On Structural Parameterizations of Load Coloring

I. Vinod Reddy

Department of Electrical Engineerng and Computer Science Indian Institute of Technology Bhilai, Raipur, India vinod@iitbhilai.ac.in,

Abstract. Given a graph G and a positive integer k, the 2-LOAD COL-ORING problem is to check whether there is a 2-coloring $f: V(G) \to \{r, b\}$ of G such that for every $i \in \{r, b\}$, there are at least k edges with both end vertices colored i. It is known that the problem is NP-complete even on special classes of graphs like regular graphs. Gutin and Jones (Inf Process Lett 114:446-449, 2014) showed that the problem is fixed-parameter tractable by giving a kernel with at most 7k vertices. Barbero et al. (Algorithmica 79:211-229, 2017) obtained a kernel with less than 4k vertices and O(k) edges, improving the earlier result.

In this paper, we study the parameterized complexity of the problem with respect to structural graph parameters. We show that 2-LOAD COLORING cannot be solved in time $f(w)n^{o(w)}$, unless ETH fails and it can be solved in time $n^{O(w)}$, where n is the size of the input graph, w is the clique-width of the graph and f is an arbitrary function of w. Next, we consider the parameters distance to cluster graphs, distance to cocluster graphs and distance to threshold graphs, which are weaker than the parameter clique-width and show that the problem is fixed-parameter tractable (FPT) with respect to these parameters. Finally, we show that 2-LOAD COLORING is NP-complete even on bipartite graphs and split graphs.

1 Introduction

Given a graph G and a positive integer c, the load distribution of a c-coloring $f: V(G) \to [c]$ is a tuple (f_1, \ldots, f_c) , where f_i is the number of edges with at least one end point colored with i. The c-LOAD COLORING problem is to find a coloring f such that the function $\ell_f(G) = \max\{f_i : i \in [c]\}$ is minimum. We denote this minimum by $\ell(G)$. Ahuja et al [1] showed that the problem is NP-hard on general graphs when c = 2. They also gave a polynomial time algorithm for 2-LOAD COLORING on trees.

In a 2-coloring $f: V(G) \to \{r, b\}$, an edge is called red (resp. blue) if both end vertices are colored with r (resp. b). We use r_f and b_f to denote the number of red and blue edges in a 2-coloring f of G. Let $\mu_f(G) = \min\{r_f, b_f\}$ and $\mu(G)$ is the maximum of $\mu_f(G)$ over all possible 2-colorings of G. Ahuja et al. [1] showed that the 2-LOAD COLORING problem is equivalent to maximizing $\mu(G)$ over all possible 2-colorings of G, in particular they showed that $\ell(G) = |E(G)| - \mu(G)$. **Input:** A graph G = (V, E) and an integer k**Question:** Does there exists a coloring $f : V(G) \to \{r, b\}$ such that $\mu(G) \ge k$? (i.e, $r_f \ge k$ and $b_f \ge k$)

The above version of the load coloring problem has been studied from the parameterized complexity perspective. Gutin et al. [9] proved that the problem admits a polynomial kernel (with at most 7k vertices) parameterized by k. They also showed that the problem is fixed-parameter tractable when parameterized by the tree-width of the input graph. More recently, Barbero et al. [2] obtained a kernel for the problem with at most 4k vertices improving the result of [9].

In this paper, we study the following variant of the load coloring problem.

2-LOAD COLORING **Input:** A graph G = (V, E) and integers k_1 and k_2 **Question:** Does there exists a coloring $f : V(G) \to \{r, b\}$ such that $r_f \ge k_1$ and $b_f \ge k_2$?

Our contributions. In this paper, we study the 2-LOAD COLORING problem from the viewpoint of parameterized complexity. A parameterized problem with input size n and parameter k is called fixed-parameter tractable (FPT) if it can be solved in time $f(k)n^{O(1)}$, where f is a function only depending on the parameter k (for more details on parameterized complexity refer to the text [5]). There are many possible parameterizations for 2-LOAD COLORING. One such parameter is the size of the solution. The problem admits a linear kernel [9] with respect to the size of the solution. In this paper, we study the 2-LOAD COLORING problem with respect to various structural graph parameters. These parameters measure the complexity of the input rather than the problem itself. Tree-width is one of the well-known structural graph parameters. The 2-LOAD COLORING problem is FPT when parameterized by tree-width [9] of the input graph.

Even though tree-width is a widely used graph parameter for sparse graphs, it is not suitable for dense graphs, even if they have a simple structure. In Section 3, we consider the graph parameter clique-width introduced by Courcelle and Olariu [4], which is a generalization of the parameter tree-width. We show that 2-LOAD COLORING can be solved in time $n^{O(w)}$, and cannot be solved in $f(w)n^{o(w)}$ unless ETH fails, where w is the clique-width of the n vertex input graph and f is an arbitrary function of w. Next, we consider the parameters distance to cluster graphs, distance to co-cluster graphs, and distance to threshold graphs. These parameters are weaker than the parameter clique-width in the sense that they are a subclass of bounded clique-width graphs. Thus studying the parameterized complexity of 2-LOAD COLORING with respect to these parameters reduces the gap between tractable and intractable parameterizations. In Section 4, we show that 2-LOAD COLORING is fixed-parameter tractable with

Parameter	Results
size of the solution	Linear kernel [2,9]
tree-width	FPT [9]
clique-width(w)	$n^{O(w)}$ algorithm (Theorem 1)
	no $f(w)n^{o(w)}$ algorithm (Theorem 2)
distance to cluster graphs	FPT (Theorem 3)
distance to co-cluster graphs	FPT (Theorem 4)
distance to threshold graphs	FPT (Theorem 5)
bipartite graphs	para-NP-hard (Theorem 6)
split graphs	para-NP-hard (Theorem 7)

Table 1: Known and new parameterized results for 2-LOAD COLORING

respect to the parameters distance to cluster, distance to co-cluster and distance to threshold graphs. Finally in Section 5, we show that 2-LOAD COLORING is NP-complete on bipartite graphs and split graphs. Table 1 gives an overview of our results.

2 Preliminaries

In this section, we introduce some basic notation and terminology related to graph theory and parameterized complexity. For $k \in \mathbb{N}$, we use [k] to denote the set $\{1, 2, \ldots, k\}$. If $f : A \to B$ is a function and $C \subseteq A$, $f|_C$ denotes the restriction of f to C, that is $f|_C : C \to B$ such that for all $x \in C$, $f|_C(x) = f(x)$ All graphs we consider in this paper are undirected, connected, finite and simple. For a graph G = (V, E), by V(G) and E(G) we denote the vertex set and edge set of G respectively. We use n to denote the number of vertices and m to denote the number of edges of a graph. An edge between vertices x and y is denoted as xy for simplicity. For a subset $X \subseteq V(G)$, the graph G[X] denotes the subgraph of G induced by vertices of X and $E_G[X]$ denote the set of edges having both end vertices in the set X. For subsets $X, Y \subseteq V(G)$, $E_G[X, Y]$ denote the set of edges connecting X and Y.

For a vertex set $X \subseteq V(G)$, we denote G - X, the graph obtained from Gby deleting all vertices of X and their incident edges. For a vertex $v \in V(G)$, by N(v), we denote the set $\{u \in V(G) \mid vu \in E(G)\}$ and we use N[v] to denote the set $N(v) \cup \{v\}$. The neighbourhood of a vertex set $S \subseteq V(G)$ is $N(S) = (\bigcup_{v \in V(G)} N(v)) \setminus S$. A vertex is called *universal vertex* if it is adjacent to every other vertex of the graph. For more details on standard graph-theoretic notation and terminology, we refer the reader to the text [6].

2.1 Graph classes

We now define the graph classes which are considered in this paper. A graph is *bipartite* if its vertex set can be partitioned into two disjoint sets such that no

two vertices in the same set are adjacent. A *cluster* graph is a disjoint union of complete graphs. A *co-cluster* graph is the complement graph of a cluster graph. A graph is a *split graph* if its vertices can be partitioned into a clique and an independent set. Split graphs are $(C_4, C_5, 2K_2)$ -free. A graph is a *threshold graph* if it can be constructed from the one-vertex graph by repeatedly adding either an isolated vertex or a universal vertex. The class of threshold graphs is the intersection of split graphs and cographs [10]. Threshold graphs are $(P_4, C_4, 2K_2)$ -free. We denote a split graph (resp. threshold graph) with G = (C, I) where C and I denotes the partition of G into a clique and an independent set.

For a graph class \mathcal{F} the distance to \mathcal{F} of a graph G is the minimum number of vertices that have to be deleted from G in order to obtain a graph in \mathcal{F} . The parameters distance to cluster graphs [8], distance to co-cluster graphs, distance to threshold graphs [3] can be computed in FPT time.

2.2 Clique-width

The *clique-width* of a graph G denoted by cw(G), is defined as the minimum number of labels needed to construct G using the following four operations:

- i. Introducing a vertex. $\Phi = v(i)$, creates a new vertex v with label i. G_{Φ} is a graph consisting a single vertex v with label i.
- ii. Disjoint union. $\Phi = \Phi' \oplus \Phi''$, G_{Φ} is a disjoint union of labeled graphs $G_{\Phi'}$ and $G_{\Phi''}$
- iii. Introducing edges. $\Phi = \eta_{i,j}(\Phi')$, connects each vertex with label *i* to each vertex with label *j* ($i \neq j$) in $G_{\Phi'}$.
- iv. Renaming labels. $\Phi = \rho_{i \to j}(\Phi')$: each vertex of label *i* is changed to label *j* in $G_{\Phi'}$.

An expression build from the above four operations using w labels is called as *w*-expression. In other words, the *clique-width* of a graph G, is the minimum w for which there exists a *w*-expression that defines the graph G. A *w*-expression Ψ is a *nice w*-expression of G, if no edge is introduced twice in Ψ .

3 Graphs of Bounded Clique-width

3.1 Upper bound

In this section, we present an algorithm for solving 2-LOAD COLORING which runs in time $n^{O(w)}$ on graphs of clique-width at most w.

Theorem 1. 2-LOAD COLORING can be solved in time $n^{O(w)}$, where w is the clique-width of the input graph.

Proof. The algorithm is based on a dynamic programming over the *w*-expression of the input graph G. We assume that the *w*-expression Ψ defining G is nice, that is every edge is introduced exactly once in Ψ .

For each subexpression Φ of Ψ ,

$OPT(\Phi, n_{1,r}, n_{1,b}, n_{2,r}, n_{2,b}, \dots, n_{w,r}, n_{w,b}, k_1)$

denotes maximum number of blue edges that can be obtained in a 2-coloring $f: V(G_{\Phi}) \to \{r, b\}$ of G_{Φ} with the constraint that number of red edges is at least k_1 in G_{Φ} and the number of vertices of label i in G_{Φ} that are colored with a color ℓ in G_{Φ} is $n_{i,\ell}$, where $\ell \in \{r, b\}, i \in [w]$.

If there are no colorings satisfying the constraint $OPT(\Phi, n_{1,r}, n_{1,b}, \ldots, n_{w,r}, n_{w,b}, k_1)$, then we set its value equal to $-\infty$. Observe that G is a YES-instance of 2-LOAD COLORING if and only if $OPT(\Psi, \ldots, k_1) \geq k_2$, for some 2-coloring of G_{Ψ} .

Now we give the details of calculating the values of $OPT(\Phi,)$ at each operation.

1. $\Phi = v(i)$. In this case G_{Φ} contains one vertex of label *i* and no edges. Hence $OPT(\Phi, 0, 0, \dots, n_{i,r} = 1, 0, \dots, 0, k_1 = 0) = 0$ and $OPT(\Phi, 0, 0, \dots, 0, n_{i,b} = 1, 0, \dots, 0, k_1 = 0) = 0$. Otherwise $OPT(\Phi, \dots, k_1) = -\infty$.

2.
$$\Phi = \rho_{i \to j}(\Phi')$$
.

All vertices of label *i* are relabled to *j* by $\rho_{i \to j}$ operation in G_{Φ} , hence, there are no vertices of label *i* in G_{Φ} , so $n_{i,r} = n_{i,b} = 0$. Let *c* be a coloring corresponding to an entry $OPT(\Phi, n_{1,r}, n_{1,b}, \ldots, n_{w,r}, n_{w,b}, k_1)$. Then *c* is also a coloring of $G_{\Phi'}$, corresponding to the entry $OPT(\Phi', n'_{1,r}, n'_{1,b}, \ldots, n'_{w,r}, n'_{w,b}, k_1)$, where $n'_{i,\ell} + n'_{j,\ell} = n_{j,\ell}$ for $\ell \in \{r, b\}$ and $n'_{p,\ell} = n_{p,\ell}$ for all $p \in [w] - \{i, j\}$ and $\ell \in \{r, b\}$. The number of red and blue edges in G_{Φ} is the same as that in $G_{\Phi'}$ with respect to the coloring *c*. Hence, we have the following relation.

$$OPT(\Phi, n_{1,r}, n_{1,b}, \dots, n_{w,r}, n_{w,b}, k_1) = \left\{ \max \left\{ OPT(\Phi', n'_{1,r}, \dots, n'_{w,b}, k_1) \middle| \begin{array}{l} n'_{i,\ell} + n'_{j,\ell} = n_{j,\ell}, n_{i,\ell} = 0 \text{ for each } \ell \in \{r, b\} \\ \text{and } n_{p,\ell} = n'_{p,\ell} \text{ for all } p \in [w] - \{i, j\} \\ \text{and } \ell \in \{r, b\} \end{array} \right\}$$

3. $\Phi = \Phi' \oplus \Phi''$. As this operation does not add new edges, any coloring c corresponding to $OPT(\Phi, n_{1,r}, n_{1,b}, \ldots, n_{w,r}, n_{w,b}, k_1)$ is split between two colorings $c' = c|_{V(G_{\Phi'})}$ and $c'' = c|_{V(G_{\Phi''})}$ respectively. As G_{Φ} is the disjoint union of $G_{\Phi'}$ and $G_{\Phi''}$, the number of red and blue edges edges in G_{Φ} with respect to c is a sum of number red and blue edges with respect to c' and c'' in the graphs $G_{\Phi'}$ and $G_{\Phi''}$.

$$OPT(\Phi, n_{1,r}, n_{1,b}, \dots, n_{w,r}, n_{w,b}, k_1) = \max_{\substack{n_{i,a} + n_{i,a}' = n_{i,a} \\ k_1' + k_1'' = k_1}} \{OPT(\Phi', n_{1,r}', \dots, n_{w,b}', k_1') + OPT(\Phi'', n_{1,r}'', \dots, n_{w,b}'', k_1'')\}$$

4. $\Phi = \eta_{i,j}(\Phi')$. The graph G_{Φ} is obtained from $G_{\Phi'}$ by adding the edges between each vertex of label *i* to each vertex of label *j*. Any coloring *c* of G_{Φ} is also a coloring of $G_{\Phi'}$. As given w-expression is nice, every edge is which is added by this operation was not present in $G_{\Phi'}$. Therefore $\eta_{i,j}$ operation on $G_{\varPhi'}$ creates $n_{i,r} \cdot n_{j,r}$ many red edges and $n_{i,b} \cdot n_{j,b}$ blue edges. Hence, we have the following relation

$$OPT(\Phi, n_{1,r}, \dots, n_{w,b}, k_1) = OPT(\Phi', n_{1,r}, \dots, n_{w,b}, k_1 - n_{i,r} \cdot n_{j,r}) + n_{i,b} \cdot n_{j,b}$$

We have described the recursive formulas for all possible cases. The correctness of the algorithm follows from the description of the procedure. The number of entries in the OPT table is at most $|\Psi| n^{O(w)}$. We can compute each entry of the OPT table in $n^{O(w)}$ time. The maximum number of blue edges that can be obtained in G_{Ψ} is equals to $\max_{n_{1,r},\ldots,n_{w,b}} OPT(\Psi, n_{1,r}, n_{1,b}, \ldots, n_{w,r}, n_{w,b}, k_1)$ which can be computed in $n^{O(w)}$ time. This proves that 2-LOAD COLORING can be solved in time $n^{O(w)}$ on graphs of clique-width at most w. П

Lower Bound 3.2

We now show the lower bound complementing with the corresponding upper bound result of the previous section. To prove our result we give a linear FPT reduction from the MINIMUM BISECTION problem. In the MINIMUM BISECTION problem, we are given a graph G with an even number of vertices and a positive integer k, and the goal is to determine whether there is a partition of V(G) into two sets V_1 and V_2 of equal size such that $|E_G[V_1, V_2]| \leq k$. Fomin et al. [7] showed that MINIMUM BISECTION cannot be solved in time $f(w)n^{o(w)}$ unless ETH fails.

Theorem 2. The 2-LOAD COLORING problem cannot be solved in time $f(w)n^{o(w)}$ unless ETH fails. Here, w is the clique-width of n vertex input graph.

Proof. We give a reduction from the MINIMUM BISECTION problem to the 2-LOAD COLORING problem. Let (G, k) be an instance of MINIMUM BISECTION. We construct a graph H as follows.

- 1. For every vertex $v \in V(G)$, we introduce two vertices a_v, b_v in H. Let A = $\{a_{v_1}, \ldots, a_{v_n}\}$ and $B = \{b_{v_1}, \ldots, b_{v_n}\}$ 2. For every edge $uv \in E(G)$, we add the edges $a_u a_v$ and $b_u b_v$ to H.
- 3. Finally, for every vertex $a_{v_i} \in A$ and every vertex $b_{v_i} \in B$ add the edge $a_{v_i}b_{v_i}$ to H.

It is easy to see that the graph H has 2n vertices and $2m + n^2$ edges and the construction of H can be done in polynomial time. Moreover, if cw(G) = wwith w-expression Φ_G , then we can construct a (w+1)-expression Φ_H of H by taking two disjoint copies of Φ_G and relabel every vertex in the first copy with the label w + 1 and every vertex in the second copy with some arbitrary label $\ell \in [w]$ and finally add edges between both the copies using $\eta_{w+1,\ell}$. This shows that $cw(H) \leq w+1$. Let us set $k_1 = k_2 = m - k + n^2/4$. We now show that (G, k) is a YES instance of MINIMUM BISECTION if and only if (H, k_1, k_2) is a YES instance of 2-LOAD COLORING.

Forward direction. Let (V_1, V_2) be a partition of V(G) such that $|E_G[V_1, V_2]| \leq k$ and $|V_1| = |V_2|$. We construct a 2-coloring $f : V(H) \to \{r, b\}$ of H as follows. For each $v \in V(G)$, $f(a_v) = r$ and $f(b_v) = b$ if $v \in V_1$ and $f(a_v) = b$ and $f(b_v) = r$ if $v \in V_2$. Let $A_r = \{a_v : f(a_v) = r\}, A_b = \{a_v : f(a_v) = b\}$ and $B_r = \{b_v : f(b_v) = r\}, B_b = \{b_v : f(b_v) = b\}$. It is easy to see that $|A_r| = |A_b| = |B_r| = |B_b| = n/2$.

$$r_f = |E_H[A_r]| + |E_H[B_r]| + |E_H[A_r, B_r]| = m - k + n^2/4 = k_1$$

Similarly, we can show that $b_f = k_2$. Therefore (H, k_1, k_2) is a YES instance of 2-LOAD COLORING.

Reverse direction. Let $f: V(H) \to \{r, b\}$ be a 2-coloring of H such that $r_f = k_1$ and $b_f = k_2$. Let $A_r = \{a_v : f(a_v) = r\}, A_b = \{a_v : f(a_v) = b\}$ and $B_r = \{b_v : f(b_v) = r\}, B_b = \{b_v : f(b_v) = b\}$. Let $V_r = A_r \cup B_r$ and $V_b = A_b \cup B_b$. Then we have $|E_H[V_r, V_b]| = 2m + n^2 - r_f - b_f = 2k + n^2/2$.

Let $V_1 := \{v : f(a_v) = r\}$ and $V_2 := \{v : f(a_v) = b\}$. We show that $|E_G(V_1, V_2)| \le k$ and $|V_1| = |V_2|$. Let $|A_b| = p$ and $|B_b| = q$ such that $p+q = |V_b|$. Then we have $|A_r| = n - p$ and $|B_r| = n - q$.

We know that

$$|E_H[V_r]| + |E_H[V_b]| = k_1 + k_2 = 2m - 2k + n^2/2$$

 $|E_{H}[A_{r}]| + |E_{H}[B_{r}]| + |E_{H}[A_{r}, B_{r}]| + |E_{H}[A_{b}]| + |E_{H}[B_{b}]| + |E_{H}[A_{b}, B_{b}]| = 2m - 2k + n^{2}/2$ $|E_{H}[A_{r}]| + |E_{H}[B_{r}]| + (n - p)(n - q) + |E_{H}[A_{b}]| + |E_{H}[B_{b}]| + pq = 2m - 2k + n^{2}/2$

After simplifying we get $(n-p)(n-q) + pq = n^2/2$. Which implies p = q = n/2, that is $|A_r| = |A_b| = |B_r| = |B_b| = n/2$. Hence $|V_1| = |V_2| = n/2$, that is (V_1, V_2) is a bisection of G.

Finally, we show that $|E_G[V_1, V_2]| \leq k$. We know that $|E_H[V_r, V_b]| = 2k + n^2/2$.

$$|E_H[V_r, V_b]| = |E_H[A_r, A_b]| + |E_H[B_r, B_b]| + |E_H[A_r, B_b]| + |E_H[A_b, B_r]| = 2k + n^2/2$$

By the construction of H we have $|E_H[A_r, A_b]| = |E_H[B_r, B_b]|$. Therefore we get

$$2|E_H[A_r, A_b]| + n^2/4 + n^2/4 = 2k + n^2/2$$

Which implies $|E_H[A_r, A_b]| = k$ and hence $|E_G[V_1, V_2]| = k$. Therefore (G, k) is a YES instance of MINIMUM BISECTION. This concludes the proof.

4 Parameterized Algorithms

Clique-width of cluster graphs, co-cluster graphs and threshold graphs is at most two. Hence, from the Theorem 1, 2-LOAD COLORING is polynomial time solvable on these graph classes. In this section, we show that 2-LOAD COLORING is FPT parameterized by distance to cluster graphs, distance to co-cluster graphs and distance to threshold graphs.

4.1 Distance to Cluster Graphs

Theorem 3. 2-LOAD COLORING is fixed-parameter tractable parameterized by the distance to cluster graphs.

Proof. Let (G, X, k_1, k_2) be a 2-LOAD COLORING instance, where $X \subseteq V(G)$ of size k such that G - X is a disjoint union of cliques C_1, C_2, \ldots, C_ℓ . We first guess the colors of vertices in X in an optimal 2-coloring of G. This can be done in $O(2^k)$ time. Let $h: X \to \{r, b\}$ be such a coloring. Without loss of generality we assume that X is an independent set in G. Otherwise, let e_r and e_b be the number of red and blue edges in the coloring h having both end vertices in the set X. We build the new instance (G', X, k'_1, k'_2) of 2-LOAD COLORING, where G' is the graph obtained from G by deleting the edges having both their end vertices in X and $k'_1 = k_1 - e_r$ and $k'_2 = k_2 - e_b$. It is easy to see that (G, X, k_1, k_2) is a YES instance of 2-LOAD COLORING iff there exists a 2-coloring g of G' such that $g|_X = h$ and $r_g \ge k'_1$ and $b_g \ge k'_2$. Hence, we assume that X is an independent set in G.

For each $i \in [\ell]$, let $G_i = G[X \cup C_i]$ be the subgraph of G induced by the vertices of the clique C_i and the set X.

The algorithm is based on dynamic programming technique, which has two phases. In phase-1, given a graph G_i and non-negative integers q and n_i^r , we find a 2-coloring of G_i that maximizes the number of blue edges with the constriant that there are at least q red edges in G_i and n_i^r red vertices in C_i . In phase-2, for each $t \in [\ell]$, and a non-negative integer p, we find a 2-coloring of $\widehat{G}_t = G[C_1 \cup, \ldots, \cup C_t \cup X]$ that maximizes the number of blue edges with the constraint that number of red edges is at least p.

Phase-1. For each $i \in [\ell]$, $q \in [|E(G_i)|] \cup \{0\}$ and $n_i^r \in [|C_i|] \cup \{0\}$, let $b[G_i, n_i^r, q]$ be the maximum number of blue edges that can be attained in a 2-coloring g of G_i satisfying the following constraints.

- 1. $g|_X = h$.
- 2. number of red edges in G_i is at least q.
- 3. number of red vertices in C_i is equal to n_i^r .

If the constraint cannot be satisfied, then we let $b[G_i, n_i^r, q] = -\infty$. From the definition of b[,], we can see that $b[G_i, 0, 0]$ gives the number of blue edges in G_i when all vertices of C_i are colored blue. $b[G_i, 0, q] = -\infty$ for q > 0, $b[G_i, n_i^r, 0] = -\infty$ for $n_i^r > 1$. For a given values of i, n_i^r and q, the computation of $b[G_i, n_i^r, q]$ is described as follows.

Let H_i be the graph obtained from G_i by deleting the edges inside the clique C_i . It is easy to see that X is a vertex cover of the graph H_i . If g colors n_i^r vertices red in the clique C_i then we get $\binom{n_i^r}{2}$ red edges and $\binom{n_b^i}{2}$ blue edges inside C_i , where $n_i^r + n_i^b = |C_i|$. Hence we get the following relation.

$$b\left[G_i, n_i^r, q\right] = b\left[H_i, n_i^r, q - \binom{n_i^r}{2}\right] + \binom{n_i^b}{2}$$

For $v \in H_i - X$, let $r^i(v)$ and $b^i(v)$ denote the number of red and blue vertices in $N(v) \cap X$ respectively. For a vertex $v \in H_i$, we use $H_i - \{v\}$ to denote the graph obtained from H_i by deleting the vertex v and its incident edges. Let $q' = q - \binom{n_i^r}{2}$. Using this notation, we get the following recurrence.

$$b\Big[H_i, n_i^r, q'\Big] = \max\left\{b\Big[H_i - \{v\}, n_i^r - 1, \max\{q' - r^i(v), 0\}\Big], b\Big[H_i - \{v\}, n_i^r, q'\Big] + b^i(v)\right\}$$

If v is colored red, then we get $r^i(v)$ red edges between v and the neighbors of v in X. If v is colored blue, then we get $b^i(v)$ blue edges between v and neighbors of v in X.

The size of the DP table is at most $O(mn^2)$ and each entry can be computed in O(n) time. Hence the running time of Phase-1 is $O(n^3m)$.

Phase-2. Let $\widehat{G_t}$ be the subgraph of G induced by the cliques C_1, \dots, C_t and the set X. Let OPT[t, p] be the maximum number of blue edges that can be attained in a 2-coloring f of $\widehat{G_t}$ with the constraint that number of red edges is at least p and $f|_X = h$. If the constraint cannot be satisfied, then we let $OPT[t, p] = -\infty$. From the definition of OPT, we have OPT[0, 0] = 0 and $OPT[0, p] = -\infty$ for p > 0. For t > 0 we have:

$$OPT[t,p] = \max_{\substack{q = 0, \dots, |E(G_t)| \\ n_t^r = 0, \dots, |C_t|}} \left\{ OPT[t-1, \max\{p-q, 0\}] + b[G_t, n_t^r, q] \right\}$$

If there are q red edges in G_t and n_t^r red vertices in C_t in the coloring f, then we get $b[G_t, n_t^r, q]$ blue edges in G_t . We consider all possible values for q and n_t^r and pick the values that maximizes the OPT.

Observe that (G, X, k_1, k_2) is a YES instance if and only if $OPT[\ell, k_1] \ge k_2$. There are $O(\ell k_1)$ subproblems, each of which can be solved in $O(n^2m)$ time. As $\ell \le n, k_1 \le m$, the running time of phase-2 is $O(\ell k_1 n^2 m) = O(n^3 m^2)$. The overall running time of the algorithm is $O(2^k n^3 m^2)$, where $O(2^k)$ is the time required for guessing the coloring of X in an optimal coloring of G.

4.2 Distance to Co-cluster Graphs

Theorem 4. 2-LOAD COLORING is fixed-parameter tractable parameterized by the distance to co-cluster graphs.

Proof. Let (G, X, k_1, k_2) be a 2-LOAD COLORING instance, where $X \subseteq V(G)$ of size k such that G - X is a co-cluster graph. The graph G - X is either an independent set or a connected graph. If G - X is an independent set then X is a vertex cover of G, hence we apply the algorithm of [9] to solve this instance in FPT time.

Hence, we assume that G - X is connected. As G - X is a complement of a cluster graph we can partition the co-cluster G - X into maximal independent

sets I_1, I_2, \ldots, I_ℓ . Note that for any $i, j \in [\ell]$ and $i \neq j$, every vertex of I_i is adjacent to every vertex of I_j . We guess the colors of vertices in X in an optimal 2-coloring of G. This can be done in $O(2^k)$ time. Let $h: X \to \{r, b\}$ be such a coloring. We assume that X is an independent set in G, otherwise we count the number of red and blue edges inside X and adjust the parameters k_1, k_2 and delete all the edges inside X.

For each $i \in [\ell]$, let $G_i = G[X \cup I_i]$ be the subgraph of G induced by the vertices of the independent set I_i and the set X.

The algorithm is similar to the algorithm of distance to cluster described in Theorem 3. It is based on dynamic programming technique, which has two phases. In phase-1, given a graph G_i and non-negative integers q and n_i^r , we find a 2-coloring of G_i that maximizes the number of blue edges with the constriant that there are at least q red edges in G_i and n_i^r red vertices in I_i . In phase-2, for each $t \in [\ell]$, and a non-negative integer p, we find a 2-coloring of $\widehat{G}_t = G[I_1 \cup \ldots \cup I_t \cup X]$ that maximizes the number of blue edges with the constraint that number of red edges is at least p.

Phase-1. We can observe that X is a vertex cover of G_i of size k. For each $i \in [\ell], q \in [|E(G_i)|] \cup \{0\}$ and $n_i^r \in [|I_i|] \cup \{0\}$, let $b[G_i, n_i^r, q]$ be the maximum number of blue edges that can be attained in a 2-coloring g of G_i satisfying the following constraints.

- 1. $g|_X = h$.
- 2. number of red edges in G_i is at least q.
- 3. number of red vertices in I_i is equal to n_i^r .

If the constraint cannot be satisfied, then we let $b[G_i, n_i^r, q] = -\infty$. From the definition of b[,], we can see that $b[G_i, 0, 0]$ gives the number of blue edges in G_i when all vertices of I_i are colored blue. $b[G_i, 0, q] = -\infty$ for q > 0, $b[G_i, n_i^r, 0] = -\infty$ for $n_i^r > 1$. For $v \in G_i - X$, let $r^i(v)$ and $b^i(v)$ denote the number of red and blue vertices in $N(v) \cap X$ respectively. For a given values of i, n_i^r and q, the computation of $b[G_i, n_i^r, q]$ is described as follows.

$$b\Big[G_i, n_i^r, q\Big] = \max\left\{b\Big[G_i - \{v\}, n_i^r - 1, \max\{q - r^i(v), 0\}\Big], b\Big[G_i - \{v\}, n_i^r, q\Big] + b^i(v)\right\}$$

If v is colored red, then we get $r^i(v)$ red edges between v and the neighbors of v in X. If v is colored blue, then we get $b^i(v)$ blue edges between v and neighbors of v in X.

The size of the DP table is at most $O(mn^2)$ and each entry can be computed in O(n) time. Hence the running time of Phase-1 is $O(n^3m)$.

Phase-2. Let $\widehat{G_t}$ be the subgraph of G induced by the independent sets I_1, \dots, I_t and the set X. Let $OPT[t, N_t^r, p]$ be the maximum number of blue edges that can be attained in a 2-coloring f of $\widehat{G_t}$ satisfying the following constrains.

1.
$$f|_X = h$$

2. number of red edges in \widehat{G}_t is at least p.

3. number of red vertices in $\widehat{G_t} - X$ is N_t^r .

If the constraint cannot be satisfied, then we let $OPT[t, N_t^r, p] = -\infty$. From the definition of OPT, we have OPT[0, 0, 0] = 0 and $OPT[0, N_r^t, p] = -\infty$ for p > 0, $OPT[t, 0, p] = -\infty$ for p > 0, For t > 0 we have:

$$OPT[t, N_t^r, p] = \max_{\substack{q = 0, \dots, |E(G_t)| \\ n_t^r = 0, \dots, |I_t|}} \left\{ OPT[t-1, N_t^r - n_t^r, \max\{p - q - n_t^r(N_t^r - n_t^r), 0\}] + b[G_t, n_t^r, q] + (N_t^b - |V(G_t)| + n_t^r)(|V(G_t)| - n_t^r) \right\}$$

Where $N_t^b = |V(\widehat{G_t})| - N_t^r$. If there are q red edges in G_t and n_t^r red vertices in I_t in the coloring f, then we get $b[G_t, n_t^r, q]$ blue edges in G_t . Also, we get $n_t^r(N_t^r - n_t^r)$ red edges and $(N_b^t - |V(G_t)| + n_t^r)(|V(G_t)| - n_t^r)$ blue edges between the sets I_t and $I_1 \cup \ldots, I_{t-1}$ respectively. We consider all possible values for number of red edges in G_t and red vertices in I_t and pick the one that maximizes the OPT.

Observe that (G, X, k_1, k_2) is a YES instance if and only if $OPT[\ell, N_{\ell}^r, k_1] \ge k_2$ for some non-negative integer N_{ℓ}^r . There are $O(\ell n k_1)$ subproblems, each of which can be solved in $O(n^2m)$ time. As $\ell \le n, k_1 \le m$, the overall running time of this algorithm is $O(2^k \ell k_1 n^3 m) = O(2^k n^4 m^2)$.

4.3 Distance to Threshold Graphs

Theorem 5. 2-LOAD COLORING is fixed-parameter tractable parameterized by the distance to threshold graphs.

Proof. Let (G, X, k_1, k_2) be a 2-LOAD COLORING instance, where $X \subseteq V(G)$ of size k such that G - X is a threshold graph. We guess the coloring of X in an optimal 2-coloring of G in $O(2^k)$ time. Let $h: X \to \{r, b\}$ be such a coloring. We assume that X is an independent set in G, if not we count the number of red edges e_r and blue edges e_b inside X and replace the parameters k_1 and k_2 with $k_1 - e_r$ and $k_2 - e_b$ respectively. Then finally we delete all the edges inside X. For a vertex v in G - X, we use $n_X^r(v)$ (resp. $n_X^b(v)$) to denote the number of neighbors of the vertex v in the set X which are colored red (resp. blue). Let v_1, v_2, \ldots, v_ℓ be the ordering of the vertices of the threshold graph G - Xobtained from its construction (i.e., v_i is added before v_{i+1} to the graph G - X).

For $t \in [\ell]$, let $V_t = \{v_1, \ldots, v_t\}$ and G_t be the graph induced by the vertices $V_t \cup X$. Using the definition of threshold graphs, we can see that, for each $t \in [\ell]$, the vertex v_t is either a universal vertex or an isolated vertex in the graph $G_t - X$. We use n_t^r and n_t^b denote the number of red and blue vertices in a 2-coloring of G_t respectively.

Let $OPT[t, n_t^r, p]$ be the maximum number of blue edges that can be obtained in a 2-coloring g of G_t with the constraint that $g|_X = h$ and V_t has n_t^r vertices of color red, G_t has at least p red edges. If the constraint cannot be satisfied, then we let $OPT[t, n_t^r, p] = -\infty$. From the definition of OPT, we have OPT[0, 0, 0] = 0, $OPT[0, n_t^r, p] = -\infty$ for p > 0 and $OPT[t, 0, p] = -\infty$ for p > 0.

For t > 0, we have two cases based on whether v_t is a universal vertex or an isolated vertex in the graph $G_t - X$. If v_t is a universal vertex, then it is adjacent to all vertices of $G_t - X$. Hence we get the following relation.

 $OPT[t, n_t^r, p] =$

$$\max\left\{OPT[t-1, n_t^r - 1, \max\{p-(n_t^r - 1) - n_X^r(v_t), 0\}\right\}, OPT[t-1, n_t^r, p] + (n_t^b - 1 + n_X^b(v_t))\right\}$$

If v_t is colored red, then we get $(n_t^r - 1)$ red edges between v_t and neighbors of v_t in $G_t - X$ and $n_X^r(v_t)$ red edges between v_t and its neighbors in X. If v_t is colored blue, then we get $n_t^b - 1$ blue edges between v_t and neighbors of v_t in $G_t - X$ and $n_X^b(v_t)$ blue edges between v_t and its neighbors in X.

If v_t is an isolated vertex, then it is not adjacent to any vertex of $G_t - X$. Then we get the following relation.

$$OPT[t, n_t^r, p] = \max \left\{ OPT[t-1, n_t^r - 1, \max\{p - n_X^r(v_t), 0\}], OPT[t-1, n_t^r, p] + n_X^b(v_t) \right\}$$

If v_t is colored red, then we get $n_X^r(v_t)$ red edges between v_t and its neighbors in X. If v_t is colored blue, then we get $n_X^b(v_t)$ blue edges between v_t and its neighbors in X.

Observe that (G, X, k_1, k_2) is a YES instance of 2-LOAD COLORING if and only if $OPT[\ell, n_t^r, k_1] \ge k_2$ for some integer n_t^r . There are $O(n^2k_1)$ subproblems, each of which can be solved in O(n) time. As $k_1 \le m$, the overall running time of this algorithm is $O(2^k n^3 m)$.

5 Special Graph Classes

In this section, we show that 2-LOAD COLORING is NP-complete on bipartite graphs and split graphs.

Theorem 6. 2-LOAD COLORING is NP-complete on bipartite graphs.

Proof. We give a reduction from 2-LOAD COLORING on general graphs. Let (G, k_1, k_2) be an instance of 2-LOAD COLORING. Without loss of generality, we assume that $k_2 \ge k_1$. We construct a bipartite graph $H = ((X \cup Z) \cup Y), E)$ as follows. For every vertex $v \in V(G)$, we introduce a vertex $x_v \in X$. For every edge $e \in E(G)$, we introduce a vertex $y_e \in Y$, and if e = uv, then y_e is adjacent to x_u and x_v in H. For every edge $e \in E(G)$, we also introduce additional m^2 vertices $z_e^1, z_e^2, \ldots, z_e^{m^2}$ in Z. For each $e \in E(G)$ and $i \in [m^2]$ introduce an edge between z_e^i and y_e in H. This completes the construction of the bipartite graph H and clearly can be performed in polynomial time. We set $k'_1 = 2k_1 + k_1m^2$, $k'_2 = (m + k_2 - k_1) + (m - k_1)m^2$. We argue that (G, k_1, k_2) is a YES instance

of 2-LOAD COLORING if and only if (H, k'_1, k'_2) is a YES instance of 2-LOAD COLORING.

Forward direction. Let $f: V(G) \to \{r, b\}$ is a 2-coloring of G such that $r_f = k_1$ and $b_f = k_2$. Then we define a coloring $g: V(H) \to \{r, b\}$ of H as follows: $g(x_v) = f(v)$ for all $x_v \in X$. For every edge $e = uv \in E(G)$, if f(u) = f(v) = r, then $g(y_e) = f(u)$, else $g(y_e) = b$. For each $e \in E(G)$ and $i \in [m^2], g(z_e^i) = g(y_e)$. Note that for each red (resp blue) edge e = uv in $G, m^2 + 2$ edges (namely $y_e x_v, y_e x_u, z_e^i x_u$) are red (resp. blue) with respect to g. For each edge $e = uv \in E(G)$ with $f(u) \neq f(v)$ we get $m^2 + 1$ blue edges with respect to g. Therefore, $r_g = (m^2 + 2)k_1 = 2k_1 + m^2k_1$ and $b_g = (m^2 + 2)k_2 + (m - k_1 - k_2)(m^2 + 1) = (m + k_2 - k_1) + (m - k_1)m^2$. Hence (H, k'_1, k'_2) is a YES instance of 2-LOAD COLORING.

Reverse direction. Let $g: V(H) \to \{r, b\}$ be a 2-coloring of H such that $r_g = 2k_1 + k_1m^2$ and $b_g = (m + k_2 - k_1) + (m - k_1)m^2$. We assume that $g(z_e^i) = g(y_e)$ for each $e \in E(G)$ and $i \in [m^2]$, otherwise we recolor the vertices z_e^i with the color of y_e , as it only increases the number of red and blue edges. Now, we argue that the 2-coloring g of H when restricted to the vertices of G gives a 2-coloring f of G such that $r_f = k_1$ and $b_f = k_2$.

We partition the vertices of Y in H into four sets based on their coloring in g as follows.

$$P_r = \{y_e : e = uv, \ g(y_e) = g(x_u) = g(x_v) = r\}$$
$$Q_r = \{y_e : e = uv, \ g(y_e) = r \text{ and } g(x_u) \neq g(x_v)\}$$
$$P_b = \{y_e : e = uv, \ g(y_e) = g(x_u) = g(x_v) = b\}$$
$$Q_b = \{y_e : e = uv, \ g(y_e) = b \text{ and } g(x_u) \neq g(x_v)\}$$

Let p_r , q_r , p_b and q_b denote the sizes of the sets P_r , Q_r , P_b and Q_b respectively. Using this notation, we have $r_g = 2p_r + q_r + m^2(p_r + q_r)$, $b_g = 2p_b + q_b + m^2(p_b + q_b)$. Also, it is easy to see that $p_r + q_r + p_b + q_b = m$. By using above two equations, we get

$$r_g + b_g = p_r + p_b + m + m^3$$

 $p_r + p_b = r_g + b_g - m - m^3 = 2k_1 + k_1m^2 + (m + k_2 - k_1) + (m - k_1)m^2 - m - m^3 = k_1 + k_2$

Claim. $p_r = k_1$ and $p_b = k_2$.

Suppose $p_r = k_1 - \ell$ and $p_b = k_2 + \ell$ for some $\ell \ge 1$. Then

1

$$r_g = 2p_r + q_r + m^2(p_r + q_r) = 2(k_1 - \ell) + q_r + m^2(k_1 - \ell + q_r)$$

Since $r_g = 2k_1 + k_1m^2$, by substituting in the above equation, we get $q_r = \left(\frac{m^2+2}{m^2+1}\right)\ell \geq \ell + 1$. That implies $q_b \leq m - (k_1 - \ell) - (k_2 + \ell) - (\ell + 1) = (m - k_1 - k_2 - \ell - 1)$. Then

$$b_g = 2p_b + q_b + m^2(p_b + q_b)$$

=2(k₂ + l) + (m - k₁ - k₂ - l - 1) + m²(k₂ + l + (m - k₁ - k₂ - l - 1))
<(m + k₂ - k₁) + m²(m - k₁)

This is a contradiction as $b_g = (m+k_2-k_1)+m^2(m-k_1)$. Therefore $p_r = k_1$ and $p_b = k_2$. Define $f : V(G) \to \{r, b\}$ as $f(u) = g(x_u)$. Since $p_r = k_1$ and $p_b = k_2$, we get $r_f \ge k_1$ and $b_f \ge k_2$. Hence, by restricting the coloring g of Hto the vertices of G, we get a coloring f of G with at least k_1 red edges and k_2 blues edges. Therefore (G, k_1, k_2) is a YES instance of 2-LOAD COLORING.

Theorem 7. 2-LOAD COLORING is NP-complete on split graphs.

Proof. The proof is similar to the case of bipartite graphs except that we add all possible edges between the vertices of Y to make it a clique in the graph H. We set $k'_1 = 2k_1 + m^2k_1 + \binom{k_1}{2}$ and $k'_2 = (m + k_2 - k_1) + (m - k_1)m^2 + \binom{m-k_1}{2}$.

6 Conclusion

In this paper, we have studied the parameterized complexity of 2-LOAD COLOR-ING. We showed that 2-LOAD COLORING (a) cannot be solved in time $f(w)n^{o(w)}$, unless ETH fails, (b) can be solved in time $n^{O(w)}$, where w is the clique-width of the graph. We have shown that the problem is FPT parameterized by (a) distance to cluster graphs (b) distance to co-cluster graphs and (c) distance to threshold graphs. We also studied the complexity of the problem on special classes of graphs. We have shown that the problem is NP-complete on bipartite graphs and split graphs. We conclude with the following open questions.

- As the problem admits a linear kernel with respect to the solution size, it is natural to study the kernelization complexity of the problem with respect to structural graph parameters.
- Finally, we do not know if 2-LOAD COLORING can be solved in polynomial time on interval graphs or permutation graphs.

References

- Ahuja, N., Baltz, A., Doerr, B., Přívětivý, A., Srivastav, A.: On the minimum load coloring problem. Journal of Discrete Algorithms 5(3), 533–545 (2007)
- 2. Barbero, F., Gutin, G., Jones, M., Sheng, B.: Parameterized and approximation algorithms for the load coloring problem. Algorithmica 79(1), 211–229 (2017)
- Cai, L.: Fixed-parameter tractability of graph modification problems for hereditary properties. Information Processing Letters 58(4), 171–176 (1996)
- 4. Courcelle, B., Olariu, S.: Upper bounds to the clique width of graphs. Discrete Applied Mathematics 101(1), 77–114 (2000)
- Cygan, M., Fomin, F.V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized Algorithms. Springer (2015)
- 6. Diestel, R.: Graph theory. 2005. Grad. Texts in Math (2005)
- Fomin, F.V., Golovach, P.A., Lokshtanov, D., Saurabh, S.: Algorithmic lower bounds for problems parameterized by clique-width. In: Proceedings of the twentyfirst annual ACM-SIAM symposium on Discrete Algorithms. pp. 493–502. SIAM (2010)

- 8. Guo, J.: A more effective linear kernelization for cluster editing. Theoretical Computer Science 410(8-10), 718–726 (2009)
- 9. Gutin, G., Jones, M.: Parameterized algorithms for load coloring problem. Information Processing Letters 114(8), 446–449 (2014)
- Mahadev, N.V., Peled, U.N.: Threshold graphs and related topics, vol. 56. Elsevier (1995)