Finite-Word Hyperlanguages

Borzoo Bonakdarpour^a, Sarai Sheinvald^b

^aDepartment of Computer Science and Engineering, Michigan State University, USA ^bDepartment of Software Engineering, ORT Braude College, Israel

Abstract

Formal languages are in the core of models of computation and their behavior. A rich family of models for many classes of languages have been widely studied. *Hyperproperties* lift conventional trace-based languages from a set of execution traces to a set of sets of executions. Hyperproperties have been shown to be a powerful formalism for expressing and reasoning about information-flow security policies and important properties of cyber-physical systems. Although there is an extensive body of work on formal-language representation of trace properties, we currently lack such a general characterization for hyperproperties.

We introduce *hyperlanguages* over finite words and models for expressing them. Essentially, these models express multiple words by using assignments to quantified *word variables*. Relying on the standard models for regular languages, we propose *hyperregular expressions* and *finite-word hyperautomata* (*NFH*), for modeling the class of *regular hyperlanguages*. We demonstrate the ability of regular hyperlanguages to express hyperproperties for finite traces. We explore the closure properties and the complexity of the fundamental decision problems such as nonemptiness, universality, membership, and containment for various fragments of NFH.

1. Introduction

Formal languages, along with the models that express them, are in the core of modeling, specification, and verification of computing systems. Execution traces are formally described as words, and various families of automata are used for modeling systems of different types. *Regular languages* are a classic formalism for finite traces and when the traces are infinite, ω -regular languages are used.

There are well-known connections between specification logics and formal languages. For example, LTL [33] formulas can be translated to ω -regular expressions, and CTL* [17] formulas can be expressed using tree automata. Accordingly, many verification techniques that exploit these relations have been developed. For instance, in the automata-theoretic approach to verification [36, 37], the model-checking problem

Email addresses: borzoo@msu.edu (Borzoo Bonakdarpour), sarai@braude.ac.il (Sarai Sheinvald)

is reduced to checking the nonemptiness of the product automaton of the model and the complement of the specification.

Hyperproperties [14] generalize the traditional trace properties [4] to *system properties*, i.e., a set of sets of traces. A hyperproperty prescribes how the system should behave in its entirety and not just based on its individual executions. Hyperproperties have been shown to be a powerful tool for expressing and reasoning about information-flow security policies [14] and important properties of cyber-physical systems [38] such as *sensitivity* and *robustness*, as well as consistency conditions in distributed computing such as *linearizability* [9]. While different types of logics have been suggested for expressing hyperproperties, their formal-language counterparts and the models that express them are currently missing.

In this paper, we establish a formal-language theoretical framework for *hyperlanguages*, that are sets of sets of words, which we term *hyperwords*. Our framework is based on an underlying standard automata model for formal languages, augmented with quantified *word variables* that are assigned words from a set of words in the hyperlanguage. This formalism is in line with logics for hyperproperties (e.g., HyperLTL [13] and HyperPCTL [2, 1]). These logics express the behavior of infinite trace systems. However, a basic formal model for expressing general hyperproperties for finite words has not been defined yet.

To begin with the basics, we focus this paper on a regular type of hyperlanguages of sets consisting of finite words, which we call *regular hyperlanguages*. The models we introduce and study are based on the standard models for regular languages, namely regular expressions and finite-word automata.

1.1. Motivation and Applications

Hyperlanguages based on finite words have many practical applications. Let us first explain the idea of hyperlanguages with two examples.

Example 1. Consider the following *hyperregular expression* (HRE) over the alphabet $\{a\}$.

$$r_1 = \forall x. \exists y. \underbrace{\left(\{a_x, a_y\}^* \{\#_x, a_y\}^*\right)}_{\hat{r}_1}$$

The HRE r_1 uses two word variables x and y, which are assigned words from a hyperword. The HRE r_1 contains an underlying regular expression \hat{r}_1 , whose alphabet is $(\{a\} \cup \{\#\})^{\{x,y\}}$, and whose (regular) language describes different word assignments to x and y, where # is used for padding at the end if the words assigned to x and y are of different lengths. In a word in the language of \hat{r}_1 , the *i*'th letter describes both *i*'th letters in the words assigned to x and y. For example, the word $\{a_x, a_y\}\{a_x, a_y\}\{\#_x, a_y\}$ describes the assignment $x \mapsto aa, y \mapsto aaa$. The regular expression \hat{r}_1 requires that the word assigned to y be longer than the word assigned to x. The quantification condition $\forall x.\exists y$ of r_1 requires that for every word in a hyperword S in the hyperlanguage of r_1 , there exists a longer word in S. This holds iff S contains infinitely many words. Therefore, the hyperlanguage of r_1 is the set of all infinite hyperwords over $\{a\}$.

Example 2. Path planning objectives for robotic systems often stipulate the *existence* of one or more *finite* paths that stand out from *all* other paths. For example, robotics applications are often concerned with finding the shortest path that reaches a goal g, starting from an initial location i. The shortest path requirement can be expressed by the following HRE over an alphabet Σ :

$$r_2 = \exists x. \forall y. \{i_x, i_y\} \{\bar{g}_x, \bar{g}_y\}^* (\{g_x, \bar{g}_y\} \mid \{g_x, g_y\}) \{\#_x, \$_y\}^*$$

where $\bar{g} \in \Sigma - \{g\}$ and $\$ \in \Sigma$. That is, there exists a path x that is shorter than any other path y in reaching g.

Another interesting application in robotics is in *adversarial* settings, where some robots may interfere (e.g., act as moving obstacles) with a set of controllable robots. In this scenario, given any behavior of the adversarial robots, the controllable robots should be able to achieve their operation objectives. This specification is in general of the following form:

$$r_3 = \underbrace{\forall x_1.\forall x_2\ldots\forall x_n}_{\text{advarsaries}} \cdot \underbrace{\exists y_1.\exists y_2\ldots\exists y_m}_{\text{controllable}} \hat{r}$$

where words $x_1 \cdots x_n$ express the behavior of the adversaries, words $y_1 \cdots y_m$ describe the behavior of the controllable robots and regular expression \hat{r} specifies the control objectives.

1.2. Contributions

Although there is an ongoing line of research on model-checking hyperproperties [27, 5, 15], the work on finite-trace hyperproperties is limited to [18], where the authors construct a finite-word representation for the class of regular k-safety hyperproperties. We make the following contributions:

- Introduce regular hyperlanguages and HREs, and demonstrate the ability of HREs to express important information-flow security policies such as different variations of *noninterference* [29] and *observational determinism* [39].
- Present nondeterministic finite-word hyperautomata (NFH), an automata-based model for expressing regular hyperlanguages.
- Conduct a comprehensive study of the properties of regular hyperlanguages (see Table 1):
 - We show that regular hyperlanguages are *closed* under union, intersection, and complementation.
 - We consider the *nonemptiness* problem for NFH:
 - * We prove that the nonemptiness problem is in general undecidable for NFH.
 - * However, for the alternation-free fragments (which only allow one type of quantifier), as well as for the ∃∀ fragment (in which the quantification condition is limited to a sequence of ∃ quantifiers followed by a sequence of ∀ quantifiers), nonemptiness is decidable.

Property	Result	
Closure	Complementation, Union, Intersection (Theorems 1, 2, 3)	
Nonemptiness	A33	Undecidable (Theorem 4)
	∃* / ∀* / ∃** / ∀**	NL-complete (Theorems 5, 10)
	$\exists^* \forall^*$	PSPACE-complete (Theorem 6)
	∃*∀**	EXPSPACE-complete (Theorem 11)
Bounded Nonemptiness	NFH	PSPACE-complete (Theorem 9)
Universality	AAE	Undecidable (Theorem 12)
	\exists^* / \forall^*	PSPACE-complete (Theorem 12)
	$\forall^* \exists^*$	EXPSPACE (Theorem 12)
Finite membership	NFH	PSPACE (Theorem 13)
	$O(\log(k)) \ \forall$	NP-complete (Theorem 13)
Regular membership	Decidable (Theorem 14)	
Containment	NFH	Undecidable (Theorem 15)
	$\exists_* \subset A_* \setminus A_* \subset \exists_*$	PSPACE-complete (Theorem 16)
	$\exists^* \forall^* \subseteq \forall^* \exists^*$	EXPSPACE (Theorem 16)

Table 1: Summary of results on properties of hyperregular languages.

- * As another positive result in the area of nonemptiness, we show that the *bounded nonemptiness problem*, in which we decide whether an NFH accepts a hyperword of bounded size, is PSPACE-complete.
- * We consider the construction of HRE and NFH with *wild card letters*, which allow expressing the assignment to only a subset of the variables, by assigning a wild card letter to the rest of the variables. We show that adding wild cards does not alter the complexity of the nonemptiness for the alternation-free fragments, while it does increase the complexity of this problem for the ∃∀ fragment.
- * We describe a semi-algorithm for deciding the nonemptiness of NFH with a $\forall \exists$ quantification condition. The procedure begins with the largest potential hyperword, and iteratively prunes it in a consistent way in case it is not accepted. Since the problem is undecidable, there are inputs for which our semi-algorithm does not halt. However, in case it does halt, it is guaranteed to return a correct answer. Since $\forall \exists$ is a useful fragment, our procedure can be a useful tool.
- We study the *universality*, *membership* and *containment* problems. These results are aligned with the complexity of HyperLTL model checking for tree-shaped and general Kripke structures [5]. This shows that the complexity results in [5] mainly stem from the nature of quantification over finite words and depend on neither the full power of the temporal operators nor the infinite nature of HyperLTL semantics.

Comparison to the conference version. This article substantially extends the results of our original conference submission [10] by the following new contributions.

• An upper and lower bound of the bounded nonempitness problem.

- Upper and lower bounds for the nonemptiness problem for the various fragments of NFH in the presence of wild-card letters.
- A semi-algorithm for deciding the nonemptiness for the $\forall \exists$ fragment.
- A detailed discussion on related work.

In summary, the material in Sections 6.2, 6.4 6.3, and 8 is all new. Finally, all proof sketches are now extended to revised and detailed full proofs.

1.3. Organization

The rest of the paper is organized as follows. Preliminary concepts are presented in Section 2. We introduce the notion of HRE and NFH in Sections 3 and 4, while their properties and our complexity results are studied in Sections 5, 6, and 7. Related work is discussed in Section 8. Finally, we make concluding remarks and discuss future work in Section 9.

2. Preliminaries

An *alphabet* is a nonempty finite set Σ of *letters*. A *word* over Σ is a finite sequence of letters from Σ . The *empty word* is denoted by ϵ , and the set of all words is denoted by Σ^* . A *language* is a subset of Σ^* . We assume that the reader is familiar with the syntax and semantics of regular expressions (RE). We use the standard notations $\{\cdot, |, *\}$ for concatenation, union, and Kleene star, respectively, and denote the language of an RE r by $\mathcal{L}(r)$. A language L is *regular* if there exists an RE r such that $\mathcal{L}(r) = L$.

Definition 1. A nondeterministic finite-word automaton (NFA) is a tuple $A = \langle \Sigma, Q, Q_0, \delta, F \rangle$, where Σ is an alphabet, Q is a nonempty finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $F \subseteq Q$ is a set of accepting states, and $\delta \subseteq Q \times \Sigma \times Q$ is a transition relation.

Given a word $w = \sigma_1 \sigma_2 \cdots \sigma_n$ over Σ , a run of A on w is a sequence of states (q_0, q_1, \ldots, q_n) , such that $q_0 \in Q_0$, and for every $0 < i \le n$, it holds that $(q_{i-1}, \sigma_i, q_i) \in \delta$. The run is *accepting* if $q_n \in F$. We say that A *accepts* w if there exists an accepting run of A on w. The *language* of A, denoted $\mathcal{L}(A)$, is the set of all words that A accepts. It is well-known that a language L is regular iff there exists an NFA A such that $\mathcal{L}(A) = L$.

3. Hyperregular Expressions

Definition 2. A hyperword over Σ is a set of words over Σ and a hyperlanguage over Σ is a set of hyperwords over Σ .

Before formally defining hyperregular expressions, we explain the idea behind them. A hyperregular expression (HRE) over Σ uses a set of word variables $X = \{x_1, x_2, \ldots, x_k\}$. When expressing a hyperword S, these variables are assigned words from S. An HRE r is composed of a quantification condition α over X, and an underlying RE \hat{r} , which represents word assignments to X. An HRE r defines a hyperlanguage $\mathfrak{L}(r)$. The condition α defines the assignments that should be in $\mathcal{L}(\hat{r})$. For example, $\alpha = \exists x_1. \forall x_2$ requires that there exists a word $w_1 \in S$ (assigned to x_1), such that for every word $w_2 \in S$ (assigned to x_2), the word that represents the assignment $x_1 \mapsto w_1, x_2 \mapsto w_2$, is in $\mathcal{L}(\hat{r})$. The hyperword S is in $\mathfrak{L}(r)$ iff S meets these conditions.

We represent an assignment $v: X \to S$ as a *word assignment* w_v , which is a word over the alphabet $(\Sigma \cup \{\#\})^X$ (that is, assignments from X to $\Sigma \cup \{\#\}$), where the *i*'th letter of w_v represents the *k i*'th letters of the words $v(x_1), \ldots, v(x_k)$ (in case that the words are not of equal length, we "pad" the end of the shorter words with # symbols). We represent these *k i*'th letters as an assignment denoted $\{\sigma_{1x_1}, \sigma_{2x_2}, \ldots, \sigma_{kx_k}\}$, where x_j is assigned σ_j . For example, the assignment $v(x_1) = aa$ and $v(x_2) = abb$ is represented by the word assignment $w_v = \{a_{x_1}, a_{x_2}\}\{a_{x_1}, b_{x_2}\}\{\#_{x_1}, b_{x_2}\}$.

Definition 3. A hyperregular expression is a tuple $r = \langle X, \Sigma, \alpha, \hat{r} \rangle$, where $\alpha = \mathbb{Q}_1 x_1 \cdots \mathbb{Q}_k x_k$, where $\mathbb{Q}_i \in \{\exists, \forall\}$ for every $i \in [1, k]$, and where \hat{r} is an RE over $\hat{\Sigma} = (\Sigma \cup \{\#\})^X$.

Let S be a hyperword and let $v: X \to S$ be an assignment of the word variables of r to words in S. We denote by $v[x \mapsto w]$ the assignment obtained from v by assigning the word $w \in S$ to $x \in X$. We represent v by w_v . We now define the membership condition of a hyperword S in the hyperlanguage of r. We first define a relation \vdash for S, \hat{r} , a quantification condition α , and an assignment $v: X \to S$, as follows.

- For $\alpha = \epsilon$, define $S \vdash_v (\alpha, \hat{r})$ if $w_v \in \mathcal{L}(\hat{r})$.
- For $\alpha = \exists x.\alpha'$, define $S \vdash_v (\alpha, \hat{r})$ if there exists $w \in S$ s.t. $S \vdash_{v[x \mapsto w]} (\alpha', \hat{r})$.
- For $\alpha = \forall x.\alpha'$, define $S \vdash_v (\alpha, \hat{r})$ if $S \vdash_{v[x \mapsto w]} (\alpha', \hat{r})$ for every $w \in S$.¹

Since all variables are under the scope of α , membership is independent of v, and so if $S \vdash (\alpha, \hat{r})$, we denote $S \in \mathfrak{L}(r)$. The hyperlanguage of r is $\mathfrak{L}(r) = \{S \mid S \in \mathfrak{L}(r)\}$.

Definition 4. We call a hyperlanguage \mathfrak{L} a *regular hyperlanguage* if there exists an HRE r such that $\mathfrak{L}(r) = \mathfrak{L}$.

Application of HRE in Information-flow Security

Noninterference [29] requires high-secret commands to be removable without affecting observations of users holding low clearances:

$$\varphi_{\mathsf{n}\mathsf{i}} = \forall x. \exists y \{l_x, l\lambda_y\}^*,$$

where *l* denotes a low state and $l\lambda$ denotes a low state such that all high commands are replaced by a dummy value λ .

¹In case that α begins with \forall , membership holds vacuously with an empty hyperword. We restrict the discussion to nonempty hyperwords.

Observational determinism [39] requires that if two executions of a system start with low-security-equivalent events, they should remain low equivalent:

$$\varphi_{\mathsf{od}} = \forall x. \forall y. \left(\{l_x, l_y\}^+ \mid \{\bar{l}_x, \bar{l}_y\} \{\$_x, \$_y\}^* \mid \{l_x, \bar{l}_y\} \{\$_x, \$_y\}^* \mid \{\bar{l}_x, l_y\} \{\$_x, \$_y\}^* \right)$$

where l denotes a low event, $\overline{l} \in \Sigma \setminus \{l\}$, and $\$ \in \Sigma$. We note that similar policies such as *Boudol and Castellani's noninterference* [28] can be formulated in the same fashion.²

Generalized noninterference (GNI) [32] allows nondeterminism in the low-observable behavior, but requires that low-security outputs may not be altered by the injection of high-security inputs:

$$\varphi_{\mathsf{gni}} = \forall x. \forall y. \exists z. \left(\{h_x, l_y, hl_z\} \mid \{\bar{h}_x, l_y, \bar{h}l_z\} \mid \{h_x, \bar{l}_y, h\bar{l}_z\} \mid \{\bar{h}_x, \bar{l}_y, \bar{h}\bar{l}_z\} \right)^*$$

where h denotes the high-security input, l denotes the low-security output, $\bar{l} \in \Sigma \setminus \{l\}$, and $\bar{h} \in \Sigma \setminus \{h\}$.

Declassification [34] relaxes noninterference by allowing leaking information when necessary. Some programs must reveal secret information to fulfill functional requirements. For example, a password checker must reveal whether the entered password is correct or not:

$$\varphi_{\mathsf{dc}} = \forall x. \forall y. \{li_x, li_y\} \{pw_x, pw_y\} \{lo_x, lo_y\}^+$$

where li denotes low-input state, pw denotes that the password is correct, and lo denotes low-output states. We note that for brevity, φ_{dc} does not include behaviors where the first two events are not low or, in the second event, the password is not valid.

Termination-sensitive noninterference requires that for two executions that start from low-observable states, information leaks are not permitted by the termination behavior of the program (here, l denotes a low state and $\$ \in \Sigma$):

$$\begin{split} \varphi_{\mathsf{tsni}} &= \forall x. \forall y. \Big(\{l_x, l_y\} \{\$_x, \$_y\}^* \{l_x, l_y\} \mid \{\bar{l}_x, \bar{l}_y\} \{\$_x, \$_y\}^* \mid \\ & \{l_x, \bar{l}_y\} \{\$_x, \$_y\}^* \mid \{\bar{l}_x, l_y\} \{\$_x, \$_y\}^* \Big) \end{split}$$

4. Nondeterminsitic Finite-Word Hyperautomata

We now present a model for regular hyperlanguages, namely *finite-word hyperautomata*. A hyperautomaton is composed of a set X of word variables, a quantification condition, and an underlying finite-word automaton that accepts representations of assignments to X.

Definition 5. A nondeterministic finite-word hyperautomaton (NFH) is a tuple $\mathcal{A} = \langle \Sigma, X, Q, Q_0, F, \delta, \alpha \rangle$, where Σ, X and α are as in Definition 3, and where $\langle \hat{\Sigma}, Q, Q_0, F, \delta \rangle$ forms an underlying NFA over $\hat{\Sigma} = (\Sigma \cup \{\#\})^X$.

²This policy states that every two executions that start from bisimilar states (in terms of memory low-observability), should remain bisimilarly low-observable.

$$\begin{array}{c} \{a_x, a_y\}, \\ \{b_x, b_y\} \\ \forall x \forall y \end{array} \overbrace{\{b_x, \#_y\}} \left\{ \begin{array}{c} \{\#_x, b_y\} \\ \{b_x, \#_y\} \\ \{b_x, \#_y\} \end{array} \right\} \\ fightharpoonup \\ \{b_x, \#_y\} \end{array} \overbrace{\{b_x, \#_y\}} \left\{ \begin{array}{c} \{a_x, a_y\} \\ \{\#_x, a_y\} \\ \{\#_y\} \\ \{\#_y\}$$

Figure 1: The NFH A_1 (left) and A_2 (right).

The acceptance condition for NFH, as for HRE, is defined with respect to a hyperword S, the NFH \mathcal{A} , the quantification condition α , and an assignment $v : X \to S$. For the base case of $\alpha = \epsilon$, we define $S \vdash_v (\alpha, \mathcal{A})$ if $\hat{\mathcal{A}}$ accepts w_v . The cases where α is of the type $\exists x.\alpha'$ and $\forall x.\alpha'$ are defined similarly as for HRE, and if $S \vdash (\alpha, \mathcal{A})$, we say that \mathcal{A} accepts S.

Definition 6. Let \mathcal{A} be an NFH. The *hyperlanguage* of \mathcal{A} , denoted $\mathfrak{L}(\mathcal{A})$, is the set of all hyperwords that \mathcal{A} accepts.

Example 3. Consider the NFH A_1 in Figure 1 (left), whose alphabet is $\Sigma = \{a, b\}$, over two word variables x and y. The NFH A_1 contains an underlying standard NFA \hat{A}_1 . For two words w_1, w_2 that are assigned to x and y, respectively, \hat{A}_1 requires that (1) w_1, w_2 agree on their a (and, consequently, on their b) positions, and (2) once one of the words has ended (denoted by #), the other must only contain b letters. Since the quantification condition of A_1 is $\forall x_1.\forall x_2$, in a hyperword S that is accepted by A_1 , every two words agree on their a positions. As a result, all the words in S must agree on their a positions. The hyperlanguage of A_1 is then all hyperwords in which all words agree on their a positions.

Example 4. The NFH A_2 of Figure 1 (right) depicts the translation of the HRE of Example 1 to an NFH.

Since regular expressions are equivalent to NFA, we can translate the underlying regular expression \hat{r} of an HRE r to an equivalent NFA, and vice versa – translate the underlying NFA \hat{A} of an NFH A to a regular expression. It is then easy to see that every HRE has an equivalent NFH over the same set of variables with the same quantification condition.

We consider several fragments of NFH, which limit the structure of the quantification condition α . HRE_{\forall} is the fragment in which α contains only \forall quantifiers, and similarly, in HRE_{\exists}, α contains only \exists quantifiers. In the fragment HRE_{$\exists\forall$}, α is of the form $\exists x_1 \cdots \exists x_i \forall x_{i+1} \cdots \forall x_k$.

4.1. Additional Terms and Notations

We present several terms and notations which we use throughout the paper. Recall that we represent an assignment $v : X \to S$ as a word assignment w_v . Conversely, a word w over $(\Sigma \cup \{\#\})^X$ represents an assignment $v_w : X \to \Sigma^*$, where $v_w(x_i)$ is formed by concatenating the letters of Σ that are assigned to x_i in the letters of w. We denote the set of all such words $\{v_w(x_1), \ldots, v_w(x_k)\}$ by S(w). Since we only allow padding at the end of a word, if a padding occurs in the middle of w, then w does not represent a legal assignment. Notice that this occurs iff w contains two consecutive letters $w_i w_{i+1}$ such that $w_i(x) = \#$ and $w_{i+1}(x) \neq \#$ for some $x \in X$. We call w*legal* if v_w represents a legal assignment from X to Σ^* .

Consider a function $g : A \to B$ where A, B are some sets. The *range* of g, denoted range(g) is the set $\{g(a)|a \in A\}$.

A sequence of g is a function $g' : A \to B$ such that $\operatorname{range}(g') \subseteq \operatorname{range}(g)$. A permutation of g is a function $g' : A \to B$ such that $\operatorname{range}(g') = \operatorname{range}(g)$. We extend the notions of sequences and permutations to word assignments. Let w be a word over $\hat{\Sigma}$. A sequence of w is a word w' such that $S(w') \subseteq S(w)$, and a permutation of w is a word w' such that S(w') = S(w).

Throughout the paper, when we use a general NFH A, we assume that its ingredients are as in Definition 5.

5. Closure Properties of Regular Hyperlanguages

We now consider closure properties of regular hyperlanguages. We show, via constructions on NFH, that regular hyperlanguages are closed under all the Boolean operations.

Theorem 1. Regular hyperlanguages are closed under complementation.

Proof. Let \mathcal{A} be an NFH. The NFA $\hat{\mathcal{A}}$ can be complemented with respect to its language over $\hat{\Sigma}$ to an NFA $\overline{\hat{\mathcal{A}}}$. Then, for every assignment $v : X \to S$, it holds that $\hat{\mathcal{A}}$ accepts \mathbf{w}_v iff $\overline{\hat{\mathcal{A}}}$ does not accept \mathbf{w}_v . Let $\overline{\alpha}$ be the quantification condition obtained from α by replacing every \exists with \forall and vice versa. We can prove by induction on α that $\overline{\mathcal{A}}$, the NFH whose underlying NFA is $\overline{\hat{\mathcal{A}}}$, and whose quantification condition is $\overline{\alpha}$, accepts $\mathfrak{L}(\overline{\mathcal{A}})$. The size of $\overline{\mathcal{A}}$ is exponential in $|\hat{\mathcal{A}}|$, due to the complementation construction for $\hat{\mathcal{A}}$ and complementing the set of transitions in δ .

Theorem 2. Regular hyperlanguages are closed under union.

Proof. let $\mathcal{A}_1 = \langle \Sigma, X, Q, Q_0, \delta_1, F_1, \alpha_1 \rangle$ and $\mathcal{A}_2 = \langle \Sigma, Y, P, P_0, \delta_2, F_2, \alpha_2 \rangle$ be two NFH with |X| = k and |Y| = k' variables, respectively.

We construct an NFH $\mathcal{A}_{\cup} = \langle \Sigma, X \cup Y, Q \cup P \cup \{p_1, p_2\}, Q_0 \cup P_0, \delta, F_1 \cup F_2 \cup \{p_1, p_2\}, \alpha \rangle$, where $\alpha = \alpha_1 \alpha_2$ (that is, we concatenate the two quantification conditions), and where δ is defined as follows.

- For every $(q_1 \xrightarrow{f} q_2) \in \delta_1$ we set $(q_1 \xrightarrow{f \cup g} q_2) \in \delta$ for every $g \in (\Sigma \cup \{\#\})^Y$.
- For every $(q_1 \xrightarrow{f} q_2) \in \delta_2$ we set $(q_1 \xrightarrow{f \cup g} q_2) \in \delta$ for every $g \in (\Sigma \cup \{\#\})^X$.
- For every $q \in F_1$, we set $(q \xrightarrow{\{\#\}^X \cup g} p_1), (p_1 \xrightarrow{\{\#\}^X \cup g} p_1) \in \delta$ for every $g \in (\Sigma \cup \{\#\})^Y$.
- For every $q \in F_2$, we set $(q \xrightarrow{g \cup \{\#\}^Y} p_2), (p_2 \xrightarrow{g \cup \{\#\}^Y} p_2) \in \delta$ for every $g \in (\Sigma \cup \{\#\})^X$.

Let S be a hyperword. For every $v : (X \cup Y) \to S$, it holds that if $w_{v|_X} \in \mathcal{L}(\hat{\mathcal{A}}_1)$, then $w_v \in \mathcal{L}(\hat{\mathcal{A}}_{\cup})$. Indeed, according to our construction, every word assigned to the Y variables is accepted in the \mathcal{A}_1 component of the construction, and so it satisfies both types of quantifiers. A similar argument holds for $v|_Y$ and \mathcal{A}_2 .

Also, according to our construction, for every $v : (X \cup Y) \to S$, if $w_v \in \mathcal{L}(\hat{\mathcal{A}}_{\cup})$, then either $w_{v|_X} \in \mathcal{L}(\hat{\mathcal{A}}_1)$, or $w_{v|_Y} \in \mathcal{L}(\hat{\mathcal{A}}_2)$. As a conclusion, we have that $\mathfrak{L}(\mathcal{A}_{\cup}) = \mathfrak{L}(\mathcal{A}_1) \cup \mathfrak{L}(\mathcal{A}_2)$.

The state space of \mathcal{A}_{\cup} is linear in the state spaces of $\mathcal{A}_1, \mathcal{A}_2$. However, the size of the alphabet of \mathcal{A}_{\cup} may be exponentially larger than that of \mathcal{A}_1 and \mathcal{A}_2 , since we augment each letter with all functions from Y to $\Sigma \cup \{\#\}$ (in \mathcal{A}_1) and from X to $\Sigma \cup \{\#\}$ (in \mathcal{A}_2).

Theorem 3. Regular hyperlanguages are closed under intersection.

Proof. The proof follows from the closure of regular hyperlanguages under union and complementation. However, we also offer a direct translation, which avoids the need to complement.

let $\mathcal{A}_1 = \langle \Sigma, X, Q, Q_0, \delta_1, F_1, \alpha_1 \rangle$ and $\mathcal{A}_2 = \langle \Sigma, Y, P, P_0, \delta_2, F_2, \alpha_2 \rangle$ be two NFH with |X| = k and |Y| = k' variables, respectively.

We construct an NFH $\mathcal{A}_{\cap} = \langle \Sigma, X \cup Y, (Q \cup \{q\}) \times (P \cup \{p\}), (Q_0 \times P_0), \delta, (F_1 \cup \{q\}) \times (F_2 \cup \{p\}), \alpha_1 \alpha_2 \rangle$, where δ is defined as follows.

• For every $(q_1 \xrightarrow{f} q_2) \in \delta_1$ and every $(p_1 \xrightarrow{g} p_2) \in \delta_2$, we have

$$\left((q_1, p_1) \xrightarrow{f \cup g} (q_2, p_2)\right) \in \delta$$

• For every $q_1 \in F_1, (p_1 \xrightarrow{g} p_2) \in \delta_2$ we have

$$\left((q_1, p_1) \xrightarrow{\{\#\}^X \cup g} (q, p_2)\right), \left((q, p_1) \xrightarrow{\{\#\}^k \cup g} (q, p_2)\right) \in \delta$$

• For every $(q_1 \xrightarrow{f} q_2) \in \delta_1$ and $p_1 \in F_2$, we have

$$\left((q_1, p_1) \xrightarrow{f \cup \{\#\}^Y} (q_2, p) \right), \left((q_1, p) \xrightarrow{f \cup \{\#\}^Y} (q_2, p) \right) \in \delta$$

Intuitively, the role of q, p is to keep reading $\{\#\}^X$ and $\{\#\}^Y$ after the word read by $\hat{\mathcal{A}}_1$ or $\hat{\mathcal{A}}_2$, respectively, has ended.

The NFH $\hat{\mathcal{A}}_{\cap}$ simultaneously reads two word assignments that are read along $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$, and accepts iff both word assignments are accepted. The correctness follows from the fact that for $v : (X \cup Y) \to S$, we have that w_v is accepted by $\hat{\mathcal{A}}$ iff $w_{v|_X}$ and $w_{v|_Y}$ are accepted by $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$, respectively. This construction is polynomial in the sizes of \mathcal{A}_1 and \mathcal{A}_2 .

6. Nonemptiness of NFH.

The *nonemptiness problem* is to decide, given an NFH \mathcal{A} , whether $\mathfrak{L}(\mathcal{A}) = \emptyset$. The complexity of the nonemptiness problem affects the complexity of various other decision problems, such as universality and containment. In this section, we extensively study various versions of this problem for various fragments of NFH. First, we show that the problem for general NFH is undecidable. Then, we show that nonemptiness is decidable for various fragments of NFH, with varying complexities.

We then study the *bounded nonemptiness* problem, in which we ask whether an NFH accepts a hyperword of bounded size.

Finally, we study the nonemptiness problem in the presence of *wild-card letters*, which represent free assignments to a variable. Wild-card letters can exponentially decrease the number of transitions of an NFH. We show that for the alternation-free fragments of NFH, wild-card letters do not increase the complexity of the nonemptiness problem, while for the fragment of NFH_{$\exists\forall$}, the smaller representation comes with an exponential blow-up in complexity.

6.1. General Nonemptiness Results

We begin with the nonemptiness problem for general NFH.

Theorem 4. The nonemptiness problem for NFH is undecidable.

Proof. In [19], a reduction from the *Post correspondence problem* is used for proving the undecidability of HyperLTL satisfiability. We mimic the proof ideas of [19] to show that the nonemptiness problem for NFH is, in general, undecidable. A PCP instance is a collection C of dominoes of the form:

$$\left\{ \left[\frac{u_1}{v_1}\right], \left[\frac{u_2}{v_2}\right], \dots, \left[\frac{u_k}{v_k}\right] \right\}$$

where for all $i \in [1, k]$, we have $v_i, u_i \in \{a, b\}^*$. The problem is to decide whether there exists a finite sequence of the dominoes of the form

$$\left[\frac{u_{i_1}}{v_{i_1}}\right] \left[\frac{u_{i_2}}{v_{i_2}}\right] \cdots \left[\frac{u_{i_m}}{v_{i_m}}\right]$$

where each index i_j is in [1, k], such that the upper and lower finite strings of the dominoes are equal, i.e.,

$$u_{i_1}u_{i_2}\cdots u_{i_m}=v_{i_1}v_{i_2}\cdots v_{i_m}$$

For example, if the set of dominoes is

$$C_{\mathsf{exmp}} = \left\{ \left[\frac{ab}{b} \right], \left[\frac{ba}{a} \right], \left[\frac{a}{aba} \right] \right\}$$

Then, a possible solution is the following sequence of dominoes from C_{exmp} :

$$\operatorname{sol} = \left[\frac{a}{aba}\right] \left[\frac{ba}{a}\right] \left[\frac{ab}{b}\right].$$

Given an instance C of PCP, we encode a solution as a word w_{sol} over the following alphabet:

$$\Sigma = \left\{ \frac{\sigma}{\sigma'} \mid \sigma, \sigma' \in \{a, b, \dot{a}, \dot{b}, \$\} \right\}.$$

Intuitively, $\dot{\sigma}$ marks the beginning of a new domino, and \$ marks the end of a sequence of the upper or lower parts of the dominoes sequence.

We note that w_{sol} encodes a legal solution iff the following conditions are met:

- 1. For every $\frac{\sigma}{\sigma'}$ that occurs in w_{sol} , it holds that σ, σ' represent the same domino letter (both *a* or both *b*, either dotted or undotted).
- 2. The number of dotted letters in the upper part of w_{sol} is equal to the number of dotted letters in the lower part of w_{sol} .
- 3. w_{sol} starts with two dotted letters, and the word u_i between the *i*'th and i + 1'th dotted letters in the upper part of w_{sol} , and the word v_i between the corresponding dotted letters in the lower part of w_{sol} are such that $\left[\frac{u_i}{v_i}\right] \in C$, for every *i*.

We call a word that represents the removal of the first k dominoes from w_{sol} a partial solution, denoted by $w_{sol,k}$. Note that the upper and lower parts of $w_{sol,k}$ are not necessarily of equal lengths (in terms of a and b sequences), since the upper and lower parts of a domino may be of different lengths, and so we use letter \$ to pad the end of the encoding in the shorter of the two parts.

We construct an NFH \mathcal{A} , which, intuitively, expresses the following ideas: (1) There exists an encoding w_{sol} of a solution to C, and (2) For every $w_{sol,k} \neq \epsilon$ in a hyperword S accepted by \mathcal{A} , the word $w_{sol,k+1}$ is also in S.

 $\mathfrak{L}(\mathcal{A})$ is then the set of all hyperwords that contain an encoded solution w_{sol} , as well as all its suffixes obtained by removing a prefix of dominoes from w_{sol} . This ensures that w_{sol} indeed encodes a legal solution. For example, a matching hyperword S (for the solution sol discussed earlier) that is accepted by \mathcal{A} is:

$$S = \{w_{sol} = \frac{\dot{a}}{\dot{a}}\frac{b}{b}\frac{a}{a}\frac{\dot{a}}{\dot{b}}\frac{b}{b}, w_{sol,1} = \frac{b}{\dot{a}}\frac{\dot{a}}{\dot{b}}\frac{\dot{a}}{\$}\frac{b}{\$}, w_{sol,2} = \frac{\dot{a}}{\dot{b}}\frac{b}{\$}, w_{sol,3} = \epsilon\}$$

Thus, the quantification condition of \mathcal{A} is $\alpha = \forall x_1 \exists x_2 \exists x_3$, where x_1 is to be assigned a potential partial solution $w_{sol,k}$, and x_2 is to be assigned $w_{sol,k+1}$, and x_3 is to be assigned w_{sol} .

During a run on a hyperword S and an assignment $v : \{x_1, x_2, x_3\} \rightarrow S$, the NFH \mathcal{A} checks that the upper and lower letters of w_{sol} all match. In addition, \mathcal{A} checks that the first domino of $v(x_1)$ is indeed in C, and that $v(x_2)$ is obtained from $v(x_1)$ by removing the first tile. \mathcal{A} performs the latter task by checking that the upper and lower parts of $v(x_2)$ are the upper and lower parts of $v(x_1)$ that have been "shifted" back appropriately. That is, if the first tile in $v(x_2)$ is the encoding of $[\frac{w_i}{v_i}]$, then \mathcal{A} uses states to remember, at each point, the last $|w_i|$ letters of the upper part of $v(x_2)$ and the last $|v_i|$ letters of the lower part of $v(x_2)$, and verifies, at each point, that the next letter in $v(x_1)$ matches the matching letter remembered by the state.

Next, we show that for the alternation-free fragments, a simple reachability test suffices to decide nonemptiness.

Theorem 5. The nonemptiness problem for NFH_{\exists} and NFH_{\forall} is NL-complete.

Proof. The lower bound for both fragments follows from the NL-hardness of the nonemptiness problem for NFA.

We turn to the upper bound, and begin with NFH_∃. Let \mathcal{A}_{\exists} be an NFH_∃. We claim that \mathcal{A}_{\exists} is nonempty iff $\hat{\mathcal{A}}_{\exists}$ accepts some legal word \mathbf{w} . The first direction is trivial. For the second direction, let $\mathbf{w} \in \mathcal{L}(\hat{\mathcal{A}}_{\exists})$. By assigning $v(x_i) = v_{\mathbf{w}}(x_i)$ for every $x_i \in X$, we get $\mathbf{w}_v = \mathbf{w}$, and according to the semantics of \exists , we have that \mathcal{A}_{\exists} accepts $S(\mathbf{w})$. To check whether $\hat{\mathcal{A}}_{\exists}$ accepts a legal word, we can run a reachability check on-the-fly, while advancing from a letter σ to the next letter σ' only if σ' assigns # to all variables for which σ assigns #. While each transition $T = q \xrightarrow{f} p$ in $\hat{\mathcal{A}}$ is of size k, we can encode T as a set of size k of encodings of transitions of type $q \xrightarrow{(x_i, \sigma_i)} p$ with a binary encoding of p, q, σ_i , as well as i, t, where t marks the index of T within the set of transitions of $\hat{\mathcal{A}}$. Therefore, the reachability test can be performed within space that is logarithmic in the size of \mathcal{A}_{\exists} .

Now, let \mathcal{A}_{\forall} be an NFH $_{\forall}$ over X. We claim that \mathcal{A}_{\forall} is nonempty iff \mathcal{A}_{\forall} accepts a hyperword of size 1. For the first direction, let $S \in \mathfrak{L}(\mathcal{A}_{\forall})$. Then, by the semantics of \forall , we have that for every assignment $v : X \to S$, it holds that $\mathbf{w}_v \in \mathcal{L}(\hat{\mathcal{A}}_{\forall})$. Let $u \in S$, and let $v_u(x_i) = u$ for every $x_i \in X$. Then, in particular, $\mathbf{w}_{v_u} \in \mathcal{L}(\hat{\mathcal{A}}_{\forall})$. Then for every assignment $v : X \to \{u\}$ (which consists of the single assignment v_u), it holds that $\hat{\mathcal{A}}_{\forall}$ accepts \mathbf{w}_v , and therefore \mathcal{A}_{\forall} accepts $\{u\}$. The second direction is trivial.

To check whether \mathcal{A}_{\forall} accepts a hyperword of size 1, we restrict the reachability test on $\hat{\mathcal{A}}_{\forall}$ to letters over $\hat{\Sigma}$ that represent fixed functions.

For NFH_{$\exists\forall$}, we show that the problem is decidable, by checking the nonemptiness of an exponentially larger equi-empty NFA.

Theorem 6. The nonemptiness problem for $NFH_{\exists\forall}$ is PSPACE-complete.

Proof. Let \mathcal{A} be an NFH_{$\exists\forall$} with k quantifiers and m \exists -quatifiers. We begin with a PSPACE upper bound.

Let $S \in \mathfrak{L}(\mathcal{A})$. Then, according to the semantics of the quantifiers, there exist $w_1, \ldots, w_m \in S$, such that for every assignment $v : X \to S$ in which $v(x_i) = w_i$ for every $1 \le i \le m$, it holds that $\hat{\mathcal{A}}$ accepts w_v . Let $v : X \to S$ be such an assignment. Then, $\hat{\mathcal{A}}$ accepts $w_{v'}$ for every sequence v' of v that agrees with v on its assignments to x_1, \ldots, x_m , and in particular, for such sequences whose range is $\{w_1, \ldots, w_m\}$. Therefore, by the semantics of the quantifiers, we have that $\{w_1, \ldots, w_m\}$ is in $\mathfrak{L}(\mathcal{A})$. The second direction is trivial.

We call $w_{v'}$ as described above a *witness to the nonemptiness of* \mathcal{A} . We construct an NFA A based on $\hat{\mathcal{A}}$ that is nonempty iff $\hat{\mathcal{A}}$ accepts a witness to the nonemptiness of \mathcal{A} .

Let Γ be the set of all functions of the type $\zeta : [1, k] \to [1, m]$ such that $\zeta(i) = i$ for every $i \in [1, m]$, and such that range $(\zeta) = [1, m]$. For a letter assignment $f = \{\sigma_{1_{x_1}}, \ldots, \sigma_{k_{x_k}}\}$, we denote by f_{ζ} the letter assignment $\{\sigma_{\zeta(1)_{x_1}}, \ldots, \sigma_{\zeta(k)_{x_k}}\}$. For every function $\zeta \in \Gamma$, we construct an NFA $A_{\zeta} = \langle \hat{\Sigma}, Q, Q_0, \delta_{\zeta}, F \rangle$, where for every $q \xrightarrow{g} q'$ in δ , we have $q \xrightarrow{f} q'$ in δ_{ζ} , for every f that occurs in $\hat{\mathcal{A}}$ for which $f_{\zeta} = g$. Intuitively, for every run of A_{ζ} on a word w there exists a similar of $\hat{\mathcal{A}}$ on the sequence of w that matches ζ . Therefore, $\hat{\mathcal{A}}$ accepts a witness w to the nonemptiness of \mathcal{A} iff $w \in \mathcal{L}(A_{\zeta})$ for every $\zeta \in \Gamma$.

We define $A = \bigcap_{\zeta \in \Gamma} A_{\zeta}$. Then $\hat{\mathcal{A}}$ accepts a witness to the nonemptiness of \mathcal{A} iff A is nonempty.

Since $|\Gamma| = m^{k-m}$, the state space of A is of size $O(n^{m^{k-m}})$, where n = |Q|, and its alphabet is of size $|\hat{\Sigma}|$. Notice that for A to be nonempty, δ must be of size at least $|(\Sigma \cup \#)|^{(k-m)}$, to account for all the sequences of letters in the words assigned to the variables under \forall quantifiers (otherwise, we can immediately return "empty"). Therefore, $|\hat{A}|$ is $O(n \cdot |\Sigma|^k)$. We then have that the size of A is $O(|\hat{A}|^k)$. If the number k-m of \forall quantifiers is fixed, then m^{k-m} is polynomial in k. However, now $|\hat{A}|$ may be polynomial in n, k, and $|\Sigma|$, and so in this case as well, the size of A is $O(|\hat{A}|^k)$.

Since the nonemptiness problem for NFA is NL-complete, the problem for NFH_{$\exists\forall$} can be decided in space of size that is polynomial in $|\hat{\mathcal{A}}|$.

For the lower bound, we show a reduction from a polynomial version of the *corridor tiling problem*, defined as follows. We are given a finite set T of tiles, two relations $V \subseteq T \times T$ and $H \subseteq T \times T$, an initial tile t_0 , a final tile t_f , and a bound n > 0. We have to decide whether there is some m > 0 and a tiling of a $n \times m$ -grid such that (1) The tile t_0 is in the bottom left corner and the tile t_f is in the top right corner, (2) A horizontal condition: every pair of horizontal neighbors is in H, and (3) A vertical condition: every pair of vertical neighbors is in V. When n is given in unary notation, the problem is known to be PSPACE-complete. Given an instance C of the tiling problem, we construct an NFH_{$\exists \forall d$} that is nonempty iff C has a solution. We encode a solution to C as a word $w_{sol} = w_1 \cdot w_2 \cdot w_m$ \$ over $\Sigma = T \cup \{1, 2, ..., n, \$\}$, where the word w_i , of the form $1 \cdot t_{1,i} \cdot 2 \cdot t_{2,i}, \ldots n \cdot t_{n,i}$, describes the contents of row i.

To check that w_{sol} indeed encodes a solution, we need to make sure that:

- 1. w_1 begins with t_0 and w_m ends with t_f .
- 2. w_i is of the correct form.
- 3. Within every w_i , it holds that $(t_{j,i}, t_{j+1,i}) \in H$.
- 4. For w_i, w_{i+1} , it holds that $(t_{j,i}, t_{j,i+1}) \in V$ for every $j \in [1, n]$.

Verifying items 1 - 3 is easy via an NFA of size O(n|H|). The main obstacle is item 4.

We describe an NFH_{$\exists\forall$} $\mathcal{A} = \langle T \cup \{0, 1, 2, \dots, n, \$\}, \{y_1, y_2, y_3, x_1, \dots, x_{\log(n)}\}, Q, \{q_0\}, \delta, F, \alpha \rangle$ that is nonempty iff there exists a word that satisfies items 1 - 4. The quantification condition α is $\exists y_1 \exists y_2 \exists y_3 \forall x_1 \dots \forall x_{\log(n)}$. The NFH \mathcal{A} only proceeds on letters whose assignments to y_1, y_1, y_3 is r, 0, 1, respectively, where $r \in T \cup \{1, \dots, n, \$\}$. Notice that this means that \mathcal{A} requires the existence of the words $0^{|w_{sol}|}$ and $1^{|w_{sol}|}$ (the 0-word and 1-word, henceforth). \mathcal{A} makes sure that the word assigned to y_1 matches a correct solution w.r.t. items 1 - 3 described above. We proceed to describe how to handle the requirement for V. We need to make sure that for every position j in a row, the tile in position j in the next row matches the current one w.r.t. V. We can use a state q_j to remember the tile in position j, and compare it to the tile in the next occurrence of j. The problem is avoiding having to check all positions simultaneously, which would require exponentially many states. To this end, we use $\log(n)$ copies of the 0- and 1-words to form a binary encoding of the position j that is to be remembered. The $\log(n) \forall$ conditions make sure that every position within 1 - n is checked.

We limit the checks to words in which $x_1, \ldots x_{\log(n)}$ are the 0- or 1-words, by having \hat{A} accept every word in which there is a letter that is not over 0, 1 that is assigned to the x variables. This takes care of accepting all cases in which the word assigned to y_1 is also assigned to one of the x variables.

To check that $x_1, \ldots x_{\log(n)}$ are the 0- or 1-words, \mathcal{A} checks that the letter assignments to these variables remain constant throughout the run. In these cases, upon reading the first letter, $\hat{\mathcal{A}}$ remembers the value j that is encoded by the constant assignments to $x_1, \ldots x_{\log(n)}$ in a state, and makes sure that throughout the run, the tile that occurs in the assignment to y_1 in position j in the current row matches the tile in position j in the next row.

We construct a similar reduction for the case that the number of \forall quantifiers is fixed: instead of encoding the position by $\log(n)$ bits, we can directly specify the position by a word of the form j^* , for every $j \in [1, n]$. Accordingly, we construct an NFH_{$\exists \forall$} over $\{x, y_1, \ldots, y_n, z\}$, with a quantification condition $\alpha = \exists x \exists y_1 \ldots \exists y_n \forall z$. The NFA $\hat{\mathcal{A}}$ advances only on letters whose assignments to y_1, \ldots, y_n are always $1, 2, \ldots n$, respectively, and checks only words assigned to z that are some constant $1 \leq j \leq n$. Notice that the fixed assignments to the y variables leads to δ of polynomial size. In a hyperword accepted by \mathcal{A} , the word assigned to x is w_{sol} , and the word assigned to z specifies which index should be checked for conforming to V.

6.2. Bounded nonemptiness

The bounded nonemptiness problem is to decide, given an NFH \mathcal{A} and $m \in \mathbb{N}$, whether \mathcal{A} accepts a hyperword of size at most m. Notice that some nonempty NFH only accept infinite hyperwords (for example, \mathcal{A}_2 of Figure 1), and so they do not accept a hyperword of size m, for every $m \in \mathbb{N}$.

We show that the bounded nonemptiness problem is decidable for all of NFH.

Theorem 7. The bounded nonemptiness problem for NFH is in PSPACE.

Proof. Let \mathcal{A} be an NFH with a quantification condition α with k quantifiers, and let $m \in \mathbb{N}$. Intuitively, we construct an NFA A in which a single run simultaneously follows all runs of \mathcal{A} on the possible assignments of a potential hyperword S of size m to the variables of \mathcal{A} . Then, A accepts a set of such legal assignments (represented as a single word) iff \mathcal{A} accepts a hyperword of size at most m.

The assignment tree for α and m is defined as follows. The tree T has k + 1 levels, where the root is at level 0. For $0 < i \le k$, if $\mathbb{Q}_i = \forall$, then every node in level i - 1 has m children. If $\mathbb{Q}_i = \exists$, then every node in level i - 1 has a single child. Every node v in T is associated with an encoding in $[1, m]^*$ that matches the path from the root to v. For example, if v is in level 2, and α begins with $\exists \forall$, and v is the second child, then

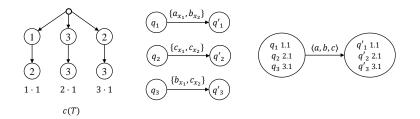


Figure 2: The labeled assignment tree c(T) (left), transitions in \mathcal{A} (middle), and their depiction in A (right).

the position of v is encoded by $1 \cdot 2$. The leaves of T are then all encoded by elements of $[1, m]^k$.

A labeling c of T labels every node (except for the root) by some value in [1, m]. For $0 < i \le k$, if $\mathbb{Q}_i = \forall$, then the m children of every node in level i - 1 are labeled 1 to m. If $\mathbb{Q}_i = \exists$, then the child of every node in level i - 1 is labeled by some value in [1, m].

Consider a hyperword $S = \{w_1, w_2, \dots, w_m\}$. Every path p along c(T) matches an assignment of the words in S to the variables in X: the variable x_i is assigned w_j , where j is the labeling of the node in level i in p. Then, c(T) matches a possible set of assignments of the words of S to the variables in X. Given p, we denote this assignment by f_p . According to the semantics of NFH, we have that \mathcal{A} accepts a hyperword of size m iff there a labeling c(T) such that for every path p of c(T), the underlying NFA $\hat{\mathcal{A}}$ accepts the word assignment for f_p .

We construct A such that a single run of A simultaneously follows every assignment f_p in a labeling c(T), letter by letter.

Let C be the set of all labelings of T, and let L be the set of all indices of leaves of T. We define the NFA A as follows. The alphabet of A is $(\Sigma \cup \{\#\})^m$. The set of states of A is $Q^L \times C$. The set of initial states is $Q_0^L \times C$, and the set of accepting states is $F^L \times C$.

The transition relation of A is as follows. We add a transition labeled $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ from $(((q_1, l_1), \ldots, (q_{|L|}, l_{|L|})), c)$ to $(((q'_1, l_1), \ldots, (q'_{|L|}, l_{|L|})), c')$ if c = c', and for every $1 \le r \le |L|$, there is a transition in δ labeled by $\{\sigma'_{1_{x_1}}, \sigma'_{2_{x_2}}, \ldots, \sigma'_{k_{x_k}}\}$ from q_r to q'_r , where $\sigma'_{i_{x_1}} = \sigma_j$, where j is the labeling of the node in level i in the path to l_r in c.

For example, consider the labeled assignment tree c(T) of Figure 2 for the quantification condition $\forall x_1 \exists x_2$, and m = 3. Then T has three leaves, labeled $1 \cdot 1, 2 \cdot 1$, and $3 \cdot 1$. The labeling c(T) assigns the nodes of T values in [1,3] as described in Figure 2. The three transitions in \mathcal{A} from q_1, q_2, q_3 are then translated to the transition from $s = (((q_1, 1 \cdot 1), (q_2, 2 \cdot 1), (q_3, 3 \cdot 1)), c(T))$ labeled (a, b, c), which means that the transition associates label 1 with a, label 2 with b, and label 3 with c, matching the transitions from q_1, q_2 , and q_3 , when they are associated with the leaves as in s.

The size of T (and hence, the size of L) is $O(m^{k'})$, where k' is the number of \forall quantifiers in α . Accordingly, the size of C is $O(m^{m^{k'}})$. Therefore, the state space of A is of size $O(n^{m^{k'}} \cdot m^{m^{k'}})$, where n is the number of states in A.

According to our construction, we have that \mathcal{A} accepts a hyperword of size m iff A is nonempty, when considering only paths that are legal assignments, that is, once a value i is assigned the letter #, it continues to be assigned #. Checking A for such nonemptiness can be done on-the-fly in space that is logarithmic in the size of A. Notice, as mentioned in the proof of theorem 6, that for m > 1, the size of the transition relation of \mathcal{A} must be exponential in the size of k', to account for the different assignments to the \forall -quantifiers (otherwise, \mathcal{A} is empty and we can return "false"). Therefore, the size of each state of A is polynomial in the size of \mathcal{A} , and a PSPACE upper bound follows.

A PSPACE lower bound for the bounded nonemptiness problem for NFH directly follows from the nonemptiness problem for NFH_{$\exists\forall$}, since, as we prove in Theorem 6, an NFH_{$\exists\forall$} \mathcal{A} with $k' \exists$ -quantifiers is nonempty iff it accepts a hyperword of size k'. However, we prove PSPACE-hardness for a $\forall x \exists y$ quantification condition, showing that this problem is PSPACE-hard even for a fixed number of \forall and \exists quantifiers.

Theorem 8. *The bounded nonemptiness problem for NFH with* $\alpha = \forall x \exists y \text{ is PSPACE-hard.}$

Proof. We reduce from the problem of deciding the nonemptiness of the intersection of k given NFA, which is known to be PSPACE-hard.

Let A_1, A_2, \ldots, A_k be NFA, where $A_i = \langle \Sigma, Q_i, Q_0^i, \delta_i, F_i \rangle$. We construct an NFH $\mathcal{A} = \langle \Sigma', \{x, y\}, Q, Q_0, F, \delta, \forall x \exists y \rangle$ that accepts a hyperword whose size is at most k iff there exists a word w such that $w \in \mathcal{L}(A_i)$ for every $i \in [1, k]$.

The set of states Q of \mathcal{A} is $\bigcup_i Q_i \times Q_{(i+1) \text{mod}k}$, and $\Sigma' = \bigcup_i Q_i \times \Sigma \times Q_i$. The set of accepting states is $\bigcup_i F_i \times F_{(i+1) \text{mod}k}$, and the set of initial states Q_0 is $\bigcup_i Q_0^i \times Q_0^{(i+1) \text{mod}k}$. The transitions are as follows. For every $i \in [1, k]$, every $\sigma \in \Sigma$, and every two transitions $(q, \sigma, q') \in \delta_i, (p, \sigma, p') \in \delta_{(i+1) \text{mod}k}$, we set $((q, p), \{(q, \sigma, q')_x, (p, \sigma, p')_y\}, (q', p')) \in \delta$. Notice that the size of \mathcal{A} is polynomial in the sizes of A_1, \ldots, A_k . Every word assignment w that is read along $\hat{\mathcal{A}}$ describes the parallel run of A_i and $A_{(i+1) \text{mod}k}$ on the same word w. The word assignment w is accepted by $\hat{\mathcal{A}}$ iff w is accepted by both A_i and $A_{(i+1) \text{mod}k}$.

If there exists a word w that is accepted by all NFA, then the hyperword S that describes all the matching accepting runs on w by the different NFA is accepted by A. Indeed, for the accepting run on w by A_i there is a matching accepting run on w by $A_{(i+1) \text{mod}k}$.

Conversely, if there exists a hyperword of size (at most) k that is accepted by \mathcal{A} , then it contains descriptions of runs of $A_1, \ldots A_k$ on words. By the way we have defined \mathcal{A} , if there exists $r \in S$ that describes the accepting run of A_i on a word w, then there must exist $r' \in S$ that describes the accepting run of $A_{(i+1) \text{mod}k}$ on w. As a result, and combined with the size of S, we have that S must contain an accepting run of every NFA in the set, and these runs must all be on the same word w. Therefore, the intersection of $A_1, \ldots A_k$ is nonempty.

As a conclusion from Theorems 7 and 8, we have the following.

Theorem 9. The bounded nonemptiness problem for NFH is PSPACE-complete.

6.3. NFH with Wild Card Letters

When constructing an HRE or an NFH, every letter must include an assignment to all variables. However, an HRE may only need to describe the assignment to a subset of the variables at each step. For example, the HRE

$$\exists x \exists y \{a_x\} \{b_y\}$$

describes hyperwords in which there exist two words, where the first word starts with a, and the second word has b in its second position. Since the first letter and the second letter of the second and first words, respectively, do not matter, there is no need to express them. Therefore, we can define a more general and useful notion of HRE in which the letters are *partial* functions from X to Σ .

To translate the notion of partial functions to NFH, we simply add a wild-card letter \star which can stand for every letter assignment to the variables. For example, the letter $\{a_x, \star_y\}$ stands for all the assignments to x, y in which x is assigned a.

The size of the alphabet $\hat{\Sigma}$ of an underlying NFA must be exponential in the size of the number of \forall -quantifiers, to account for all the assignments of letters to all the variables under \forall -quantifiers. Otherwise, the language of the NFH is empty. Using wild-card letters, such transitions can be replaced by a single transition in which every variable under \forall is assigned \star . Thus, using wild-card letters can lead to exponentially smaller NFH.

We define NFH with wild cards accordingly. An NFH with wild card letters (NFH^{*}) is a tuple $\mathcal{A} = \langle \Sigma, X, Q, Q_0, F, \delta, \alpha \rangle$ whose underlying NFA $\hat{\mathcal{A}}$ is over the alphabet $\hat{\Sigma} = (\Sigma \cup \{\#, \star\})^X$. The semantics of NFH^{*} is similar to that of NFH. The only difference is that now, w_v contains all possible word assignments in which the letters in Σ may also be replaced with \star in the assignments to the variables.

Obviously, every NFH* can be translated to an NFH with an exponential blow-up in the number of transitions. The constructions for intersection, union, and complementation can all be adjusted to handle the wild cards. Due to the exponential decrease in size, the complexity of the various decision procedures for NFH* may, in the worst case, increase exponentially. Since the nonemptiness problem is at the core of most decision procedures, we study its complexity for the various fragments of NFH*.

We begin with NFH^{\pm} and NFH^{\pm}, and show that for these fragments, adding wildcard letters does not change complexity of the nonemptiness problem.

According to the proof of Theorem 5, a simple reachability test on the underlying NFA suffices to determine nonemptiness for these fragments. We notice that this holds also in the presence of wild-card letters. Indeed, an NFH^{*}_{\forall} is nonempty iff it accepts a hyperword of size 1. The proof of Theorem 5 locates such a word by following an accepting path in the underlying NFH in which all variables are equally assigned at every step. It is easy to see that such a path suffices also when some of the variables are assigned wild-card letters. Similarly, an accepting path in an NFH_{\exists} induces a finite accepted hyperword, and the same holds also when traversing transitions with wild-card letters. Therefore, we have the following.

Theorem 10. The nonemptiness problem for NFH_{\exists}^* and NFH_{\forall}^* is NL-complete.

We turn to study the fragment of NFH $_{\exists\forall}^{*}$. Recall that in the proof for the lower bound of Theorem 6, we argue that the size of the transition relation of a nonempty NFH $_{\exists\forall}$ must be exponential in its number of \forall -quantifiers, which affects the space complexity analysis of the size of the NFA that we construct. For an NFH $_{\exists\forall}^{*} \mathcal{A}$, this argument no longer holds. While we can construct a similar NFA and check its nonemptiness, its size may now be exponential in that of \mathcal{A} , conforming to an EXPSPACE upper bound. We prove a matching lower bound, and conclude that in contrast to the alternation-free fragments, adding wild-card letters hardens the nonemptiness problem for NFH $_{\exists\forall}$.

Theorem 11. The nonemptiness problem for $NFH^*_{\exists\forall}$ is EXPSPACE-complete.

Proof. Let \mathcal{A} be an NFH_{$\exists\forall$}. Consider the NFA A constructed in the proof of Theorem 6. A similar NFA can be constructed to decide the nonemptiness of \mathcal{A} . The only difference is the need to consider the intersection of letters which carry wild-card letters. These can be easily computed: the intersection letter of $\{\sigma_{1_{x_1}}, \sigma_{2_{x_2}}, \ldots, \sigma_{k_{x_k}}\}$ and $\{\sigma'_{1_{x_1}}, \sigma'_{2_{x_2}}, \ldots, \sigma'_{k_{x_k}}\}$ is $\{\gamma_{1_{x_1}}, \gamma_{2_{x_2}}, \ldots, \gamma_{k_{x_k}}\}$, where $\gamma_i = \sigma_i$ if $\sigma'_i = \star$, and $\gamma_i = \sigma'_i$ if $\sigma_i = \star$, and otherwise it must hold that $\gamma_i = \sigma_i = \sigma'_i$.

The size of A is, as in the proof of Theorem 6, $O(n^{m^{k-m}})$, where n is the number of states in \mathcal{A} , and m is the number of \exists -quantifiers in α . Since the nonemptiness problem for NFA is NL-complete, an EXPSPACE upper bound follows.

We turn to the lower bound. As in the proof of Theorem 6, we reduce from the corridor tiling problem: we are given an input C which consists of a finite set T of tiles, two relations $V \subseteq T \times T$ and $H \subseteq T \times T$, an initial tile t_0 , a final tile t_f , and a bound n > 0. In the exponential version of this problem, we need to decide whether there exists a legal tiling of a $2^n \times m$ for some m > 0 (in contrast to the polynomial version which we use for NFH_{$\exists\forall$}). This problem is known to be EXPSPACE-complete.

We use a similar idea as for NFH_{$\exists\forall$}, and encode the legal solution as a word, while using the 0- and 1-words under \forall as memory. However, the exponential length of each row in the tiling poses two main obstacles. First, we can no longer use a state to remember the index in the row that we need to check in order to verify the vertical condition. Second, we can no longer use numbered letters to mark the index in every row, and using binary encoding requires verifying that the encoding is correctly ordered. We describe how we overcome these two obstacles by using wild-card letters.

We encode a solution $w_{sol} = \$w_1 \cdot w_2 \cdot w_m\$$ over $\Sigma = T \cup \{0, 1, \$, \&\}$, where the word w_i , of the form $b_0 \cdot t_{0,i} \cdot b_1 \cdot t_{2,i}, \ldots b_{2^n-1} \cdot t_{2^n-1,i}$, describes the contents of row i, where b_j is the *n*-bit binary encoding of index j. Additionally, we use the 0-word which only consists of 0 letters, and similarly we use the 1-word. Here, we precede the sequence of bits with &.

We construct an NFH^{*}_{$\exists\forall$} \mathcal{A} with a quantification condition $\alpha = \exists s \exists x_0 \exists x_1 \forall u \forall y_1 \dots \forall y_n \forall z_1 \dots \forall z_n$ that is nonempty iff C has a solution. Intuitively, as in the proof of Theorem 6, the assignment to s must be w_{sol} , and the assignment to x_0 and x_1 must be the 0- and the 1-words, respectively. The assignment to u must be equal to the assignment of either s, x_0 , or x_1 . Notice that since u is under \forall , then if \mathcal{A} is nonempty then the only hyperword it can accept is $\{w_{sol}, 0, 1\}$. Therefore, the rest of the variables must always be assigned one of these three words in order for \mathcal{A} to accept.

When the assignments to $y_1 \dots y_n$ are the 0- and 1- words, their binary values are used for encoding a single index j that verifies that every two consecutive tiles in position j satisfy V, as we describe below. \hat{A} accepts all runs in which one of the yvariables is assigned w_{sol} . To this end, the transition relation δ of \hat{A} uses transitions from the initial state labeled by letters in which one of $y_1, \dots y_n$ is assigned \$\$ and the rest are assigned \star , leading to accepting runs for these cases.

To match the encoding of the y variables with the correct index j in w_{sol} , the transition relation δ of \mathcal{A} describes the n bits of j in cycles of length n + 1, where in each cycle, the i'th bit of j is specified in the i'th step, and the rest of the values are represented as \star . In each cycle, the i'th bit is compared with the i'th bit in w_{sol} . In cycles in which all n index bits in w_{sol} match those of $y_1 \dots y_n$, the tile in the letter that follows the encoding (the n + 1'th letter in the cycle) is matched with the previous tile, remembered by a state, to verify that they satisfy V.

For example, for n = 3, the encoding 101 would be as follows.

$$\begin{pmatrix} y_1 &= \& & 1 & \star & \star & 1 & \star & \star \cdots \\ y_2 &= \& & \star & 0 & \star & \star & 0 & \star \cdots \\ y_3 &= \& & \star & \star & 1 & \star & \star & \star & 1 \cdots \end{pmatrix}$$

Notice that (considering only y variables), only 2n + 2 letters are needed to describe this encoding: two for every value of the *i*'th bit, one of all wild-cards, and one for all &. Specifying all bits in a single letter would require exponentially many letters.

We now describe how to verify that the index encoding along w_{sol} is correct. We use the z variables in a similar way to the y variables, to encode the successor position of the one encoded in the y variables. To check that they are indeed successors, it suffices to check, within the first cycle, that all bits up to some $1 \le i < n$ are equal, that $z_i = 1$ and $y_i = 0$, and that $y_{i+1} \dots y_n = 1$ and $z_{i+1} \dots z_n = 0$ (the only exception is for $2^n + 1$ and 0, in which we only need to check that all y bits are 1 and all z bits are 0). Runs of \hat{A} in which the encoding in the z variables is not the successor of the encoding of the y variables, or in which one of the z variables is assigned w_{sol} , are accepting. Otherwise, whenever the encoding of the position in w_{sol} is equal to that of the y variables (we check this bit by bit), we check that the encoding of the position in the next cycle is equal to that of the z variables.

For example, for checking the successor of 101, the assignments to the y and z variables would be as follows.

 $\begin{pmatrix} y_1 &= \& & 1 & \star & \star & \star & 1 & \star & \star \cdots \\ y_2 &= \& & \star & 0 & \star & \star & * & 0 & \star \cdots \\ y_3 &= \& & \star & \star & 1 & \star & \star & \star & 1 \cdots \\ z_1 &= \& & 1 & \star & \star & \star & 1 & \star & \star \cdots \\ z_2 &= \& & \star & 1 & \star & \star & \star & 1 & \star \cdots \\ z_3 &= \& & \star & \star & 0 & \star & \star & 0 \cdots \end{pmatrix}$

Since the y and z variables are under \forall , all positions along w_{sol} are checked over all runs of $\hat{\mathcal{A}}$ on the different assignments to the y and z variables. It is left to check that the first position in w_{sol} is 0^n , and the last position is 1^n , which can be done via states.

Checking the horizontal condition itself can be done by comparing every two consecutive tiles in the same row. These tiles are n letters apart, and so this can be done via the states and does not require using the variables as memory. The rest of the checks, i.e, the identity of the first and last tiles, and the correct form of w_{sol} , can also be easily checked by the states.

In every letter of $\hat{\mathcal{A}}$ (other than the first in the run, in which all y and z variables are assigned &), there are at most six non-wild card letters: the assignments to s, x_0, x_1 and u, and y_i and z_i for some $1 \leq i \leq n$, and additionally the letters in which one of the y or z variables is assigned with a word that starts with \$. Therefore, the alphabet of \mathcal{A} is polynomial in the input. The number of states needed for the various checks is also polynomial, and therefore the size of \mathcal{A} is polynomial in |C|.

6.4. A semi-algorithm for deciding the nonemptiness for $\forall \exists$

The nonemptiness problem for NFH is undecidable already for the fragment of $\forall \exists$, as shown in Theorem 4. However, this fragment is of practical use in expressing finite-word properties, as shown in Section 3. We now describe a semi-algorithm for testing the nonemptiness of an NFH with a quantification condition of the type $\forall \exists$. Intuitively, this procedure aims at finding the largest hyperword that is accepted by the NFH.

The procedure first considers the set L_0 of all the words that can be assigned to x_1 , and checks whether this set subsumes the matching assignments for the \exists quantifier. If so, then L_0 is a suitable hyperword. Otherwise, L_0 is pruned to the largest potential hyperword by omitting from L_0 all words that are not assigned to the variable under \exists , and the procedure continues to the next round. In case that the procedure does not find an accepted hyperword, or conversely if the procedure does not reach an empty set, it does not halt.

We describe our procedure with more detail. Let $\mathcal{A} = \langle \Sigma, \{x, y\}, Q, Q_0, F, \delta, \forall x \exists y \rangle$ be an NFH. Let $L^0_{\forall} = \{u | \exists v : \mathbf{w}_{x \mapsto u, y \mapsto v} \in \mathcal{L}(\hat{\mathcal{A}})\}$, and let $L^0_{\exists} = \{v | \exists u : \mathbf{w}_{x \mapsto u, y \mapsto v} \in \mathcal{L}(\hat{\mathcal{A}})\}$. We denote the NFA obtained from $\hat{\mathcal{A}}$ by restricting the transitions to assignments to x by $\hat{\mathcal{A}}_x$, and similarly define $\hat{\mathcal{A}}_y$. It is easy to see that $A^0_{\forall} = \hat{\mathcal{A}}_x$ is an NFA for L^0_{\forall} .

for L^0_{\forall} , and $A^0_{\exists} = \hat{\mathcal{A}}_y$ is an NFA for L^0_{\exists} . If $L^0_{\exists} \subseteq L^0_{\forall}$, then by the semantics of NFH, we have that L^0_{\forall} is accepted by \mathcal{A} . If $L^0_{\exists} \cap L^0_{\forall} = \emptyset$, then by the semantics of NFH, we have that \mathcal{A} is empty. Otherwise, there exists a word in L^0_{\exists} that is not in L^0_{\forall} , and vice versa.

We define $L_{\exists}^{1} = L_{\exists}^{0} \cap L_{\forall}^{0}$. Notice that L_{\exists}^{1} is regular, and an NFA A_{\exists}^{1} for L_{\exists}^{1} can be calculated by the intersection construction for A_{\forall}^{0} and A_{\exists}^{0} . Now $L_{\exists}^{1} \subseteq L_{\forall}^{0}$. However, it may be the case that there exists a word $u \in L_{\forall}^{0}$ for which there exists no matching $v \in L_{\exists}^{1}$. Therefore, we restrict L_{\forall}^{0} to a set $L_{\forall}^{1} = \{u | \exists v \in L_{\exists}^{1} : w_{x \mapsto u, y \mapsto v} \in \mathcal{L}(\hat{\mathcal{A}})\}$. We calculate an NFA A_{\forall}^{1} for L^{1} , as follows. Let $A_{\exists}^{0} = \langle P, \Sigma, p_{0}, \delta_{0}, F_{0} \rangle$. We define $\hat{A}^{1} = \langle Q \times P, (\Sigma \cup \{\#\})^{\{x,y\}}, (q_{0}, p_{0}), \delta_{1}, F_{1} \times F_{2} \rangle$, where $\delta_{1} = \{((q, p), \{\sigma'_{x}, \sigma_{y}\}, (q', p') | \sigma, \sigma' \in \Sigma, (q, \{\sigma_{x}, \sigma'_{y}\}, q') \in \delta, (p, \sigma', p') \in \delta_{0}\}$. That is, $\hat{\mathcal{A}}^{1}$ is roughly the intersection construction of $\hat{\mathcal{A}}$ and A_{\exists}^{0} , when considering only the letter assignments to y. We denote this construction by \cap_{y} . Finally, we set $A_{\forall}^{1} = \hat{\mathcal{A}}_{x}^{1}$.

Now, if $\mathcal{L}(A_{\exists}^1) \subseteq \mathcal{L}(A_{\forall}^1)$, then $\mathcal{L}(A_{\forall}^1)$ is accepted by \mathcal{A} , and if $\mathcal{L}(A_{\exists}^1) \cap \mathcal{L}(A_{\forall}^1) = \emptyset$, then $\mathcal{L}(\mathcal{A}) = \emptyset$. Otherwise, we repeat the process above with respect to $\hat{\mathcal{A}}^1, A_{\forall}^1, A_{\exists}^1$.

Algorithm 1 describes the procedure.

Algorithm 1: Nonemptiness test for $\forall \exists$ Input: A. **Output:** $\mathfrak{L}(\mathcal{A}) \neq \emptyset$? 1 $A_{\forall} = \hat{\mathcal{A}}_x, A_{\exists} = \hat{\mathcal{A}}_y$ 2 while true do if $\mathcal{L}(A_{\exists}) \subseteq \mathcal{L}(A_{\forall})$ then 3 return tt 4 else if $\mathcal{L}(A_{\exists}) \cap \mathcal{L}(A_{\forall}) = \emptyset$ then 5 return ff 6 $A_{\exists} = A_{\exists} \cap A_{\forall}$ 7 $\hat{\mathcal{A}} = \hat{\mathcal{A}} \cap_{u} \mathcal{A}_{\exists}$ 8 $A_{\forall} = \hat{\mathcal{A}}_x$ 9 10 endwhile

7. Additional decision procedures

The *universality problem* is to decide whether a given NFH \mathcal{A} accepts every hyperword over Σ . Notice that \mathcal{A} is universal iff $\overline{\mathcal{A}}$ is empty. Since complementing an NFH involves an exponential blow-up, we conclude the following from the results in Section 6, combined with the PSPACE lower bound for the universality of NFA.

Theorem 12. The universality problem for

- 1. NFH is undecidable,
- 2. NFH_{\exists} and NFH_{\forall} is PSPACE-complete, and
- 3. NFH_{$\forall \exists$} is in EXPSPACE.

We turn to study the membership problem for NFH: given an NFH \mathcal{A} and a hyperword S, is $S \in \mathfrak{L}(\mathcal{A})$? When S is finite, so is the set of assignments from X to S, and so the problem is decidable. We call this case the *finite membership problem*.

Theorem 13. • The finite membership problem for NFH is in PSPACE.

 The finite membership problem for a hyperword of size k and an NFH with O(log(k)) ∀ quantifiers is NP-complete.

Proof. Let S be a finite hyperword, and let \mathcal{A} be an NFH with k variables. We can decide the membership of S in $\mathfrak{L}(\mathcal{A})$ by iterating over all relevant assignments from X to S, and for every such assignment v, checking on-the-fly whether w_v is accepted by $\hat{\mathcal{A}}$. This algorithm uses space of size that is polynomial in k and logarithmic in $|\mathcal{A}|$.

In the case that the number of \forall quantifiers is $O(\log k)$, an NP upper bound is met by iterating over all assignments to the variables under \forall , while guessing assignments to the variables under \exists . For every such assignment v, checking whether $w_v \in \mathcal{L}(\hat{\mathcal{A}})$ can be done on-the-fly.

We show NP-hardness for this case by a reduction from the Hamiltonian cycle problem. Given a graph $G = \langle V, E \rangle$ where $V = \{v_1, v_2, \ldots, v_n\}$ and |E| = m, we construct an NFH_∃ \mathcal{A} over $\{0, 1\}$ with n states, n variables, δ of size m, and a hyperword S of size n, as follows. $S = \{w_1, \ldots, w_n\}$, where w_i is the word over $\{0, 1\}$ in which all letters are 0 except for the i'th. The structure of $\hat{\mathcal{A}}$ is identical to that of G, and we set $Q_0 = F = \{v_1\}$. For the transition relation, for every $(v_i, v_j) \in E$, we have $(v_i, \varphi_i, v_j) \in \delta$, where φ_i assigns 0 to all variables except for x_i . Intuitively, the i'th letter in an accepting run of $\hat{\mathcal{A}}$ marks traversing v_i . Assigning w_j to x_i means that the j'th step of the run traverses v_i . Since the words in w make sure that every $v \in V$ is traversed exactly once, and that the run on them is of length n, we have that \mathcal{A} accepts S iff there exists some ordering of the words in S that matches a Hamiltonian cycle in G.

remark To account for all the assignments to the \forall variables, δ – and therefore, $\hat{\mathcal{A}}$ – must be of size at least $2^{k'}$ (otherwise, we can return "no"). We then have that if k = O(k'), then space of size k is logarithmic in $|\hat{\mathcal{A}}|$, and so the problem in this case can be solved within logarithmic space. A matching NL lower bound follows from the membership problem for NFA.

When S is infinite, it may still be finitely represented, allowing for algorithmic membership testing. We now address the problem of deciding whether a regular language \mathcal{L} (given as an NFA) is accepted by an NFH. We call this *the regular membership* problem for NFH. We show that this problem is decidable for the entire class of NFH.

Theorem 14. The regular membership problem for NFH is decidable.

Proof. Let $A = \langle \Sigma, P, P_0, \rho, F \rangle$ be an NFA, and let $\mathcal{A} = \langle \Sigma, \{x_1, \dots, x_k\}, Q, Q_0, \delta, \mathcal{F}, \alpha \rangle$ be an NFH.

First, we construct an NFA $A' = \langle \Sigma \cup \{\#\}, P', P'_0, \rho', F' \rangle$ by extending the alphabet of A to $\Sigma \cup \{\#\}$, adding a new and accepting state p_f to P with a self-loop labeled by #, and transitions labeled by # from every $q \in F$ to p_f . The language of A' is then $\mathcal{L}(A) \cdot \#^*$. We describe a recursive procedure (iterating over α) for deciding whether $\mathcal{L}(A) \in \mathfrak{L}(A)$.

For the case that k = 1, if $\alpha = \exists x_1$, then $\mathcal{L}(A) \in \mathfrak{L}(\mathcal{A})$ iff $\mathcal{L}(A) \cap \mathcal{L}(\hat{\mathcal{A}}) \neq \emptyset$. Otherwise, if $\alpha = \forall x_1$, then $\mathcal{L}(A) \in \mathfrak{L}(\mathcal{A})$ iff $\mathcal{L}(A) \notin \mathfrak{L}(\overline{\mathcal{A}})$, where $\overline{\mathcal{A}}$ is the NFH for $\overline{\mathfrak{L}(\mathcal{A})}$. The quantification condition for $\overline{\mathcal{A}}$ is $\exists x_1$, conforming to the base case.

For k > 1, we construct a sequence of NFA $A_k, A_{k-1} \dots, A_1$ as follows. Initially, $A_k = \hat{\mathcal{A}}$. Let $A_i = \langle \Sigma_i, Q_i, Q_i^0, \delta_i, \mathcal{F}_i \rangle$. If $\mathbb{Q}_i = \exists$, then we construct A_{i-1} as follows. The set of states of A_{i-1} is $Q_i \times P$, and the set of initial states is $Q_i^0 \times P_0$. The set of accepting states is $\mathcal{F}_i \times F$. For every $(q \xrightarrow{f} q') \in \delta_i$ and every $(p \xrightarrow{f(x_i)} p') \in \rho$, we have $((q, p) \xrightarrow{f \setminus \{\sigma_{i_{x_i}}\}} (q', p')) \in \delta_{i-1}$. We denote this construction by $A \cap_{x_i} A_i$. Then, A_{i-1} accepts a word assignment w_v iff there exists a word $u \in \mathcal{L}(A)$, such that A_i accepts $w_{v \cup \{x_i \mapsto u\}}$.

If $\mathbb{Q}_i = \forall$, then we set $A_{i-1} = A \cap_{x_i} \overline{A_i}$ Notice that A_{i-1} accepts a word assignment w_v iff for every $u \in \mathcal{L}(A)$, it holds that A_i accepts $w_{v \cup \{x_i \mapsto u\}}$.

For $i \in [1, k]$, let \mathcal{A}_i be the NFH whose quantification condition is $\alpha_i = \mathbb{Q}_1 x_1 \cdots \mathbb{Q}_i x_i$, and whose underlying NFA is A_i . Then, according to the construction of A_{i-1} , we have that $\mathcal{L}(A) \in \mathfrak{L}(\mathcal{A}_i)$ iff $\mathcal{L}(A) \in \mathfrak{L}(\mathcal{A}_{i-1})$.

The NFH A_1 has a single variable, and we can now apply the base case.

Every \forall quantifier requires complementation, which is exponential in |Q|. Therefore, in the worst case, the complexity of this algorithm is $O(2^{2^{\dots^{|Q||A|}}})$, where the tower is of height k. If the number of \forall quantifiers is fixed, then the complexity is $O(|Q||A|^k)$.

The *containment problem* is to decide, given NFH A_1 and A_2 , whether $\mathfrak{L}(A_1) \subseteq \mathfrak{L}(A_2)$. Since we can reduce the nonemptiness problem to the containment problem, we have the following as a result of Theorem 4.

Theorem 15. The containment problem for NFH is undecidable.

However, the containment problem is decidable for various fragments of NFH.

Theorem 16. The containment problem of $NFH_{\exists} \subseteq NFH_{\forall}$ and $NFH_{\forall} \subseteq NFH_{\exists}$ is *PSPACE-complete. The containment problem of* $NFH_{\exists\forall} \subseteq NFH_{\forall\exists}$ *is in EXPSPACE*

Proof. A lower bound for all cases follows from the PSPACE-hardness of the containment problem for NFA. For the upper bound, for two NFH A_1 and A_2 , we have that $\mathfrak{L}(A_1) \subseteq \mathfrak{L}(A_2)$ iff $\mathfrak{L}(A_1) \cap \overline{\mathfrak{L}(A_2)} = \emptyset$. We can compute an NFH $\mathcal{A} = \mathcal{A}_1 \cap \overline{\mathcal{A}_2}$ (Theorems 1, 3), and check its nonemptiness. Complementing A_2 is exponential in its number of states, and the intersection construction is polynomial.

If $\mathcal{A}_1 \in NFH_{\exists}$ and $\mathcal{A}_2 \in NFH_{\forall}$ or vice versa, then \mathcal{A} is an NFH_{\exists} or NFH_{\forall}, respectively, whose nonemptiness can be decided in space that is logarithmic in $|\mathcal{A}|$.

The quantification condition of an NFH for the intersection may be any interleaving of the quantification conditions of the two intersected NFH. (Theorem 3). Therefore, for the rest of the fragments, we can construct the intersection such that \mathcal{A} is an NFH_{$\exists\forall$}. The exponential blow-up in complementing \mathcal{A}_2 , along with The PSPACE upper bound of Theorem 6 gives an EXPSPACE upper bound for the rest of the cases.

8. Related Work

It is well-known that classic specification languages like regular expressions and LTL cannot express hyperproperties. The study of specific hyperproperties, such as noninterference, dates back to the seminal work by Goguen and Meseguer [29] in the 1980s. The first systematic study of hyperproperties is due to Clarkson and Schneider [14]. Subsequently, temporal logics HyperLTL and HyperCTL* were introduced [13] to give formal syntax and semantics to hyperproperties. HyperLTL was recently extended to A-HLTL [6] to capture *asynchronous* hyperproperties, where some execution traces can stutter while others advance.

There has been much recent progress in automatically verifying [27, 26, 25, 15, 31] and monitoring [3, 24, 11, 9, 23, 35, 30] HyperLTL specifications. HyperLTL is also supported by a growing set of tools, including the model checkers HyperQube [31],

MCHyper [27, 15], the satisfiability checkers EAHyper [22] and MGHyper [20], and the runtime monitoring tool RVHyper [23].

Related to the nonemptiness problem in this paper is the *satisfiability* problem for HyperLTL, which was shown to be decidable for the $\exists^*\forall^*$ fragment, and undecidable for any fragment that includes a $\forall\exists$ quantifier alternation [19]. The hierarchy of hyperlogics beyond HyperLTL has been studied in [16]. Furthermore, our other results are aligned with the complexity of HyperLTL model checking for tree-shaped and general Kripke structures [5], which encode finite traces. In particular, our membership results are in line with the results on the complexity of verification in [5]. This shows that the complexity results in [5] mainly stem from the nature of quantification over finite words and depend on neither the full power of the temporal operators nor the infinite nature of HyperLTL semantics.

The *synthesis* problem has shown to be undecidable in general, and decidable for the \exists^* and $\exists^*\forall$ fragments. While the synthesis problem becomes, in general, undecidable as soon as there are two universal quantifiers, there is a special class of universal specifications, called the linear \forall^* -fragment, which is still decidable [21]. The linear \forall^* -fragment corresponds to the decidable *distributed synthesis* problems. The *bounded synthesis* problem considers only systems up to a given bound on the number of states. Bounded synthesis from hyperproperties is studied in [21], and has been successfully applied to small examples such as the dining cryptographers [12]. Program repair and controller synthesis for HyperLTL have been studied in [7, 8]. Our results on bounded nonemptiness complement the known results, as it resembles the complexity of bounded synthesis.

9. Discussion and Future Work

We have introduced and studied *hyperlanguages* and a framework for their modeling, focusing on the basic class of regular hyperlanguages, modeled by HRE and NFH. We have shown that regular hyperlanguages are closed under set operations and are capable of expressing important hyperproperties for information-flow security policies over finite traces. We have also investigated fundamental decision procedures for various fragments of NFH, conscentrating mostly on the important decision problem of nonemptiness. Some gaps, such as the precise lower bound for the universality and containment problems for NFH_{$\exists\forall$}, are left open.

Since our framework does not limit the type of underlying model, it can be lifted to handle hyperwords consisting of infinite words, with an underlying model designed for such languages, such as *nondeterministic Büchi automata*, which model ω -regular languages. Just as Büchi automata can express LTL, such a model can express the entire logic of HyperLTL [13].

As future work, we plan on studying non-regular hyperlanguages (e.g., contextfree), and object hyperlanguages (e.g., trees). Another direction is designing learning algorithms for hyperlanguages, by exploiting known canonical forms for the underlying models, and basing on existing learning algorithms for them. The main challenge would be handling learning sets and a mechanism for learning word variables and quantifiers.

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