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Odd Wheels Are Not Odd-Distance Graphs

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Abstract
An odd wheel graph is a graph formed by connecting a new vertex to all vertices of an odd cycle. We answer a question of Rosenfeld and Le by showing that odd wheels cannot be drawn in the plane so that the lengths of the edges are odd integers.

Keywords Distance geometry · Odd-distance graphs · Forbidden subgraphs

Mathematics Subject Classification 51K05

1 Introduction

A *geometric graph* is a graph drawn in the plane so that the vertices are represented by distinct points and the edges are represented by possibly intersecting straight line segments connecting the corresponding points. A *unit-distance graph* is a geometric graph where all edges are represented by segments of length 1. The study of the chromatic number of unit-distance graphs started with the question of Edward Nelson, who raised the problem of determining the minimum number of colors that are needed to color the points of the plane so that no two points unit distance apart are assigned the same color. This number is known as the *chromatic number of the plane*. Until recently the best lower bound was 4, but it was improved by Aubrey de Grey [8], who constructed a unit-distance graph that cannot be colored with four colors. The best upper bound is 7. For more details on unit-distance graphs see for example [19].

Erdős [6] raised the problem of determining the maximal number of edges in a unit-distance graph on n vertices and this question became known as the Erdős Unit Distance Problem. Erdős and Rosenfeld [2] asked analogous questions for odd distances. An

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odd-distance graph is a geometric graph in which edges are represented by segments whose length is an odd integer. G^{odd} is the graph whose vertex set is the plane and in which two vertices are connected if their distance is an odd integer. Note that odd-distance graphs are subgraphs of G^{odd}. In [2] the authors asked to determine the chromatic number of G^{odd} (denoted by $\chi(G^{\text{odd}})$), and the maximum number of edges of an odd-distance graph on n vertices.

Since there are odd-distance graphs whose chromatic number is 5 [2,8], we have $5 \leq \chi(G^{\text{odd}})$. However, unlike the case of unit distances, no finite upper bound is known for $\chi(G^{\text{odd}})$. If in addition we require the color classes to be measurable sets, the chromatic number is infinite [4,20].

There do not exist four points in the plane with pairwise odd distances [13], hence K_4 is not an odd-distance graph. It follows from Turán's theorem that the complete tripartite graph $K_{n,n,n}$ has the maximal number of edges among K_4-free graphs. Piepmeyer [17] showed that $K_{n,n,n}$, and therefore any 3-colorable graph, is an odd-distance graph. This settles the second question of Erdős and Rosenfeld.

Let W_n be the *wheel graph* formed by connecting a new vertex to all vertices of a cycle on n vertices.[1] W_{2k} is 3-colorable, hence it is an odd-distance graph. Rosenfeld and Le [18] showed that having K_4 (which is isomorphic to W_3) as a subgraph is not the only obstruction for being an odd-distance graph, since W_5 is not an odd-distance graph either. This motivated the following question from [18]. Is it true that W_{2k+1} is not a subgraph of G^{odd} for any k? We answer this affirmatively.

Theorem 1.1 *If n is odd, then W_n is not an odd-distance graph.*

This result is also connected to the chromatic number of G^{odd}. Take any vertex v of G^{odd} and let G' be the graph that is induced by the neighborhood of v. Theorem 1.1 is equivalent to saying that G' is 2-colorable for any v.

We prove Theorem 1.1 by a careful analysis of the angles that appear around the center of the wheel. In Sect. 2 we study drawings of wheel graphs in a more general setting, without assuming that the edge lengths are odd numbers. In order to understand which angles can appear around the center and how these angles behave when we sum them up, we develop a number of useful lemmas. These tools might prove useful for related questions. For example, there are a number of interesting results and questions about geometric graphs where we allow the edges to be represented by segments of arbitrary integer length [3,9,11,21]. One of these questions is Harborth's conjecture [10] stating that all planar graphs admit a planar drawing with integer edge lengths. There are a number of constructions showing that any wheel graph admits such a drawing, see for example [3, Sect. 5.11]. Since the maximal planar graphs are the triangulations, they contain many wheels. Hence, understanding the possible drawings of wheels is vital for solving the conjecture. In Sect. 3 we refine our tools by assuming that the distances are odd integers. Finally, in Sect. 4, we prove Theorem 1.1.

[1] There is some discrepancy in the literature, since some authors prefer to denote by W_n the wheel graph on n vertices.

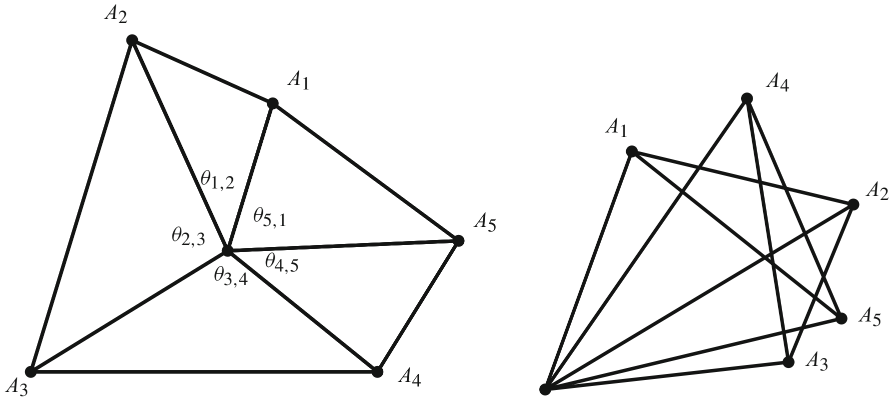

Fig. 1 Two embeddings of the wheel graph W_5

2 Wheels with Integer Edge Lengths

2.1 Embeddings of Wheel Graphs

Throughout this paper we will assume that the center of the wheel is embedded at O and the other points are $A_1, A_2, \ldots, A_n$, following the order of the vertices in the defining cycle of the wheel (see Fig. 1). In the following notations the indices are mod n. We will call the n triangles OA_iA_{i+1} for $i \in \{1, \ldots, n\}$ the *triangles of the embedding*. We will use the notations $r_i = |OA_i|$, $r_{i,i+1} = |A_iA_{i+1}|$, and $\theta_{i,i+1} = \angle A_iOA_{i+1}$.

That is, the i-th triangle has sides of length r_i, r_{i+1}, and $r_{i,i+1}$, and its inner angle is $\theta_{i,i+1}$. Note that $\theta_{i,i+1}$ is a directed angle. We do not assume planarity or even general position of the points. For example, crossings are allowed, and O does not need to be in the interior of the cycle (see Fig. 1).

2.2 Geometry of a Triangle

Let us recall some classical results from elementary geometry. Let $T(a, b, c)$ denote a triangle with sides a, b, c and let α denote the angle opposite to a. By the law of cosines we have

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \tag{1}$$

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2bc}. \tag{2}$$

Let A denote the area of $T(a, b, c)$. Then

$$A = \frac{bc\sin\alpha}{2} = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{4}. \tag{3}$$

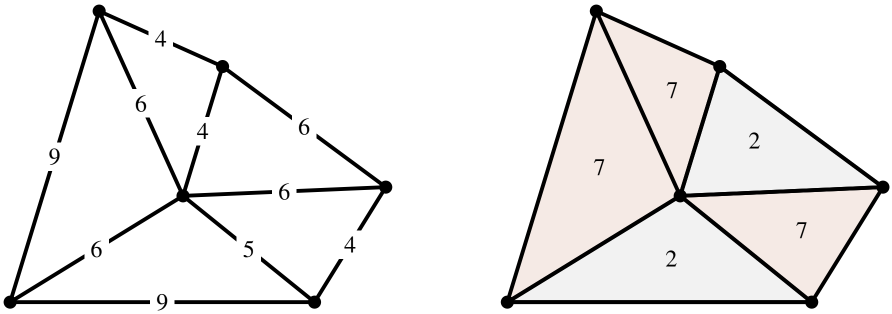

Fig. 2 Characteristics of the triangles in an embedding

Using these formulas we will introduce two notions, the *characteristic* of a triangle and the *residual* of an angle. Strictly speaking we will only need residuals for the proof of Theorem 1.1, but there is a strong connection to the characteristics of triangles so they are worth mentioning.

2.3 Characteristic of a Triangle

From (3) we can see that if a, b, and c are integers, then we can write the area of $T(a, b, c)$ as $r\sqrt{D}$ for some rational number r and a square-free integer D. If the area is 0, then $r = 0$ and D can be any square-free integer. If the area is non-zero, then D must be the square-free part of $4b^2c^2 - (b^2 + c^2 - a^2)^2$. In this case D is called the *characteristic* of the triangle.

We say that a point set in the plane is *integral* if the pairwise distances between its points are integers. The characteristic of triangles is a useful tool to study and algorithmically generate integral point sets (see for example [11]). The following statement is folklore, for a proof see [12].

Lemma 2.1 *The triangles spanned by any three non-collinear points in a plane integral point set have the same characteristic.*

Consider an embedding of a wheel graph such that the edges have integer lengths. The rest of the distances might be non-integer, thus the n triangles of the embedding can have different characteristics. (See for example Fig. 2). At first one might hope that the number of different characteristics corresponding to a drawing of any wheel graph is bounded. This turns out not to be true, however the characteristics still have some useful properties. Later in this section we will show the following statement.

Lemma 2.2 *Consider an embedding of a wheel graph with integer edge lengths. Then for any fixed characteristic D the sum of those angles $\theta_{i,i+1}$ that correspond to triangles with characteristic D is an integer multiple of π.*

2.4 Residual of an Angle

Considering (1) and (3) we obtain that the characteristic of a triangle is connected to the sine values of the angles in the following sense. If a triangle with integer side lengths has characteristic D, then the sine of its angles is of the form $q\sqrt{D}$ for some rational number q, and the cosine of its angles is rational. Hence we will say that an angle θ *has residual* D if D is square-free, $\sin\theta = q\sqrt{D}$ for some rational number q, and $\cos\theta$ is rational.

Most angles in general do not have any residual. Integer multiples of π have residual D for any square-free integer D, but other angles have at most one residual. If the residual is unique, it will be called *the residual of the angle*. For example, the residual of $\pi/2$ is 1, the residual of $\pi/3$ is 3, but $\pi/6$ does not have any residual. As the characteristic of triangles, the residual of the angles is a useful tool, in [7] it was used to find trisectible angles in triangles with integer side lengths.

We can easily see from the trigonometric addition formulas that the set of angles with residual D is closed under addition. Also, for any ϕ the following angles have the same set of residuals: $\phi, -\phi, \pi+\phi, \pi-\phi$.

2.5 Angles Whose Squared Trigonometric Functions Are Rational

Conway et al. [5] studied angles whose squared trigonometric functions are rational. They called θ a *pure geodetic angle* if the square of its sine is rational, and showed the following theorem.

Splitting Theorem 2.3 *Let $\theta_1, \ldots, \theta_n$ be pure geodetic angles, let $r_1, \ldots, r_n$ be rational numbers, and let D be a positive integer. Let $\theta_{i_1}, \ldots, \theta_{i_l}$ be those angles whose tangents are rational multiples of $\sqrt{D}$. If $\sum_{i=1}^{n} r_i\theta_i$ is a rational multiple of π, then $\sum_{j=1}^{l} r_{i_j}\theta_{i_j}$ is also a rational multiple of π.*

Clearly, angles with residual D are pure geodetic angles and have tangents that are rational multiples of $\sqrt{D}$. (It is not hard to see that α is pure geodetic if and only if 2α has a residual). Therefore, Theorem 2.3 applies to them, but we can even strengthen it in some sense. Note that in the next theorem we consider simple sums instead of rational linear combinations.

Splitting Theorem 2.4 (for angles that have residual) *Let $\theta_1, \ldots, \theta_n$ be angles that have a residual and let D be a square-free positive integer. Let $\theta_{i_1}, \ldots, \theta_{i_l}$ be those angles whose residual is D. If $\sum_{i=1}^{n} \theta_i$ is a rational multiple of π, then $\sum_{j=1}^{l} \theta_{i_j}$ is also a rational multiple of π. Furthermore, $\sum_{j=1}^{l} \theta_{i_j}$ is an integer multiple of $\pi/3$ or $\pi/2$.*

Proof The first part is clear from Theorem 2.3. For the second part we recall Niven's Theorem [14]:

Theorem 2.5 *Consider the angles in the range $0 \le \theta \le \pi/2$. The only values of θ such that both θ/π and $\cos\theta$ are rational are 0, $\pi/3$, and $\pi/2$.*

Since angles corresponding to a given residual are closed under addition, the sum $\sum_{j=1}^{l}\theta_{i_j}$ gives us an angle that has residual D. But angles that have a residual have rational cosine, so we can apply Theorem 2.5 to them. □

Now we are ready to prove Lemma 2.2. Since triangles with characteristic D have angles that have residual D, it is enough to show the following residual version.

Lemma 2.6 *Consider an embedding of a wheel graph with integer edge lengths. Then for any fixed residual D the sum of those angles $\theta_{i,i+1}$ that have residual D is an integer multiple of π.*

Proof Clearly $\sum_{i=1}^{n}\theta_{i,i+1}$ is an integer multiple of 2π. Hence we can apply Theorem 2.4 for the angles $\theta_{i,i+1}$. Suppose that the angles corresponding to residual D add up to θ. From Theorem 2.4 it is clear that θ is either an integer multiple of $\pi/2$ or an integer multiple of $\pi/3$. Let $\theta' = \theta \bmod \pi$. Then $\theta' = 0, \pi/3, 2\pi/3$, or $\pi/2$. Note that since θ is the sum of some angles that have residual D, it also has residual D. Hence θ' also has residual D.

From $\sin(\pi/3) = \sin(2\pi/3) = \sqrt{3}/2$ we can see that if $\theta' = \pi/3$ or $\theta' = 2\pi/3$, then $D = 3$. Similarly, if $\theta' = \pi/2$, then we have $D = 1$. Therefore, if we group the terms of $\sum_{i=1}^{n}\theta_{i,i+1}$ based on the residuals, every group will sum up to an integer multiple of π except maybe the ones corresponding to $D = 1$ and $D = 3$. (Some $\theta_{i,i+1}$ might not have a unique residual but those are themselves integer multiples of π, thus we can pick an arbitrary residual for them). Since the whole sum should be an integer multiple of π, the exceptional cases together must add up to an integer multiple of π. This can only happen if both of them add up to an integer multiple of π, since $\pi/3 + \pi/2$ and $2\pi/3 + \pi/2$ are not integer multiples of π. Hence, every sum corresponding to a given residual is an integer multiple of π. □

3 Wheels with Odd Edge Lengths

In the previous section we considered wheels with arbitrary integer edge lengths. Now we are ready to turn our attention to drawings of odd wheels with odd integer edge lengths.

Lemma 3.1 *If a, b, and c are odd numbers and the characteristic of the triangle $T(a, b, c)$ is D, then $D \equiv 3 \bmod 8$.*

Proof It follows from (3) that the characteristic of the triangle is the square-free part of $4a^2b^2 - (a^2 + b^2 - c^2)^2$. Since squares of odd numbers are congruent to 1 modulo 8, we have $4a^2b^2 - (a^2 + b^2 - c^2)^2 \equiv 3 \bmod 8$. Since the square part of $4a^2b^2 - (a^2 + b^2 - c^2)^2$ is the square of an odd number, it is congruent to 1 modulo 8. Hence $D \equiv 3 \bmod 8$. □

This means that if we have an embedding of a wheel graph with odd edge lengths, then each $\theta_{i,i+1}$ has a unique residual that is congruent to 3 modulo 8. The next idea is to classify the angles whose residual is congruent to 3 modulo 8.

Lemma 3.2 *Suppose that $D \equiv 3 \bmod 8$ and ϕ is an angle that has residual D. Then $\cos\phi$ can be written as $m/(2p)$, where $p \equiv 1 \bmod 8$ and m is an integer. Furthermore the remainder of m modulo 8 is determined by the angle, and it is 1, 2, 3, 5, 6, or 7.*

We will call the remainder of m modulo 8 the *class* of ϕ.

Proof By the definition of having a residual, $\cos\phi$ is rational. Since $D \equiv 3 \bmod 8$ the value of $\cos\phi$ is non-zero. Hence $\cos\phi = a/b$ for some non-zero integers a, b such that $\gcd(a, b) = 1$. There are two cases.

- First, suppose that a and b are odd. Odd numbers have an inverse in $\mathbb{Z}_8$. So, if b is odd, then there is an odd number k such that $bk \equiv 1 \bmod 8$. Hence $\cos\phi = 2ak/(2bk)$. Now ak is odd, therefore $2ak$ is not divisible by 4. Thus, we can choose $m = 2ak$ and $p = bk$.
- Second, suppose that a or b is even. Since $\gcd(a, b) = 1$, one of them is even and the other one is odd. Consider that $\sin\phi = \pm\sqrt{1 - a^2/b^2} = \pm\sqrt{b^2 - a^2}/b$. The square-free part of $b^2 - a^2$ is D, and $b^2 - a^2$ is odd, so $b^2 - a^2 \equiv 3 \bmod 8$. Since the only quadratic residuals modulo 8 are 0, 1, and 4, the only possibility is that $b^2 \equiv 4 \bmod 8$ and $a^2 \equiv 1 \bmod 8$. Since $b^2 \equiv 4 \bmod 8$, $b' = b/2$ is odd and, similarly to the previous case, there is an odd k such that $kb' \equiv 1 \bmod 8$. Since $a^2 \equiv 1 \bmod 8$, a must be odd. Thus, we can choose $m = ak$, $p = b'k$.

It is also easy to see that an angle cannot fall into two classes, notice that $m_1/(2p_1) = m_2/(2p_2)$ implies $m_1 p_2 \equiv m_2 p_1 \bmod 8$. □

The aim of the next lemma is to answer the following question. Suppose that angle θ is of class m_1 and angle ϕ is of class m_2. What is the class of $\theta + \phi$, assuming that it exists?

Lemma 3.3 *If* $\cos\theta = m_1/(2p_1)$, $\cos\phi = m_2/(2p_2)$, *and* $\cos(\theta+\phi) = m_3/(2p_3)$ *for some integers* p_1, p_2, p_3 *that are congruent to* 1 *modulo* 8 *and integers* m_1, m_2, m_3, *then*

$$m_1^2 + m_2^2 + m_3^2 - m_1 m_2 m_3 - 4 \equiv 0 \mod 8. \tag{4}$$

Proof Using the cosine addition formula $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$, we get

$$\begin{aligned}
&\frac{m_3}{2p_3} = \frac{m_1}{2p_1} \cdot \frac{m_2}{2p_2} - \left(\pm\frac{\sqrt{4p_1^2 - m_1^2}}{2p_1}\right) \cdot \left(\pm\frac{\sqrt{4p_2^2 - m_2^2}}{2p_2}\right),\\
&(2m_3 p_1 p_2 - m_1 m_2 p_3)^2 = p_3^2(4p_1^2 - m_1^2)(4p_2^2 - m_2^2),\\
&4m_3^2 p_1^2 p_2^2 + m_1^2 m_2^2 p_3^2 - 4m_1 m_2 m_3 p_1 p_2 p_3\\
&\quad = 16 p_1^2 p_2^2 p_3^2 - 4p_1^2 m_2^2 p_3^2 - 4m_1^2 p_2^2 p_3^2 + m_1^2 m_2^2 p_3^2,\\
&p_1^2 p_2^2 m_3^2 - p_1 p_2 p_3 m_1 m_2 m_3 - 4p_1^2 p_2^2 p_3^2 + p_1^2 m_2^2 p_3^2 + m_1^2 p_2^2 p_3^2 = 0.
\end{aligned}$$

Since $p_1 \equiv p_2 \equiv p_3 \equiv 1 \bmod 8$ we get (4). □

Consider the solutions of (4) in $\mathbb{Z}_8$. Clearly, every triple (m_1, m_2, m_3) that is not a solution of this equation encodes a forbidden change in the class when we take the sum of two angles. For example, since $(1,2,3)$ is not a solution, adding an angle of class 1 and an angle of class 2 cannot result in an angle of class 3. The equation is symmetric in m_1, m_2, and m_3. We will be later interested in solutions where one of the m_is is 1, 3, 5, or 7. Checking every triple we find that these solutions are the following ones and the re-orderings of these: $(1,1,2)$, $(1,1,7)$, $(1,2,5)$, $(1,3,5)$, $(1,3,6)$, $(1,6,7)$, $(2,3,3)$, $(2,3,7)$, $(2,5,5)$, $(2,7,7)$, $(3,3,7)$, $(3,5,6)$, $(5,5,7)$, $(5,6,7)$, $(7,7,7)$.

4 Proof of the Main Theorem

The idea of the proof is simple, we aim to show that $\sum_{i=1}^{n}\theta_{i,i+1}$ is not a multiple of 2π using the fact that each $\theta_{i,i+1}$ is an angle of a triangle whose sides have odd length. Note that it is important that the triangles in a wheel embedding share sides with their neighbors. The statement is not true for an an arbitrary set of triangles, as shown by the following example. Both $\pi/3$ and $2\pi/3$ appear in triangles with odd sides and $\pi/3+\pi/3+\pi/3+\pi/3+2\pi/3=2\pi$.

Proof of Theorem 1.1 Suppose that there is a counterexample to Theorem 1.1. It follows from Lemma 3.1 that each $\theta_{i,i+1}$ has a unique residual. Let $\phi_1,\dots,\phi_n$ be a reordering of the angles $\theta_{1,2},\theta_{2,3},\dots,\theta_{n,1}$ in such a way that the angles of given residuals are consecutive. In general an arbitrary angle might not have any residual. The advantage of this ordering is that $\sum_{i=1}^{\ell}\phi_i$ has a residual for each $\ell\in\{0,1,\dots,n\}$. To see this suppose that the residual of ϕ_ℓ is D. From Lemma 2.6 we see that the ϕ_is before ϕ_ℓ whose residual is not D sum up to an integer multiple of π. Thus, they do not affect the residual of $\sum_{i=1}^{\ell}\phi_i$. Since angles with residual D are closed under addition, the rest sums up to an angle that has residual D.

Further, it follows from Lemma 3.1 that $D\equiv 3 \bmod 8$. Hence by Lemma 3.2 we obtain that $\sum_{i=1}^{\ell}\phi_i$ has a class for each $\ell\in\{0,1,\dots,n\}$. Consider how the class changes as ℓ goes from 0 to n.

In each step we increase the angle by some $\theta_{j,j+1}$. We have

$$\cos\theta_{j,j+1}=\frac{r_j^2+r_{j+1}^2-r_{j,j+1}^2}{2r_jr_{j+1}}=\frac{(r_j^2+r_{j+1}^2-r_{j,j+1}^2)r_jr_{j+1}}{2r_j^2r_{j+1}^2}.$$

Since r_j, r_{j+1}, and $r_{j,j+1}$ are odd numbers, $r_j^2+r_{j+1}^2-r_{j,j+1}^2\equiv 1 \bmod 8$ and $r_j^2r_{j+1}^2\equiv 1 \bmod 8$. Therefore the class of $\theta_{j,j+1}$ is the remainder of r_jr_{j+1} modulo 8, which is either 1, 3, 5, or 7. We will use this fact in the following form. If $r_jr_{j+1}\equiv 1 \bmod 4$, then the class of $\theta_{j,j+1}$ is either 1 or 5, and if $r_jr_{j+1}\equiv 3 \bmod 4$, then the class of $\theta_{j,j+1}$ is either 3 or 7. Therefore, as we increase ℓ the angle $\sum_{i=1}^{\ell}\phi_i$ changes either by an angle whose class is 1 or 5, or by an angle whose class is 3 or 7, depending on the remainder of r_jr_{j+1} divided by 4.

Now we are ready to use Lemma 3.3. The solutions of (4) have an underlying structure, which we can use. This is depicted in Fig. 3. We create a graph G whose vertex set is $\{1,2,3,5,6,7\}$. For solutions of the form $(1,x,y)$ and $(5,x,y)$ we connect x and y by a dashed edge. For solutions of the form $(3,x,y)$ and $(7,x,y)$

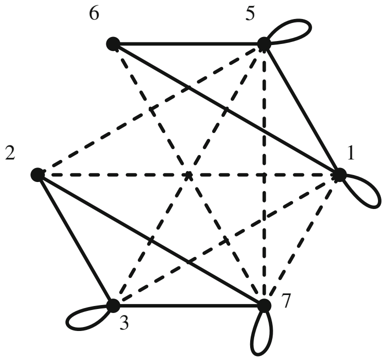

Fig. 3 Possible changes in the class when adding an angle of class 1, 3, 5, or 7

we connect them by a solid edge, allowing loop edges. These two sets of edges are disjoint. Note that the dashed edges form a bipartite graph such that the solid edges connect vertices inside the two parts. This allows us to use a parity argument, as any closed trail in this graph must use an even number of dashed edges.

Now we are ready to finish the proof. Let T be the trail of length $n+1$ in G whose ℓ-th vertex is the class of $\sum_{i=1}^{\ell-1}\phi_i$. Since $\sum_{i=1}^{n}\phi_i$ is an integer multiple of 2π, we have $\cos\sum_{i=1}^{n}\phi_i = 2/(2\cdot 1)$. Hence the trail starts and ends at 2. By Lemma 3.3, when the class of ϕ_i is 1 or 5, we follow one of the solid edges, if the class is 3 or 7, we follow a dashed edge.

Finally, we show that we followed a dashed edge an odd number of times. Considering the equation $(r_1r_2)(r_2r_3)\cdots(r_{n-1}r_n)(r_nr_1) = (\prod r_i)^2 \equiv 1 \bmod 4$ we have $r_ir_{i+1} \equiv 3 \bmod 4$ for an even number of is. Since n is odd, this implies that we have an odd number of is when $r_ir_{i+1} \equiv 1 \bmod 4$. Hence the trail contains an odd number of dashed edges. Since the dashed edges form a bipartite graph and the solid edges connect vertices inside the two parts, the trail cannot end where it started, a contradiction. This shows that a counterexample to Theorem 1.1 cannot exists. $\square$

5 Final Remarks

We note that some parts of the proof can be replaced by other arguments. For example, Lemma 3.3 also follows from the analysis of Cayley–Menger determinants.

The main goal of understanding odd-distance graphs is to determine the chromatic number of G^{odd}. Odd wheels are the simplest graphs that are not 3-colorable, yet they are not odd-distance graphs. Our proof relies on the fact that a wheel graph contains many triangles. An other interesting question of Rosenfeld and Le [18] is the following. Are there triangle-free graphs that are not odd-distance graphs?

Piepmeyer's construction which shows that $K_{n,n,n}$ is an odd-distance graph comes from an integral point set [17]. Recently Le Tien Nam and Nguyen Tho Tung showed a similar construction [15]. Naturally, one might be tempted to look for odd-distance graphs with high chromatic number in a similar way. Take an integral point set and then consider the odd-distance graph given by the edges of odd length. We note that this method is not sufficient, the chromatic number of graphs constructed this way is at most 3. We leave the proof of this statement to the interested readers.

We can also consider the following analog of Harborth's conjecture. Which planar graphs have a planar drawing with odd edge lengths? Take for example a maximal planar graph, in other words a triangulation. If it contains an odd wheel, it is not an odd-distance graph. On the other hand if it does not contain an odd wheel, it is 3-colorable. Hence it is an odd distance graph, but this does not imply that we can find a plane drawing. Is it true that all 3-colorable planar graphs have an embedding without crossings with odd integer edge lengths?

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