_1 \succ v_2$, $v_2 \succ v_3$, and $v_1 \succ v_4$. Here, v_1 is comparable to all other agents, v_2 and v_3 are only incomparable to v_4 , and v_4 is incomparable to v_2 and v_3 , and we have $\kappa(\succ) = 2$. Note that if \bar{G}_{\succ} is the incomparability graph of the poset (V, \succ) (i.e., the graph whose vertex set is V and there is an edge between $v, w \in V$ if and only if $v \perp w$), then $\Delta(\bar{G}_{\succ}) = \kappa(\succ)$, where $\Delta(\bar{G}_{\succ})$ is the maximum vertex degree in \bar{G}_{\succ} . If \succ is a weak order, then the parameter $\kappa(\succ)$ equals the maximum size of a tie.

Dilworth's Theorem [14] states that the width of a poset is the minimum number of chains such that each element of the poset is contained in one of these chains.

Having defined posets, we now show the connection to MULTIDIMENSIONAL STABLE ROOMMATES by using preferences derived from a poset of agents.

Definition 1. Given a set A of agents, a poset (A, \succ_{ML}) (called the *master poset*), and an integer d , a preference list \succ_a on $\binom{A \setminus \{a\}}{d-1}$ is derived from \succ_{ML} if whenever a_1, \dots, a_{d-1} and b_1, \dots, b_{d-1} with $a_i \succeq_{\text{ML}} b_i$ for all $i \in [d-1]$, then $\{a_1, \dots, a_{d-1}\} \succeq_v \{b_1, \dots, b_{d-1}\}$.

Example 3. Let $a_1 \succ a_2 \succ a_3 \succ a_4 \succ a_5$ be a master poset. Then a_1 has one of the following two preferences: $\{a_2, a_3\} \succ_{a_1} \{a_2, a_4\} \succ_{a_1} \{a_2, a_5\} \succ_{a_1} \{a_3, a_4\} \succ_{a_1} \{a_3, a_5\} \succ_{a_1} \{a_4, a_5\}$ or $\{a_2, a_3\} \succ_{a_1} \{a_2, a_4\} \succ_{a_1} \{a_3, a_4\} \succ_{a_1} \{a_2, a_5\} \succ_{a_1} \{a_3, a_5\} \succ_{a_1} \{a_4, a_5\}$.

For the master poset \succ with $a_2 \succ a_3$, $a_2 \succ a_4$, $a_3 \succ a_1$, and $a_4 \succ a_1$ (and $a_3 \perp a_4$), agent a_1 has one of the following preferences: $\{a_2, a_3\} \succ_{a_1} \{a_2, a_4\} \succ_{a_1} \{a_3, a_4\}$ or $\{a_2, a_4\} \succ_{a_1} \{a_2, a_3\} \succ_{a_1} \{a_3, a_4\}$.

The formal definition of MDSR-POSET reads as follows.

MDSR-POSET

Input: An MDSR instance $\mathcal{I} = (A, (\succ_a)_{a \in A}, d)$ and a master poset \succeq_{ML} such that the preferences \succ_a of each agent a are derived from \succeq_{ML} .

Task: Decide whether there exists a stable matching in \mathcal{I} .

Again, we denote by ℓ -DSR-POSET the problem MDSR-POSET restricted to instances with $d = \ell$.

Parameterized Complexity. A *parameterized language* L over a finite alphabet Σ is a subset $L \subseteq \Sigma^* \times \mathbb{N}$. A parameterized language L is *fixed-parameter tractable* if there exists an algorithm which correctly decides for every instance $(I, k) \in \mathbb{N}$ whether $(I, k) \in L$ in FPT-time, that is, $f(k) \cdot |(I, k)|^{O(1)}$, where $|(I, k)| := |I| + k$ with $|I|$ being the size of I and f being a computable function. The class of all fixed-parameter tractable parameterized languages is denoted by FPT.

To indicate that a parameterized language is not fixed-parameter tractable, one usually uses the notions of parameterized reductions and W[1]-hardness. A *parameterized reduction* from a parameterized language L to a parameterized language L' is an algorithm \mathcal{A} which computes, given an instance (I, k) of L , an instance $(I', k') \in L'$ with $k' \leq g(k)$ for some computable function g such that $(I, k) \in L$ if and only if $(I', k') \in L'$ and runs in FPT-time. W[1] is the class of parameterized problems which are equivalent to CLIQUE parameterized by solution size under parameterized reductions. To show W[1]-hardness of a parameterized problem L , one usually gives a parameterized reduction to CLIQUE or another W[1]-hard problem. Under the complexity-theoretic assumption $\text{FPT} \neq \text{W}[1]$, W[1]-hardness of a parameterized problem implies that this problem is not in FPT. For more details on parameterized complexity, we refer to standard textbooks [10, 15, 17, 37].

3 Three-dimensional stable roommates with master list of 2-sets

In this section, we consider the case that the preferences are complete and derived from a master list of $(d-1)$ -sets. In Section 3.1, we give a small instance with six agents not admitting a stable matching. We use this to show in Section 3.2 that already for dimension $d = 3$ and preferences derived from a master list of 2-sets, deciding whether an instance admits a stable matching is NP-complete.

Table 2: A blocking 3-set for each matching in instance $\mathcal{I}_{\text{instable}}$ from Observation 3.1.

Matching	Blocking 3-set	Matching	Blocking 3-set
$\{a, b, c\}, \{d, e, f\}$	$\{a, d, e\}$	$\{a, c, e\}, \{b, d, f\}$	$\{a, b, e\}$
$\{a, b, d\}, \{c, e, f\}$	$\{a, c, e\}$	$\{a, c, f\}, \{b, d, e\}$	$\{a, b, e\}$
$\{a, b, e\}, \{c, d, f\}$	$\{b, c, d\}$	$\{a, d, e\}, \{b, c, f\}$	$\{a, b, e\}$
$\{a, b, f\}, \{c, d, e\}$	$\{a, c, d\}$	$\{a, d, f\}, \{b, c, e\}$	$\{a, b, e\}$
$\{a, c, d\}, \{b, e, f\}$	$\{a, b, e\}$	$\{a, e, f\}, \{b, c, d\}$	$\{a, c, d\}$

3.1 A 3-DSR-ML instance not admitting a stable matching

Subsequently, we present a 3DSR-ML-SETS instance $\mathcal{I}_{\text{instable}}$ with six agents not admitting a stable matching, showing that stable matchings do not have to exist even in the presence of master lists. This somewhat surprising observation will be crucial for our NP-hardness proof in Section 3.2.

The instance $\mathcal{I}_{\text{instable}}$ has six agents a, b, c, d, e , and f . The master list is: $\{a, b\} \succ \{a, c\} \succ \{a, d\} \succ \{a, f\} \succ \{b, e\} \succ \{c, d\} \succ \{a, e\} \succ \{b, f\} \succ \{c, e\} \succ \{b, d\} \succ \{d, e\} \succ \{b, c\} \succ \{c, f\} \succ \{d, f\} \succ \{e, f\}$.

Observation 3.1. *Instance $\mathcal{I}_{\text{instable}}$ does not admit a stable matching.*

Proof. Table 2 presents for each of the $\binom{6}{3} = 10$ matchings a blocking 3-set. \square

3.2 NP-completeness of 3-DSR-ML

Using the instance $\mathcal{I}_{\text{instable}}$ from Section 3.1, we show NP-completeness of 3DSR-ML-SETS, reducing from the NP-complete problem 1-IN-3 POSITIVE 3-OCCURRENCE-SAT [20].

1-IN-3 POSITIVE 3-OCCURRENCE-SAT

Input: A boolean formula in conjunctive normal form, where each clause contains *exactly* three *pairwise different* variables, and each variable appears exactly three times and only non-negatedly in the formula.

Task: Decide whether there exists a truth assignment satisfying exactly one literal from each clause.

The basic idea of the reduction is the following. For each clause C_j , we have two agents c_j and d_j . For each variable x_i , we have three agents x_i^1, x_i^2 , and x_i^3 , one for each occurrence of the variable. Additionally, there are six agents $z_i^{k,\ell}$, $\ell \in [6]$, for each literal of a clause. In any stable matching the agents c_j and d_j are matched to an agent x_i^ℓ corresponding to a variable occurring in clause C_j . Now consider a variable x_i . If agent x_i^k (corresponding to the k -th occurrence of x_i) is matched to a 2-set $\{c_j, d_j\}$ (corresponding to clause C_j) for all $k \in [3]$, then a stable matching can match $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$. If agent x_i^k (corresponding to occurrences of variable x_i) is not matched to a 2-set $\{c_j, d_j\}$ (corresponding to clause C_j) for all $k \in [3]$, then we can match $\{x_i^1, x_i^2, x_i^3\}$ and then again match $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$. If, however, agent x_i^k is matched to a 2-set of the form $\{c_j, d_j\}$ for one or two values of k , then an agent x_i^j which is not matched to a 2-set $\{c_j, d_j\}$ will together with $z_i^{k,1}, z_i^{k,2}, z_i^{k,3}, z_i^{k,4}$, and $z_i^{k,5}$ form a subinstance of six agents which does not admit a stable matching, and, thus, the resulting matching will not be stable.

Summarizing, any stable matching matches c_j and d_j to exactly one variable occurring in clause C_j , and for each variable x_i , either all or none of the agents x_i^k are matched to such 2-sets $\{c_j, d_j\}$. In other words, setting those variables x_i such that x_i^k is matched to a 2-set $\{c_j, d_j\}$ to true and all other variables to false, we get a solution of the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance from each stable matching.

“Inverting” this process (i.e., matching each clause to its true variable, and then matching the variable gadgets as described above), allows to construct a stable matching from a solution to the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance.

Note that MDSR is in NP as the size of the input is $\Omega(\binom{n}{d-1})$, where n is the number of agents, as the master preference list contains $\binom{n}{d-1}$ sets of size $d-1$, and, thus, stability can be verified in time polynomial in the input size by just checking for each d -set whether it is blocking.

We now formally describe the reduction and prove its correctness.

3.2.1 The reduction

Let x_1, \dots, x_n be the variables and let C_1, \dots, C_m be the clauses of a 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance \mathcal{I} . We construct a 3-DSR-ML instance $\mathcal{I}' = (A, (\succ_a)_{a \in A})$ as follows.

For each clause C_j , we add two agents c_j and d_j to A . For the k -th occurrence ($k \in [3]$) of a variable x_i in a clause, we add an agent x_i^k . We refer to the agent corresponding to the ℓ -th literal of clause C_j as y_j^ℓ . The k -th occurrence of x_i is also the ℓ -th literal of some clause C_j , and we will denote the agent x_i^k also by y_j^ℓ , i.e., $x_i^k = y_j^\ell$. For each agent x_i^k , we add six agents $z_i^{k,1}, \dots, z_i^{k,6}$ to A .

For each $j \in [m]$, we define \mathcal{A}_j to be the following part of the master list:

$$\{c_j, d_j\} \succ \{y_j^1, d_j\} \succ \{y_j^3, c_j\} \succ \{y_j^2, d_j\} \succ \{y_j^2, c_j\} \succ \{y_j^3, d_j\} \succ \{y_j^1, c_j\}.$$

For each agent x_i^k , we define \mathcal{B}_i^k to be the following part of the master list (note that (by renaming x_i^k to a , agent $z_i^{k,1}$ to b , agent $z_i^{k,2}$ to c , \dots , and $z_i^{k,5}$ to f) this contains the instance $\mathcal{I}_{\text{instable}}$ from Observation 3.1; this ensures that x_i^k has to be matched to a 2-set which is before \mathcal{B}_i^k in the master list):

$$\begin{aligned} & \{x_i^k, z_i^{k,1}\} \succ \{x_i^k, z_i^{k,2}\} \succ \{x_i^k, z_i^{k,3}\} \succ \{x_i^k, z_i^{k,5}\} \succ \{z_i^{k,1}, z_i^{k,4}\} \succ \{z_i^{k,2}, z_i^{k,3}\} \\ & \succ \{x_i^k, z_i^{k,4}\} \succ \{z_i^{k,1}, z_i^{k,5}\} \succ \{z_i^{k,2}, z_i^{k,4}\} \succ \{z_i^{k,1}, z_i^{k,3}\} \succ \{z_i^{k,3}, z_i^{k,4}\} \\ & \succ \{z_i^{k,1}, z_i^{k,2}\} \succ \{z_i^{k,2}, z_i^{k,5}\} \succ \{z_i^{k,3}, z_i^{k,5}\} \succ \{z_i^{k,4}, z_i^{k,5}\} \succ \{x_j^i, z_i^{k,6}\} \\ & \succ \{z_i^{k,1}, z_i^{k,6}\} \succ \{z_i^{k,2}, z_i^{k,6}\} \succ \{z_i^{k,3}, z_i^{k,6}\} \succ \{z_i^{k,4}, z_i^{k,6}\} \succ \{z_i^{k,5}, z_i^{k,6}\}. \end{aligned}$$

We extend this to a sublist \mathcal{C}_i as follows:

$$\{x_i^1, x_i^2\} \succ \{x_i^2, x_i^3\} \succ \{x_i^1, x_i^3\} \succ \mathcal{B}_i^1 \succ \mathcal{B}_i^2 \succ \mathcal{B}_i^3.$$

The complete master list looks as follows.

$$\mathcal{A}_1 \succ \dots \mathcal{A}_m \succ \mathcal{C}_1 \succ \dots \succ \mathcal{C}_n \succ \dots, \text{ where the rest is arbitrarily ordered.}$$

We call the constructed 3-DSR-ML instance \mathcal{I}' .

3.2.2 Proof of the forward direction

We show how to construct a stable matching from a solution to a 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance.

Lemma 3.2. *Let $f : \{x_i\} \rightarrow \{\text{true}, \text{false}\}$ be a solution to the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance \mathcal{I} . Then \mathcal{I}' admits a stable matching.*

Proof. We construct a stable matching M as follows. We start with $M = \emptyset$.

Denote by $S := \{i \in [n] : f(x_i) = \text{true}\}$ the set of indices such that the corresponding variables are set to **true** by truth assignment f .

For a clause c_j , let $y_j^{\ell_j}$ be the variable in c_j set to **true** by f . For each $j \in [m]$, we add the 3-set $\{c_j, d_j, y_j^{\ell_j}\}$ to M . For all $i \in [n] \setminus S$, we add the 3-set $\{x_i^1, x_i^2, x_i^3\}$ to M . Finally, for each pair $(i, k) \in [n] \times [3]$, we add the 3-sets $\{z_i^{k,1}, z_i^{k,2}, z_i^{k,3}\}$ and $\{z_i^{k,4}, z_i^{k,5}, z_i^{k,6}\}$.

Since f assigns exactly one true variable to each clause, M is indeed a matching. It remains to show that M is stable. We do so by showing for each agent that it is not contained in a blocking 3-set.

Claim 1. *For all $j \leq m$, neither c_j nor d_j is contained in a blocking 3-set.*

Proof of Claim: We prove the claim by induction on j . For $j = 0$, there is nothing to show, as there are no agents c_0 or d_0 .

For the induction step, first note that we can ignore all 2-sets containing an agent c_p or d_p for some $p < j$, as we already know that they are not contained in a blocking 3-set. Thus, we consider the sublist ML' of ML arising by deleting all such 2-sets. The first 2-set of ML' is $\{c_j, d_j\}$. Let $y_j^{\ell_j} = x_i^k$ such that $\{c_j, d_j, y_j^{\ell_j}\} \in M$. The variable $y_j^{\ell_j}$ is not contained in any blocking 3-set, as it is matched to the first 2-set of sublist ML' . If c_j is contained in a blocking 3-set, then the blocking 3-set is $\{c_j, y_j^p, d_j\}$ for some $p < \ell_j$, as $\{y_j^p, d_j\}$ for $p < \ell_j$ are the only 2-sets c_j prefers to $\{y_j^{\ell_j}, d_j\}$. However, d_j does not prefer $\{y_j^p, c_j\}$ to $\{y_j^{\ell_j}, c_j\}$, and thus, $\{c_j, y_j^p, d_j\}$ is not a blocking 3-set. By symmetric arguments, d_j also cannot be contained in a blocking 3-set. ■

Claim 2. *No agent x_i^k or $z_i^{k,p}$ for $i \leq n$, $k \in [3]$ and $p \in [6]$ is contained in a blocking 3-set.*

Proof of Claim: We prove the claim by induction on i . For $i = 0$, there is nothing to show.

Note that all 2-sets from $\bigcup_{\ell \in [m]} \mathcal{A}_\ell$ contain an agent of the type c_ℓ or d_ℓ , and thus, no blocking 3-set contains a 2-set from $\bigcup_{\ell \in [m]} \mathcal{A}_\ell$ by Claim 1. Furthermore, by the induction hypothesis, no 2-set from \mathcal{C}_q for $q < i$ can be contained in a blocking 3-set. Thus, it is enough to consider the sublist ML_i of the master list arising through the deletion of \mathcal{A}_j for all $j \in [m]$ and \mathcal{C}_q for $q < i$.

If x_i is set to **true**, then all agents x_i^k are matched better than any 2-set from ML_i , and thus, cannot be part of a blocking 3-set. Otherwise, all of x_i^1 , x_i^2 , and x_i^3 are matched to the first 2-set not containing themselves in ML_i , and thus are not contained in a blocking 3-set.

Considering the sublist ML'_i arising from ML_i by deleting all 2-sets containing an agent x_i^k , one can check that the first 15 sets of two agents of this sublist only consider agents of $z_i^{k,p}$ for $p \in [6]$, and all agents $z_i^{k,p}$ are matched to one of these 15 sets of size two. Thus, any blocking 2-set containing an agents $z_i^{k,p}$ consists only of agents from $\{z_i^{k,q} : q \in [6]\}$. By enumerating all 20 such 3-sets, one easily verifies that none of them is blocking. ■

Since the set A of agents in \mathcal{I}' is $\{c_j, d_j : j \in [m]\} \cup \{x_i^k, z_i^{k,p} : i \in [n], k \in [3], p \in [6]\}$, the lemma now directly follows from Claims 1 and 2. □

3.2.3 Proof of the backward direction

Now, we show how to construct a solution to the 1-IN-3 POSITIVE 3-OCCURRENCE-SAT instance \mathcal{I} from a stable matching. First, we identify small subsets A' of agents such that \mathcal{I}' restricted to the agents from A' does not admit a stable matching, implying that at least one agent of A' must be matched to at least one vertex outside A' .

Lemma 3.3. *For any $i \in [n]$ and $k \in [3]$, the subinstance \mathcal{I}_i^k of \mathcal{I}' , which results from \mathcal{I}' by deleting all but the agents x_i^k and $\{z_i^{k,p} : p \in [6]\}$, does not admit a stable matching.*

Proof. Assume for the sake of contradiction that there exists a stable matching M . Note that deleting $z_i^{k,6}$ from \mathcal{I}_i^k results in an instance identical to the instance $\mathcal{I}_{\text{instable}}$ (this can be seen by renaming agent x_i^k to a , agent $z_i^{k,p}$ to b , agent $z_i^{k,2}$ to c , agent $z_i^{k,3}$ to d , agent $z_i^{k,4}$ to e , and agent $z_i^{k,5}$ to f). Thus, Observation 3.1 implies that M matches agent $z_i^{k,6}$ to a 2-set $\{x, y\}$. As all 2-sets containing $z_i^{k,6}$ appear on the end of the preference lists, x and y together with any unmatched agent form a blocking 3-set, a contradiction to the stability of M . \square

We now show that the two agents c_j and d_j created for clause C_j have to be matched to an agent corresponding to a literal in this clause; indeed, we will later see that this literal satisfies the clause C_j in the found solution.

Lemma 3.4. *In any stable matching M and for each $j \in [m]$, there is an $\ell \in [3]$ such that $\{c_j, d_j, y_j^\ell\} \in M$.*

Proof. We prove the lemma by induction on j .

Base case: If M does not contain the 2-set $\{c_1, d_1, y_1^\ell\}$ for all $\ell \in [3]$, then $\{c_1, d_1, y_1^\ell\}$ is a blocking 2-set for every $\ell \in [3]$.

Induction step: By the induction hypothesis, no agent c_p or d_p for $p < j$ is matched to c_j , d_j , or some y_j^ℓ . Hence, neither c_j nor d_j nor y_j^ℓ is matched to a 2-set which comes before \mathcal{A}_j in the master list. Thus, if M does not contain $\{c_j, d_j, y_j^\ell\}$ for all $\ell \in [3]$, then $\{c_j, d_j, y_j^\ell\}$ is a blocking 3-set for every $\ell \in [3]$, contradicting the stability of M . \square

We now want to show that for any $i \in [n]$, a stable matching contains either $\{x_i^1, x_i^2, x_i^3\}$ or matches x_i^k to 2-sets of the form $\{c_j, d_j\}$. In order to do so, we first show that for any $i \in [n]$ and $k \in [3]$, agents $z_i^{k,1}, \dots, z_i^{k,6}$ are matched to two 3-sets in any stable matching M unless at least one agent $z_i^{k,p}$ is matched to a 2-set containing an agent c_j , d_j , $x_{i'}^{k'}$, or $z_{i'}^{k',q}$ for some $j \in [m]$, $(i', k') \in [n] \times [3]$ with $i' < i$ or $i' = i$ and $k' \leq k$, and $q \in [6]$.

Lemma 3.5. *Let M be any stable matching, and let $i \in [n]$ and $k \in [3]$. Let $X := \{c_j, d_j : j \in [m]\} \cup \{x_{i'}^{k'}, z_{i'}^{k',q} : i' < i, k' \in [3], q \in [6]\} \cup \{x_{i'}^{k'}, z_{i'}^{k',q} : k' \leq k, q \in [6]\}$, and let $Z_i^k := \{z_i^{k,p} : p \in [6]\}$. If no agent $z_i^{k,p}$ is matched to a 2-set containing an agent from X , then M contains two 3-sets t_1 and t_2 which are subsets of Z_i^k .*

Proof. Note that every 2-set before \mathcal{B}_i^k in the master list contains an agent from X . List \mathcal{B}_i^k contains all 2-sets $\{z_i^{k,p}, z_i^{k,q}\}$ for $p, q \in [6]$ with $p \neq q$ (as well as some 2-sets containing $x_i^k \in X$). Now assume for a contradiction that the lemma does not hold, i.e., there exists some $z \in Z$ which is not matched to a 2-set $\{z', z''\}$ with $z', z'' \in Z_i^k$. Then there exist three such agents $z_1, z_2, z_3 \in Z_i^k$. Hence, z_1, z_2 , and z_3 are matched to 2-sets which appear after \mathcal{B}_i^k in the master list. Consequently, $\{z_1, z_2, z_3\}$ is a blocking 3-set, contradicting the stability of M . \square

Now we can show the following structural statement about the agents x_i^1 , x_i^2 , and x_i^3 , essentially stating that if one of these agents is matched to a 2-set $\{c_j, d_j\}$ (corresponding to setting variable x_i to true), then all three of them are. From this statement, the backward direction of the correctness proof for the reduction will then easily follow.

Lemma 3.6. *Let M be any stable matching. Then for all $i \in [n+1]$, the following holds:*

For all $i^ < i$ either $\{x_{i^*}^1, x_{i^*}^2, x_{i^*}^3\} \in M$ or for each $k \in [3]$, there exists some $j \in [m]$ such that $\{x_{i^*}^k, c_j, d_j\} \in M$. Furthermore, for each $k \in [3]$, matching M contains two 3-sets t_1 and t_2 with $t_1, t_2 \subseteq \{z_{i^*}^{k,p} : p \in [6]\}$.*

Proof. We prove the lemma by induction on i . For $i = 1$, there is nothing to show.

By the induction hypothesis, for every $q < i$, no agent x_q^k or $z_q^{k,p}$ is matched to a 2-set containing an agent x_q^r or $z_q^{r,s}$ with $r \neq q$, and by Lemma 3.4, no agent $z_i^{k,p}$ is matched to a 2-set containing an agent c_j or d_j .

Let $S := \{k \in [3] : \exists j \in [m] \text{ s.t. } \{x_i^k, c_j, d_j\} \in M\}$ be the set of indices $k \in [3]$ such that x_i^k is matched to a 2-set of the form $\{c_j, d_j\}$.

Case 1: $|S| = [3]$.

By induction on k , we can apply Lemma 3.5 for every $k \in [3]$, showing that M contains two 3-sets $t_1, t_2 \subseteq \{z_i^{k,p} : p \in [6]\}$.

Case 2: $S = \emptyset$.

Then M contains $\{x_i^1, x_i^2, x_i^3\}$, as all 2-sets before sublist \mathcal{C}_i contain an agent which is not matched to an agent x_i^k in M . As in Case 1, induction on k together with Lemma 3.5 implies that for every $k \in [3]$, matching M contains two 3-sets t_1 and t_2 with $t_1, t_2 \subseteq \{z_i^{k,p} : p \in [6]\}$.

Case 3: $|S| \in \{1, 2\}$.

We show that this case leads to a contradiction and, therefore, cannot occur.

Case 3 (a): No 3-set in M contains two agents of the form x_i^k for $k \in [3]$.

If there is no $j \in [m]$ such that $\{x_i^1, c_j, d_j\} \in M$, then the agents $x_i^1, z_i^{1,1}, \dots, z_i^{1,5}$ have to be matched to two 3-sets, as all agents appearing in a 2-set before \mathcal{B}_i^1 in the master list cannot be matched to x_i^1 or $z_i^{1,p}$ because of Lemma 3.4 and the induction hypothesis. By Lemma 3.3, this implies that M contains a blocking 3-set inside $x_i^1, z_i^{1,1}, \dots, z_i^{1,5}$, contradicting the stability of M . Thus, there exists some $j \in [m]$ such that $\{x_i^1, c_j, d_j\} \in M$. Then M contains two 3-sets t_1 and t_2 with $t_1, t_2 \subseteq \{z_i^{1,p} : p \in [6]\}$ by Lemma 3.5.

We can conclude then by the same argument that x_i^2 is matched to a 2-set $\{c_{j'}, d_{j'}\}$ for some $j' \in [m]$, and from this that x_i^3 is matched to a 2-set $\{c_{j''}, d_{j''}\}$ for some $j'' \in [m]$, a contradiction.

Case 3 (b): There is a 3-set $\{x_i^{k_1}, x_i^{k_2}, z\} \in M$ with $z \notin \{x_i^k\}$.

If $\{x_i^1, c_j, d_j\} \in M$ for some $j \in [m]$, then M contains two tuples inside $z_i^{1,1}, \dots, z_i^{1,6}$ by Lemmas 3.4 and 3.5 and the induction hypothesis. It follows that there exists a blocking 3-set inside $x_i^2, z_i^{2,1}, \dots, z_i^{2,5}$ by Lemma 3.3.

Consequently, we can assume in the following that $\{x_i^1, c_j, d_j\} \notin M$ for all $j \in [m]$. We may also assume that $k_1 = 1$ or $k_2 = 1$, since otherwise $\{x_i^1, x_i^2, x_i^3\}$ was a blocking 3-set. Then x_i^1 prefers to be matched to the 2-set $\{z_i^{1,1}, z_i^{1,2}\}$ or $\{z_i^{1,3}, z_i^{1,5}\}$. Every 2-set which agent $z_i^{1,1}$ respectively $z_i^{1,2}$ prefers to $\{x_i^1, z_i^{1,2}\}$ respectively $\{x_i^1, z_i^{1,1}\}$ is one of the 2-sets $\{x_i^1, x_i^2\}$, $\{x_i^1, x_i^3\}$, or $\{x_i^2, x_i^3\}$, contains an agent x_k^j with $j < i$, or contains an agent c_j or d_j for some $j \in [m]$. Thus, unless $z_i^{1,1} = z$ or $z_i^{1,2} = z$ holds, $z_i^{1,1}$ and $z_i^{1,2}$ also prefer to be matched by $\{x_i^1, z_i^{1,1}, z_i^{1,2}\}$. If $z = z_i^{1,1}$ or $z = z_i^{1,2}$, then $z_i^{1,3} \neq z$ and $z_i^{1,5} \neq z$, and by arguments symmetrically to the above ones, $\{x_i^1, z_i^{1,3}, z_i^{1,5}\}$ is blocking. \square

The backward direction now easily follows.

Lemma 3.7. *If there exists a stable matching M , then there is a truth assignment for \mathcal{I} satisfying exactly one literal in each clause.*

Proof. Consider the assignment $f : \{x_i\} \rightarrow \{\text{true}, \text{false}\}$, with $f(x_i) := \text{true}$ if and only if all x_i^k are matched to a 2-set of the form $\{c_j, d_j\}$.

Assume that f is not a solution to \mathcal{I} . We distinguish two cases.

Case 1: There is a clause C_j which is not satisfied by f .

By Lemma 3.4, for each $j \in [m]$, there exists some $\ell \in [3]$ such that $\{c_j, d_j, y_j^\ell\} \in M$. Let $y_j^\ell = x_i^k$. By Lemma 3.6, for every $k' \in [3]$, agent $x_i^{k'}$ is matched to 2-set $\{c_p, d_p\}$ for some $p \in [m]$. Thus, the clause C_j is satisfied by f , a contradiction.

Case 2: There is a clause C_j which is satisfied by at least two variables x_i and $x_{i'}$.

Matching M can only contain one 3-set containing c_j and d_j , and so without loss of generality x_i^k is not matched to $\{c_j, d_j\}$ for any $k \in [3]$. By Lemma 3.4, literal $x_i^{k'}$ contained in C_j does not match to any 2-set $\{c_q, d_q\}$ for $q \in [m]$, and thus, we have $f(x_i) = \text{false}$, a contradiction.

Altogether, we conclude that f is a solution to \mathcal{I} . \square

Finally, we are ready to prove the main theorem of this section.

Theorem 3.8. *3-DSR-ML is NP-complete.*

Proof. Observe that 3-DSR-ML is in NP (the stability of a matching can be checked in $O(n^3)$ time by enumerating all 3-sets, where n is the number of agents).

The reduction described in Section 3.2.1 can be performed in linear time. Thus, the NP-completeness of 3-DSR-ML follows directly from Lemma 3.2 and Lemma 3.7. \square

We have seen that the strong restriction that the preferences of each agent are complete and derived from a common master list of d -sets presumably does not lead to an efficient algorithm, even for $d = 2$. Thus, in order to get tractable cases, other restrictions of the preferences are needed. One possibility would be to additionally require that every agent has consistent preferences. In this case, the master list implies that the preferences of every agent is indeed derived from a master poset which is a strict order, and we will see in the beginning of Section 4 that this implies that there exists a unique stable matching which can be found efficiently.

4 Master poset of agents

In this section, we consider the case when there does not exist a master list of $(d - 1)$ -sets of agents, but a master poset \succ_{ML} of single agents; in other words, we study the complexity of the problem MDSR-POSET. Each agent can derive its preferences from this master list, meaning that if for two $(d - 1)$ -sets $t \neq t'$, one can find a bijection σ from the elements of t to the elements of t' such that $a \succeq_{\text{ML}} \sigma(a)$ for all $a \in t$, then any agent (not occurring in t or t') shall prefer t to t' . We show that this problem is easily polynomial-time solvable if the master poset is a strict order (Section 4.1). Afterwards, following the approach of distance-from-triviality parameterization [21, 37], we generalize this result by showing fixed-parameter tractability for the parameter κ , the “maximum number of agents incomparable to a single agent” (Section 4.2.1). On the contrary, for the stronger parameter width of the poset, we show W[1]-hardness (Section 4.2.2), leaving open whether it can be solved in polynomial time for constant width (in parameterized complexity known as the question for containment in XP). Afterwards, in Section 4.3, again employing a distance-from-triviality parameterization we show W[1]-hardness of MDSR-POSET parameterized by the number of agents one needs to delete in order to have the preferences derived from a strict order. Finally, we show that the variation of MDSR-POSET where agents may declare an arbitrary part of $(d - 1)$ -sets as unacceptable

(that is, this agent may not be matched to such a $(d-1)$ -set) is NP-complete even if the master poset is a strict order (Section 4.4). Note that in order to distinguish from the master list of $(d-1)$ -sets in Section 3, we will always refer to the master poset as a poset, even if it is a strict order.

4.1 Strict orders

We consider the case that the master poset is a strict order. Then, an easy algorithm solves the problem: Just match the first d agents from the master poset together, delete them, and recurse. Note that the preferences of any agent cannot be directly derived from the master poset, as e.g. an agent may prefer either $\{a_1, a_4\}$ to $\{a_2, a_3\}$ or $\{a_2, a_3\}$ to $\{a_1, a_4\}$. Thus, the input contains the complete preferences of all agents, and the input size is $\Theta(d \binom{n}{d-1})$. In this sense, the running time of our algorithm is sublinear.

Proposition 4.1. *If \succeq_{ML} is a strict order, then any MDSR-POSET instance admits a uniquely determined stable matching. Assuming that the poset is given as a ranking $a_1 \succ_{\text{ML}} a_2 \succ_{\text{ML}} \dots \succ_{\text{ML}} a_n$, this stable matching can be found in $O(n)$ time, where n is the number of agents. If the poset is given via pairwise comparisons, then the unique stable matching can be found in $O(n^2)$ time.*

Proof. We number the agents in such a way that $a_1 \succ_{\text{ML}} a_2 \succ_{\text{ML}} \dots \succ_{\text{ML}} a_n$.

We claim that $M := \{\{a_{d(i-1)+1}, a_{d(i-1)+2}, \dots, a_{di}\} : 1 \leq i \leq \lfloor \frac{n}{d} \rfloor\}$ is a stable matching. We prove this claim by contradiction, so assume that there is a blocking d -set $\{a_{i_1}, a_{i_2}, \dots, a_{i_d}\}$ with $i_1 < i_2 < \dots < i_d$. Let $\{a_{i_1}, b_2, b_3, \dots, b_d\} \in M$ be the d -set containing a_{i_1} . Note that such a d -set exists as M leaves at most the $d-1$ last agents of the master poset unmatched, and the agents a_{i_j} , $j \in \{2, 3, \dots, d\}$, are ranked after a_{i_1} in the master poset. Since a_{i_j} is after a_{i_1} in the master poset, we have $b_j \succeq_{\text{ML}} a_{i_j}$ for all $j \in \{2, 3, \dots, d\}$. Thus, a_{i_1} cannot prefer $\{a_{i_2}, \dots, a_{i_d}\}$ to M , a contradiction. If the master poset is given as the order $a_1 \succ_{\text{ML}} \dots \succ_{\text{ML}} a_n$, then M can clearly be constructed in $O(n)$ time. If the master poset is given via pairwise comparisons, then we compute the strict order $a_1 \succ_{\text{ML}} \dots \succ_{\text{ML}} a_n$ in $O(n^2)$ time using a sorting algorithm, and from this, we can compute M in $O(n)$ time.

It remains to show that M is uniquely determined. Assume that there is a stable matching $M' \neq M$. Let i be the smallest index such that a_i is matched differently in M and M' . Since only the at most $d-1$ agents with highest index are unmatched in M , we get that a_i is matched in M . As either all or no agent from a d -set $t \in M$ are matched differently in M and M' , and all d -sets from M are of the form $\{a_{d(j-1)+1}, a_{d(j-1)+2}, \dots, a_{dj}\}$, it follows that $i = d(j-1) + 1$ for some $j \in [\lfloor \frac{n}{d} \rfloor]$.

We claim that $t := \{a_i, a_{i+1}, a_{i+2}, \dots, a_{i+d-1}\}$ is a blocking d -set for M' . By the definition of i , matching M' does not contain a d -set with one agent with index smaller than i and one agent with index at least i , and we have that $t \notin M'$. Thus, for any $a \in t$ with $t'_a := M'(a) \neq \emptyset$, the bijection $\sigma_a : t \setminus \{a\} \rightarrow t'_a \setminus \{a\}$ matching the agent with the j -th-lowest index in $t \setminus \{a\}$ to the agent with the j -th-lowest index in $t'_a \setminus \{a\}$ satisfies $b \succeq_{\text{ML}} \sigma(b)$ for all $b \in t \setminus \{a\}$. Since the preferences are derived from \succeq_{ML} , it follows that a prefers $t \setminus \{a\}$ to t'_a . Thus, t is a blocking d -set. \square

The polynomial-time solvability of the special case from Proposition 4.1 motivates three distance-from-triviality parameterizations studied in the following.

4.2 Posets

In two-dimensional stable (or popular) matching problems with master lists, also reflecting the needs of typical real-world applications, the master list usually contains ties [4, 26, 29, 38, 40].

In the following, we allow the master list not only to contain ties, but to be an arbitrary poset. In this case, the problem clearly is NP-complete, as the poset where each agent is incomparable to each other agent does not pose any restrictions on the preferences of the agents [36]. Hence, we consider several parameters measuring the similarity of the poset to a strict order—this is our polynomial-time solvable special case of the previous section. Thus, for the parameter “maximum number of agents incomparable to a single agent”, we show fixed-parameter tractability in Section 4.2.1, and for the stronger parameter width of the poset, we show W[1]-hardness in Section 4.2.2. Here, by “stronger” we mean that there are posets for which the width is bounded, while the maximum number of agents incomparable to a single agent is not, while the converse is not true (as an antichain of size k implies an agent with $k - 1$ agents incomparable to it). Consider for example the poset \succ over a set of $2n$ agents $a_1, \dots, a_n, b_1, \dots, b_n$ with $a_i \succ a_j$ and $b_i \succ b_j$ for all $i < j$ as well as $a_i \perp b_j$ for all $i, j \in [n]$. Then the width of this poset is two (as it can be decomposed in the two chains $a_1 \succ a_2 \succ \dots \succ a_n$ and $b_1 \succ b_2 \succ \dots \succ b_n$), while every agent is incomparable to n other agents.

4.2.1 Maximum number of agents incomparable to a single agent

In this section, we show that MDSR-POSET is fixed-parameter tractable when parameterized by $\kappa(\succ_{\text{ML}})$ (recall that $\kappa(\succ_{\text{ML}})$ denotes the maximum number of agents incomparable to a single agent in the master poset \succ_{ML}). As a first step of the algorithm, we show how to “approximate” the given poset by a strict order, meaning that for any two agents a and b with a being before b in the strict order, we have $a \succ_{\text{ML}} b$ or $a \perp_{\text{ML}} b$, and if a is “much earlier” in the strict order than b , we have that $a \succ_{\text{ML}} b$.

Lemma 4.2. *For any poset (A, \succeq) , there is an order a_1, a_2, \dots, a_n of A such that (i) for all $i < j$, we have that $a_i \succ a_j$ or $a_i \perp a_j$, and (ii) for all $j > i + 2\kappa(\succeq)$, we have $a_i \succ a_j$. Moreover, such an order can be found in $O(|A|^2)$ time.*

Proof. We prove the statement by induction on $|A|$. If $|A| = 1$, then there is nothing to show. So assume $|A| > 1$.

Let $a_1 \in A$ be an element such that $a_1 \succeq a$ or $a_1 \perp a$ for all $a \in A$. Such an element a_1 has to exist in any poset. By induction, we can find an order $a_2, \dots, a_{|A|}$ of $A \setminus \{a_1\}$ satisfying the lemma; note that $\kappa(A) \geq \kappa(A \setminus \{a_1\})$. We then add a_1 at the beginning of $a_2, \dots, a_{|A|}$. Let A' be the set of elements incomparable with a_1 . It remains to show that the elements from A' are among the $2\kappa(\succeq)$ first elements. Note that there is no $a' \in A'$ and $a \in A \setminus A'$ with $a \succ a'$, as otherwise $a_1 \succeq a \succ a'$ and thus $a_1 \succ a'$, but by the definition of A' we have $a_1 \perp a'$. Thus, for any $a' \in A'$, there are at most $|A'| + \kappa_{\succ}(a') \leq 2\kappa(\succeq)$ elements before a' , and thus, $a_1, \dots, a_{|A|}$ satisfies the lemma.

Since we can compute an element a_1 with $a_1 \succeq a$ or $a_1 \perp a$ for all $a \in A \setminus \{a_1\}$ in linear time, the order can be found in quadratic time. \square

For the remainder of Section 4.2.1, we fix an instance $\mathcal{I} = (A, (\succeq_a)_{a \in A}, \succeq_{\text{ML}})$ of MDSR-POSET, and an order a_1, \dots, a_n of the agents in A fulfilling the conditions of Lemma 4.2 for the poset (A, \succeq_{ML}) . Let $\kappa := \kappa(\succeq_{\text{ML}})$. Furthermore, let $A[\leq i] := \{a_1, \dots, a_i\}$, let $A[i, j] := \{a_i, a_{i+1}, \dots, a_j\}$, and let $A[\geq i] := \{a_i, a_{i+1}, \dots, a_n\}$ for any $i \in [n]$.

We now show that the agents contained in a d -set of a stable matching are close to each other in the order a_1, \dots, a_n .

Lemma 4.3. *Let $\mathcal{I} = (A, (\succeq_a)_{a \in A}, \succeq_{\text{ML}})$ be an MDSR-POSET-instance and let a_1, \dots, a_n be an order of the agents in A such that this order fulfills Lemma 4.2 for the poset (A, \succeq_{ML}) .*

For any stable matching M and any d -set $\{a_{i_1}, a_{i_2}, \dots, a_{i_d}\} \in M$ with $i_1 < i_2 < \dots < i_d$, it holds that $i_{j+1} - i_j \leq 2\kappa d^2 + 4\kappa + 3d + 1$ for all $j \in [d - 1]$.

Proof. Let M be a stable matching, and $\{a_{i_1}, a_{i_2}, \dots, a_{i_d}\} \in M$ be a d -set contained in M . We assume $i_1 < i_2 < \dots < i_d$, and fix some $j \in [d-1]$.

Let \mathcal{T}^+ be the set of d -sets in M containing at least one agent from $A[i_j + 2\kappa + 1, i_{j+1} - 2\kappa - 1]$ and at least one agent from $A[\geq i_{j+1} - 2\kappa]$ and let \mathcal{T}^- be the set of d -sets in M containing at least one agent from $A[i_j + 2\kappa + 1, i_{j+1} - 2\kappa - 1]$, and at least one agent from $A[\leq i_j + 2\kappa]$. We now give an example for the definitions of \mathcal{T}^+ and \mathcal{T}^- .

Example 4. Let $d = 4$, $\kappa = 5$, and M be a stable matching. Assume that M contains the 4-set $\{a_3, a_{14}, a_{50}, a_{157}\}$. Thus, $i_1 = 3$, $i_2 = 14$, $i_3 = 50$, and $i_4 = 157$. For instance, taking $j = 3$, the set \mathcal{T}^+ contains all 4-sets from M containing an agent from $\{a_{61}, a_{62}, \dots, a_{146}\}$, an agent from $\{a_{147}, a_{148}, \dots, a_n\}$, and two more arbitrary agents. The set \mathcal{T}^- contains all 4-sets from M containing an agent from $\{a_1, a_2, \dots, a_{60}\}$, an agent from $\{a_{61}, a_{62}, \dots, a_{146}\}$, and two more arbitrary agents.

Now, let t be a d -set from \mathcal{T}^+ . We claim that for every d -set $t' \in \mathcal{T}^+$ other than t , there exist agents $a \in t$ and $a' \in t'$ with $a \perp_{\text{ML}} a'$. Assume for a contradiction that there are two d -sets $t, t' \in \mathcal{T}^+$ such that there do not exist $a \in t$ and $a' \in t'$ with $a \perp_{\text{ML}} a'$. Let t^* contain the d agents from $t \cup t'$ with minimum index. By the definition of \mathcal{T}^+ , any d -set from \mathcal{T}^+ contains an agent from $A[\leq i_{j+1} - 2\kappa - 1]$ and one agent from $A[\geq i_{j+1} - 2\kappa]$. Therefore, at least one agent of t^* is contained in t , and at least one agent of t^* is contained in t' . For any agent $a_p \in t \setminus t^*$ and any $a_q \in t' \cap t^*$, it holds by the definition of t^* that $q < p$. By Lemma 4.2, it follows that $a_q \succ_{\text{ML}} a_p$ or $a_q \perp_{\text{ML}} a_p$. However, the latter is not possible, since we assumed that there are no two agents $a \in t$ and $a' \in t'$ with $a \perp_{\text{ML}} a'$. Thus, we have that each $a \in t \cap t'$ prefers t^* to t , and by symmetric arguments also each $a' \in t' \cap t^*$ prefers t^* to t' . It follows that the d -set t^* is blocking, contradicting the assumption that M is stable.

As any agent is incomparable to at most κ other agents, it follows that $|\mathcal{T}^+| \leq \kappa d + 1$. By analogous arguments, one can show that $|\mathcal{T}^-| \leq \kappa d + 1$.

Any d -set $s \in M$ consisting solely of agents from $A[i_j + 2\kappa + 1, i_{j+1} - 2\kappa - 1]$ directly implies a blocking d -set $\{a_{i_1}, \dots, a_{i_j}\} \cup s_{d-j}$, where s_{d-j} is an arbitrary subset of s containing exactly $d - j$ agents.

Hence, M contains at most $2(\kappa d + 1)$ sets containing an agent from $A[i_j + 2\kappa + 1, i_{j+1} - 2\kappa - 1]$, implying that $(i_{j+1} - 2\kappa - 1) - (i_j + 2\kappa + 1) \leq d \cdot 2(\kappa d + 1) + d - 1$, where $d - 1$ is added since there can be at most $d - 1$ unmatched agents. It follows that $i_{j+1} - i_j \leq 2\kappa d^2 + 4\kappa + 3d + 1$. \square

Lemma 4.3 implies that in order to find a stable matching, we only have to consider matchings M such that for every d -set $t \in M$, we have for any two agents $a_i, a_j \in t$ that $|i - j| \leq d(2\kappa d^2 + 4\kappa + 3d + 1)$. We will call such matchings *local*. We now develop a dynamic program which decides whether there is a local and stable matching using agents a_1, \dots, a_i for every $i \in [n]$, resulting in an FPT-algorithm for the combined parameter $\kappa + d$.

Proposition 4.4. MDSR-POSET can be solved in $O(n^2) + (\kappa d^4)^{O(\kappa d^4)} n$ time, where κ is the maximum number of agents incomparable to a single agent, d is the dimension (i.e., the group size), and n is the number of agents.

Proof. We first apply Lemma 4.2 to the poset (A, \succeq_{ML}) to get an order v_1, \dots, v_n of the agents in $O(n^2)$ time. Let $k := 2d(d-1)(2\kappa d^2 + 4\kappa + 3d + 1)$.

Our dynamic programming table τ has an entry $\tau[i, M]$ for each $i \in [n]$ and each local matching M such that any d -set $t \in M$ contains at least one agent of a_i, \dots, a_{i+k} . This entry shall be true if and only if M can be extended to a local matching M^* not admitting a blocking d -set consisting solely of agents from a_1, \dots, a_{i+k} . By Lemma 4.3, there exists a stable matching if and only if $\tau[n - k, M] = \text{true}$ for some local matching M . Thus, it remains to show how to compute these values.

For $i = 1$, we set $\tau[1, M] := \text{true}$ if and only if M does not contain a blocking d -set inside a_1, \dots, a_k .

To compute $\tau[i, M]$ for $i > 1$, we need to determine whether we can extend M to a local matching M^* on a_1, \dots, a_{i+k} such that every blocking d -set involves an agent a_j with $j > i + k$. Any such extension M^* induces a matching M_{i-1} by taking all d -sets containing an agent from $a_{i-1}, \dots, a_{i-1+k}$, and M^* also witnesses that $\tau[i-1, M_{i-1}] = \text{true}$. However, given a matching M_{i-1} with $\tau[i-1, M_{i-1}] = \text{true}$ and $M_{i-1}(a_j) = M(a_j)$ for all $j \in \{i, i+1, \dots, i+k-1\}$ does not imply that $\tau[i, M] = \text{true}$, as there might be a blocking d -set involving a_{i+k} and $d-1$ other agents from a_1, \dots, a_{i-1+k} . Since M_{i-1} does not store how a_1, \dots, a_{i-2} are matched, we cannot just enumerate all such d -sets and check whether they are blocking, but only can do this for d -sets consisting solely of agents from a_{i-1}, \dots, a_{i+k} . We will show that for any matching M_{i-1}^* witnessing that $\tau[i-1, M_{i-1}] = \text{true}$ that no blocking d -set containing a_{i+k} and at least one agent from a_1, \dots, a_{i-1+k} can occur; therefore, it is enough to check for blocking d -sets containing only agents from a_{i-1}, \dots, a_{i+k} . Thus, in order to compute $\tau[i, M]$, we look up whether there exists a local matching M_{i-1} with $\tau[i-1, M_{i-1}] := \text{true}$ and $M(a_j) = M_{i-1}(a_j)$ for all $j \in [i, i+k-1]$ such that $M_{i-1} \cup M$ does not admit a blocking d -set consisting of agents from a_{i-1}, \dots, a_{i+k} . If this is the case, then we set $\tau[i, M] = \text{true}$, and otherwise we set $\tau[i, M] = \text{false}$.

Since there are at most $k^{O(k)}$ partitions of a k -element set [12], table τ contains at most $nk^{O(k)}$ entries. Each entry can be computed in $k^{O(k)}$ time, resulting in an overall running time of $k^{O(k)}n = (\kappa d^4)^{O(\kappa d^4)}n$.

It remains to show the correctness of this dynamic program, which we do by induction. For $i = 1$, the values $\tau[i, M]$ are computed correctly by definition. Let $i > 1$. First assume that $\tau[i, M] = \text{true}$. Let M_{i-1} be the local matching with $\tau[i-1, M_{i-1}] = \text{true}$ and $M(a_j) = M_{i-1}(a_j)$ for all $j \in [i, i+k-1]$ and $M_{i-1} \cup M$ not admitting a blocking d -set consisting of agents from a_{i-1}, \dots, a_{i+k} . By the induction hypothesis, $\tau[i-1, M_{i-1}]$ was computed correctly, implying that there exists a local matching M_{i-1}^* on a_1, \dots, a_{i+k-1} such that for each $j \in [i-1, i+k-1]$, we have $M_{i-1}(a_j) = M_{i-1}^*(a_j)$, and no blocking d -set consists solely of agents from a_1, \dots, a_{i+k-1} . We define $M^* := M_{i-1}^* \cup M$. Note that M^* is indeed a (local) matching since a_{i-1} cannot be matched to agents with index at least $i+k$ and a_{i+k} cannot be matched to agents with index at most $i-1$ by the locality of M_{i-1}^* and M . Clearly, it holds that $M(a_j) = M^*(a_j)$ for all $j \in [i, i+k]$. Next, we show that M^* contains no blocking d -set consisting solely of agents from $A[\leq i+k]$. Any such d -set t must contain a_{i+k} , since M_{i-1}^* does not admit blocking d -sets consisting solely of agents from a_1, \dots, a_{i-1+k} . Furthermore, it must contain at least one agent a_j with $j < i-1$ since we checked for all d -sets of agents from a_{i-1}, \dots, a_{i+k} whether they are blocking. Since $k = 2d(d-1)2\kappa d^2 + 4\kappa + 3d + 1$, it follows that there exists some $\ell \in [i + (d-1)2\kappa d^2 + 4\kappa + 3d + 1, i+k - (d-1)2\kappa d^2 + 4\kappa + 3d + 1]$ such that t contains no agent from $A[\ell - (d-1)(2\kappa d^2 + 4\kappa + 3d + 1), \ell + (d-1)(2\kappa d^2 + 4\kappa + 3d + 1)]$, and a_ℓ is matched in M^* (note that such an agent exists since at most $d-1$ agents from $A[\leq i+k-1]$ can be unmatched in M^*). Let $t' \in M^*$ with $a_\ell \in t'$, and let t^* contain the d agents with minimum index from $t \cup t'$. Every agent $a \in t^* \cap t$ prefers every agent $b \in t^* \cap t'$ to every agent $c \in t \setminus t^*$ since the index of b is at least 2κ positions before c in the order a_1, \dots, a_n . Similarly, every agent $a \in t^* \cap t'$ prefers every agent $b \in t^* \cap t$ to every agent $c \in t' \setminus t^*$ since the index of b is at least 2κ positions before c in the order a_1, \dots, a_n . Hence, t^* forms a blocking d -set consisting solely of agents from $A[\leq i+k-1]$, a contradiction to the definition of M_{i-1}^* .

Now assume that the dynamic program computed $\tau[i, M]$ to be false. We assume for a contradiction that the correct value of $\tau[i, M]$ is true. Let M^* be a local matching witnessing that the correct value of $\tau[i, M]$ is true, i.e., we have $M^*(a_j) = M(a_j)$ for all $j \in [i, i+k]$, and M^* does not admit a blocking d -set consisting solely of agents from a_1, \dots, a_{i+k} . Let M_{i-1} be

the restriction of M^* to the d -sets containing $a_{i-1}, \dots, a_{i+k-1}$. The matching M^* also witnesses that $\tau[i-1, M_{i-1}] = \text{true}$, and this value is correctly computed by induction. Extending M_{i-1} with $M(a_{i+k})$ does not lead to a blocking d -set containing only agents from a_{i-1}, \dots, a_{i+k} , as M^* does not contain such a blocking d -set. It follows that the algorithm computed $\tau[i, M]$ to be true, a contradiction. \square

Getting rid of parameter d , we now extend Proposition 4.4 to an FPT-algorithm for the single parameter κ . To do so, we show that if κ is much smaller than d , then there always exists a stable matching. Note that we prove this by giving an efficient algorithm for *finding* a stable matching.

Lemma 4.5. *If $4\kappa 2^{4\kappa} \leq d$, then there exists a stable matching.*

Proof. We apply Lemma 4.2 to the poset (A, \succeq_{ML}) , getting an order a_1, \dots, a_n of the agents in A .

Claim. *There exists a d -set t^* such that, for any matching containing t^* , no blocking d -set contains an agent from t^* .*

Proof of Claim: For each $i \in [d - 2\kappa]$, let $t_i := \{a_i\} \cup t$, where t is the first $(d - 1)$ -set in the preferences of a_i . By Lemma 4.2, it holds that t_i contains a_j for each $j \in [d - 2\kappa]$, and all agents of t_i are from $A[\leq d + 2\kappa]$. Thus, there are at most $\binom{4\kappa}{2\kappa} \leq 2^{4\kappa} - 2$ different classes. Since $d - 4\kappa \geq 4\kappa(2^{4\kappa} - 1) > 4\kappa(2^{4\kappa} - 2)$, there exists a d -set t^* with $t^* = t_i$ for at least 4κ agents a_i .

Consider any matching M containing t^* , and assume that there is a blocking d -set t containing an agent $a^* \in t^*$. Let $t \setminus t^* = \{a_{i_1}, \dots, a_{i_r}\}$ with $i_1 < i_2 < \dots < i_r$ and $t^* \setminus t = \{a_{j_1}, \dots, a_{j_r}\}$ with $j_1 < j_2 < \dots < j_r$. Each agent from $t \setminus t^*$ is contained in $A[\geq d - 2\kappa + 1]$ (because $A[\leq d - 2\kappa] \subseteq t^*$), and $t \setminus t^*$ contains at most 2κ agents from $A[d - 2\kappa + 1, d + 2\kappa]$ (as the other 2κ agents of this set are contained in t^*). Thus, we have that $i_p \geq d - 2\kappa + p$ for all $p \in [r]$, and $i_p \geq d + 2\kappa + (p - 2\kappa) = d + p$ for all $p > 2\kappa$. Every agent a_i with $i \in [d - 2\kappa]$ such that $t_i = t^*$ is matched to its first choice and thus not contained in t . Thus, we have $j_{4\kappa} \leq d - 2\kappa$ and therefore, for all $p \in [4\kappa]$, we have that $j_p \leq d - 2\kappa - (4\kappa - p) = d - 6\kappa + p < d - 4\kappa + p \leq i_p - 2\kappa$. For $p > 4\kappa$ we have that $j_p \leq d + 2\kappa$, while $i_p \geq d + p > j_p + 2\kappa$. By Lemma 4.2 it follows that $a_{j_p} \succ_{\text{ML}} a_{i_p}$ for all $p \in [r]$. Thus, a^* prefers t^* to t , contradicting that t is a blocking d -set. \blacksquare

From the claim, the lemma follows easily: We start with an empty matching $M = \emptyset$ and as long as there are at least d unmatched agents, we successively compute such a d -set t^* , add t^* to M , delete the agents from t^* , and repeat. The resulting matching is clearly stable, as the agents from the d -sets added to M are not part of a blocking d -set. \square

Finally, it directly follows that MDSR-POSET is fixed-parameter tractable even when parameterized solely by κ .

Theorem 4.6. *MDSR-POSET can be solved in $O(n^2) + (\kappa^5 2^{16\kappa})^{O(\kappa^5 2^{16\kappa})} n$ time, where κ is the maximum number of agents an agent is incomparable to, and n is the total number of agents.*

Proof. If $4\kappa 2^{4\kappa} \leq d$, then we can safely answer yes by Lemma 4.5. Otherwise we have $d \leq 4\kappa 2^{4\kappa}$ and thus, Proposition 4.4 yields an algorithm running in $h(\kappa) + O(n^2)$ time with $h(\kappa) = f(\kappa, 4\kappa 2^{4\kappa})$ where $f(\kappa, d) = (\kappa d^4)^{O(\kappa d^4)}$. \square

So far, we considered the decision version of MDSR-POSET. However, the algorithms from Proposition 4.4 and Lemma 4.5 also solve (in the case of Proposition 4.4 after the straightforward

modification of storing a local matching in the dynamic program instead of true) the search version of MDSR-POSET, i.e., find a stable matching if one exists.

There is also a natural generalization of STABLE MARRIAGE to dimension d , namely d -DIMENSIONAL STABLE MARRIAGE, and it is natural to ask whether our algorithm carries over to this setting. In d -DIMENSIONAL STABLE MARRIAGE, the set A of agents is partitioned into d sets A^1, \dots, A^d of agents, and each agent of A^i has preferences over all $(d-1)$ -sets containing exactly one agent from A^j for all $j \in [d] \setminus \{i\}$. This problem is also fixed-parameter tractable parameterized by $\kappa+d$: The master poset of agents can then be decomposed into d master posets of agents, one for each set A^i . Then, one can apply Lemma 4.2 to each of these d master posets to get a strict order for the agents from $A^i = \{a_1^i, \dots, a_n^i\}$. Similarly to Lemma 4.3, one can show that, for any stable matching M and any d -set $\{a_{i_1}^1, \dots, a_{i_d}^d\}$ (w.l.o.g. we have $i_j \leq i_{j+1}$), it holds that $i_{j+1} \leq i_j + O(\kappa d^2)$. Now one can apply an algorithm similar to Proposition 4.4 (sweeping over the sets A^1, \dots, A^d from top to bottom, considering any matching on $k = f(\kappa, d)$ consecutive agents in each set A^i) to get an FPT-algorithm parameterized by $\kappa + d$. However, Lemma 4.5 does not seem to generalize to this case: for $d = 3$, there exists a small instance with $|A_1| = |A_2| = |A_3| = 3$ without a stable matching. “Cloning” the agents from one of the sets, say A_3 , an arbitrary number of times will result in an instance of unbounded d but $\kappa = 3$. It remains therefore unclear whether Theorem 4.6 generalizes to d -DIMENSIONAL STABLE MARRIAGE.

Remark 1. Until now, we assumed that the input is encoded naively, i.e., for each agent, its complete preference list is given as part of the input. However, this list is of length $\Omega(n^{d-1})$, which would result in a total input size of $O(n^d)$. Thus, it might be more reasonable to assume that the input is given by an oracle, which can answer queries about the preferences. In fact, the FPT-algorithm with the combined parameter κ and d only needs one type of queries, namely given two $(d-1)$ -sets t and t' and an agent a , the oracle tells whether a prefers t to t' . Thus, our FPT-algorithm parameterized only by κ also works when only using this query; however, in the case that κ is much smaller than d , it cannot compute a stable matching, but only state its existence. In order to also compute a stable matching efficiently, the algorithm would also need to be able to query, given an agent a and a set X of agents, what is the first $(d-1)$ -set in a 's preference list not containing an agent from X .

Having shown that MDSR-POSET is fixed-parameter tractable for the parameter κ , in Section 4.2.2 we turn to a stronger parameter, the width of the master poset.

4.2.2 Width of the master poset

Reducing from MULTICOLORED INDEPENDENT SET parameterized by solution size, we next show that 3DSR-POSET is $W[1]$ -complete parameterized by the width of the master poset.

In this section, at several points it does not matter how the preferences between a set of 2-sets look, as long as the preferences are derived from the poset (which will be described later in this section). Thus, whenever we describe the preferences of an agent, and these preferences contain a set \mathcal{X} of 2-sets, then this means that one gets the preferences of the agent through replacing \mathcal{X} by an arbitrary strict order $\succ_{\mathcal{X}}$ of the 2-sets contained in \mathcal{X} such that $\succ_{\mathcal{X}}$ is derived from the master poset. Furthermore, we only describe the beginning of the preferences of an agent, followed by “ $\succ \overset{(\text{rest})}{\dots}$ ”. For example, if we write $\{\{v, w\} : v, w \in \{v_1, v_2, v_3\}\} \succ \overset{(\text{rest})}{\dots}$ and the master poset is $v_1 \succ_{\text{ML}} v_i$ for $i \in \{2, 3, 4\}$, then the preferences start with $\{v_1, v_2\} \succ \{v_1, v_3\} \succ \{v_2, v_3\}$ or $\{v_1, v_3\} \succ \{v_1, v_2\} \succ \{v_2, v_3\}$, and end with $\{v_1, v_4\} \succ \{v_2, v_4\} \succ \{v_3, v_4\}$ or $\{v_1, v_4\} \succ \{v_3, v_4\} \succ \{v_2, v_4\}$. Then, all 2-sets not listed before $\succ \overset{(\text{rest})}{\dots}$ can be added in an arbitrary way obeying the master poset.

The reduction. We provide a parameterized reduction from MULTICOLORED INDEPENDENT SET parameterized by solution size, which is W[1]-hard [15, 42].

MULTICOLORED INDEPENDENT SET

Input: A k -partite graph $G = (V^1 \dot{\cup} V^2 \dot{\cup} \dots \dot{\cup} V^k, E)$ with $|V^i| = n$ for all $i \in [k]$.
Task: Decide whether G contains an independent set I such that $I \cap V^i \neq \emptyset$ for all $i \in [k]$.

Let $V^i = \{v_1^i, \dots, v_n^i\}$ for every $i \in [k]$. The basic idea of the reduction is as follows. For every V^i , we add a *vertex-selection gadget*, encoding which vertex from V^i shall be part of the multicolored independent set. For every edge $e \in E$, we add an *edge gadget*, ensuring that not both end vertices can be “selected” to be part of the multicolored independent set by the corresponding vertex-selection gadget. Thereby, the edge gadgets ensure that the vertices “selected” to be part of a multicolored independent set indeed form an independent set. Both the vertex-selection gadget and the edge gadget use a third kind of gadget, which we call *cut-off gadget*. A cut-off gadget for a given agent v and a 2-set t ensures that agent v has to be matched at least as good as t in every stable matching.

We now describe these three gadgets, starting with the cut-off gadget.

Cut-off gadget. For many agents, the reduction contains a *cut-off gadget*. The cut-off gadget for an agent a “cuts off” the preference list of agent a after a specific 2-set p_a , and enforces a to be matched to a 2-set p with $p \succeq_a p_a$.

Given an instance \mathcal{I} of MDSR-POSET with master poset \succ_{ML} , the cut-off gadget for an agent a and a 2-set p_a , denoted by CO_a , contains six agents z_a^1, \dots, z_a^6 . The basic idea is that deleting all agents but a and z_a^1, \dots, z_a^5 results in the instance $\mathcal{I}_{\text{instable}}$ (see Section 3.1). Because no stable matching will match agent z_a^r to a 2-set which z_a^r prefers to every 2-set $\{x, y\}$ with $x, y \in \{a, z_a^1, z_a^2, \dots, z_a^6\}$, agent a has to be matched better than every 2-set $\{z_a^r, z_a^{r'}\}$ for $r, r' \in [6]$ in every stable matching.

Given a 3DSR-POSET-instance $\mathcal{I} = (A, (\succ_a)_{a \in A}, d, \succ_{\text{ML}})$, we construct an instance \mathcal{I}' arising from \mathcal{I} by adding a cut-off gadget for an agent $a \in A$ and a 2-set p_a if we add six agents z_a^1, \dots, z_a^6 to \mathcal{I} , and the preferences of the agents in \mathcal{I}' are as follows (see Example 5 for an example). The preferences of any agent z_a^r start with all 2-sets in $\binom{B}{2} \cup (W \times \{z_a^\ell : \ell \in [6]\})$, where $B := \{b, z_b^\ell : b \succ_{\text{ML}} a, \ell \in [6]\}$, and are then followed by (ignoring all 2-sets containing the agent itself)

$$\begin{aligned} \mathcal{A} := & \{a, z_a^1\} \succ \{a, z_a^2\} \succ \{a, z_a^3\} \succ \{a, z_a^5\} \succ \{z_a^1, z_a^4\} \succ \{z_a^2, z_a^3\} \succ \{a, z_a^4\} \succ \{z_a^1, z_a^5\} \\ & \succ \{z_a^2, z_a^4\} \succ \{z_a^1, z_a^3\} \succ \{z_a^3, z_a^4\} \succ \{z_a^1, z_a^2\} \succ \{z_a^2, z_a^5\} \succ \{z_a^3, z_a^5\} \succ \{z_a^4, z_a^5\} \\ & \succ \{a, z_a^6\} \succ \{z_a^1, z_a^6\} \succ \{z_a^2, z_a^6\} \succ \{z_a^3, z_a^6\} \succ \{z_a^4, z_a^6\} \succ \{z_a^5, z_a^6\} \succ \dots \end{aligned}$$

The preferences of a in \mathcal{I}' arise from the preferences of a in \mathcal{I} by inserting after the 2-set p_a all 2-sets $\{z_a^r, z_a^{r'}\}$ with $r, r' \in [6]$ in the same order as in \mathcal{A} , and appending all other 2-sets containing an agent z_a^r at the end of the preferences of a . The preferences in \mathcal{I}' of an agent $b \in A \setminus \{a\}$ arise from b 's preferences in \mathcal{I} as follows: If $a \succ_{\text{ML}} w$, then we add all 2-sets containing an agent from the cut-off gadget at the beginning of their preference, while otherwise we add all 2-sets containing an agent z_a^r at the end of their preferences in an arbitrary order (consistent with the master poset).

Example 5. Consider the 3DSR-POSET-instance \mathcal{I} with four agents a_1, a_2, a_3 , and a_4 , master poset $a_1 \succ_{\text{ML}} a_2 \succ_{\text{ML}} a_3 \succ_{\text{ML}} a_4$, and the following preferences:

$$a_1 : \{a_2, a_3\} \succ \{a_2, a_4\} \succ \{a_3, a_4\},$$

$$\begin{aligned}
a_2 &: \{a_1, a_3\} \succ \{a_1, a_4\} \succ \{a_3, a_4\}, \\
a_3 &: \{a_1, a_2\} \succ \{a_1, a_4\} \succ \{a_2, a_4\}, \\
a_4 &: \{a_1, a_2\} \succ \{a_1, a_3\} \succ \{a_2, a_3\}.
\end{aligned}$$

The instance \mathcal{I}' arising from \mathcal{I} by adding a cut-off gadget for $a := a_2$ and $p_a := \{a_1, a_4\}$ has the following preferences (where for z_a^r all 2-sets containing z_a^r shall be deleted):

$$\begin{aligned}
a_1 &: \{a_2, a_3\} \succ \{a_2, a_4\} \succ \{a_3, a_4\} \dots \succ \overset{(\text{rest})}{\dots}, \\
a_2 &: \{a_1, a_3\} \succ \{a_1, a_4\} \succ \{z_a^1, z_a^4\} \succ \{z_a^2, z_a^3\} \succ \{z_a^1, z_a^5\} \succ \{z_a^2, z_a^4\} \succ \{z_a^1, z_a^3\} \succ \{z_a^3, z_a^4\}, \\
&\succ \{z_a^1, z_a^2\} \succ \{z_a^2, z_a^5\} \succ \{z_a^3, z_a^5\} \succ \{z_a^4, z_a^5\} \succ \{a, z_a^6\} \succ \{z_a^1, z_a^6\} \succ \{z_a^2, z_a^6\} \succ \{z_a^3, z_a^6\}, \\
&\succ \{z_a^4, z_a^6\} \succ \{z_a^5, z_a^6\} \succ \{a_3, a_4\}, \\
a_3 &: \{\{z_a^r, z_a^{r'}\} : r, r' \in [6], r \neq r'\} \succ \{a_1, a_2\} \succ \{a_1, a_4\} \succ \{a_2, a_4\} \succ \overset{(\text{rest})}{\dots}, \\
a_4 &: \{\{z_a^r, z_a^{r'}\} : r, r' \in [6], r \neq r'\} \succ \{a_1, a_2\} \succ \{a_1, a_3\} \succ \{a_2, a_3\} \succ \overset{(\text{rest})}{\dots}, \\
z_a^r &: \{a, z_a^1\} \succ \{a, z_a^2\} \succ \{a, z_a^3\} \succ \{a, z_a^5\} \succ \{z_a^1, z_a^4\} \succ \{z_a^2, z_a^3\} \succ \{a, z_a^4\} \succ \{z_a^1, z_a^5\} \\
&\succ \{z_a^2, z_a^4\} \succ \{z_a^1, z_a^3\} \succ \{z_a^3, z_a^4\} \succ \{z_a^1, z_a^2\} \succ \{z_a^2, z_a^5\} \succ \{z_a^3, z_a^5\} \succ \{z_a^4, z_a^5\} \\
&\succ \{a, z_a^6\} \succ \{z_a^1, z_a^6\} \succ \{z_a^2, z_a^6\} \succ \{z_a^3, z_a^6\} \succ \{z_a^4, z_a^6\} \succ \{z_a^5, z_a^6\} \succ \overset{(\text{rest})}{\dots}.
\end{aligned}$$

We now show that adding for every agent a a cut-off gadget for a and a 2-set p_a enforces that a is matched at least as good as p_a in every stable matching.

Lemma 4.7. *Let \mathcal{I}_0 be an MDSR-POSET-instance with set A of agents, and for each $a \in A$, let p_a be a 2-set not containing a . Let $A = \{a_1, \dots, a_n\}$ with $a_i \succ a_j$ or $a_i \perp a_j$ for all $i < j$. Let \mathcal{I}_s arise from \mathcal{I}_{s-1} by adding a cut-off gadget for agent a_s and 2-set p_{a_s} . Let \mathcal{M}_s be the set of stable matchings M in \mathcal{I}_0 with $M(a) \succ_a p_a$ for all $a \in \{a_1, \dots, a_s\}$.*

Then, for any matching $M \in \mathcal{M}_s$, the matching $M_s := M \cup \{\{z_a^1, z_a^2, z_a^3\}, \{z_a^4, z_a^5, z_a^6\} : a \in \{a_1, \dots, a_s\}\}$ is stable in \mathcal{I}_s . Furthermore, for every matching M' which is stable in \mathcal{I}_s , we have that $M' \cap \binom{A}{3} \in \mathcal{M}_s$.

Proof. We prove the lemma by induction on s . For $s = 0$, there is nothing to show. Fix $s > 0$.

First, we show that for any stable matching $M \in \mathcal{I}_0$ with $M(a) \succeq_a p_a$ for all $a \in \{a_1, \dots, a_s\}$, it holds that $M_s := M \cup \{\{z_b^1, z_b^2, z_b^3\}, \{z_b^4, z_b^5, z_b^6\} : b \in \{a_1, \dots, a_s\}\}$ is a stable matching in \mathcal{I}_s . Let $a := a_s$. No blocking 3-set consists solely of agents from $\{z_a^r : r \in [6]\}$; this can be seen by checking all 20 such 3-sets. Any blocking 3-set needs to contain at least one agent z_a^r (else it would already be a blocking 3-set in \mathcal{I}_{s-1} , contradicting the induction hypothesis). Agent a is not part of such a blocking 3-set, as it ranks all 2-sets containing an agent z_a^r after $M(a) \succeq_a p_a$. All other 2-sets preferred by agent z_a^r contain an agent a' or z_a^q with $a' \succ_{\text{ML}} a$. However, a' is matched to a 2-set $M(a')$ it prefers to $p_{a'}$, and therefore a' does not prefer any 2-set containing z_a^r to $M(a')$. Thus, a' is not part of a blocking 3-set. Consequently, the blocking 3-set contains solely agents from cut-off gadgets for agents a' with $a' \succ_{\text{ML}} a$. However, no agent $z_{a_i}^r$ prefers any pair $\{z_{a_i}^{i'}, z_{a_i}^{i''}\}$ with $i' \geq i$ and $i'' > i$. Thus, the blocking 3-set cannot contain agents from different cut-off gadgets, a contradiction.

To see the second part of the lemma, let M_s be a stable matching in \mathcal{I}_s . Let M be the matching arising from M_s by deleting all 3-sets containing an agent $z_{a_i}^r$. We need to show that M is stable and $M(a) \succeq_a p_a$ for all $a \in \{a_1, \dots, a_s\}$. If there exists some $a \in \{a_1, \dots, a_s\}$ such that $M(a) \succeq_a p$ does not hold, then let a_i be the agent of minimal index for which $p_{a_i} \succ M(a_i)$. Then for all $j < i$, matching M matches the agents from $\{z_{a_j}^r : r \in [6]\}$ in two 3-sets: These agents are not matched to $a_{j'}$ with $j' \leq j$ since $a_{j'}$ is matched at least as good as $p_{a_{j'}}$, and if $z_{a_j}^r$

is not matched solely to agents from CO_{a_j} , then it forms a blocking 3-set together with any two other agents of CO_{a_j} which are not matched solely to agents from CO_{a_j} . Therefore, no agent from CO_{a_i} including a_i is matched to a 2-set which it prefers to any 2-set consisting of agents from CO_{a_i} . By Lemma 3.3, it follows that there is a blocking 3-set in $\{a_i\} \cup \{z_{a_i}^r : r \in [5]\}$, contradicting the stability of M . Thus, $M(a) \succeq_a p$ holds for all $a \in \{a_1, \dots, a_s\}$.

It remains to show that M is stable in \mathcal{I}_0 . First note that the above argument shows that M_s matches every agent $z_{a_j}^r$ from cut-off gadget CO_{a_j} to a 2-set consisting solely of agents from CO_{a_j} . Thus, every agent $a \in A$ is matched the same in M_s and M . Hence, any blocking 3-set for M is also a blocking 3-set for M_s , and thus, M is stable. \square

Having shown that the cut-off gadgets have the desired effect, we now show how we can model the selection of one vertex being part of the independent set.

Vertex-Selection Gadget. A vertex-selection gadget has $6n$ agents a_p, b_p ($p \in [n]$), c_q, \bar{c}_q ($q \in [n-1]$), d_r, \bar{d}_r ($r \in [n+1]$). The intuitive idea is the following. The agents a_p and b_p want to be matched to 2-sets $\{c_q, \bar{c}_q\}$. As c_q prefers $\{a_p, \bar{c}_q\}$ to $\{b_{p'}, \bar{c}_q\}$ while \bar{c}_q prefers $\{b_{p'}, c_q\}$ to $\{a_p, c_q\}$, we can match the $n-1$ sets of size two of the form $\{c_q, \bar{c}_q\}$ to the agents $\{a_p : p < \ell\} \cup \{b_p : p < n+1-\ell\}$ for any $\ell \in [n]$, corresponding to selecting the vertex v_ℓ to be part of the independent set. The agents $\{a_p : p \geq \ell\} \cup \{b_p : p \geq n+1-\ell\}$ are then matched to the 2-sets $\{d_r, \bar{d}_r\}$, and can form blocking 3-sets with the edge gadgets (which are described later).

Formally, the preferences look as follows.

$$\begin{aligned}
a_p &: \{\{c_q, \bar{c}_q\} : q, q' \in [n-1]\} \succ \{\{d_r, \bar{d}_r\} : r, r' \in [n+1]\} \succ \text{CO}_{a_p}, \\
b_p &: \{\{c_q, \bar{c}_q\} : q, q' \in [n-1]\} \succ \{\{d_r, \bar{d}_r\} : r, r' \in [n+1]\} \succ \text{CO}_{b_p}, \\
c_q &: \{\{a_1, \bar{c}_q\} : q' \in [n-1]\} \succ \{\{a_2, \bar{c}_q\} : q' \in [n-1]\} \succ \dots \succ \{\{a_n, \bar{c}_q\} : q' \in [n-1]\} \\
&\succ \{\{b_1, \bar{c}_q\} : q' \in [n-1]\} \succ \{\{b_2, \bar{c}_q\} : q' \in [n-1]\} \succ \dots \succ \{\{b_n, \bar{c}_q\} : q' \in [n-1]\} \\
&\succ \text{CO}_{c_q}, \\
\bar{c}_q &: \{\{b_1, c_q\} : q' \in [n-1]\} \succ \{\{b_2, c_q\} : q' \in [n-1]\} \succ \dots \succ \{\{b_n, c_q\} : q' \in [n-1]\} \\
&\succ \{\{a_1, c_q\} : q' \in [n-1]\} \succ \{\{a_2, c_q\} : q' \in [n-1]\} \succ \dots \succ \{\{a_n, c_q\} : q' \in [n-1]\} \\
&\succ \text{CO}_{\bar{c}_q}, \\
d_r &: \{\{a_1, \bar{d}_r\} : r' \in [n+1]\} \succ \{\{a_2, \bar{d}_r\} : r' \in [n+1]\} \succ \dots \succ \{\{a_n, \bar{d}_r\} : r' \in [n+1]\} \\
&\succ \{\{b_1, \bar{d}_r\} : r' \in [n+1]\} \succ \{\{b_2, \bar{d}_r\} : r' \in [n+1]\} \succ \dots \succ \{\{b_n, \bar{d}_r\} : r' \in [n+1]\} \\
&\succ \text{CO}_{d_r}, \\
\bar{d}_r &: \{\{b_1, d_r\} : r' \in [n-1]\} \succ \{\{b_2, d_r\} : r' \in [n-1]\} \succ \dots \succ \{\{b_n, d_r\} : r' \in [n-1]\} \\
&\succ \{\{a_1, d_r\} : r' \in [n+1]\} \succ \{\{a_2, d_r\} : r' \in [n+1]\} \succ \dots \succ \{\{a_n, d_r\} : r' \in [n+1]\} \\
&\succ \text{CO}_{\bar{d}_r}.
\end{aligned}$$

In the master poset, we have $x_p \succ x_q$ for $p < q$ for all $x \in \{a, b, c, \bar{c}, d, \bar{d}\}$ as well as $x_i \perp y_j$ for all $x \neq y$. See Figure 1 for an example of the 3-sets $\{v_1, v_2, v_3\}$ such that $\{v_1, v_2, v_3\} \setminus \{v_p\}$ is before the cut-off gadget of v_p in the preferences of v_p for all $p \in [3]$.

Lemma 4.8. *Let \mathcal{I} be a MDSR-POSET-instance containing a vertex-selection gadget such that every agent v in the vertex-selection gadget except for $\{a_i, b_i : i \in [3n'+1]\}$ prefers all 2-sets consisting of two agents in the vertex-selection gadget to a 2-set containing an agent outside the vertex-selection gadget.*

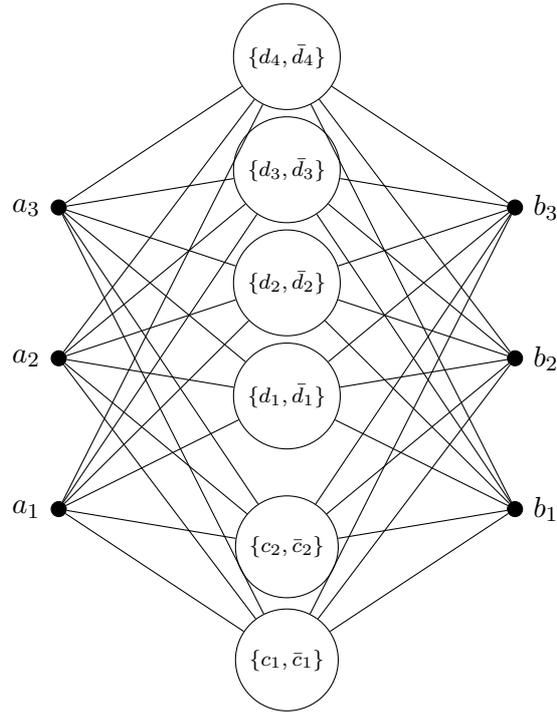


Figure 1: The acceptable 3-sets (i.e., 3-sets $\{x_1, x_2, x_3\}$ such that $\{x_1, x_2, x_3\} \setminus \{x_i\}$ is before the cut-off gadget of x_i in the preferences of x_i for all $i \in [3]$) of a vertex-selection gadget. For the sake of readability, we ignored 3-sets containing the agents c_i and \bar{c}_j or d_i and \bar{d}_j for $i \neq j$. Each edge corresponds to a 3-set containing the one endpoint a_i or b_i and the two vertices contained in the circle of the other endpoint of the edge. For example, the edge between a_1 and $\{c_1, \bar{c}_1\}$ indicates that $\{a_1, c_1, \bar{c}_1\}$ is an acceptable 3-set.

For every stable matching M in \mathcal{I} , there exists some $p^* \in [n]$ such that $M(a_p) \in \{\{c_q, \bar{c}_{q'}\} : q, q' \in [n-1]\}$ for all $p < p^*$, $M(a_p) \in \{\{d_r, \bar{d}_{r'}\} : r, r' \in [n+1]\}$ for $p \geq p^*$, $M(b_p) \in \{\{c_q, \bar{c}_{q'}\} : q, q' \in [n-1]\}$ for $p \leq n - p^*$, and $M(b_p) \in \{\{d_r, \bar{d}_{r'}\} : r, r' \in [n+1]\}$ for $i > n - p^*$.

Proof. Consider any stable matching M . Due to the cut-off gadgets, any agent c_q is matched to a 2-set $\{a_p, \bar{c}_{q'}\}$ or $\{b_p, \bar{c}_{q'}\}$ for some $p \in [n]$ and $q' \in [n-1]$, and any agent d_r is matched to a 2-set $\{a_p, \bar{d}_{r'}\}$ or $\{b_p, \bar{d}_{r'}\}$ for some $p \in [n]$ and $r' \in [n+1]$. Since $|\{a_p, b_p : p \in [n]\}| = 2n = |\{c_q : q \in [n-1]\}| + |\{d_r : r \in [n+1]\}|$, it follows that any agent a_p or b_p is matched to a 2-set $\{c_q, \bar{c}_{q'}\}$ or $\{d_r, \bar{d}_{r'}\}$.

Assume for a contradiction that there exists some $p \in [n]$ such that $\{a_p, c_q, \bar{c}_{q'}\} \notin M$ for all $q, q' \in [n-1]$ but $\{a_{p+1}, c_s, \bar{c}_{s'}\} \in M$ for some $s, s' \in [n-1]$ (the case that there exists such b_p and b_{p+1} is symmetric). Then $\{a_p, c_s, \bar{c}_{s'}\}$ is a blocking 3-set, contradicting the stability of M . \square

We now show that it is indeed possible to select an arbitrary vertex in any vertex-selection gadget.

Lemma 4.9. *Let \mathcal{I} be an MDSR-POSET-instance containing a vertex-selection gadget.*

For any $p^ \in [n]$, the matching $M_{p^*} := \{\{a_p, c_p, \bar{c}_{n-p^*+p}\} : p < p^*\} \cup \{\{a_p, d_{p-p^*+1}, \bar{d}_{p+1}\} : p \geq p^*\} \cup \{\{b_p, c_{p+p^*-1}, \bar{c}_p\} : p \leq n - p^*\} \cup \{\{b_p, d_{p+1}, \bar{d}_{p-(n-p^*)}\} : p > n - p^*\}$ contains no blocking 3-set solely consisting from agents in the vertex-selection gadget.*

Proof. Let $p^* \in [n]$ and assume that M_{p^*} contains a blocking 3-set t inside the vertex-selection gadget. Then t contains two agents c_q and $\bar{c}_{q'}$ or d_r and $\bar{d}_{r'}$, and an agent a_p or b_p .

Case 1: $t = \{a_p, c_q, \bar{c}_{q'}\}$. If $p > p^*$, then $\bar{c}_{q'}$ does not prefer $t \setminus \{\bar{c}_{q'}\}$ to $M(\bar{c}_{q'})$, so assume $p \leq p^*$, implying that $\{a_p, c_p, \bar{c}_{n-p^*+i}\} \in M_{p^*}$. If $p \leq \min\{q, q' + p^* - n\}$, then a_p does not prefer $t \setminus \{a_p\}$ to $M_{p^*}(a_p)$. If $q \leq \min\{p, q' + p^* - n\}$, then $\{a_q, c_q, \bar{c}_{n-p^*+q}\} \in M_{p^*}$ and c_q does not prefer $t \setminus \{c_q\}$ to $M_{p^*}(c_q)$. If $q' \leq \min\{p, q\} + n - p^*$, then $\bar{c}_{q'}$ does not prefer $t \setminus \{\bar{c}_{q'}\}$ to $M_{p^*}(\bar{c}_{q'})$.

Case 2: $t = \{a_p, d_r, \bar{d}_{r'}\}$. If $p \leq \min\{r + p^* - 1, r' - 1\}$, then a_p does not prefer $t \setminus \{a_p\}$ to $M_{p^*}(a_p)$. If $r \leq \min\{p + 1, r'\} - p^*$, then $\{a_{r+p^*-1}, d_r, d_{r+p^*}\} \in M_{p^*}$ and d_r does not prefer $t \setminus \{d_r\}$ to $M_{p^*}(d_r)$. If $r' \leq \min\{p, r + p^* - 1\} + 1$, then $\bar{d}_{r'}$ does not prefer $t \setminus \{\bar{d}_{r'}\}$ to $M_{p^*}(\bar{d}_{r'})$.

Case 3: $t = \{b_p, c_q, \bar{c}_{q'}\}$. If $p > n - p^*$, then c_q does not prefer $t \setminus \{c_q\}$ to $M_{p^*}(c_q)$, so assume $p \leq n - p^*$, implying that $\{b_p, c_{p+p^*-1}, \bar{c}_p\} \in M_{p^*}$. If $p \leq \min\{q - p^* + 1, q'\}$, then b_p does not prefer $t \setminus \{b_p\}$ to $M_{p^*}(b_p)$. If $q \leq \min\{p, q'\} + p^* - 1$, then c_q does not prefer $t \setminus \{c_q\}$ to $M_{p^*}(c_q)$. If $q' \leq \min\{p, q - p^* + 1\}$, then $\bar{c}_{q'}$ does not prefer $t \setminus \{\bar{c}_{q'}\}$ to $M_{p^*}(\bar{c}_{q'})$.

Case 4: $t = \{b_p, d_r, \bar{d}_{r'}\}$. If $p \leq \min\{r - 1, r' + n - p^*\}$, then b_p does not prefer $t \setminus \{b_p\}$ to $M_{p^*}(b_p)$. If $r \leq \min\{p, r' + n - p^*\} + 1$, then d_r does not prefer $t \setminus \{d_r\}$ to $M_{p^*}(d_r)$. If $r' \leq \min\{p, r - 1\} - (n - p^*)$, then $\{b_{r'+n-p^*}, d_{r'+n-p^*+1}, \bar{d}_{r'}\} \in M_{p^*}$ and $\bar{d}_{r'}$ does not prefer $t \setminus \{\bar{d}_{r'}\}$ to $M_{p^*}(\bar{d}_{r'})$.

In each of the four cases, we have a contradiction, finishing the proof of the lemma. \square

The parameterized reduction will create a vertex-selection gadget S^i for each set V^i , $i \in [k]$. We refer to agent a_p (respectively b_p, c_q, \bar{c}_q, d_r , or \bar{d}_r) from the i -th vertex-selection gadget via a_p^i (respectively $b_p^i, c_q^i, \bar{c}_q^i, d_r^i$, or \bar{d}_r^i).

We say that a matching M selects the vertex $v_{p^*}^i$ in vertex-selection gadget S^i if, for all $p < p^*$, we have $M(a_p) = \{c_q, \bar{c}_{q'}\}$ for some $q, q' \in [n-1]$, and for all $p \geq p^*$, we have $M(a_p) = \{d_r, \bar{d}_{r'}\}$ for some $r, r' \in [n+1]$.

We now turn to the edge gadgets, which model the edges of G .

Edge gadget. Fix a pair $(i, j) \in [k]^2$ with $i < j$, and let $E^{i,j} := \{\{v, w\} \in E(G) : v \in V^i, w \in V^j\}$ be the set of edges between V^i and V^j . For each edge $e = \{v_{r_i}^i, v_{r_j}^j\} \in E^{i,j}$, our reduction contains an edge gadget between S^i and S^j , containing the 15 agents $h_a^{e,i}, h_b^{e,i}, h_a^{e,j}, h_b^{e,j}, g_1^e, g_2^e, g_3^e, \bar{g}_1^e, \bar{g}_2^e, \bar{g}_3^e, f^e, \bar{f}^e, \alpha_1^e, \alpha_2^e, \alpha_3^e$.

The intuitive function of the edge gadget is the following. By Lemma 4.8, in any stable matching M there exists some i^* such that S^i selects $v_{i^*}^i$, i.e., a_ℓ^i is matched to a pair $\{d_p, \bar{d}_q\}$ if and only if $\ell \geq i^*$. Given an edge e , for any $\ell \geq i^*$, agent a_ℓ^i prefers being matched to the edge gadget (more specifically, to the 2-set $\{h_a^{e,i}, \alpha_1^e\}$) to being matched to M . Similarly, b_ℓ for $\ell > n - i^*$ prefers to be matched to the edge gadget, namely to the 2-set $\{h_b^{e,i}, \alpha_1^e\}$ to being matched to M . The agent $h_a^{e,i} [h_b^{e,i}]$ prefers the 2-set $\{a_{i^*}^i, \alpha_1^e\} [\{b_{n+1-i^*}^j, \alpha_1^e\}]$ to the 2-set $\{f^e, \bar{f}^e\}$ if $r_i \leq i^* [r_i \geq i^*]$. Thus, if $v_{r_i}^i = v_{i^*}^i$, i.e., if the vertex selected by S^i is an endpoint of e , then both $h_a^{e,i}$ and $h_b^{e,i}$ cannot be matched to $\{f^e, \bar{f}^e\}$. The edge gadget is now designed in such a way that an arbitrary of these vertices $h_x^{e,y}$ ($x \in \{a, b\}$ and $y \in \{i, j\}$) has to be matched to $\{f^e, \bar{f}^e\}$, which is possible if and only if e is not an edge between the two vertices selected by the vertex-selection gadgets, i.e., $e \neq \{v_{i^*}^i, v_{j^*}^j\}$.

We now give a formal description of the edge gadget. We fix an arbitrary order $\succ^{i,j}$ of the edges from $E^{i,j}$, and denote by $E^{\succeq e}$ the set of edges $e' \in E^{i,j}$ with $e' \succeq^{i,j} e$. In the master poset, an agent x^e is before $y^{e'}$ for $x \in \{h_a^{e,i}, h_a^{e,j}, h_b^{e,i}, h_b^{e,j}, g_1, g_2, g_3, \bar{g}_1, \bar{g}_2, \bar{g}_3, f, \bar{f}, \alpha_1, \alpha_2, \alpha_3\}$ if and only if $x = y$ and e is before e' in the order of edges from $E^{i,j}$. Let $e = \{v_{r_i}^i, v_{r_j}^j\}$. The preferences of the agents look as follows.

$$\begin{aligned}
h_a^{e,i} &: \{\{g_p^{e'}, \bar{g}_q^{e''}\} : p, q \in [3], e', e'' \in E^{\succeq e}\} \succ \{\{a_p^i, \alpha_1^e\} : p \leq r_i, e' \in E^{\succeq e}\} \\
&\succ \{f^{e'}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{h_a^{e,i}}, \\
h_a^{e,j} &: \{\{g_p^{e'}, \bar{g}_q^{e''}\} : p, q \in [3], e', e'' \in E^{\succeq e}\} \succ \{\{a_p^j, \alpha_1^e\} : p \leq r_j, e' \in E^{\succeq e}\} \\
&\succ \{f^{e'}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{h_a^{e,j}}, \\
h_b^{e,i} &: \{\{g_p^{e'}, \bar{g}_q^{e''}\} : p, q \in [3], e', e'' \in E^{\succeq e}\} \succ \{\{b_p^i, \alpha_1^e\} : p \leq n + 1 - r_i, e' \in E^{\succeq e}\} \\
&\succ \{f^{e'}, \bar{f}^{e''} : e' \in E^{\succeq e}\} \succ \text{CO}_{h_b^{e,i}}, \\
h_b^{e,j} &: \{\{g_p^{e'}, \bar{g}_q^{e''}\} : p, q \in [3], e', e'' \in E^{\succeq e}\} \succ \{\{b_p^j, \alpha_1^e\} : p \leq n + 1 - r_j, e' \in E^{\succeq e}\} \\
&\succ \{f^{e'}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{h_b^{e,j}}, \\
g_p^e &: \{\{h_a^{e',i}, \bar{g}_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \{\{h_b^{e',i}, \bar{g}_p^{e''}\} : e', e'' \in E^{\succeq e}\} \\
&\succ \{\{h_a^{e',j}, \bar{g}_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \{\{h_b^{e',j}, \bar{g}_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{g_p^e}, \\
\bar{g}_p^e &: \{\{h_b^{e',j}, g_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \{\{h_a^{e',j}, g_p^{e''}\} : e', e'' \in E^{\succeq e}\} \\
&\succ \{\{h_b^{e',i}, g_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \{\{h_a^{e',i}, g_p^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{\bar{g}_p^e}, \\
f^e &: \{h_a^{e',i}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \{h_b^{e',i}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \\
&\succ \{h_a^{e',j}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \{h_b^{e',j}, \bar{f}^{e''} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{f^e}, \\
\bar{f}^e &: \{h_b^{e',j}, f^{e''} : e', e'' \in E^{\succeq e}\} \succ \{h_a^{e',j}, f^{e''} : e', e'' \in E^{\succeq e}\} \\
&\succ \{h_b^{e',i}, f^{e''} : e', e'' \in E^{\succeq e}\} \succ \{h_a^{e',i}, f^{e''} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{\bar{f}^e}, \\
\alpha_1^e &: \{\{a_p^i, h_a^{e',i}\}, \{b_p^i, h_a^{e',i}\}, \{a_p^j, h_a^{e',j}\}, \{b_p^j, h_a^{e',j}\} : p \in [n] : e' \in E^{\succeq e}\} \\
&\succ \{\{\alpha_2^e, \alpha_3^e\} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{\alpha_1^e}, \\
\alpha_2^e &: \{\{\alpha_1^e, \alpha_3^e\} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{\alpha_2^e},
\end{aligned}$$

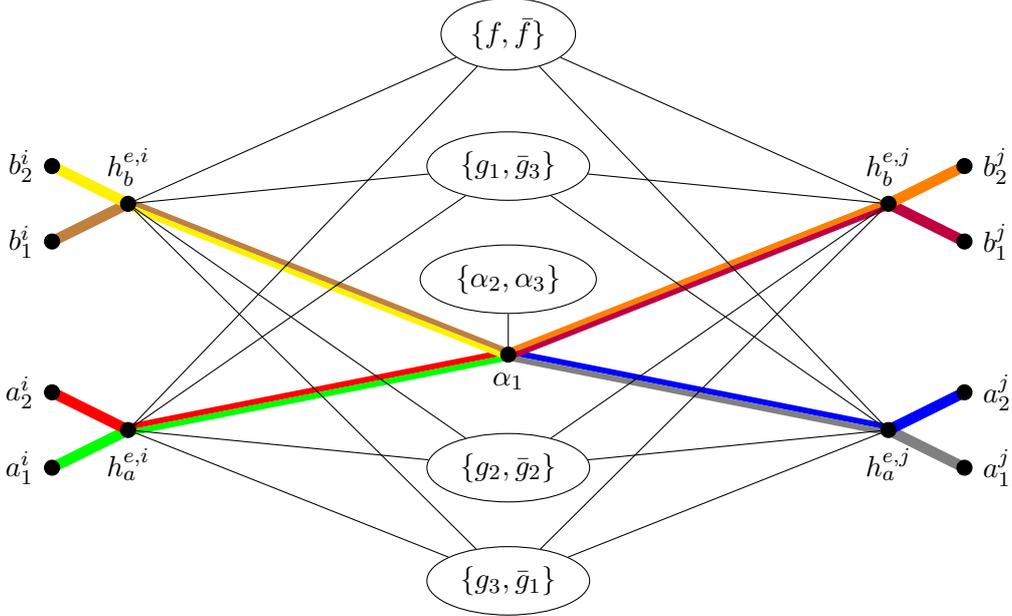


Figure 2: The acceptable 3-sets (i.e., 3-sets which are preferred over the cut-off gadget by all agents they contain) of the edge gadget for the edge $\{v_1^i, v_2^j\}$. Acceptable 3-sets are drawn in two ways: If they contain one of the ellipses, then they consist of the two agents inside the ellipse and the other endpoint of an edge incident to the ellipse (for example, the edge between $h_b^{e,i}$ and $\{f, \bar{f}\}$ corresponds to the acceptable 3-set $\{h_b^{e,i}, f, \bar{f}\}$). Otherwise, they are marked by the bold colored paths of length two (for example, the red path $a_2^i - h_a^{e,i} - \alpha_1$ corresponds to the 3-set $\{a_2^i, h_a^{e,i}, \alpha_1\}$).

$$\alpha_3^e: \{\{\alpha_1^{e'}, \alpha_2^{e''}\} : e', e'' \in E^{\succeq e}\} \succ \text{CO}_{\alpha_3^e}.$$

Figure 2 visualizes the 3-sets $t = \{x_1, x_2, x_3\}$ such that x_i prefers $t \setminus \{x_i\}$ to the 2-sets containing agents from its cut-off gadget for all $i \in [3]$.

Furthermore, we extend the preferences of a_p^ℓ by inserting the 2-sets $\{h_a^{e,\ell}, \alpha_1^e\}$ for all $e \in E^{i,j}$ directly after $\{c_{n-1}^\ell, \bar{c}_{n-1}^\ell\}$ (and before $\{d_1^\ell, \bar{d}_1^\ell\}$), and the preferences of b_p^ℓ by the 2-sets $\{h_b^{e,\ell}, \alpha_1^e\}$ for all $e \in E^{i,j}$ directly after $\{c_{n-1}^\ell, \bar{c}_{n-1}^\ell\}$ for every $p \in [n]$ and $\ell \in \{i, j\}$.

We now show that if $E^{i,j}$ contains an edge between the two vertices selected by S^i and S^j for a matching M , then M is not stable.

Lemma 4.10. *Let \mathcal{I} consist of two vertex-selection gadgets S^i and S^j and the edge gadget for $E^{i,j}$ between S^i and S^j .*

If for a matching M in \mathcal{I} and an edge $e = \{v_r^i, v_s^j\} \in E^{i,j}$ vertex-selection gadgets S^i and S^j select the vertices v_r^i and v_s^j , then matching M contains a blocking 3-set.

Proof. We need to show that M contains a blocking 3-set. By the cut-off gadgets, we know that any 3-set from M contains three agents belonging to the same edge. By the cut-off gadgets for f^e and \bar{f}^e , we know that M must contain a 3-set $\{h_x^{e,\ell}, f^e, \bar{f}^e\}$ for some $\ell \in \{i, j\}$ and $x \in \{a, b\}$. We assume $\ell = i$; the case $\ell = j$ is symmetric. We make a case distinction on whether $x = a$ or $x = b$.

If $x = a$, then we claim that $\{a_r^i, h_a^{e,i}, \alpha_1^e\}$ is a blocking 3-set. Agent a_r^i prefers $\{h_a^{e,i}, \alpha_1^e\}$ to $M(a_r^i) = \{d_p^i, \bar{d}_q^i\}$. Agent $h_a^{e,i}$ prefers $\{a_r^i, \alpha_1^e\}$ to $M'(h_a^{e,i}) = \{f^e, \bar{f}^e\}$ as v_r^i is an endpoint of e .

By the cut-off gadget for α_2^e , we have $M'(\alpha_1^e) = \{\alpha_2^e, \alpha_3^e\}$, and therefore, α_1^e prefers $\{a_r^i, d_a^{e,i}\}$ to M' .

If $x = b$, then we claim that $\{b_{n-r+1}^i, h_b^{e,i}, \alpha_1^e\}$ is a blocking 3-set. Agent b_{n-r+1}^i prefers $\{h_b^{e,i}, \alpha_1^e\}$ to $M(b_{n-r+1}^i) = \{d_p^i, \bar{d}_q^i\}$ for some $p, q \in [n+1]$. Also agent $h_b^{e,i}$ prefers $\{b_{n-r+1}^i, \alpha_1^e\}$ to $M'(h_a^{e,i}) = \{f^e, \bar{f}^e\}$ as v_r^i is an endpoint of e . By the cut-off gadget for α_2^e , we have $M'(\alpha_1^e) = \{\alpha_2^e, \alpha_3^e\}$, and therefore, α_1^e prefers $\{a_r^i, d_a^{e,i}\}$ over M' . \square

We now turn to the reverse direction, i.e., if S^i and S^j select two vertices which are not connected by an edge from $E^{i,j}$, then we can find a stable matching inside the edge gadget.

Lemma 4.11. *Let \mathcal{I} consist of two vertex-selection gadgets S^i and S^j and the edge gadget for every edge $e \in E^{i,j}$ between S^i and S^j .*

Given a matching M inside the vertex-selection gadgets such that at least one endpoint of e is not selected by a vertex-selection gadget for every $e \in E^{i,j}$, then there exists a matching M' containing M such that there is no blocking 3-set containing an agent from an edge gadget for an edge $e \in E^{i,j}$.

Proof. Assume that S^i selects $v_{i^*}^i$ and S^j selects $v_{j^*}^j$. Let $e = \{v_r^i, v_s^j\} \in E^{i,j}$. Since $e \neq \{v_{i^*}^i, v_{j^*}^j\}$, we have $r \neq i^*$ or $s \neq j^*$. We assume that $r \neq i^*$ (the case $s \neq j^*$ is symmetric by switching the roles of i and j). If $r < i^*$, then we extend M to the edge gadget by adding the 3-sets $\{h_a^{e,i}, f^e, \bar{f}^e\}$, $\{h_b^{e,i}, g_1^e, \bar{g}_1^e\}$, $\{h_a^{e,j}, g_2^e, \bar{g}_2^e\}$, $\{h_b^{e,j}, g_3^e, \bar{g}_3^e\}$, and $\{\alpha_1^e, \alpha_2^e, \alpha_3^e\}$. If $r > i^*$, then we extend M to the edge gadget by adding the 3-sets $\{h_a^{e,i}, g_1^e, \bar{g}_1^e\}$, $\{h_b^{e,i}, f^e, \bar{f}^e\}$, $\{h_a^{e,j}, g_2^e, \bar{g}_2^e\}$, $\{h_b^{e,j}, g_3^e, \bar{g}_3^e\}$, and $\{\alpha_1^e, \alpha_2^e, \alpha_3^e\}$.

It remains to show that this does not lead to a blocking 3-set. First, note that no blocking 3-set can contain two agents belonging to different edge gadgets (because for two edges $e, e' \in E^{i,j}$ with $e \succ^{i,j} e'$, every agent x from the edge gadget for e' prefers $M(x)$ to any 2-set containing an agent from the edge gadget for e'). We assume again without loss of generality that $r \neq i^*$. None of the agents g_p^e, \bar{g}_p^e, f^e , and \bar{f}^e is contained in a blocking 3-set (any blocking 3-set containing g_p^e also contains \bar{g}_p^e ; however, \bar{g}_p^e ranks the acceptable triples in reverse order to g_p^e ; a symmetric argument applies to f^e and \bar{f}^e).

If $r < i^*$, then agents $h_b^{e,i}, h_a^{e,j}$, and $h_b^{e,j}$ are not part of a blocking 3-set because all 2-sets which they prefer to $\{g_p^e, \bar{g}_p^e\}$ contain an agent $g_{p'}^e$. The only remaining possible blocking 3-set is $\{a_p^i, \alpha_1^e, h_a^{e,i}\}$ for $p \leq r < i^*$. However, $M(a_p^i) = \{c_q^i, \bar{c}_{q'}^i\}$ and thus a_p^i prefers $M(a_p^i)$ to $\{\alpha_1^e, h_a^{e,i}\}$.

If $r > i^*$, then agents $h_a^{e,i}, h_a^{e,j}$, and $h_b^{e,j}$ are not part of a blocking 3-set because all 2-sets which they prefer to $\{g_p^e, \bar{g}_p^e\}$ contain an agent $g_{p'}^e$. The only remaining possible blocking 3-set is thus $\{b_p^i, \alpha_1^e, h_b^{e,i}\}$ for $p \leq n - r + 1$. However, since $n - r + 1 \leq n - i^*$, it follows that $M(b_p^i) = \{c_q^i, \bar{c}_{q'}^i\}$ and therefore b_p^i prefers $M(b_p^i)$ to $\{\alpha_1^e, h_b^{e,i}\}$. \square

Having described the reduction and the crucial properties of the gadgets, the correctness of the reduction now easily follows.

Proof of the forward direction. We split the proof of correctness into two parts and start by showing that a multicolored independent set implies a stable matching.

Lemma 4.12. *If G contains a multicolored independent set I , then there exists a stable matching.*

Proof. We construct a stable matching as follows. Each vertex-selection gadget S^i selects the vertex from $I \cap V^i$. This matching is extended to the edge gadgets and cut-off gadgets as described in Lemmas 4.7 and 4.11. The stability of the constructed matching follows from Lemmas 4.7, 4.9 and 4.11. \square

Proof of the backward direction. We now turn to the backward direction, showing that a stable matching implies a multicolored independent set.

Lemma 4.13. *If there exists a stable matching, then G contains a multicolored independent set.*

Proof. By Lemma 4.8, every vertex-selection gadget selects a vertex. By Lemma 4.10, no two selected vertices are adjacent. Thus, the selected vertices form a multicolored independent set. \square

The parameter. It remains to show that the preferences of the constructed MDSR-POSET-instance can be derived from a poset \succ_{ML} of bounded width.

The poset \succ_{ML} looks as follows. For each vertex-selection gadget S^i , we have $a_j^i \succ_{\text{ML}} a_{j'}^i$, $b_j^i \succ_{\text{ML}} b_{j'}^i$, $c_j^i \succ_{\text{ML}} c_{j'}^i$, $\bar{c}_j^i \succ_{\text{ML}} \bar{c}_{j'}^i$, $d_j^i \succ_{\text{ML}} d_{j'}^i$, and $\bar{d}_j^i \succ_{\text{ML}} \bar{d}_{j'}^i$ if and only if $j < j'$. Furthermore, we have for every $q \in [6]$ that $z_{a_j^i}^q \succ_{\text{ML}} z_{a_{j'}^i}^q$, $z_{b_j^i}^q \succ_{\text{ML}} z_{b_{j'}^i}^q$, $z_{c_j^i}^q \succ_{\text{ML}} z_{c_{j'}^i}^q$, $z_{\bar{c}_j^i}^q \succ_{\text{ML}} z_{\bar{c}_{j'}^i}^q$, $z_{d_j^i}^q \succ_{\text{ML}} z_{d_{j'}^i}^q$, and $z_{\bar{d}_j^i}^q \succ_{\text{ML}} z_{\bar{d}_{j'}^i}^q$.

For each $i, j \in [k]$ with $i < j$, each $e, e' \in E^{i,j}$ and $p \in [3]$, we have $h_a^{e,i} \succ_{\text{ML}} h_a^{e',i}$, $h_b^{e,i} \succ_{\text{ML}} h_b^{e',i}$, $h_a^{e,j} \succ_{\text{ML}} h_a^{e',j}$, $h_b^{e,j} \succ_{\text{ML}} h_b^{e',j}$, $h_b^{e,i} \succ_{\text{ML}} h_b^{e',i}$, $g_p^e \succ_{\text{ML}} g_p^{e'}$, $f^e \succ_{\text{ML}} f^{e'}$, and $\alpha_p^e \succ_{\text{ML}} \alpha_p^{e'}$ if and only if $e \succ^{i,j} e'$. Furthermore, we have for every $p \in [3]$ and $q \in [6]$ that $z_{h_a^{e,i}}^q \succ_{\text{ML}} z_{h_a^{e',i}}^q$, $z_{h_b^{e,i}}^q \succ_{\text{ML}} z_{h_b^{e',i}}^q$, $z_{h_a^{e,j}}^q \succ_{\text{ML}} z_{h_a^{e',j}}^q$, $z_{h_b^{e,j}}^q \succ_{\text{ML}} z_{h_b^{e',j}}^q$, $z_{h_b^{e,i}}^q \succ_{\text{ML}} z_{h_b^{e',i}}^q$, $z_{g_p^e}^q \succ_{\text{ML}} z_{g_p^{e'}}^q$, $z_{f^e}^q \succ_{\text{ML}} z_{f^{e'}}^q$, and $z_{\alpha_p^e}^q \succ_{\text{ML}} z_{\alpha_p^{e'}}^q$ if and only if $e \succ^{i,j} e'$.

We arrive at the following observation.

Observation 4.14. *The preferences are derived from a poset of width at most $O(k^2)$.*

Proof. It is easy to verify that the preferences are derived from \succ_{ML} , so it remains to show that \succ_{ML} has width $O(k^2)$. Note that \succ_{ML} decomposes in $O(k^2)$ chains, from which the observation follows by Dilworth's Theorem [14]: For every vertex-selection gadget S^i , we have 42 chains $a_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} a_n^i$, $b_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} b_n^i$, $c_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} c_n^i$, $\bar{c}_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} \bar{c}_n^i$, $d_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} d_n^i$, and $\bar{d}_1^i \succ_{\text{ML}} \dots \succ_{\text{ML}} \bar{d}_n^i$ as well as for every $q \in [6]$, chains $z_{a_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{a_n^i}^q$, $z_{b_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{b_n^i}^q$, $z_{c_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{c_n^i}^q$, $z_{\bar{c}_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{\bar{c}_n^i}^q$, $z_{d_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{d_n^i}^q$, and $z_{\bar{d}_1^i}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{\bar{d}_n^i}^q$. For every $i, j \in [k]$ with $i < j$, let $E^{i,j} = \{e_1, \dots, e_s\}$ such that $e_r \succ^{i,j} e_{r+1}$ for every $r \in [s-1]$. We have 77 chains $h_a^{e_1,i} \succ_{\text{ML}} \dots \succ_{\text{ML}} h_a^{e_s,i}$, $h_b^{e_1,i} \succ_{\text{ML}} \dots \succ_{\text{ML}} h_b^{e_s,i}$, $h_a^{e_1,j} \succ_{\text{ML}} \dots \succ_{\text{ML}} h_a^{e_s,j}$, $h_b^{e_1,j} \succ_{\text{ML}} \dots \succ_{\text{ML}} h_b^{e_s,j}$, $h_b^{e_1,i} \succ_{\text{ML}} \dots \succ_{\text{ML}} h_b^{e_s,i}$, $g_p^{e_1} \succ_{\text{ML}} \dots \succ_{\text{ML}} g_p^{e_s}$ for every $p \in [3]$, $f^{e_1} \succ_{\text{ML}} \dots \succ_{\text{ML}} f^{e_s}$, and $\alpha_p^{e_1} \succ_{\text{ML}} \alpha_p^{e_s}$ for every $p \in [3]$ as well as for every $q \in [6]$, chains $z_{h_a^{e_1,i}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{h_a^{e_s,i}}^q$, $z_{h_b^{e_1,i}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{h_b^{e_s,i}}^q$, $z_{h_a^{e_1,j}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{h_a^{e_s,j}}^q$, $z_{h_b^{e_1,j}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{h_b^{e_s,j}}^q$, $z_{h_b^{e_1,i}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{h_b^{e_s,i}}^q$, $z_{g_p^{e_1}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{g_p^{e_s}}^q$ for every $p \in [3]$, $z_{f^{e_1}}^q \succ_{\text{ML}} \dots \succ_{\text{ML}} z_{f^{e_s}}^q$, and $z_{\alpha_p^{e_1}}^q \succ_{\text{ML}} z_{\alpha_p^{e_s}}^q$ for every $p \in [3]$. \square

We now have all ingredients to obtain the following main result.

Theorem 4.15. *3DSR-POSET parameterized by poset width is W[1]-hard.*

Proof. The reduction clearly runs in polynomial time. Lemmas 4.12 and 4.13 prove its correctness, and Observation 4.14 shows that the width of the poset is bounded by $O(k^2)$. Thus, we found a parameterized reduction from MULTICOLORED INDEPENDENT SET to 3DSR-POSET parameterized by the width of the poset, proving W[1]-hardness. \square

After an FPT-algorithm for the parameter maximum number of agents incomparable to a single agent in Section 4.2.1, we have seen $W[1]$ -hardness for the stronger parameter width of the master poset. It remains open whether 3DSR-POSET parameterized by the width of the master poset lies in XP .

We now investigate a third parameter measuring similarity to a strictly ordered master poset, namely the (agent) deletion distance to a strictly ordered master poset.

4.3 Deletion distance to a strictly ordered master poset

We saw that MDSR-POSET is fixed-parameter tractable when parameterized by the maximum number of agents incomparable to a single agent, but it is $W[1]$ -hard when parameterized by the width of the poset. We now consider another natural parameter measuring the similarity to a strict order, namely the deletion distance to a strict order, i.e., the minimum number of agents which need to be deleted such that the resulting preferences are derived from a strict order (note that this does not pose any condition on the preferences of the deleted agents). Notably, this parameter is orthogonal to the two parameters investigated before (width of the poset and maximum number of agents incomparable to an agent), as can be seen from the following two examples: If the master poset is the weak order $a_1 \perp_{ML} a_2 \succ_{ML} a_3 \perp_{ML} a_4 \succ_{ML} a_5 \perp_{ML} a_6 \succ_{ML} \dots \succ_{ML} a_{n-1} \perp_{ML} a_n$, thus $\kappa(ML) = 2$, while one has to delete $n/2$ agents in order to obtain a strict order. If the preferences of all but one agent are derived from a strict order, and the last agent's preferences are derived from the inverse of this strict order, then the deletion distance is one while any master poset from which this preferences are derived contains a tie of size at least $n - 1$ and thus has width at least $n - 1$. In this section, reducing from MULTICOLORED CLIQUE we show that MDSR is $W[1]$ -hard parameterized by the deletion distance to a strictly ordered master poset. First, we formally introduce the parameter.

Definition 2. For an MDSR-instance \mathcal{I} , let $\lambda(\mathcal{I})$ denote the minimum number of agents such that the preferences of the instance arising through the deletion of these agents are derived from a strict order.

Note that the agents which were deleted to arrive at a strictly ordered master poset may have arbitrary preferences. We now present an example for this parameter.

Example 6. Consider an MDSR instance \mathcal{I} with $d = 3$ and the following preferences.

$$\begin{aligned} a_1 &: \{a_2, a_3\} \succ \{a_3, a_4\} \succ \{a_2, a_5\} \succ \{a_2, a_4\} \succ \{a_3, a_5\} \succ \{a_4, a_5\}, \\ a_2 &: \{a_1, a_3\} \succ \{a_3, a_4\} \succ \{a_1, a_5\} \succ \{a_1, a_4\} \succ \{a_3, a_5\} \succ \{a_4, a_5\}, \\ a_3 &: \{a_1, a_2\} \succ \{a_1, a_4\} \succ \{a_2, a_5\} \succ \{a_2, a_4\} \succ \{a_1, a_5\} \succ \{a_4, a_5\}, \\ a_4 &: \{a_1, a_5\} \succ \{a_1, a_2\} \succ \{a_2, a_5\} \succ \{a_1, a_3\} \succ \{a_3, a_5\} \succ \{a_2, a_3\}, \\ a_5 &: \{a_2, a_3\} \succ \{a_3, a_4\} \succ \{a_1, a_2\} \succ \{a_2, a_4\} \succ \{a_1, a_3\} \succ \{a_1, a_4\}. \end{aligned}$$

After the deletion of a_5 , the preferences are

$$\begin{aligned} a_1 &: \{a_2, a_3\} \succ \{a_3, a_4\} \succ \{a_2, a_4\}, \\ a_2 &: \{a_1, a_3\} \succ \{a_3, a_4\} \succ \{a_1, a_4\}, \\ a_3 &: \{a_1, a_2\} \succ \{a_1, a_4\} \succ \{a_2, a_4\}, \\ a_4 &: \{a_1, a_2\} \succ \{a_1, a_3\} \succ \{a_2, a_3\}. \end{aligned}$$

which are derived from the strict order $a_1 \succ a_2 \succ a_3 \succ a_4$. Hence, $\lambda(\mathcal{I}) = 1$.

To show that 3-DSR parameterized by $\lambda(\mathcal{I})$ is W[1]-hard, we give a parameterized reduction from MULTICOLORED CLIQUE, which is W[1]-complete [15, 42].

MULTICOLORED CLIQUE

Input: A k -partite graph $G = (V^1 \dot{\cup} V^2 \dot{\cup} \dots \dot{\cup} V^k, E)$.

Task: Decide whether G contains a clique C with $C \cap V^i \neq \emptyset$ for all $i \in [k]$.

The sets V^1, \dots, V^k are called *color classes*. Let $E^{i,j}$ be the set of edges with one endpoint in V^i and one endpoint in V^j . By adding vertices and edges, we may assume without loss of generality that there exists some $n' \in \mathbb{N}$ such that $|V^i| = 3n' + 1$ for all $i \in [k]$, and that there exists some $m' \in \mathbb{N}$ such that $|E^{i,j}| = 3m' + 1$ for all $i, j \in [k]$. For each color class V^i , we fix an arbitrary order of the vertices, i.e., $V^i = \{v_1^i, v_2^i, \dots, v_{3n'+1}^i\}$. Furthermore, for each $i < j \in [k]$, we fix an arbitrary order of $E^{i,j} = \{e_1^{i,j}, \dots, e_{3m'+1}^{i,j}\}$, and we set $e_\ell^{j,i} := e_\ell^{i,j}$. For a vertex $v \in V(G)$, we denote by $\delta(v)$ the set of all edges incident to v .

As in Section 4.2.2, we only describe the beginning of the preferences of an agent, followed by $\succ^{(\text{rest})}$. The remaining acceptable 2-sets can be added in an arbitrary way obeying the strictly ordered master poset (extended to the $\lambda(\mathcal{I})$ agents not contained in this poset).

The basic idea of the reduction is that we create for every $i \in [k]$ a vertex-selection gadget S^i and for every $i < j \in [k]$ an incidence-checking gadget. The vertex-selection gadget S^i then encodes the selection of a vertex from V^i to be part of the multicolored clique, and the incidence-checking gadgets ensure that all vertices selected by the vertex-selection gadgets are indeed incident.

We begin by describing the agents in the MDSR-POSET instance constructed by the reduction, and the master poset. Afterwards, we describe the gadgets used in the reduction.

4.3.1 Master poset

Every vertex-selection gadget S^i has $3n' + 3$ agents, namely s^i , \overleftarrow{s}^i , and every vertex from V^i will also be an agent in S^i , i.e., S^i also contains agent $v_1^i, \dots, v_{3n'+1}^i$. Every incidence-checking gadget $B^{i,j}$ contains $3(3m' + 1) + 6$ agents: $b_\ell^{i,j}$ and $b_\ell^{j,i}$ for $\ell \in [3m' + 1]$, agent $c_\ell^{i,j}$ for $\ell \in [3m']$, and agents $x^{i,j}$, $\overleftarrow{x}^{i,j}$, $x^{j,i}$, $\overleftarrow{x}^{j,i}$, $z_1^{i,j}$, $z_2^{i,j}$, and $z_3^{i,j}$. Additionally, there are $O(k^2)$ agents contained in so-called ‘‘cut-off gadgets’’ (see Section 4.3.2); however, all these agents will be deleted in order to have the preferences derived from a strictly ordered master poset, and thus, are not considered here. Furthermore, in order to arrive at a strict order as master poset, we delete agents s^i and \overleftarrow{s}^i for every $i \in [k]$ as well as agents $x^{i,j}$, $\overleftarrow{x}^{i,j}$, $x^{j,i}$, $\overleftarrow{x}^{j,i}$, $z_1^{i,j}$, $z_2^{i,j}$, and $z_3^{i,j}$ for every $i < j \in [k]$. Inside the vertex-selection gadget S^i , the master poset has the form $\mathcal{A}^i := v_1^i \succ_{\text{ML}} v_2^i \succ_{\text{ML}} \dots \succ_{\text{ML}} v_{3n'+1}^i$. Inside incidence-checking gadget $B^{i,j}$, the master poset has the form $\mathcal{B}^{i,j} := b_1^{i,j} \succ b_1^{j,i} \succ c_1^{i,j} \succ b_e^{i,j} \succ b_2^{j,i} \succ c_2^{i,j} \succ \dots \succ c_{3m'}^{i,j} \succ b_{3m'+1}^{i,j} \succ b_{3m'+1}^{j,i}$. The complete master poset looks as follows.

$$\begin{aligned} \mathcal{A}^1 \succ_{\text{ML}} \mathcal{A}^2 \succ_{\text{ML}} \dots \succ_{\text{ML}} \mathcal{S}^k \succ_{\text{ML}} \mathcal{B}^{1,2} \succ_{\text{ML}} \mathcal{B}^{1,3} \succ_{\text{ML}} \dots \succ_{\text{ML}} \mathcal{B}^{1,k} \\ \succ_{\text{ML}} \mathcal{B}^{2,3} \succ_{\text{ML}} \mathcal{B}^{2,4} \succ_{\text{ML}} \dots \succ_{\text{ML}} \mathcal{B}^{2,k} \succ_{\text{ML}} \mathcal{B}^{3,4} \succ_{\text{ML}} \dots \succ_{\text{ML}} \mathcal{B}^{k-1,k}. \end{aligned}$$

We continue by describing the gadgets used in the reduction.

4.3.2 Cut-off gadget

As in Section 4.2.2, we have a cut-off gadget ensuring for a given agent a and a 2-set p that a is matched to a 2-set it likes at least as much as p . Indeed, we use the cut-off gadget

from Section 4.2.2. Since we will use only $O(k^2)$ many cut-off gadgets, adding these agents may increase the deletion distance to a strict master poset by at most $O(k^2)$; thus, we can choose the preferences arbitrarily and do not have to consider the master poset.

We already showed in Lemma 4.7 that cut-off gadgets work as desired. Having described the cut-off gadgets, we can now describe the remaining gadgets, namely the vertex-selection gadget and the incidence-checking gadget. We start with the vertex-selection gadget.

4.3.3 Vertex-selection gadget

For each color class V^i , the reduction adds a vertex selection gadget S^i . This vertex-selection gadget contains an agent for each vertex v_p^i with $p \in [3n' + 1]$; we identify the agent and the vertex and call both v_p^i . Furthermore, the vertex-selection gadget contains two agents s^i and \overleftarrow{s}^i . The vertex-selection gadget also contains a cut-off gadget for each of the three agents s^i , \overleftarrow{s}^i , and $v_{3n'+1}^i$.

The intuitive idea behind the vertex-selection gadget is as follows. Selecting a vertex v_ℓ^i to be part of a multicolored clique corresponds to matching this vertex to $\{s^i, \overleftarrow{s}^i\}$, and partitioning the remaining agents from $\{v_1^i, \dots, v_{3n'+1}^i\} \setminus \{v_\ell^i\}$ into n' 3-sets which are contained in the stable matching. Since s^i prefers being matched to $\{\overleftarrow{s}^i, v_\ell^i\}$ with small ℓ while \overleftarrow{s}^i prefers being matched to $\{s^i, v_\ell^i\}$ with large ℓ , how much s^i and \overleftarrow{s}^i like their partners in a matching encodes which vertex is selected. The cut-off gadget for $v_{3n'+1}^i$ actually implies that in every stable matching M , there has to exist some $\ell \in [3n' + 1]$ such that $\{v_\ell^i, s^i, \overleftarrow{s}^i\} \in M$ (see Lemma 4.23). This implies that every vertex has to be matched this way, and thus, the vertex-selection gadget has to select a vertex in each stable matching.

The preferences of an agent $v_p^i \in V^i$ start with all 2-sets containing an agent from a vertex-selection gadget S^j with $j < i$, and continue with $\{s^i, \overleftarrow{s}^i\} \succ \{\{v_1^i, v_q^i\} : q \in [3n' + 1]\} \succ \{\{v_2^i, v_q^i\} : q \in [3n' + 1]\} \succ \dots \succ \{\{v_{3n'+1}^i, v_q^i\} : q \in [3n' + 1]\} \succ \dots$ (rest).

For the agent $v_{3n'+1}^i$, a cut-off gadget follows; for the other agents, the preferences are extended in an arbitrary way obeying the master poset.

The preferences of s^i and \overleftarrow{s}^i are as follows (the agents $x^{i,j}$, $\overleftarrow{x}^{i,j}$, and $b_\ell^{i,j}$ are contained in incidence-checking gadgets (see Section 4.3.4)).

$$\begin{aligned} s^i: & \{\overleftarrow{s}^i, v_1^i\} \succ \{\{b_\ell^{i,j}, \overleftarrow{x}^{i,j}\} : j \in [k] \setminus \{i\}, e_\ell^{i,j} \in E^{i,j} \cap \delta(v_1^i)\} \succ \{\overleftarrow{s}^i, v_2^i\} \\ & \succ \{\{b_\ell^{i,j}, \overleftarrow{x}^{i,j}\} : j \in [k] \setminus \{i\}, e_\ell^{i,j} \in E^{i,j} \cap \delta(v_2^i)\} \succ \{\overleftarrow{s}^i, v_3^i\} \succ \dots \succ \{\overleftarrow{s}^i, v_{3n'+1}^i\} \succ \text{CO}_{s^i} \\ \overleftarrow{s}^i: & \{s^i, v_{3n'+1}^i\} \succ \{\{b_\ell^{i,j}, x^{i,j}\} : j \in [k] \setminus \{i\}, e_\ell^{i,j} \in E^{i,j} \cap \delta(v_{3n'+1}^i)\} \succ \{s^i, v_{3n'}^i\}, \\ & \succ \{\{b_\ell^{i,j}, x^{i,j}\} : j \in [k] \setminus \{i\}, e_\ell^{i,j} \in E^{i,j} \cap \delta(v_{3n'}^i)\} \succ \{s^i, v_{3n'-1}^i\} \succ \dots \succ \{s^i, v_1^i\} \succ \text{CO}_{\overleftarrow{s}^i}. \end{aligned}$$

Next, we turn to the incidence-selection gadget.

4.3.4 Incidence-checking gadget

For each pair (V^i, V^j) of color classes with $i < j$, we add an incidence-checking gadget $B^{i,j}$. For each edge $\ell \in [3m' + 1]$, the incidence-checking gadget $B^{i,j}$ contains two agents $b_\ell^{i,j}$ and $b_\ell^{j,i}$. Furthermore, there are $3m'$ agents $c_1^{i,j}, \dots, c_{3m'}^{i,j}$. The gadget also contains seven agents $x^{i,j}$, $\overleftarrow{x}^{i,j}$, $x^{j,i}$, $\overleftarrow{x}^{j,i}$, $z_1^{i,j}$, $z_2^{i,j}$, and $z_3^{i,j}$, which do not have the master preferences, and a cut-off gadget for $z_2^{i,j}$.

The idea of the gadget is as follows: Every stable matching has to contain $\{b_\ell^{i,j}, b_\ell^{j,i}, c_\ell^{i,j}\}$ or $\{b_\ell^{i,j}, b_\ell^{j,i}, c_{\ell-1}^{i,j}\}$ for all $\ell \in [3m' + 1] \setminus \{r\}$ for some $r \in [3m' + 1]$, while for the remaining r , agent $b_r^{i,j}$ must be matched to $\{x^{i,j}, \overleftarrow{x}^{i,j}\}$ and $b_r^{j,i}$ must be matched to $\{x^{j,i}, \overleftarrow{x}^{j,i}\}$. Thus, the

better $x^{i,j}$ is matched, the worse $\overleftarrow{x}^{i,j}$ is matched. Unless e_r is incident to the vertex selected by vertex-selection gadget S^i , it follows that either $x^{i,j}$ and s^i or $\overleftarrow{x}^{i,j}$ and \overleftarrow{s}^i are part of a blocking 3-set together with the vertex selected by S^i . By symmetric arguments, e_r is incident to the vertex selected by S^j in any stable matching, and thus, the vertices selected by the vertex-selection gadgets form a clique.

Let $Z^{i,j}$ be the set of agents which are before $b_1^{i,j}$ in the master poset. As the agents from $Z^{i,j}$ are before every agent from $B^{i',j'}$ in the master poset, we have to add them in the beginning of the preferences of agent $b_\ell^{i,j}$ and $b_\ell^{j,i}$. Let $X := \{x^{i,j}, \overleftarrow{x}^{i,j}, x^{j,i}, \overleftarrow{x}^{j,i}\}$ and let A denote the set of all agents.

The preferences of $b_\ell^{i,j}$ (resp. $b_\ell^{j,i}$) are as follows (deleting all 2-sets containing $b_\ell^{i,j}$ (resp. $b_\ell^{j,i}$)).

$$\begin{aligned} & \{\{z, a\} : z \in Z^i, a \in A\} \succ \{\{a, x\} : a \in \{s^i, \overleftarrow{s}^i, s^j, \overleftarrow{s}^j\}, x \in X\} \\ & \succ \{\{x, x'\} : x, x' \in X\} \succ \{\{x, b_q^{i,j}\}, \{x, b_q^{j,i}\} : x \in X, q \in [3m' + 1]\} \\ & \succ \{\{b_q^{i,j}, z_1^{i,j}\}, \{b_q^{j,i}, z_1^{j,i}\} : q \in [\ell - 1]\} \succ \left(\frac{\{b_q^{i,j}, b_q^{j,i}, c_q^{i,j} : q \in [\ell]\}}{2} \right) \\ & \succ \{\{b_q^{i,j}, z_1^{i,j}\}, \{b_q^{j,i}, z_1^{j,i}\} : q \in [\ell, 3m' + 1]\} \succ \overset{(\text{rest})}{\dots}. \end{aligned}$$

The preferences of $c_q^{i,j}$ are arbitrary preferences obeying the master poset.

The preferences of $z_1^{i,j}$ are as follows. $\{\{b_p^{i,j}, b_q^{j,i}\} : p \in [3m' + 1], q \in [3m' + 1]\} \succ \{z_2^{i,j}, z_3^{i,j}\} \succ \overset{(\text{rest})}{\dots}$.

The preferences of $z_2^{i,j}$ are as follows. $\{z_1^{i,j}, z_3^{i,j}\} \succ \text{CO}_{z_2^{i,j}}$.

The preferences of $z_3^{i,j}$ are as follows. $\{z_1^{i,j}, z_2^{i,j}\} \succ \overset{(\text{rest})}{\dots}$.

To describe the preferences of $x^{i,j}$ and $\overleftarrow{x}^{i,j}$, we define sublists \mathcal{C}_ℓ^α and $\overleftarrow{\mathcal{C}}_\ell^\alpha$ for $\ell \in [3n' + 1]$ and $\alpha \in \{i, j\}$. The sublist \mathcal{C}_ℓ^i contains the 2-sets $\{\overleftarrow{x}^{i,j}, b_r^{i,j}\}$ for all $e_r^{i,j} \in E^{i,j} \cap \delta(v_\ell^i)$, ordered increasingly by r . The sublist $\overleftarrow{\mathcal{C}}_\ell^i$ contains the 2-sets $\{x^{i,j}, b_r^{i,j}\}$ for $e_r^{i,j} \in E^{i,j} \cap \delta(v_\ell^i)$, but is ordered *decreasingly* by r . Similarly, sublist \mathcal{C}_ℓ^j contains the 2-sets $\{\overleftarrow{x}^{j,i}, b_r^{j,i}\}$ for all $e_r^{j,i} \in E^{i,j} \cap \delta(v_\ell^j)$, ordered increasingly by r , and sublist $\overleftarrow{\mathcal{C}}_\ell^j$ contains 2-sets $\{x^{j,i}, b_r^{j,i}\}$ ordered decreasingly by r . The preferences of $x^{i,j}$ and $\overleftarrow{x}^{i,j}$ look as follows.

$$\begin{aligned} x^{i,j} : & \mathcal{C}_{3n'+1}^i \succ \{\overleftarrow{s}^i, v_i^{3n'+1}\} \succ \mathcal{C}_{3n'}^i \succ \{\overleftarrow{s}^i, v_i^{3n'}\} \succ \dots \succ \mathcal{C}_1^i \succ \{\overleftarrow{s}^i, v_i^1\} \succ \text{CO}_{x^{i,j}}, \\ \overleftarrow{x}^{i,j} : & \overleftarrow{\mathcal{C}}_1^i \succ \{s^i, v_i^1\} \succ \overleftarrow{\mathcal{C}}_2^i \succ \{s^i, v_i^2\} \succ \dots \succ \overleftarrow{\mathcal{C}}_{3n'+1}^i \succ \{s^i, v_i^{3n'+1}\} \succ \text{CO}_{\overleftarrow{x}^{i,j}}, \\ x^{j,i} : & \mathcal{C}_{3n'+1}^j \succ \{\overleftarrow{s}^j, v_j^{3n'+1}\} \succ \mathcal{C}_{3n'}^j \succ \{\overleftarrow{s}^j, v_j^{3n'}\} \succ \dots \succ \mathcal{C}_1^j \succ \{\overleftarrow{s}^j, v_j^1\} \succ \text{CO}_{x^{j,i}}, \\ \overleftarrow{x}^{j,i} : & \overleftarrow{\mathcal{C}}_1^j \succ \{s^j, v_j^1\} \succ \overleftarrow{\mathcal{C}}_2^j \succ \{s^j, v_j^2\} \succ \dots \succ \overleftarrow{\mathcal{C}}_{3n'+1}^j \succ \{s^j, v_j^{3n'+1}\} \succ \text{CO}_{\overleftarrow{x}^{j,i}}. \end{aligned}$$

4.3.5 The reduction

Given an instance (G, k) of MULTICOLORED CLIQUE, we construct an MDSR-instance \mathcal{I}' as follows. Instance \mathcal{I}' contains k vertex-selection gadgets S^i , one for each color class V^i . Between each pair (S^i, S^j) of vertex-selection gadgets with $i < j$, there is an incidence-checking gadget $B^{i,j}$.

We first show that our parameter λ is indeed bounded by $O(k^2)$ for the constructed instance \mathcal{I}' .

Lemma 4.16. $\lambda(\mathcal{I}') = O(k^2)$.

Proof. The master poset is described in Section 4.3.1. It is a strict order and contains all agents but the $O(k^2)$ agents $s^i, \overleftarrow{s}^i, x^{i,j}, \overleftarrow{x}^{i,j}, z^{i,j}, z_1^{i,j}, z_2^{i,j}$ for $i, j \in [k]$ with $i \neq j$, and all agents contained in cut-off gadgets. It is easy to verify that the preferences obey this master poset. \square

4.3.6 Proof of the forward direction

We prove that if G contains a clique of size k , then \mathcal{I}' admits a stable matching. So let $\{v_{p_1}^1, \dots, v_{p_k}^k\}$ be a multicolored clique. We construct a stable matching M as follows.

For the vertex-selection gadget S^i , we add the 3-set $\{v_i^{p_i}, s^i, \overleftarrow{s}^i\}$. All other vertex agents v_i^q are matched to each other, according to their index q (i.e., we match the three agents with lowest index together, then the next three agents, and so on). Next, we consider an incidence gadget $B^{i,j}$. Assume that $\{v_{p_i}^i, v_{p_j}^j\}$ is the α -th edge in the order of $E^{i,j}$ fixed in Section 4.3.4. We add $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_\alpha^{i,j}\}$ and $\{x^{j,i}, \overleftarrow{x}^{j,i}, b_\alpha^{j,i}\}$ to M . Furthermore, we add the 3-sets $\{b_\ell^{i,j}, b_\ell^{j,i}, c_\ell^{i,j}\}$ for $\ell < \alpha$, and $\{b_\ell^{i,j}, b_\ell^{j,i}, c_{\ell-1}^{i,j}\}$ for $\ell > \alpha$. Finally, we add the 3-set $\{z_1^{i,j}, z_2^{i,j}, z_3^{i,j}\}$. The agents from the cut-off gadgets are matched as described in Lemma 4.7. We call the resulting matching M .

It remains to show that M is stable. In order to do so, we will show step by step that no agent is part of a blocking 3-set. We start with the agents s^i and \overleftarrow{s}^i .

Lemma 4.17. *For any $i \in [k]$, agents s^i and \overleftarrow{s}^i are not part of a blocking 3-set.*

Proof. We prove the lemma for \overleftarrow{s}^i ; the proof for s^i is symmetric. All 2-sets which \overleftarrow{s}^i ranks better than $\{s^i, v_{p_i}^i\}$ are of the form $\{s^i, v_\ell^i\}$ for $\ell > p_i$ or $\{\overleftarrow{x}^{i,j}, b_\ell^{i,j}\}$ for an edge $e_\ell \in E^{i,j}$ whose endpoint in V^i is v_q^i with $q > p_i$.

For the 2-sets $\{s^i, v_\ell^i\}$ with $\ell > p_i$, note that s^i does not prefer $\{\overleftarrow{s}^i, v_\ell^i\}$ to $\{\overleftarrow{s}^i, v_{p_i}^i\}$, and thus, $\{\overleftarrow{s}^i, s^i, v_\ell^i\}$ is not blocking.

For the 2-set $\{\overleftarrow{x}^{i,j}, b_\ell^{i,j}\}$, note that $\overleftarrow{x}^{i,j}$ does not prefer $\{b_\ell^{i,j}, \overleftarrow{s}^i\}$ to $\{\overleftarrow{x}^{i,j}, b_r^{i,j}\}$ for the edge $e_r = \{v_{p_i}^i, v_{p_j}^j\}$ as $q > p_i$.

Thus, \overleftarrow{s}^i is not part of a blocking 3-set. \square

We now turn to the remaining agents from vertex-selection gadgets.

Lemma 4.18. *For each $i \leq k$, no agent from V^i is part of a blocking 3-set.*

Proof. We prove the statement by induction. For $i = 0$ there is nothing to show.

Fix $i \in [k]$. Note that for all $j \in [k]$, agents $v_{p_j}^j$ are matched to their first choice, and thus not part of a blocking 2-set. By Lemma 4.17, no agent s^j or \overleftarrow{s}^j is involved in a blocking 3-set. Thus, by induction on p , one easily sees that all 2-sets which v_p^i prefers to the 2-set it is matched to in M , contain an agent about which we already know that it is not contained in a blocking 3-set, implying that also v_p^i is not contained in a blocking 3-set.

Thus, no agent of V^i is part of a blocking 3-set. \square

Next, we turn to the incidence-checking gadgets and start with agents $x^{i,j}$ and $\overleftarrow{x}^{i,j}$.

Lemma 4.19. *For any $i, j \in [k]$ with $i \neq j$, agents $x^{i,j}$ and $\overleftarrow{x}^{i,j}$ are not part of a blocking 3-set.*

Proof. A blocking 3-set cannot contain s^i , \overleftarrow{s}^i , s^j , or \overleftarrow{s}^j by Lemma 4.17. Thus, it is of the form $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_\alpha^{i,j}\}$ for some $\alpha \in [3m' + 1]$. Since $x^{i,j}$ prefers $\{\overleftarrow{x}^{i,j}, b_\alpha^{i,j}\}$ to $M(x^{i,j}) = \{\overleftarrow{x}^{i,j}, b_\beta^{i,j}\}$ for some $\beta \in [3m' + 1]$, agent $\overleftarrow{x}^{i,j}$ does not prefer $\{x^{i,j}, b_\alpha^{i,j}\}$ to $\{x^{i,j}, b_\beta^{i,j}\}$, and thus, $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_\alpha^{i,j}\}$ is not blocking. \square

Now we consider agents $z_1^{i,j}$, $z_2^{i,j}$, and $z_3^{i,j}$.

Lemma 4.20. *For every $i < j \in [k]$, agents $z_1^{i,j}$, $z_2^{i,j}$, and $z_3^{i,j}$ are not part of a blocking 3-set.*

Proof. Agents $z_2^{i,j}$ and $z_3^{i,j}$ are matched to the first 2-sets in their preference lists and thus are not part of a blocking 3-set.

All 2-sets which $z_1^{i,j}$ prefers to $\{z_2^{i,j}, z_3^{i,j}\}$ are of the form $\{b_p^\ell, b_q^{\ell'}\}$ for $\ell = i, j$ or $\ell = j, i$ and $\ell' = i, j$ or $\ell' = j, i$. We assume without loss of generality that $p \leq q$ and $\ell = i, j$; the case $q > p$ or $\ell = j$ is symmetric. However, agent $b_p^{i,j}$ is matched to $\{c_{p-1}^{i,j}, b_p^{j,i}\}$ if $p < p_i$, to $\{x^{i,j}, \overleftarrow{x}^{i,j}\}$ if $p = p_i$, and to $\{c_{p-1}^{i,j}, b_p^{j,i}\}$ if $p > p_i$. In all cases, agent $b_p^{i,j}$ prefers $M(b_p^{i,j})$ to $\{z_1^{i,j}, b_q^{\ell'}\}$, and thus, $z_1^{i,j}$ is not part of a blocking 3-set. \square

Finally, we turn to the remaining agents from the incidence-checking gadgets.

Lemma 4.21. *For every $i < j \in [k]$ and each $\ell \in [3m']$, agents $b_\ell^{i,j}$, $b_\ell^{j,i}$, and $c_\ell^{i,j}$ as well as $b_{3m'+1}^{i,j}$ and $b_{3m'+1}^{j,i}$ are not part of a blocking 3-set.*

Proof. The argument is similar to the proof of Lemma 4.18.

We show that agents $b_\ell^{i,j}$, $b_\ell^{j,i}$, and $c_\ell^{i,j}$ are not contained in a blocking 3-set via induction on $ki + j$. For $ki + j < k + 1$, there are no agents $b_\ell^{i,j}$, $b_\ell^{j,i}$, and $c_\ell^{i,j}$, and thus, there is nothing to show. So fix an incidence-checking gadget $B^{i,j}$ with $ki + j \geq k + 1$. A blocking 3-set cannot contain an agent from a vertex-selection gadget (by Lemmas 4.17 and 4.20), an agent from an incidence-checking gadget $B^{i',j'}$ with $ki' + j' < ki + j$ (by the induction hypothesis), or an agent $x^{i',j'}$ or $\overleftarrow{x}^{i',j'}$ (by Lemma 4.19). Thus, every blocking 3-set only consists of agents $b_\ell^{i,j}$, $b_\ell^{j,i}$, or $c_\ell^{i,j}$. However, the agent with minimal index does not prefer the 3-set to M , a contradiction. Therefore, no blocking 3-set contains an agent $b_\ell^{i,j}$, $b_\ell^{j,i}$, or $c_\ell^{i,j}$. \square

Now, we can conclude the stability of M .

Lemma 4.22. *Matching M is stable.*

Proof. By Lemmas 4.17 to 4.21, no agent outside a cut-off gadget is contained in a blocking 3-set. Thus, by Lemma 4.7, there is no blocking 3-set. \square

4.3.7 Proof of the backward direction

Finally, we show that if \mathcal{I}' admits a stable matching, then G contains a clique of size k . We start by showing that every stable matching selects a vertex $v_{p_i}^i$ in every vertex-selection gadget S^i , meaning that it contains the 3-set $\{s^i, \overleftarrow{s}^i, v_{p_i}^i\}$ for some $p_i \in [3n' + 1]$.

Lemma 4.23. *For $i \in [k]$, every stable matching M contains a 3-set $\{s^i, \overleftarrow{s}^i, v_{p_i}^i\}$ for some $p_i \in [3n' + 1]$, and all other agents v_q^i are matched to each other for $q \neq p_i$.*

Proof. Consider a stable matching M . We prove the statement by induction on i . For $i = 0$, there is nothing to show. So fix $i > 0$. By the induction hypothesis, we know that no agent from a vertex-selection gadget $S^{i'}$ with $i' < i$ is matched to a 2-set containing an agent from S^i . Ignoring 2-sets containing an agent from $S^{i'}$ for $i' < i$, agent v_p^i prefers most to be matched to $\{v_q^i, v_{q'}^i\}$ for $q, q' \in [3n' + 1]$, or to $\{s^i, \overleftarrow{s}^i\}$. If no agent v_p^i is matched to $\{s^i, \overleftarrow{s}^i\}$, then $\{v_{3r-2}^i, v_{3r-1}^i, v_{3r}^i\} \in M$ for all $r \in [n']$ by the same arguments as in the proof of Proposition 4.1 (as after ignoring also $\{s^i, \overleftarrow{s}^i\}$, the beginning of the preferences is derived from the strict order $v_1^i \succ v_2^i \succ \dots \succ v_{3n'+1}^i$). The cut-off gadget for $v_{3n'+1}^i$ then implies that M is not stable. Thus, there exists some $p_i \in [3n' + 1]$ such that $\{v_{p_i}^i, s^i, \overleftarrow{s}^i\} \in M$. It follows that all other agents v_q^i are matched to each other for $q \neq p_i$ by the same arguments as in the proof of Proposition 4.1. \square

Lemma 4.24. *For every stable matching M and every incidence-checking gadget $B^{i,j}$, every 3-set t containing at least one agent from $B^{i,j}$ contains three agents from $B^{i,j}$.*

Proof. Assume for a contradiction that there exists a stable matching M , an incidence-checking gadget $B^{i,j}$, and a 3-set t containing one agent b from $B^{i,j}$ and one agent v not contained in $B^{i,j}$. Assume that the incidence-checking gadget is chosen such that i is minimal, and such that j is minimal among all incidence-checking gadgets with minimal i . By Lemma 4.23, no agent from $B^{i,j}$ is matched to a 2-set containing an agent from a vertex-selection gadget. By the choice of $B^{i,j}$, no agent from $B^{i,j}$ is matched to a 2-set containing an agent from an incidence-checking gadget $B^{i',j'}$ with $i' < i$ or $i' = i$ and $j' \leq j$. Since there are $9m' + 9$ agents in the incidence-checking gadget, there are three agents a_1, a_2 , and a_3 which are matched to a 2-set containing at least one agent which is not contained in $B^{i,j}$. The cut-off gadget for $z_2^{i,j}$ implies that M contains $\{z_1^{i,j}, z_2^{i,j}, z_3^{i,j}\}$. If $x^{i,j}$ is not matched to a 2-set $\{\overleftarrow{x}^{i,j}, b_r^{i,j}\}$ for some $r \in [3m' + 1]$, then $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_r^{i,j}\}$ is blocking, a contradiction to the stability of M . By symmetric arguments, $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_r^{i,j}\} \in M$ for some $r \in [3m' + 1]$. Thus, we have that $a_r \in \{b_p^{i,j}, b_p^{j,i} : p \in [3m' + 1]\} \cup \{c_q^{i,j} : q \in [3m']\}$ for every $r \in [3]$. It follows that $\{a_1, a_2, a_3\}$ blocks M , a contradiction. \square

We now show that for any pair of vertex-selection gadgets, the vertices selected by the vertex-selection gadgets are adjacent in G .

Lemma 4.25. *Let M be a stable matching such that vertex-selection gadget S^i selects vertex v_p^i and S^j selects v_q^j . Then G contains the edge $\{v_p^i, v_q^j\}$.*

Proof. We assume for a contradiction that the edge $\{v_p^i, v_q^j\}$ is not contained in G . We need to show that M contains a blocking 3-set.

Since Lemma 4.23 implies that neither s^i nor \overleftarrow{s}^i is matched to a 2-set containing $x^{i,j}$ or $\overleftarrow{x}^{i,j}$, the cut-off gadgets for $x^{i,j}$ and $\overleftarrow{x}^{i,j}$ imply that $x^{i,j}$ and $\overleftarrow{x}^{i,j}$ are matched to an agent $b_r^{j,i}$, i.e., $\{x^{i,j}, \overleftarrow{x}^{i,j}, b_r^{j,i}\} \in M$. Similarly, $x^{j,i}$ and $\overleftarrow{x}^{j,i}$ are matched to an agent $b_s^{i,j}$.

Let $e_r^{i,j} = \{v_{r_i}^i, v_{r_j}^j\}$. If $r_i < p$, then $\{b_r^{j,i}, \overleftarrow{s}^i, x^{i,j}\}$ is a blocking 3-set. If $r_i > p$, then $\{b_r^{j,i}, s^i, \overleftarrow{x}^{i,j}\}$ is a blocking 3-set. Thus, v_p^i is an endpoint of e_r . By symmetric arguments, we get that v_q^j is an endpoint of e_s . Since $\{v_p^i, v_q^j\} \notin E^{i,j}$, it follows that $s \neq r$.

First assume $r < s$. The cut-off gadget for $z_2^{i,j}$ implies that matching M contains the 3-set $\{z_1^{i,j}, z_2^{i,j}, z_3^{i,j}\}$. First, we show by induction on ℓ that for every $\ell < r$, matching M contains $\{b_{\ell'}^{j,i}, b_{\ell'}^{i,j}, c_{\ell'}^{i,j}\}$ for every $\ell' < \ell$. For $\ell = 0$, there is nothing to show. So fix $\ell > 0$.

By the induction hypothesis, M contains $\{b_{\ell'}^{j,i}, b_{\ell'}^{i,j}, c_{\ell'}^{i,j}\}$ for every $\ell' < \ell$. Every 2-set which $b_{\ell}^{j,i}$ prefers to $\{b_{\ell}^{j,i}, c_{\ell}^{i,j}\}$ contains an agent which is before $b_1^{j,i}$ in the master poset, an agent from $X := \{x^{i,j}, \overleftarrow{x}^{i,j}, x^{j,i}, \overleftarrow{x}^{j,i}\}$, or contains an agent $b_{\ell'}^{j,i}, b_{\ell'}^{i,j}$ or $c_{\ell'}^{i,j}$ for some $\ell' < \ell$. By Lemma 4.24, agents $b_{\ell}^{j,i}, b_{\ell}^{i,j}$, and $c_{\ell}^{i,j}$ are matched to 2-sets containing only agents from the incidence checking gadget. Every 2-set which $b_{\ell}^{j,i}$ prefers to $\{b_{\ell}^{j,i}, c_{\ell}^{i,j}\}$ contains an agent outside $B^{i,j}$, an agent from $X := \{x^{i,j}, \overleftarrow{x}^{i,j}, x^{j,i}, \overleftarrow{x}^{j,i}\}$, or an agent $b_q^{j,i}, b_q^{i,j}$, or $c_q^{i,j}$ with $q < \ell$. Thus, $b_{\ell}^{j,i}$ is not matched to a 2-set it prefers to $\{b_{\ell}^{j,i}, c_{\ell}^{i,j}\}$. By symmetric arguments, we have that $b_{\ell}^{i,j}$ is not matched to a 2-set it prefers to $\{b_{\ell}^{i,j}, c_{\ell}^{i,j}\}$. Every 2-set which $c_{\ell}^{i,j}$ prefers to $\{b_{\ell}^{j,i}, b_{\ell}^{i,j}\}$ contains an agent outside $B^{i,j}$ (which is not matched to $c_{\ell}^{i,j}$ by Lemmas 4.23 and 4.24), an agent from X (which is not matched to $c_{\ell}^{i,j}$), an agent $z_r^{i,j}$ for some $r \in [3]$ (which is not matched to $c_{\ell}^{i,j}$ as $\{z_1^{i,j}, z_2^{i,j}, z_3^{i,j}\} \in M$), or an agent $b_{\ell'}^{j,i}, b_{\ell'}^{i,j}$, or $c_{\ell'}^{i,j}$ for some $\ell' < \ell$ (by the induction hypothesis on ℓ). Thus, if $\{b_{\ell}^{j,i}, b_{\ell}^{i,j}, c_{\ell}^{i,j}\} \notin M$, then $\{b_{\ell}^{j,i}, b_{\ell}^{i,j}, c_{\ell}^{i,j}\}$ blocks M , contradicting the stability of M .

We now show that the 3-set $t = \{z_1^{i,j}, b_{r+1}^{j,i}, b_r^{j,i}\}$ blocks M . Agent $z_1^{i,j}$ prefers $\{b_{r+1}^{j,i}, b_r^{j,i}\}$ to $M(z_1^{i,j}) = \{z_2^{i,j}, z_3^{i,j}\}$. Every 2-set which $b_{r+1}^{j,i}$ prefers to $\{z_1^{i,j}, b_r^{j,i}\}$ contains an agent outside $B^{i,j}$, an agent from $X = \{x^{i,j}, \overleftarrow{x}^{i,j}, x^{j,i}, \overleftarrow{x}^{j,i}\}$, or $z_1^{i,j}$. Since $b_{r+1}^{j,i}$ is not matched to a 2-set

containing any of these agents, it follows that $b_{r+1}^{i,j}$ prefers $\{z_1^{i,j}, b_r^{j,i}\}$ to $M(b_{r+1}^{i,j})$. Every 2-set which $b_r^{i,j}$ prefers to $\{z_1^{i,j}, b_{r+1}^{j,i}\}$ contains an agent outside $B^{i,j}$, an agent from X , agent $z_1^{i,j}$, or two agents from $\{b_q^{i,j}, b_q^{j,i}, c_q^{i,j}\}$. From all these agents, $b_r^{i,j}$ can only be matched to $c_r^{i,j}$. It follows that $b_r^{i,j}$ prefers $\{z_1^{i,j}, b_{r+1}^{j,i}\}$ to $M(b_r^{i,j})$. Thus, t blocks M , a contradiction to the stability of M .

If $s < r$, then symmetric arguments show that $\{z_1^{i,j}, b_s^{i,j}, b_{s+1}^{j,i}\}$ is a blocking 3-set for M . \square

It now easily follows that G contains a multicolored clique.

Lemma 4.26. *If \mathcal{I}' admits a stable matching, then G admits a clique of size k .*

Proof. By Lemma 4.23, every vertex-selection gadget selects a vertex, and by Lemma 4.25, these k vertices form a clique. \square

Theorem 4.27 now directly follows from Lemmas 4.16, 4.22 and 4.26.

Theorem 4.27. *3-DSR parameterized by $\lambda(\mathcal{I})$ is W[1]-hard, where $\lambda(\mathcal{I})$ denotes the minimum number of agents such that the preferences of the instance arising through the deletion of these agents are derived from a strict order.*

Proof. The reduction clearly runs in polynomial time. By Lemmas 4.22 and 4.26 it is correct. By Lemma 4.16, we have that $\lambda(\mathcal{I}) = O(k^2)$. Thus, we have a parameterized reduction from MULTICOLORED CLIQUE parameterized by solution size k to 3-DSR parameterized by $\lambda(\mathcal{I})$. Since MULTICOLORED CLIQUE parameterized by solution size k is W[1]-hard [15, 42], 3-DSR is W[1]-hard parameterized by $\lambda(\mathcal{I})$. \square

We have shown that 3DSR-POSET parameterized by λ is W[1]-hard. A natural question is whether there is an XP-algorithm for 3DSR-POSET.

Next, we drop the assumption that preferences are complete (i.e., every agent is allowed to be matched to any set of $d - 1$ other agents), but instead require that the master poset is a strict order.

4.4 Incomplete preferences derived from a strictly ordered master poset

Let MDSRI be the MDSR problem with incomplete preference lists, i.e., \succ_a is not a strict order of $\binom{A \setminus \{a\}}{d-1}$, but a strict order of a subset $X_a \subseteq \binom{A \setminus \{a\}}{d-1}$ for each $a \in A$. In this case, we call a d -set t *acceptable* if $t \setminus \{a\} \in X_a$ for every $a \in t$, and define a *matching* M to be a set of disjoint, acceptable d -sets.

MDSRI

Input: A set A of agents together with preference lists \succ_a over X_a for a subset $X_a \subseteq \binom{A \setminus \{a\}}{d-1}$ for each $a \in A$.

Task: Decide whether a stable matching exists.

Similarly, MDSRI-ML is the MDSRI problem restricted to instances where the preferences are derived from a strict order of agents (which we call *master poset*), and ℓ -DSRI is MDSRI for the special case $d = \ell$. Here, preferences of an agent a are derived from the master poset \succ_{ML} if they are the restriction to X_a of a preference list derived from \succ_{ML} .

MDSRI-ML

Input: An MDSRI instance, and a strict order \succ_{ML} of the agents (called *master poset*) such that for each agent a , \succ_a arises from \succ_{ML} through the deletion of some $(d - 1)$ -sets.

Task: Decide whether there exists a stable matching.

In this section, we show that 3-DSRI-ML, the restriction of MDSRI-ML to $d = 3$, is NP-complete, even if the master poset is strictly ordered. In order to do so, we reduce from PERFECT-SMTI-ML. The input of this problem is an instance of MAXIMUM STABLE MARRIAGE WITH TIES AND INCOMPLETE PREFERENCES, where the preferences of men are derived from a weak order of women with maximum tie size two (called *master list of women*) and the preferences of women are derived from a strict order of men (called *master list of men*). Here, the preferences of men (women) are derived from a master list \succ_w (\succ_m) if the preferences of each man m (woman w) arise through the deletion of a set of agents from \succ_w (\succ_m). PERFECT-SMTI-ML then asks whether there exists a perfect (weakly) stable matching, i.e., a set M of man-woman pairs such that every man and every woman is contained in exactly one pair, and there is no pair (m, w) preferring each other to their partner assigned in the matching. Note that for PERFECT-SMTI-ML, we denote the assignments of a matching (as well as blocking pairs) as pairs in order to avoid confusion with 2-sets contained in the preferences of an agent in a 3-DSRI-ML instance.

PERFECT-SMTI-ML

Input: A STABLE MARRIAGE WITH TIES AND INCOMPLETE PREFERENCES instance, where the preferences are derived from two master lists \succ_w (which is a strict order) and \succ_m (which may contain ties of size at most two).

Task: Decide whether there exists a perfect stable matching.

PERFECT-SMTI-ML is NP-complete [26].

For the rest of this section, we fix a PERFECT-SMTI-ML instance $\mathcal{I} = (G, \succ_m, \succ_w)$, where G is the acceptability graph (i.e., the graph where each agent is a vertex, and two agents are connected by an edge if and only if they are contained in each other's preference list), and \succ_m and \succ_w are the master lists of men and women, respectively. We denote the set of men by U , and the set of women by W . We assume that $|U| = |W|$. Let $W = \{w_1, \dots, w_{|W|}\}$ such that $w_i \succ_m w_{i+1}$ or $w_i \perp_m w_{i+1}$ for all $i \in [|W| - 1]$ and let $U = \{m_1, \dots, m_{|U|}\}$ such that $m_i \succ_w m_{i+1}$ for all $i \in [|U| - 1]$.

The basic idea of the reduction is as follows. For each m_i , we add an agent a_i , and for each w_j , we add an agent b_j . For every acceptable pair (m_i, w_j) , we add an agent $c_{i,j}$, and the 3-set $\{a_i, b_j, c_{i,j}\}$ will be acceptable. Intuitively, a stable 3-dimensional matching matches each agent a_i to a 2-set $\{b_j, c_{i,j}\}$, which corresponds edge (m_i, w_j) being part of a stable matching. Thus, the preferences of a_i and b_j correspond to those of m_i and w_j , i.e., a_i prefers $\{b_j, c_{i,j}\}$ to $\{b_{j'}, c_{i,j'}\}$ if m_i prefers w_j to $w_{j'}$ (but a_i preferring $\{b_j, c_{i,j}\}$ to $\{b_{j'}, c_{i,j'}\}$ does not imply m_i preferring w_j to $w_{j'}$ as m_i may tie w_j and $w_{j'}$), and b_j prefers $\{a_i, c_{i,j}\}$ to $\{a_{i'}, c_{i',j}\}$ if and only if w_j prefers m_i to $m_{i'}$. However, the preferences of men may contain ties (of size two), while the preferences of a_i must not contain ties. We will use so-called *tie gadgets* to model such ties. Finally, the reduction shall ensure that every a_i is matched (as this implies that every m_i is matched). This will be done by a *cut-off gadget*.

We now describe the two gadgets (tie gadget and cut-off gadget) used in the reduction. Afterwards, we describe the reduction in detail and prove its correctness.

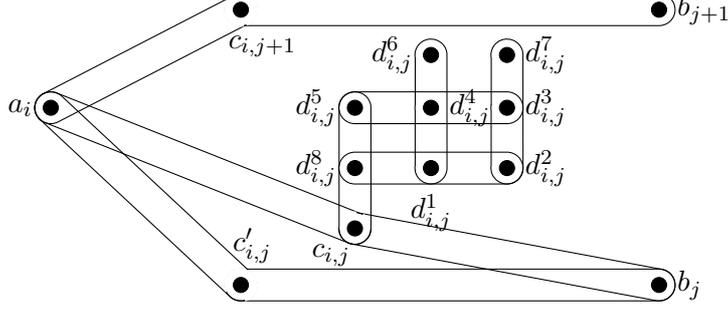


Figure 3: The acceptable 3-sets of a tie gadget T_i^j . For example, the line around a_i , $c_{i,j+1}$, and b_{j+1} indicates that the 3-set $\{a_i, c_{i,j+1}, b_{j+1}\}$ is acceptable.

4.4.1 Tie gadget

Given a man $m_i \in U$ who ties two women w_j and w_{j+1} , we construct a tie gadget T_i^j . This gadget models this tie, i.e., it allows a_i to be matched to b_j or b_{j+1} . The idea is the following: There are two stable matchings inside the gadget, one leaving $c_{i,j}$ unmatched while the other matches $c_{i,j}$. The first one allows to match m_i to w_j via the 3-set $\{a_i, b_j, c_{i,j}\}$, while the second allows to match m_i to w_{j+1} via $\{a_i, b_{j+1}, c_{i,j+1}\}$ (note that in this case $c_{i,j}$ prevents the 3-set $\{a_i, b_j, c_{i,j}\}$ from being blocking). In this case, the 3-set $\{a_i, b_j, c'_{i,j}\}$ ensures that if $\{a_i, b_{j+1}, c_{i,j+1}\}$ is not part of the matching, then the 3-set $\{a_i, b_j, c'_{i,j}\}$ can be blocking to represent the possibly blocking pair (m_i, w_j) .

We add nine agents $c'_{i,j}$ and $d_{i,j}^1, \dots, d_{i,j}^8$, together with the acceptable 3-sets $\{a_i, c'_{i,j}, b_{j+1}\}$, $\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^1, d_{i,j}^2, d_{i,j}^8\}$, $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^3, d_{i,j}^4, d_{i,j}^5\}$. See Figure 3 for an example.

The preferences of any agent arise from the following preferences through the deletion of all 2-sets which are not acceptable for an agent. $\{d_{i,j}^1, d_{i,j}^2\} \succ \{d_{i,j}^1, d_{i,j}^4\} \succ \{d_{i,j}^2, d_{i,j}^3\} \succ \{d_{i,j}^3, d_{i,j}^4\} \succ \{d_{i,j}^1, d_{i,j}^6\} \succ \{d_{i,j}^3, d_{i,j}^5\} \succ \{d_{i,j}^4, d_{i,j}^5\} \succ \{d_{i,j}^2, d_{i,j}^7\} \succ \{d_{i,j}^3, d_{i,j}^7\} \succ \{d_{i,j}^1, d_{i,j}^8\} \succ \{d_{i,j}^2, d_{i,j}^8\} \succ \{d_{i,j}^4, d_{i,j}^6\} \succ \{d_{i,j}^5, d_{i,j}^8\} \succ \{d_{i,j}^1, c_{i,j}\} \succ \{d_{i,j}^8, c_{i,j}\} \succ \{a_i, c_{i,j}\} \succ \{a_i, c_{i,j+1}\} \succ \{a_i, c'_{i,j}\} \succ \{b_j, c_{i,j}\} \succ \{b_{j+1}, c_{i,j+1}\} \succ \{b_j, c'_{i,j}\}$, which can be derived from the following strict order of agents: $d_{i,j}^1 \succ d_{i,j}^2 \succ \dots \succ d_{i,j}^8 \succ a_i \succ b_j \succ b_{j+1} \succ c_{i,j} \succ c_{i,j+1} \succ c'_{i,j}$.

The following observation shows that the tie gadget indeed models ties, i.e., it contains a stable matching which matches a_i to b_j (corresponding to matching m_i to w_j) and one which matches a_i to b_{j+1} (corresponding to matching m_i to w_{j+1}). Furthermore, given a matching M which matches a_i or both w_j and w_{j+1} to 2-sets they prefer to every 2-set of the tie gadget, we can extend M to the tie gadget without introducing a blocking 3-set.

We consider a_i , b_j , and b_{j+1} to be part of the tie gadget T_i^j . Note that a_i , b_j , and b_{j+1} may also be part of other tie gadgets. For a set X of agents, we denote by $T_i^j - X$ the instance arising from T_i^j through the deletion of all agents from X as well as every 2-sets containing an agent from X which appears in the preferences of some agent.

Observation 4.28. *Let T_i^j be a tie gadget. The matchings $M_1 = \{\{a_i, c_{i,j+1}, b_{j+1}\}, \{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}, \{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}, \{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}\}$ and $M_2 = \{\{a_i, c_{i,j}, b_j\}, \{d_{i,j}^1, d_{i,j}^2, d_{i,j}^8\}, \{d_{i,j}^3, d_{i,j}^4, d_{i,j}^5\}\}$ are stable. In $T_i^j - \{a_i\}$ or $T_i^j - \{b_j, b_{j+1}\}$, also the matching $M = \{\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}, \{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}, \{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}\}$ is stable.*

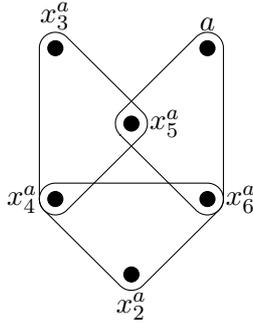


Figure 4: The acceptable 3-sets of a cut-off gadget.

Matching	Blocking 3-set
$\{a, x_5^a, x_6^a\}$	$\{x_2^a, x_4^a, x_6^a\}$
$\{x_2^a, x_4^a, x_6^a\}$	$\{x_3^a, x_4^a, x_5^a\}$
$\{x_3^a, x_4^a, x_5^a\}$	$\{a, x_5^a, x_6^a\}$

Table 3: The blocking 2-sets in the subinstance from Lemma 4.29.

4.4.2 Cut-off gadget

A cut-off gadget for an agent a consists of a together with five agents x_2^a, \dots, x_6^a . The only acceptable 3-sets are $\{a, x_5^a, x_6^a\}$, $\{x_2^a, x_4^a, x_6^a\}$, and $\{x_3^a, x_4^a, x_5^a\}$. See Figure 4 for an example.

Each agent derives its preferences from $\{x_2^a, x_4^a\} \succ \{a, x_5^a\} \succ \{a, x_6^a\} \succ \{x_3^a, x_4^a\} \succ \{x_3^a, x_5^a\} \succ \{x_2^a, x_6^a\} \succ \{x_4^a, x_5^a\} \succ \{x_4^a, x_6^a\} \succ \{x_5^a, x_6^a\}$. Note that this list can be derived from $a \succ x_2^a \succ x_3^a \succ x_4^a \succ x_5^a \succ x_6^a$.

We now observe that a cut-off gadget does not admit a stable matching, implying that one of the agents has to be matched outside the gadget. As agent a is the only agent which accepts 2-sets containing agents not contained in the cut-off gadget, it follows that a is matched to a 2-set outside the cut-off gadget.

Lemma 4.29. *A cut-off gadget does not admit a stable matching.*

Proof. Note that any acceptable 3-set contains two agents from $\{x_4^a, x_5^a, x_6^a\}$. Thus, any matching contains only one 3-set of a cut-off gadget. For each of the three possible matchings, we give a blocking 3-set in Table 3. \square

We observe that if agent a is matched outside the cut-off gadget, then the cut-off gadget does not contain a blocking 3-set.

Observation 4.30. *The cut-off gadget without a admits a stable matching, namely $\{x_3^a, x_4^a, x_5^a\}$.*

Having described the gadgets needed for the reduction, we can now describe the complete reduction.

4.4.3 The reduction

Our reduction is structured similarly to the NP-completeness proof of 3-DIMENSIONAL STABLE MARRIAGE WITH INCOMPLETE CYCLIC PREFERENCES by Biró and McDermid [5]. In both reductions, there is one agent for each man and each woman. Each such agent is forced to be

matched in any stable matching by a gadget based on a small unsolvable instance. However, modelling the ties in the preferences is a bit more complicated in our case, and is done by the tie gadget described in Section 4.4.1.

We construct a 3-DSRI-ML instance \mathcal{I}' with a strictly ordered master poset from an instance \mathcal{I} of PERFECT-SMTI-ML as follows. For each man m_i , we add an agent a_i , and for each woman w_j , we add an agent b_j . For each man m_i , and each woman w_j who is not tied with another woman in m_i 's preference list, we add an agent $c_{i,j}$. For each man m_i , and each tie $w_j \perp_m w_{j+1}$ in m_i 's preference list, we add a tie gadget T_i^j (described in Section 4.4.1).

It remains to describe the preferences. For each man m_i , we define a sublist \mathcal{A}_i as follows. Process all woman w_j adjacent to m_i by increasing j . If the woman is not tied with another woman adjacent to m_i , then add the 2-set $\{a_i, c_{i,j}\}$, followed by $\{b_j, c_{i,j}\}$. Otherwise, w_j is tied with w_{j+1} in the preference list of m_i . Then add the 2-sets $\{a_i, c_{i,j}\} \succ_{\text{ML}} \{a_i, c_{i,j+1}\} \succ_{\text{ML}} \{a_i, c'_{i,j}\} \succ_{\text{ML}} \{b_j, c_{i,j}\} \succ_{\text{ML}} \{b_j, c_{i,j+1}\} \succ_{\text{ML}} \{b_j, c'_{i,j}\}$ to \mathcal{A}_i (see Section 4.4.1).

For each man m_i , we add a cut-off gadget I_i (described in Section 4.4.2) for agent a_i .

We start by showing that the preferences are indeed derived from a strictly ordered master poset of agents.

Observation 4.31. *The master poset \succ_{ML} is derived from a strict order.*

Proof. First, we order the tie gadgets $T_{i_1}^{j_1}, T_{i_2}^{j_2}, \dots, T_{i_r}^{j_r}$ such that $i_\ell \leq i_{\ell+1}$ and if $i_\ell = i_{\ell+1}$, then $j_\ell < j_{\ell+1}$. For each tie gadget T_i^j , define the sublist \mathcal{D}_i^j via $d_{i,j}^1 \succ d_{i,j}^2 \succ \dots \succ d_{i,j}^8$. Furthermore, we define the sublist \mathcal{C}_i^j via $c_{i,j} \succ c_{i,j+1} \succ c'_{i,j}$. For every i and j such that m_i does not tie w_j with another woman but w_j is contained in the preferences of m_i , we define \mathcal{C}_i^j to be the sublist containing only $c_{i,j}$. For every i and j such that w_j is not contained in the preferences of m_i or m_i ties w_j with w_{j-1} , we define \mathcal{C} to be an empty sublist. The master poset now looks as follows:

$$\begin{aligned} & \mathcal{D}_{i_1}^{j_1} \succ \dots \succ \mathcal{D}_{i_r}^{j_r} \succ a_1 \succ a_2 \succ \dots \succ a_{|M|} \succ b_1 \succ \dots \succ b_{|W|} \\ & \succ \mathcal{C}_1^1 \succ \mathcal{C}_2^1 \succ \dots \mathcal{C}_{|U|}^1 \succ \mathcal{C}_1^2 \succ \dots \succ \mathcal{C}_{|U|}^2 \succ \mathcal{C}_1^3 \succ \dots \succ \mathcal{C}_{|U|}^{|W|}. \end{aligned}$$

It is easy to verify that the preferences are indeed derived from this master poset. \square

Notably, the preferences of every agent are not only derived from the master poset of agents, but also from a master list of 2-sets (where again every agent can declare an arbitrary set of 2-sets to be unacceptable).

Having described the construction, we continue by showing that the corresponding reduction is indeed correct.

4.4.4 Proof of the forward direction

We first show that a perfect stable matching in the PERFECT-SMTI-ML instance \mathcal{I} implies a perfect stable matching in the constructed 3-DSRI-ML instance \mathcal{I}' .

Lemma 4.32. *If the PERFECT-SMTI-ML instance \mathcal{I} admits a perfect stable matching M , then the 3-DSRI-ML instance \mathcal{I}' admits a stable matching.*

Proof. We construct a stable matching M' as follows, starting with $M' = \emptyset$. For every edge $(m_i, w_j) \in M$, we add the 3-set $\{a_i, b_j, c_{i,j}\}$ to M' . If w_j is tied with woman w_{j-1} , then we add the 3-sets $\{c_{i,\ell}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$. If w_j is tied with woman w_{j+1} , then we add the 3-sets $\{d_{i,\ell}^1, d_{i,\ell}^2, d_{i,\ell}^8\}$ and $\{d_{i,\ell}^3, d_{i,\ell}^4, d_{i,\ell}^5\}$. For each tie gadget T_i^j between m_i , w_j and w_{j+1} such that $(m_i, w_j) \notin M$ and $(m_i, w_{j+1}) \notin M$, we add the

3-sets $\{c_{i,j}, d_{i,j}^5, d_{i,j}^8\}$, $\{d_{i,j}^2, d_{i,j}^3, d_{i,j}^7\}$, and $\{d_{i,j}^1, d_{i,j}^4, d_{i,j}^6\}$. For each cut-off gadget I_i for man m_i , we add the 3-set $\{x_3, x_4, x_5\}$ to M' .

We claim that M' is a stable matching. Since M is perfect, every agent a_i is matched to a 2-set it prefers to any 2-set of agents from its cut-off gadget. By Observation 4.30, we get that no agent from cut-off gadget I_i except for a_i can be part of a blocking 3-set. For every tie gadget T_i^j , Observation 4.28 tells us that no blocking 3-set contains only agents of the form $d_{i,j}^k$. All other acceptable 3-sets are of the form $\{a_i, b_j, c_{i,j}\}$ or $\{a_i, b_j, c'_{i,j}\}$. First, we consider 3-sets of the form $\{a_i, b_j, c_{i,j}\}$. Agent a_i prefers $\{b_j, c_{i,j}\}$ to $M'(a_i) = \{b_\ell, c_{i,\ell}\}$ if and only if m_i prefers w_j to $M(m_i)$ or m_i ties w_j and w_ℓ and $\ell = j + 1$. However, in the latter case (i.e., m_i ties w_j and w_ℓ , and we have $\ell = j + 1$), we have $\{c_{i,\ell}, d_{i,j}^5, d_{i,j}^8\} \in M'$, and thus $c_{i,\ell}$ does not prefer $\{a_i, b_\ell\}$ to $M'(c_{i,\ell})$. Agent b_j prefers $\{a_i, c_{i,j}\}$ to $M'(b_j)$ if and only if w_j prefers m_i to $M(w_j)$. Thus, by the stability of M , $\{a_i, b_j, c_{i,j}\}$ is not blocking. Next, we consider an acceptable 3-set $\{a_i, b_j, c'_{i,j}\}$. Agent a_i prefers $\{b_j, c'_{i,j}\}$ to $M'(a_i)$ if and only if m_i prefers w_j to $M(m_i)$. Similarly, agent b_j prefers $\{a_i, c'_{i,j}\}$ to $M'(b_j)$ if and only if w_j prefers m_i to $M(w_j)$. Therefore, the stability of M implies that $\{a_i, b_j, c'_{i,j}\}$ is not blocking. Altogether, M' is stable. \square

4.4.5 Proof of the backward direction

We now turn to the reverse direction, i.e., showing that a stable matching in the 3-DSRI-ML instance implies a perfect stable matching in the PERFECT-SMTI-ML instance.

Lemma 4.33. *If the 3-DSRI-ML instance \mathcal{I}' admits a stable matching M' , then the PERFECT-SMTI-ML instance \mathcal{I} admits a perfect stable matching.*

Proof. Let M' be a stable matching in \mathcal{I}' . By Lemma 4.29, each agent a_i has to be matched to a 2-set outside its cut-off gadget. Any such 2-set involves an agent b_j . Thus, this defines a perfect matching $M := \{(m_i, w_j) : \exists v \text{ s.t. } \{a_i, b_j, v\} \in M'\}$.

We claim that M is stable. Assume that M admits a blocking pair (m_i, w_j) . If m_i does not tie w_j with another woman, then $\{a_i, b_j, c_{i,j}\}$ is a blocking 3-set (a_i and b_j prefer this 3-set as (m_i, w_j) is blocking, and $c_{i,j}$ as it is the only acceptable 3-set for $c_{i,j}$). If m_i ties w_j with a woman w_ℓ , then M cannot contain one of the edges (m_i, w_j) or (m_i, w_ℓ) (as else (m_i, w_j) was not blocking). Thus, M' does not contain the 3-set $t := \begin{cases} \{a_i, b_j, c_{i,j}\} & \text{if } \ell = j - 1 \\ \{a_i, b_j, c'_{i,j}\} & \text{if } \ell = j + 1 \end{cases}$. Since $c_{i,j}$ if $\ell = j - 1$ or $c'_{i,j}$ if $\ell = j + 1$ is unmatched, and (m_i, w_j) is a blocking pair, we get that t is a blocking 3-set, contradicting the stability of M' . \square

The NP-completeness of 3-DSRI-ML now easily follows.

Theorem 4.34. *3-DSRI-ML is NP-complete, even if the master poset is a strict order.*

Proof. 3-DSRI-ML is clearly in NP, as we can check the stability of a matching in $O(n^3)$ time by checking for every set of three agents whether it is blocking, where n is the number of agents. The reduction adds an agent a_i together with a cut-off gadget or b_j for every man m_i or woman w_j . Furthermore, for every acceptable pair (m_i, w_j) , we add an agent $c_{i,j}$ together with the acceptable 3-set $\{a_i, b_j, c_{i,j}\}$ or a tie gadget. As tie and cut-off gadgets have constant size, the reduction can clearly be performed in linear time. The correctness of the reduction is proven in Lemmas 4.32 and 4.33. Observation 4.31 shows that the master poset is a strict order. \square

Note that every acceptable 3-set contains exactly one agent from $A := \{a_i : i \in [|U|]\} \cup \{d_{i,j}^8, d_{i,j}^4, d_{i,j}^7\} \cup \{x_4^{a_i} : i \in [|U|]\}$, one agent from $B := \{b_j : j \in [|W|]\} \cup \{d_{i,j}^5, d_{i,j}^6, d_{i,j}^2\} \cup \{x_2^{a_i}, x_5^{a_i} :$

$i \in [|U|]$, and one agent from $C := \{c_{i,j}, c_{i,j'}, c_{i,j+1}, d_{i,j}^1, d_{i,j}^3\} \cup \{x_3^{a_i}, x_6^{a_i} : i \in [|U|]\}$. Thus, Theorem 4.34 also shows NP-completeness for the tripartite version of 3-DSRI-ML.

By “cloning” each agent corresponding to a man $d - 3$ times (and for each “acceptable 3-set”, adding the cloned men to this 3-set, and adding all $(d - 1)$ -subsets of the resulting d -set at their corresponding place in the preferences), one can derive NP-completeness of d -DSRI-ML for any fixed $d \geq 3$.

5 Conclusion

Being a fundamental problem within the fields of stable matching [35] and the analysis of hedonic games [2], our work provides a seemingly first systematic study on the parameterized complexity of the NP-hard MULTIDIMENSIONAL STABLE ROOMMATES. Focusing on the natural and well-motivated concept of master lists with the goal to identify efficiently solvable special cases, we reported partial success. While we have one main algorithmically positive result, namely fixed-parameter tractability for the parameter “maximum number of agents incomparable to a single agent”, all other (single) parameterizations led to (often surprising) hardness results (see Table 1).

As to challenges for future research, first, it remains open whether our fixed-parameter tractability result mentioned above also transfers to the setting of MULTIDIMENSIONAL STABLE MARRIAGE. Second, addressing the quest for identifying more islands of tractability, the study of further, perhaps also combined parameters is a worth-while goal. One possible parameter here would be to consider the setting that there are few strictly ordered master posets, and every agent derives its preferences of one of the master lists. Third, a natural open question is whether our W[1]-hardness results can be accompanied by XP-algorithms, or whether 3DSR-POSET is paraNP-hard for the considered parameters.

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