

Characterization and a 2D Visualization of B_0 -VPG Cocomparability Graphs

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Abstract. B_0 -VPG graphs are intersection graphs of vertical and horizontal line segments on a plane. Cohen, Golumbic, Trotter, and Wang [Order, 2016] pose the question of characterizing B_0 -VPG permutation graphs. We respond here by characterizing B_0 -VPG cocomparability graphs. This characterization also leads to a polynomial time recognition and B_0 -VPG drawing algorithm for the class. Our B_0 -VPG drawing algorithm starts by fixing any one of the many posets P whose cocomparability graph is the input graph G . The drawing we obtain not only visualizes G in that one can distinguish comparable pairs from incomparable ones, but one can also identify which among a comparable pair is larger in P from this visualization.

Keywords: Poset Visualization · Permutation graph · Cocomparability graph · B_0 -VPG · Graph drawing.

1 Introduction

Representing a graph as an intersection graph of two-dimensional geometric objects like strings, line segments, rectangles and disks is a means to depict a graph on the plane. When the graph being represented is a comparability or cocomparability graph, one can also ask whether the “direction” of the comparability relation in the associated poset can also be inferred from the drawing. In this paper we characterize cocomparability graphs which can be represented as intersection graphs of vertical and horizontal line segments in a plane. For the posets whose cocomparability graphs can be represented thus, we describe a representation from which one can also infer the direction of the comparability relation. Our drawing algorithm runs in polynomial time.

B_k -VPG graphs are intersection graphs of simple paths with at most k bends on a two-dimensional grid. Here, a path is simple if it does not pass through any grid vertex twice, and two paths are said to intersect if they share a vertex of the grid. The name B_k -VPG is an abbreviation for Vertex-intersection graphs of k -Bend Paths on a Grid. In particular, B_0 -VPG graphs are intersection graphs of vertical and horizontal line segments on a plane. The *bend number* of a graph G is the minimum k for which G belongs to B_k -VPG.

The *dimension* of a poset $P = (X, <)$ is the smallest k such that $<$ is the intersection of k total orders on X . The *comparability graph* of P is the undirected

graph on the vertex set X with edges between the pairs of elements comparable in P . A graph G is a *comparability graph* if it is the comparability graph of a poset. If two posets have the same comparability graph, then they have the same dimension [29]. Hence we can unambiguously define the *dimension of a comparability graph* G as the dimension of any poset P whose comparability graph is G . The complement of a comparability graph is a *cocomparability graph*. A *permutation graph* is a comparability graph of dimension at most two. It is known that a graph G is a permutation graph if and only if G is both comparability and cocomparability [26].

Cohen et al. [11] illustrated, via an elegant picture-proof, that if G is a comparability graph of dimension k ($k \geq 1$), the bend number of its complement \overline{G} is at most $k - 1$. In particular therefore, the bend number of a permutation graph is either 0 or 1. They posed the problem of characterizing permutation graphs with bend number 0 as an open question (Qn 4.2 in [11]). We settle this question with a stronger result. We characterize cocomparability graphs with bend number 0 as follows (Theorem 1).

The simple cycle on k vertices is denoted by C_k . A C_4 together with an additional edge e between two non-consecutive vertices of the C_4 is a *diamond* and the edge e is a *diamond diagonal*. Two vertices x and y in a graph G are *diamond related* if there exists a path from x to y in G made up of diamond diagonals alone. This is easily verified to be an equivalence relation that refines the connectivity relation in G .

Theorem 1. *A cocomparability graph G is B_0 -VPG if and only if*

- (i) *No two vertices of an induced C_4 in G are diamond related, and*
- (ii) *G does not contain an induced subgraph isomorphic to $\overline{C_6}$, the complement of C_6 .*

A poset $P = (X, \prec)$ is an *interval order* if all the elements of X can be mapped to intervals on \mathbb{R} such that $\forall x, y \in X$, $x \prec y$ if and only if the interval representing x is disjoint from and to the left of the interval representing y . Complements of the comparability graphs of interval orders form the well known class of *interval graphs*. While interval graphs are trivially B_0 -VPG, it is known that there exists interval orders of arbitrarily high dimension [3]. Hence the class of B_0 -VPG cocomparability graphs is richer than the class of B_0 -VPG permutation graphs. In fact, since permutation graphs are $\overline{C_6}$ -free, the first of the two conditions in Theorem 1 characterizes B_0 -VPG permutation graphs.

Corollary 1. *A permutation graph G is B_0 -VPG if and only if no two vertices of an induced C_4 in G are diamond related.*

A naive check for the conditions in Theorem 1 can be done in $O(n^6)$ time. Combining this with any of the known polynomial time recognition algorithms for cocomparability graphs [18,12] will give a polynomial time recognition algorithm for B_0 -VPG cocomparability graphs. We do not try to optimize the recognition algorithm here, but only note that this is in contrast to the NP-completeness of recognizing B_0 -VPG graphs.

The above algorithm starts by fixing a partial order P_G whose cocomparability graph is G and a linear extension σ of P_G . The resulting drawing D ends up being a representation of $P_G = (V(G), \prec_P)$ in the following sense. We define three binary relations $\prec_D^{v,h}$, $\prec_D^{h,v}$ and \prec_D among vertices of G based on the drawing D as follows.

Definition 1. Let x and y be two vertices in $V(G)$ and let I_x and I_y be the line segments representing them in D , respectively.

- $x \prec_D^{v,h} y$ if I_x is either vertically below I_y or if both are intersected by a horizontal line, I_x is to the left of I_y .
- $x \prec_D^{h,v} y$ if I_x is either to the left of I_y or if both are intersected by a vertical line, I_x is vertically below I_y .
- $x \prec_D y$ if and only if $x \prec_D^{v,h} y$ when I_x is horizontal and $x \prec_D^{h,v} y$ when I_x is vertical.

While the above relations are not even partial orders in general, in our drawing D , the relation \prec_D faithfully captures \prec_P . Theorem 2 states this formally and Figure 1 illustrates an example.

Theorem 2. Any poset $P_G = (V_G, \prec_P)$ whose cocomparability graph G is B_0 -VPG has a two dimensional visualization D such that $x \prec_P y$ if and only if $x \prec_D y$.

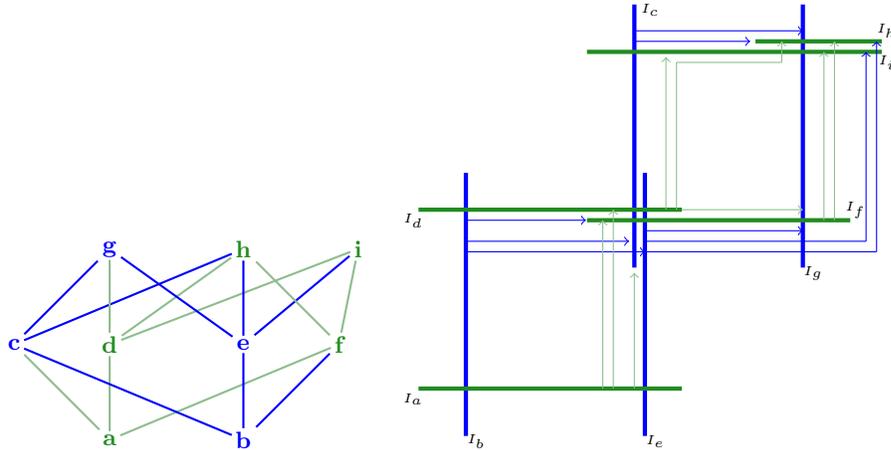


Fig. 1: Hasse diagram of a poset corresponding to a B_0 -VPG cocomparability graph and a B_0 -VPG representation D in which the covering relation is indicated by thin directed paths for clarity. The directed paths with blue color and green color respectively depict the relations $\prec_D^{h,v}$ and $\prec_D^{v,h}$.

1.1 Literature

B_k -VPG graphs were introduced by Asinowski et al. in 2012 [2] as a parameterized generalization for string graphs (intersection graphs of curves in a plane) and grid intersection graphs (bipartite graphs which are intersection graphs of vertical and horizontal segments in the plane in which all the vertices in one part are represented by vertical segments and all the vertices in the other part by horizontal line segments). Grid Intersection graphs (GIGs) are equivalent to bipartite B_0 -VPG graphs. Similarly one can show that B_k -VPG graphs with an unrestricted k , which are called VPG graphs simpliciter, are equivalent to string graphs. B_0 -VPG graphs are also equivalent to 2-DIR graphs, where a k -DIR graph is an intersection graph of line segments lying in at most k directions in the plane. All these equivalences were formally established in [2].

The NP-completeness of the recognition problem for VPG graphs follows from that of string graphs [23,27]. For B_0 -VPG graphs, it follows from that of 2-DIR graphs [24]. Chaplick et al. showed that, $\forall k \geq 0$, it is NP-complete to recognize whether a given graph G is in B_k -VPG even when G is guaranteed to be in B_{k+1} -VPG and represented as such [7]. This also shows that $\forall k \geq 0$ the classes B_k -VPG and B_{k+1} -VPG are separated. Cohen et al. showed that, $\forall k \geq 0$, there exists a cocomparability graph with bend number k (Theorem 3.1 in [11]). This shows that, $\forall k \geq 0$, the classes B_k -VPG and B_{k+1} -VPG are separated within cocomparability graphs. The question of a similar separation within chordal graphs was left open in [7] and a partial answer was given in [4].

Since the B_k -VPG representation is a kind of planar representation, the bend number of planar graphs have received special attention. Chaplick and Ueckerdt showed, disproving a conjecture in [2], that every planar graph is B_2 -VPG [8]. Every planar bipartite graph is a GIG [21] and hence B_0 -VPG. The order dimension of GIGs has also been investigated in literature [6]. A polynomial time decision algorithm for chordal B_0 -VPG graphs is developed in [5]. Characterizations for B_0 -VPG are known within the classes of split graphs, chordal bull-free graphs, chordal claw-free graphs [20] and block graphs [1].

Subclasses of cocomparability graphs within which a characterization for B_0 -VPG is known include cographs, bipartite permutation graphs, and interval graphs. Cographs, which form a subgraph of permutation graphs are B_0 -VPG if and only if they do not contain an induced W_4 [10]. A W_4 is a C_4 together with a universal fifth vertex. All bipartite permutation graphs are B_0 -VPG [11]. Interval graphs are trivially B_0 -VPG, since the interval representation itself is a B_0 -VPG representation. Theorem 1 subsumes these three results.

Towards the end of this paper, we discuss a two dimensional visualization of posets. The most common way to visualize a poset $P = (X, <)$ so far is *Hasse Diagram* (also called Order Diagram). The problem of drawing a Hasse diagram algorithmically was addressed by many algorithms *e.g. upward planar drawing* [13], *dominance drawing* [14], *confluent drawing* [15], *weak dominance drawing* [22]. The key concern here is to get a crossing-free drawing in which no two upward edges cross at a non vertex point. The first three algorithms can only handle posets of dimension at most two and a few other cases. Though

our drawing handles only those posets whose cocomparability graph is B_0 -VPG irrespective of its dimension, crossing-freeness is not a concern in our drawing since comparability is inferred from the relative position of lines.

1.2 Terminology and Notation

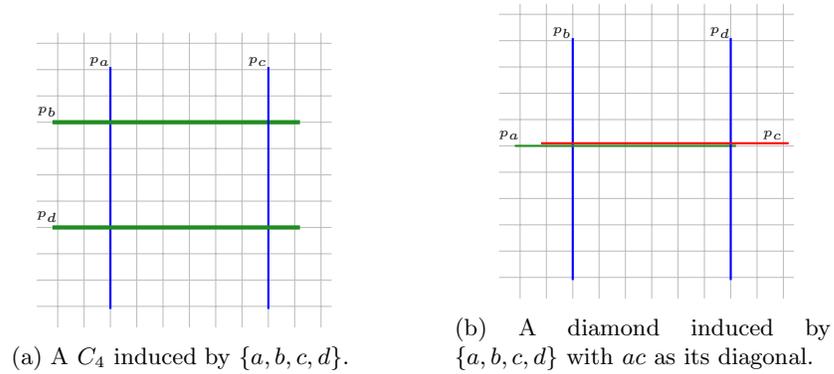
The complement of a graph G is denoted as \overline{G} . We denote a path and a cycle on n vertices, respectively, by P_n and C_n . A graph G is said to be H -free if G contains no induced subgraph isomorphic to the graph H . A poset P is said to be T -free if P contains no induced subposet isomorphic to the poset T . In this case, P is also said to *exclude* T .

The *closed neighborhood* of a vertex v is the set of neighbors of v together with v . An *Asteroidal Triple (AT)* is a set of three independent vertices such that there exists a path between each two of them not passing through any vertex from the closed neighborhood of the third. We have defined interval orders, interval, comparability, cocomparability, and permutation graphs in the introduction. We will make use of the facts that C_4 -free cocomparability graphs are interval graphs [17] and all cocomparability graphs are AT-free [19].

2 Proof of Theorem 1

The necessity of the two conditions in Theorem 1 is relatively easier to establish, and hence we do that first. A C_4 has a unique B_0 -VPG representation as shown in Figure 2(a) [2]. Notice that no two vertices of an induced C_4 can be represented by collinear paths in a B_0 -VPG representation. In contrast, one can see that in any B_0 -VPG representation of a C_3 , at least two of its three vertices have to be represented by collinear paths. Moreover, in any B_0 -VPG representation of a diamond, the endpoints of the diamond diagonal have to be represented by collinear paths as shown in Figure 2(b) [20]. Since collinearity is transitive, any two vertices which are diamond related have to be represented by collinear paths. This shows the necessity of the first condition in Theorem 1. Notice that $\overline{C_6}$ has a C_3 in which each pair of vertices is part of an induced C_4 . Being part of C_3 forces two of the corresponding three paths to be collinear which prevents a B_0 -VPG representation of the corresponding induced C_4 . Hence the necessity of the second condition.

The three step algorithm (Algorithm 1) and the proof of its correctness (Appendix A) are devoted to showing that these two necessary conditions are sufficient to construct a B_0 -VPG representation of a cocomparability graph. The construction is completed in three steps. We start with a cocomparability graph G satisfying the two conditions of Theorem 1. In the first step, we contract a subset of edges of G to obtain a bipartite minor R_G of G with a couple of additional properties. A set of vertices in G which gets represented by a single vertex in R_G after all the edge contractions is referred to as a *branch set* of R_G . We will denote the vertices in R_G by the corresponding branch sets. In the second step, for each of the subgraphs of G induced by each branch set of R_G , we find an

Fig. 2: The unique B_0 -VPG representation of C_4 and a diamond.

interval representation, again with a few additional properties. In the third and final step, we fit all the above interval representations together to get a B_0 -VPG representation of G .

Before proceeding to the algorithm, we state in the next section some of the known results which ease our construction.

2.1 Preliminaries

An ordering σ of V of a graph $G(V, E)$ is called a *cocomparability ordering* or an *umbrella-free ordering* if for all three vertices in $x <_\sigma y <_\sigma z$, adjacency of x and z implies that at least one of the other pairs are adjacent. If not, (x, y, z) is called an *umbrella* in σ .

Lemma 1 ([25]). *A graph G is a cocomparability graph if and only if there is a cocomparability ordering σ of the vertices of G .*

Definition 2. *Given a graph G and a total ordering σ of $V(G)$, a triple (u, v, w) of vertices of G where $u <_\sigma v <_\sigma w$ is called a forbidden triple if there exists a path from u to w without containing a vertex from the closed neighborhood of v .*

Lemma 2. *Any umbrella free ordering is forbidden triple free.*

Proof. Let σ be an umbrella free ordering. Assume a forbidden triple $u <_\sigma v <_\sigma w$ exists. Thus there exists a path from u to w without containing a vertex from the closed neighborhood of v . If we arrange the vertices of the path together with v in an order respecting σ , there exists two adjacent vertices u_1 and w_1 among them such that $u_1 <_\sigma v <_\sigma w_1$. Thus (u_1, v, w_1) forms an umbrella in σ which is a contradiction.

Lemma 3 ([9]). *Cocomparability is preserved under edge contraction.*

A $2+2$ is a poset containing four elements where every element is comparable with exactly one element. A $2+2$ poset corresponds to an induced C_4 in the complement of its comparability graph. Thus an interval order cannot contain a $2+2$. Similarly, if a poset does not have a $2+2$, then the complement of all of its comparability graphs will be C_4 -free cocomparability graphs.

Theorem 3 (Fishburn-1970). [[16], Theorem 6.29 in [30]] *A poset is an interval order if and only if it excludes $2+2$.*

Consider a bipartite graph $G(A \cup B, E)$. An ordering σ of A is said to have *adjacency property* if the neighborhood of every vertex of B is consecutive in σ . Here G is called *convex* if there exists an ordering σ of A with the adjacency property and *biconvex* if it is convex and there exists an ordering τ of B with the adjacency property. Bipartite permutation graphs are biconvex graphs [28].

Theorem 4 ([11]). *Bipartite permutation graphs are B_0 -VPG.*

2.2 B_0 -VPG Algorithm

We see a three-step algorithm to construct a B_0 -VPG representation for any arbitrary cocomparability graph satisfying the conditions of Theorem 1. Figure 3 helps to understand the algorithm easily. The first step is depicted in Figure 3b, 3c, 3e and the final drawing in the third step is shown in Figure 3f.

In the following definition, we assume that any self loop produced by an edge contraction is removed and any parallel edges formed by an edge contraction is represented by a single edge in the minor.

Definition 3 (dd-minor). *A dd-minor of graph G is the graph obtained by contracting every diamond diagonal in G .*

Definition 4 (Reduced dd-minor). *A reduced dd-minor of graph G is a minimal graph R_G that can be obtained by edge contractions of the dd-minor of G such that no branch-set of R_G contains more than one vertex of an induced C_4 in G .*

Remark 1. Though the dd-minor exists for every graph, a reduced dd-minor does not exist for every graph. A necessary and sufficient condition for the existence of a reduced dd-minor for a graph G is that no two vertices of an induced C_4 in G should be diamond related.

If an edge xy ($x \prec_\sigma y$) is contracted to a new vertex, then the new vertex is placed at the position of x in σ and labeled as x itself. This results in a new order σ' which is a subsequence of σ . By Lemma 3, σ' is an umbrella-free ordering. Thus after all the edge contractions to get the minimal graph R_G , we get an umbrella-free ordering σ_{R_G} of $V(R_G)$ which is a subsequence of σ . This is sufficient to say that R_G is also a cocomparability graph by Lemma 1. In fact, every vertex B of R_G is represented in σ_{R_G} by the leftmost (under σ) vertex b in the branch set B .

For any two branch sets B_i and B_j of R_G , $B_{j,i}$ denotes the vertices in B_j which have a neighbor in B_i .

Lemma 4 (Proof in Appendix A). *The following claims on the cocomparability graph R_G are true.*

1. *For any two adjacent branch sets B_1 and B_2 , the set $B_{1,2} \cup B_{2,1}$ induces a clique in G .*
2. *If B_0, B_1, B_2 form consecutive vertices of a C_3 or an induced C_4 in R_G then $B_{1,0} \cap B_{1,2} \neq \emptyset$. Moreover, if B_0, \dots, B_{k-1} is a C_3 or an induced C_4 in R_G , then there exists an induced cycle b_0, \dots, b_{k-1} in G where each $b_i \in B_i$.*
3. *R_G is a bipartite permutation graph.*

Relabeling of $V(R_G)$ It is clear from Lemma 4.3 that the reduced dd-minor R_G of the cocomparability graph G is a bipartite permutation graph. Given the umbrella-free ordering σ of G , we inherited the umbrella-free ordering σ_{R_G} for R_G which respects σ . In the algorithm, we label each branch set of the left part of R_G with B_1, B_3, \dots (odd indices) such that $i < j$ implies that in the order σ , the leftmost vertex in B_i is to the left of the leftmost vertex in B_j . Similarly we label each branch set of the right part of R_G with B_0, B_2, \dots (even indices) such that $i < j$ implies that in the order σ , the leftmost vertex in B_i is to the left of the leftmost vertex in B_j .

Henceforth, we slightly abuse the notation \prec_σ for the branch sets of the reduced dd-minor of G in the following way. For any two such branch sets B_i and B_j , $B_i \prec_\sigma B_j$ if $\forall x \in B_i, \forall y \in B_j, x \prec_\sigma y$. Thus B_i and B_j are said to be *separated in σ* if either $B_i \prec_\sigma B_j$ or $B_j \prec_\sigma B_i$.

Lemma 5 (Proof in Appendix A). *For any two branch sets B_i, B_j of the same parity if $i < j$, then $B_i \prec_\sigma B_j$.*

Lemma 6 (Proof in Appendix A). *For each branch set B_i of R_G , $G[B_i^*]$ is an interval graph. Moreover, $G[B_i^*]$ has an interval representation \mathcal{I}_i^* satisfying the following properties.*

- (i) *For all $x, y \in B_i^*$, we have the interval for x to the left of interval for y if and only if $x \prec_P y$.*
- (ii) *For each neighbor B_j of B_i , all the intervals corresponding to the vertices in $B_{j,i}$ are point intervals at a point $p_{j,i}$.*
- (iii) *If B_j and B_k are two neighbors of B_i such that $j < k$, then the point $p_{j,i}$ is to the left of the point $p_{k,i}$ in \mathcal{I}_i^* .*

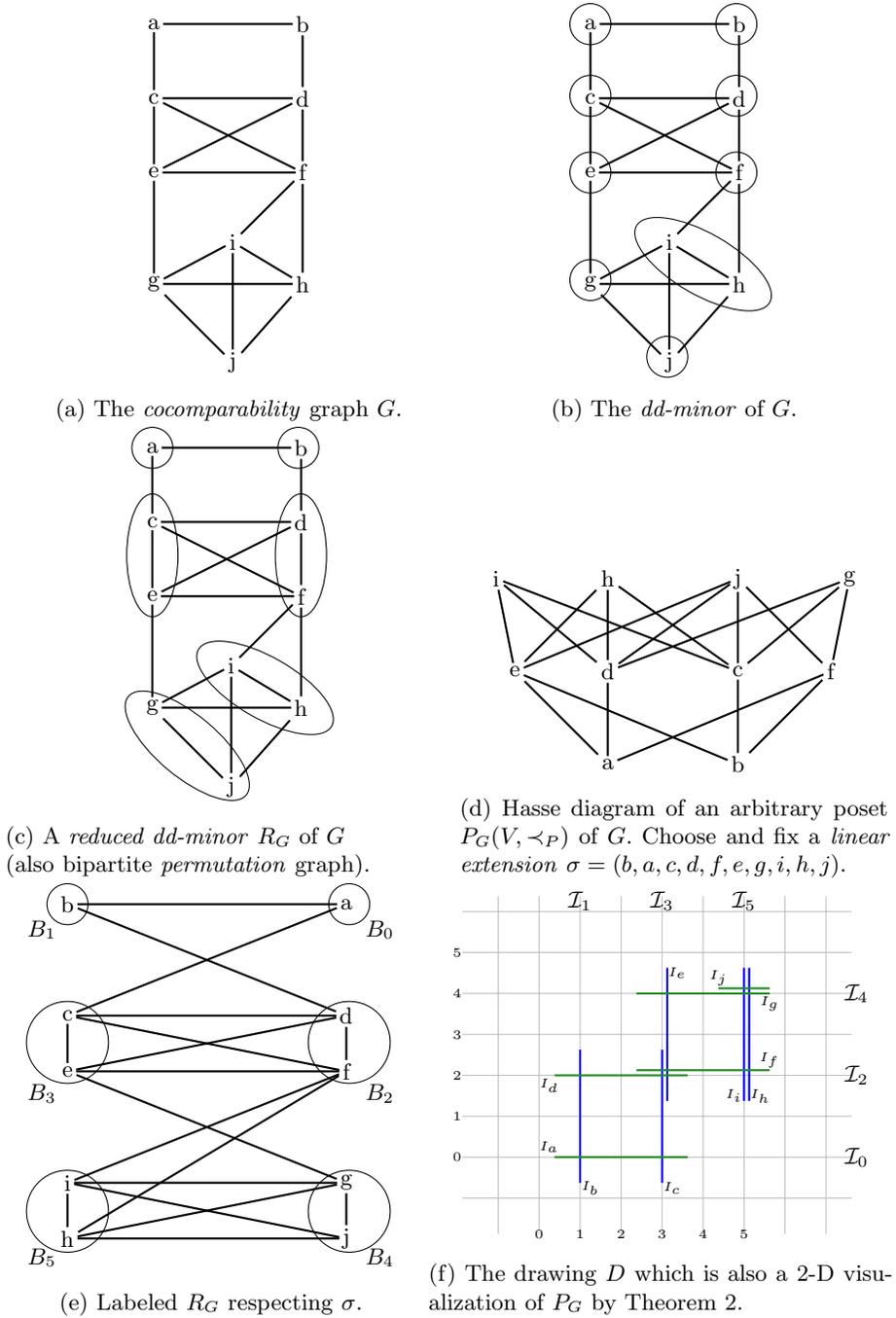


Fig. 3: Drawing a B_0 -VPG representation D of a cocomparability graph G satisfying the conditions of Theorem 1. Note that the collinear intersecting line segments in D are drawn a little apart in order to distinguish them easily.

Algorithm 1

Input. G is an arbitrary but fixed cocomparability graph satisfying the two conditions of Theorem 1.

Output. A B_0 -VPG representation D of G .

Assumptions.

- 1) $P_G(V(G), \prec_P)$ is an arbitrary but fixed partial order whose comparability graph is \overline{G} .
- 2) σ is an arbitrary but fixed linear extension of P_G and hence an umbrella-free ordering for G .

1. Step 1: Choose an arbitrary but fixed reduced dd-minor R_G of G and label $V(R_G)$ as described in the above mentioned relabeling procedure.
2. Step 2:
 - (i) For each branch set B_i of R_G , let $B_i^* = B_i \cup \{B_{j,i} : B_i B_j \in E(R_G)\}$ and we obtain an interval representation \mathcal{I}_i^* of B_i^* using Lemma 6.
 - (ii) Remove intervals of vertices in $B_i^* \setminus B_i$ from \mathcal{I}_i^* to get \mathcal{I}_i .
3. Step 3: Construction of the drawing D using the following steps. Let e and o respectively (with further subscripts if needed) denote the even and odd indices of the branch sets of R_G .
 - (i) For each odd-indexed branch set B_o , \mathcal{I}_o is drawn vertically from the point $(o, (e_1 - 0.5))$ to $(o, (e_2 + 0.5))$ where B_{e_1} and B_{e_2} are the leftmost and rightmost neighbors of B_o in σ_{R_G} .
 - (ii) Stretch or shrink the intervals in each vertical interval representation \mathcal{I}_o without changing their intersection pattern, so that for each neighbor B_e of B_o , the point $p_{e,o}$ is at (o, e) . This can be done since the intersection pattern of an interval representation with n intervals is solely determined by the order of the corresponding $2n$ endpoints.
 - (iii) For each even-indexed branch set B_e , \mathcal{I}_e is drawn horizontally from the point $((o_1 - 0.5), e)$ to $((o_2 + 0.5), e)$ where B_{o_1} and B_{o_2} are the leftmost and rightmost neighbors of B_e in σ_{R_G} .
 - (iv) Stretch or shrink the intervals in each horizontal interval representation \mathcal{I}_e without changing their intersection pattern, so that for each neighbor B_o of B_e , the point $p_{o,e}$ is at (o, e) .

Proposition 1. *The B_0 -VPG representation D is precisely a B_0 -VPG representation of the cocomparability graph G .*

The proof of the above proposition is written in Appendix A. One can easily verify that Algorithm 1 runs in polynomial time.

3 Proof of Theorem 2

In this section, we fix $P(V, \prec_P)$ as the given input poset and G as its cocomparability graph. Let D be the B_0 -VPG representation of G obtained by the construction employed in the proof of Theorem 1, where the partial order P_G

assumed in Algorithm 1 is P . We argue below that for any two vertices x and y in $V(G)$, $x \prec_D y$ (cf. Definition 1) if and only if $x \prec_P y$ and thus establish Theorem 2. First, we show that if $x \prec_P y$, then $x \prec_D y$. But a simple exchange of variables is not enough to prove the converse because \prec_D is not antisymmetric in the set of all vertical and horizontal line segments. Hence in order to complete the proof we show that the relation \prec_D is antisymmetric when restricted to the line segments in D .

3.1 If $x \prec_P y$ then $x \prec_D y$

Recall that σ is a linear extension of P_G . Thus if $x \prec_P y$, then $x \prec_\sigma y$ and x is non-adjacent to y in G .

If x and y are in the same branch set B_i of R_G then clearly $I_x \prec_{\mathcal{I}_i} I_y$ by Lemma 6.(i). Here if I_x is horizontal, clearly $x \prec_D^{v,h} y$. Otherwise, $x \prec_D^{h,v} y$. If x and y are in different branch sets, B_i and B_j respectively, of the same parity, then $i < j$ (Lemma 5). Thus in D , if both are of odd parity, B_i is drawn to the left of B_j and if both are of even parity, B_i is drawn to the bottom of B_j . Thus if I_x is vertical, then $x \prec_D^{h,v} y$ and if I_x is horizontal, then $x \prec_D^{v,h} y$.

If the parity is opposite, we have two sub-cases; that is B_i and B_j are either non-adjacent or adjacent. If non-adjacent, the branch set B_j cannot have a neighbor B_h where $h < i$. Suppose there exists such a neighbor B_h for B_j such that $h < i$. Clearly since B_h is adjacent to B_j , h and i are of the same parity and hence $B_h \prec_\sigma B_i$ (Lemma 5). Since B_h is adjacent to B_j , there exists a path from a vertex $z \in B_h$ to $y \in B_j$ in $G[B_h \cup B_j]$. Moreover since B_i is disjoint from B_h and B_j , x has no neighbor in $B_h \cup B_j$, the hence triple (z, x, y) is a forbidden triple in σ which is a contradiction as per Lemma 2. Since B_j has no opposite parity neighbor B_h for any $h \leq i$, the following property can easily be verified from our drawing. Thus if B_i is of even parity, then B_i is to the bottom of B_j in D . Otherwise, B_i to the left of B_j in D . Hence clearly if I_x is horizontal, then I_x is to the bottom of I_y , that is $x \prec_D^{v,h} y$. Similarly, if I_x is vertical, then I_x is to the left of I_y , that is $x \prec_D^{h,v} y$.

Now the remaining sub-case is that B_i and B_j are adjacent. The following observation is frequently used in the remaining part of the proof.

Observation 1. *If the interval representations \mathcal{I}_i and \mathcal{I}_j intersects, there exist at least two intersecting intervals $I_{u_1} \in \mathcal{I}_i$ and $I_{v_1} \in \mathcal{I}_j$. For any interval $I_u \in \mathcal{I}_i$, if $I_u \prec_{\mathcal{I}_i} p_{j,i}$, then $u \prec_\sigma v_1$ and if $p_{j,i} \prec_{\mathcal{I}_i} I_u$ then $v_1 \prec_\sigma u$. This is easily inferred from \mathcal{I}_i^* . We can symmetrically argue the same for I_v .*

The intervals I_x and I_y do not intersect since x and y are non-adjacent in G . If I_x contains $p_{j,i}$, then $p_{i,j}$ (geometrically coinciding with $p_{j,i}$) has to precede I_y in \mathcal{I}_j . Otherwise due to Observation 1, we get $y \prec_\sigma x$ which is a contradiction. Similarly if I_y contains $p_{i,j}$, then I_x has to precede $p_{j,i}$ in \mathcal{I}_i . In both these case, if I_x is horizontal, then $x \prec_D^{v,h} y$ and if I_x is vertical, then $x \prec_D^{h,v} y$. Henceforth we assume that neither I_x nor I_y contains the intersection point of \mathcal{I}_i and \mathcal{I}_j . In

this case, it is easy to see that x has no neighbors in B_j and y has no neighbors in B_i .

In order to rule out the following scenarios, we show the existence of a forbidden triple in σ which leads to a contradiction as per the Lemma 2.

If $I_y \prec_{\mathcal{I}_j} p_{i,j}$, then $x \prec_{\sigma} y \prec_{\sigma} u_1$ as per Observation 1. Thus (x, y, u_1) is a forbidden triple.

If $p_{j,i} \prec_{\mathcal{I}_i} I_x$, then $v_1 \prec_{\sigma} x \prec_{\sigma} y$ as per Observation 1. Thus (v_1, x, y) is a forbidden triple.

Hence $p_{i,j} \prec_{\mathcal{I}_j} I_y$ and $I_x \prec_{\mathcal{I}_i} p_{j,i}$. In this case, if I_x is horizontal, $x \prec_D^{v,h} y$ and if I_x is vertical, we get $x \prec_D^{h,v} y$. Moreover, in both these cases, I_x is to the left and to the bottom of I_y .

Thus we have proved that when $x \prec_P y$, we get that $x \prec_D^{v,h} y$ when I_x is horizontal or $x \prec_D^{h,v} y$ when I_x is vertical in the B_0 -VPG representation D of G . That is $x \prec_D y$. If x and y are incomparable in P_G , then they are adjacent in G and the corresponding intervals intersect in D .

3.2 Antisymmetry of \prec_D

Observation 2. *If two opposite parity branch sets are non-intersecting, then one of them is entirely to the bottom left of the other in D . Hence for all I_x in the bottom left branch set and for all I_y in the top right branch set, $x \prec_D y$.*

Justification. When two branch sets are non-intersecting, they are separated in σ since otherwise, there will exist a forbidden triple $u \prec_{\sigma} v \prec_{\sigma} w$ such that u and w are in the one branch set and v in the other branch set. Without loss of generality, let $B_i \prec_{\sigma} B_j$. We claim that B_i is entirely to the bottom left of B_j . Assume not. That is either B_i has an opposite parity neighbor B_k for some $k > j$ or B_j has an opposite parity neighbor B_h for some $h < i$ or both. In the first case, there exists a forbidden triple (x, y, z) for any $x \in B_i$, $y \in B_j$ and $z \in B_k$ which is a contradiction by Lemma 2. Similarly in the second case, there exists a forbidden triple (w, x, y) for any $w \in B_h$, $x \in B_i$ and $y \in B_j$ which is again a contradiction by Lemma 2.

Observation 3. *In D , there is no line segment I_b which is to the bottom right of a line segment I_t of an opposite parity branch set.*

Justification. Assume I_t is in \mathcal{I}_i and I_b is in \mathcal{I}_j . The branch sets B_i and B_j are of the opposite parity. If they are non-adjacent, then by Observation 2, either I_t has to be bottom left of I_b or I_b has to be bottom left of I_t . In both cases, I_b can not be bottom right of I_t . Now we consider the case when the branch sets are adjacent. Since I_t and I_b are non-intersecting, there exists intersecting intervals $I_{t_1} \in \mathcal{I}_i$ and $I_{b_1} \in \mathcal{I}_j$ as per Observation 1. In σ , either t precedes b or b precedes t . These result in the forbidden triple either (t, b, t_1) or (b, t, b_1) respectively leading to a contradiction by Lemma 2.

Lemma 7. *Any two non-intersecting line segments I_x and I_y in D satisfy either $x \prec_D y$ or $y \prec_D x$, but not both. In particular, the relation \prec_D is antisymmetric.*

Proof. Since D is a B_0 -VPG representation of G , x and y are nonadjacent in G and hence comparable in P . That is, either $x \prec_P y$ or $y \prec_P x$. Therefore, $x \prec_D y$ or $y \prec_D x$. Hence it is enough to verify that $x \prec_D y$ and $y \prec_D x$ cannot both be true. This is easily verified when I_x and I_y are both horizontal or both vertical. Hence we can assume without loss of generality that I_x is horizontal and I_y is vertical. If both $x \prec_D y$ and $y \prec_D x$, then I_x has to be to the bottom right of I_y . This contradicts Observation 3.

This concludes the proof of Theorem 2. For every two incomparable elements in \prec_P , the corresponding line segments intersect in D . For every two comparable elements $x, y \in V(G)$, if $x \prec_P y$, then $x \prec_D y$. By exchange of variables, if $y \prec_P x$, then $y \prec_D x$. Lemma 7 asserts that exactly one of the above is true for any two non-intersecting line segments. Hence $x \prec_D y$ only when $x \prec_P y$. Thus the relation \prec_D is isomorphic to the relation \prec_P .

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Appendix A Proof of Correctness of Algorithm 1

A.1 Step 1. Reduced dd-minor R_G of G

In this step, we contract a subset of edges of G to obtain a bipartite minor R_G of G which satisfies the additional properties listed in Lemma 4. We also label the branch-sets of R_G so that the order implied by this labeling has the adjacency property (Lemma 8).

We see a few additional properties of R_G using which we draw the B_0 -VPG representation of G . These properties are clubbed into the following lemma.

Lemma 4 Statement: *The following claims on the cocomparability graph R_G are true.*

1. For any two adjacent branch sets B_1 and B_2 , the set $B_{1,2} \cup B_{2,1}$ induces a clique in G .
2. If B_0, B_1, B_2 form consecutive vertices of a C_3 or an induced C_4 in R_G then $B_{1,0} \cap B_{1,2} \neq \emptyset$. Moreover, if B_0, \dots, B_{k-1} is a C_3 or an induced C_4 in R_G , then there exists an induced cycle b_0, \dots, b_{k-1} in G where each $b_i \in B_i$.
3. R_G is a bipartite permutation graph.

Proof. We prove the claims in the order they are listed.

Proof of Claim 1: First we need to prove that for any two adjacent branch sets B_1 and B_2 , the subgraph of G induced by $B_1 \cup B_2$ is an interval graph. Since no branch set of R_G contains more than one vertex of an induced C_4 , $G[B_1 \cup B_2]$ is a C_4 -free cocomparability graph and therefore an interval graph [17].

Now consider any two non-adjacent vertices $x, y \in B_{1,2}$ having a common neighbor $z_1 \in B_1$. Assume x and y are adjacent to a single vertex $z_2 \in B_{2,1}$. In order to avoid x and y being part of an induced C_4 , the possible edge is $z_1 z_2$. Clearly $z_1 z_2$ is a diamond diagonal edge across two branch sets. This is a contradiction.

Now we assume that there exists no common vertex z_2 in the adjacencies of x and y in $B_{2,1}$. Consider a shortest path xPy from x to y where all the intermediate vertices are in B_2 . Let x' and y' be the vertices adjacent to x and y respectively in the path. Clearly there cannot be an edge from x or y to the intermediate vertices of xPy since otherwise xPy cannot be a shortest path. Thus only possible edges in order to avoid bigger cycles due to the cocomparability property and the interval nature of the graph $G[B_1 \cup B_2]$, are from z_1 to the intermediate vertices of the path xPy . Hence $z_1 x'$ and $z_1 y'$ edges must exist. As in the previous paragraph, an edge $x' y'$ also exists. This forms a diamond diagonal $z_1 x'$ of the diamond induced by the set $\{x, x', y', z_1\}$ which is a contradiction.

Now suppose a vertex $x_1 \in B_{1,2}$ is not adjacent to a vertex $y_1 \in B_{2,1}$. Then let y_2 be a neighbor of x_1 in $B_{2,1}$ and x_2 be a neighbor of y_1 in $B_{1,2}$. The existence of x_2 and y_2 are guaranteed by the definition of $B_{i,j}$. Notice that since $G[B_{1,2}]$ and $G[B_{2,1}]$ are cliques, x_1 is adjacent to x_2 and y_1 is adjacent to y_2 . If x_2 and y_2 are non-adjacent, $G[\{x_1, y_2, y_1, x_2\}]$ is an induced C_4 in $B_1 \cup B_2$ which

is a contradiction. If x_2 and y_2 are adjacent, the edge x_2y_2 is the diagonal of the diamond $G[\{x_1, y_2, y_1, x_2\}]$ which is a contradiction.

Proof of Claim 2: First we need to prove that any induced cycle $C_n, n \geq 3$ in R_G ensures the existence of an induced cycle C_m in G where $m \geq n$. Consider any induced cycle $C_n = \{B_0, \dots, B_{n-1}\}$ in R_G . Let G_n be the induced subgraph $G[B_0 \cup \dots \cup B_{n-1}]$. For any $i \in \{0, \dots, n-1\}$, if $B_{i,i-1} \cap B_{i,i+1} \neq \emptyset$ (addition and subtraction are *modulo* n), choose any vertex b_i in that intersection. On the contrary, if $B_{i,i-1} \cap B_{i,i+1} = \emptyset$, then consider the vertices $b_{i,i-1} \in B_{i,i-1}$ and $b_{i,i+1} \in B_{i,i+1}$ such that the distance between them is the shortest. Thus there exists a shortest path $b_{i,i-1}Pb_{i,i+1}$ which cannot contain an intermediate vertex in the sets $B_{i,i-1}$ and $B_{i,i+1}$. By Claim 1, $B_{i,i-1} \cup B_{i-1,i}$ induces a clique. Similarly $B_{i,i+1} \cup B_{i+1,i}$ also induces a clique. This guarantees the adjacency between the chosen vertices of any two consecutive branch sets. Thus we get a cycle C_m in G in which each branch set B_i contributes either a vertex b_i or a path $b_{i,i-1}Pb_{i,i+1}$. Clearly there is no chord in C_m between two vertices belonging to two non-adjacent branch sets in C_n since it becomes a chord in C_n too. Similarly there is no chord in C_m between two vertices belonging to two adjacent branch sets, say B_i and B_{i+1} , in C_n since we select exactly one vertex each from $B_{i,i+1}$ and $B_{i+1,i}$. Thus if there exists an i with condition $B_{i,i-1} \cap B_{i,i+1} = \emptyset$, then we get $m > n$. Otherwise $m = n$.

Now suppose C is an induced cycle containing B_0, B_1 and B_2 as consecutive vertices. Clearly C can be either C_3 or C_4 since R_G is a cocomparability graph. Assume it is a C_3 . Then $\{B_0, B_1, B_2\}$ induces C . By the discussion in the previous paragraph, if $B_{1,0} \cap B_{1,2} = \emptyset$, then there exists a bigger induced cycle C_m in G . The only possible bigger induced cycle is a C_4 which contradicts the fact that two vertices of an induced C_4 resides inside a branch set. Now assume C is a C_4 , that is $\{B_0, B_1, B_2, B_3\}$ is an induced C_4 . By the first step, if $B_{1,0} \cap B_{1,2} = \emptyset$, then there exists a bigger induced cycle C_m in $G, m > 4$. This is a contradiction.

Thus by picking $b_i \in B_{i,i-1} \cap B_{i,i+1}$ ($0 \leq i \leq k-1$), we get the an induced k -cycle in G with $b_i \in B_i$.

Proof of Claim 3: First we need to prove the following sub-claims.

- (3.i) If B_0, B_1, B_2 induce a triangle in R_G then $B_{i,i-1} = B_{i,i+1}$ ($0 \leq i \leq 2$) where addition and subtraction are modulo 3 and hence the set $B_{0,1} \cup B_{1,2} \cup B_{2,0}$ induces a clique in G .

Proof: Using Claim 2, choose the vertices $b_0 \in B_{0,1} \cap B_{0,2}$, $b_1 \in B_{1,0} \cap B_{1,2}$ and $b_2 \in B_{2,0} \cap B_{2,1}$. Clearly $\{b_0, b_1, b_2\}$ induces a triangle. Let $i = 1$ without loss of generality. Assume $B_{1,0} \neq B_{1,2}$. Let $B_{1,0} \setminus B_{1,2} \neq \emptyset$ without loss of generality. Take a vertex $b'_1 \in B_{1,0} \setminus B_{1,2}$. Since $B_{1,0} \cup B_{0,1}$ is a clique, b'_1 is adjacent to b_0 and b_1 . Moreover since $b'_1 \notin B_{1,2}$, it is not adjacent to b_2 . Hence $\{b_0, b_1, b_2, b'_1\}$ induces a diamond with the diamond-diagonal b_0b_1 between B_0 and B_1 which is a contradiction. ■

- (3.ii) If $\{b_0, b_1, b_2, b_3\}$ is an induced C_4 in G , then $\{B_0, B_1, B_2, B_3\}$ is an induced C_4 in R_G , where for each i, B_i is the branch set containing b_i .

Proof: By the definition of R_G , it is clear that the branch sets B_0, B_1, B_2, B_3 are all distinct and $B_0B_1, B_1B_2, B_2B_3, B_1B_3$ are edges in R_G . Assume that $\{B_0, B_1, B_2, B_3\}$ is not an induced C_4 . Without loss of generality, if B_0B_2 is an edge in R_G , then R_G has a triangle $\{B_0, B_1, B_2\}$. By the sub-claim (3.i), $B_{0,1} = B_{0,2}$ and $B_{2,0} = B_{2,1}$. Hence $b_0 \in B_{0,2}$ and $b_2 \in B_{2,0}$. By Claim 1, b_0b_2 edge also exists in G which contradicts the fact that $\{b_0, b_1, b_2, b_3\}$ is an induced C_4 . Hence $\{B_0, B_1, B_2, B_3\}$ is also an induced C_4 . ■

(3.iii) Every edge of a triangle in R_G is also part of an induced C_4 in R_G .

Proof: By the minimality of R_G , any edge is contracted if the resulting branch set does not contain two vertices of an induced C_4 of G . Thus if there exists a triangle in R_G , every pair of branch sets constituting the triangle contains two vertices of an induced C_4 of G . Also by the sub-claim (3.ii), each such pair of branch sets should be adjacent vertices of an induced C_4 of R_G . Hence the statement. ■

(3.iv) R_G is a triangle-free.

Proof: By the previous sub-claim, if there exists a triangle R_3 in R_G induced by the set $\{B_0, B_1, B_2\}$, each edge of R_3 will also be part of an induced C_4 of R_G . Consider an induced subgraph R_M of R_G with *minimum* number of vertices such that it contains $V(R_3)$ and branch sets corresponding to the induced 4-cycles sharing edge with R_3 (Figure 4). Each induced C_4 sharing edge B_iB_{i+1} is denoted as $\{B_i, B_{i+1}, B_{i+1'}, B_{i''}\}$ where $i \in \{0, 1, 2\}$ and addition is modulo 3. We do not claim the adjacencies or non-adjacencies among the pair $\{B_{i'}, B_{i''}\}$. Clearly any vertex in the set $B_{i''}$ (respectively $B_{i'}$) is non-adjacent to the a vertex in B_{i+1} (respectively B_{i+2}) since they are non adjacent vertices of an induced C_4 . Similarly any vertex in the set $B_{i''}$ (respectively $B_{i'}$) is non-adjacent to the a vertex in B_{i+2} (respectively B_{i+1}) since otherwise it causes the existence of a diamond diagonal across two branch sets. For any i , if $B_{i'} \neq B_{i''}$, then $B_{i'}$ must be non-adjacent to $B_{i+1'}$ and $B_{i''}$ must be non-adjacent to $B_{i+2''}$ since we consider the minimal graph R_M .

Now in the minimal graph R_M , if $B_{i'} \neq B_{i''}$ for all $i \in \{0, 1, 2\}$, then there exists an *asteroidal triple* $\{B_{0'}, B_{1'}, B_{2'}\}$ in R_M (hence in R_G) which contradicts the fact that R_G is a cocomparability graph. If exactly two such pairs are unequal, without loss of generality $B_{0'} \neq B_{0''}$ and $B_{2'} \neq B_{2''}$, then there exists a 5-cycle induced by the set $\{B_{0''}, B_0, B_2, B_{2'}, B_{1'}\}$ in R_M (hence in R_G) which is a contradiction. If exactly one such pair is unequal, without loss of generality $B_{0'} \neq B_{0''}$, then there exists the same 5-cycle in R_M which is again a contradiction. If all such pairs are equal, then the set $\{B_{0'}, B_{1'}, B_{2'}\}$ induces a triangle. As per Sub-claim (3.i), $B_{0',1'} \cup B_{1',2'} \cup B_{2',0'}$ induces a clique. Similarly $B_{0,1} \cup B_{1,2} \cup B_{2,0}$ also induces a clique. For $i \in \{0, 1, 2\}$, choose the vertices $b_i \in B_{i,i'} \cap B_{i,i+1}$ which is nonempty by Claim (2) and $b_{i'} \in B_{i',i} \cap B_{i',i+1'}$ which is also nonempty by Claim (2). One can verify that $G[\{b_0, b_1, b_2, b_{0'}, b_{1'}, b_{2'}\}]$ is an induced $\overline{C_6}$ in G which is a contradiction since G satisfies the conditions of Theorem 1. Hence R_G is triangle free. ■

Thus by the above sub-claim, R_G is a triangle-free graph. Also R_G contains no bigger induced odd cycles since it is a cocomparability graph. Thus R_G is

a bipartite graph and hence a comparability graph. R_G is also a permutation graph since it is both cocomparability and comparability graph [26]. Hence the claim.

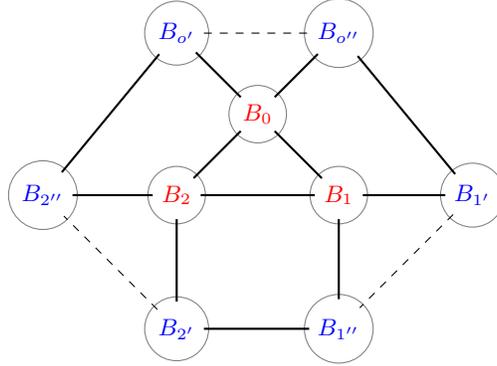


Fig. 4: Minimal graph R_M having a C_3 with each edge part of an induced C_4 .

We relabeled the branch sets of the bipartite permutation graph R_G in subsection 2.2. The following two lemmas are relevant to this labeling.

Lemma 5 Statement : For any two branch sets B_i, B_j of the same parity if $i < j$, then $B_i \prec_\sigma B_j$.

Proof. Every branch set of R_G is a connected graph since every edge contraction towards the formation of R_G assures connectivity. Suppose $\exists x \in B_i$ and $\exists y \in B_j$ such that $y \prec_\sigma x$. Then any path in $G[B_i]$ from the leftmost vertex, say z , of B_i to x does not contain a vertex from the closed neighborhood of y . Hence the triple (z, y, x) is a forbidden triple in σ which is a contradiction by Lemma 2.

Lemma 8. The relabeling of $V(R_G)$ respects the adjacency property.

Proof. We know that σ_{R_G} is an umbrella-free ordering. Moreover, the relabeling ensures that the labels in each part of R_G inherit the order of σ . In order to prove the *adjacency property* of the relabeling of $V(R_G)$, we assume that there are three branch sets B_j, B_k, B_l ($j < k < l$) of the same parity and a fourth branch set B_i of the opposite parity adjacent to B_j and B_l , but not B_k . Then either $(B_i \prec_{\sigma_{R_G}} B_k \prec_{\sigma_{R_G}} B_l)$ or $(B_j \prec_{\sigma_{R_G}} B_k \prec_{\sigma_{R_G}} B_i)$ forms an umbrella which contradicts the umbrella-freeness of σ_{R_G} .

A.2 Step 2. Interval representations of branch-sets of R_G

The *closure* of a branch set B_i of R_G is the set $B_i^* = B_i \cup \{B_{j,i} : B_i B_j \in E(R_G)\}$. That is, B_i^* consists of all the vertices in B_i together with their neighbors in G .

In order to fit together interval representations of the individual branch-sets, it turns out to be necessary to start with interval representations for the closure of each branch-set. We also need these interval representations to satisfy the properties listed in the following lemma.

Lemma 6 Statement: *For each branch set B_i of R_G , $G[B_i^*]$ is an interval graph. Moreover, $G[B_i^*]$ has an interval representation \mathcal{I}_i^* satisfying the following properties.*

- (i) *For all $x, y \in B_i^*$, we have the interval for x to the left of interval for y if and only if $x \prec_P y$.*
- (ii) *For each neighbor B_j of B_i , all the intervals corresponding to the vertices in $B_{j,i}$ are point intervals at a point $p_{j,i}$.*
- (iii) *If B_j and B_k are two neighbors of B_i such that $j < k$, then the point $p_{j,i}$ is to the left of the point $p_{k,i}$ in \mathcal{I}_i^* .*

Proof. By Lemma 4.3, R_G is a bipartite permutation graph. If $G[B_i^*]$ contains an induced C_4 , then by Lemma 4.(3.ii) (sub-claim in Appendix A), R_G contains an induced C_4 in the closed neighborhood of B_i . But since R_G is bipartite, the closed neighborhood of any vertex is a star. Hence $G[B_i^*]$ is a C_4 -free cocomparability graph and thus an interval graph [17].

(i): Let $P_G(V(G), \prec_P)$ be the poset assumed in Algorithm 1 and $P_i^* = (B_i^*, \prec_i^*)$ be the subposet of P_G induced on B_i^* . Clearly P_i^* is $(2+2)$ -free since $G[B_i^*]$ is C_4 -free. Hence P_i^* is an interval order (Theorem 3). Pick any interval representation of the interval order P_i^* and consider it as an interval representation \mathcal{I}_i^* for the interval graph $G[B_i^*]$. Thus if I_x is to the left of I_y in \mathcal{I}_i^* , that is $I_x \prec_{\mathcal{I}_i^*} I_y$, then $x \prec_i^* y$ and hence $x \prec_P y$. Similarly, if $x \prec_P y$, then $x \prec_i^* y$ and hence $I_x \prec_{\mathcal{I}_i^*} I_y$.

(ii): By Lemma 4.1, $B_{i,j} \cup B_{j,i}$ is a clique. As the vertices corresponding to the clique share a common point in the interval representation \mathcal{I}_i^* , there exists a point $p_{j,i}$ common to all the intervals corresponding to the vertices in the set $B_{j,i} \cup B_{i,j}$. The point $p_{j,i}$ is not contained in an interval corresponding to any other vertex in B_i since they are non-adjacent to the vertices in $B_{j,i}$. Since the vertices in $B_{j,i}$ have no neighbors in $G[B_i^*]$ other than $B_{j,i} \cup B_{i,j}$, we can shrink all the intervals corresponding to them to the single point $p_{j,i}$ in \mathcal{I}_i^* .

(iii): Since j and k have the same parity and $j < k$, Lemma 5 guarantees that $\forall x \in B_j, \forall y \in B_k, x \prec_\sigma y$. Since σ is the linear extension of P_G and x is non-adjacent to y in G , $x \prec_P y$ which also implies that $x \prec_i^* y$. Thus the point intervals at $p_{j,i}$ is to the left of the point intervals at $p_{k,i}$ in \mathcal{I}_i^* .

We remove the intervals other than those corresponding to the vertices of B_i from \mathcal{I}_i^* to get an interval representation \mathcal{I}_i of $G[B_i]$. But we retain the location of the point $p_{j,i}$ corresponding to each neighbor B_j of B_i .

A.3 Step 3. B_0 -VPG representation of G

In this final step, we draw a B_0 -VPG representation D of G by combining interval representations \mathcal{I}_i of each B_i in $V(R_G)$ and a B_0 -VPG representation of R_G .

The drawing procedure is rewritten here to maintain continuity for readers. Let e and o respectively (with further subscripts if needed) denote the even and odd indices of the branch sets of R_G .

- (i) For each odd-indexed branch set B_o , \mathcal{I}_o is drawn vertically from the point $(o, (e_1 - 0.5))$ to $(o, (e_2 + 0.5))$ where B_{e_1} and B_{e_2} are the leftmost and rightmost neighbors of B_o in σ_{R_G} .
- (ii) Stretch or shrink the intervals in each vertical interval representation \mathcal{I}_o without changing their intersection pattern, so that for each neighbor B_e of B_o , the point $p_{e,o}$ is at (o, e) . This can be done since the intersection pattern of an interval representation with n intervals is solely determined by the order of the corresponding $2n$ endpoints.
- (iii) For each even-indexed branch set B_e , \mathcal{I}_e is drawn horizontally from the point $((o_1 - 0.5), e)$ to $((o_2 + 0.5), e)$ where B_{o_1} and B_{o_2} are the leftmost and rightmost neighbors of B_e in σ_{R_G} .
- (iv) Stretch or shrink the intervals in each horizontal interval representation \mathcal{I}_e without changing their intersection pattern, so that for each neighbor B_o of B_e , the point $p_{o,e}$ is at (o, e) .

Proposition 1 Statement: *The B_0 -VPG representation D is precisely a B_0 -VPG representation of the cocomparability graph G .*

Proof. We need to verify the pairwise relationships of G in D . The adjacency and non-adjacency among the vertices inside each branch set B_i are guaranteed since we use the interval representation \mathcal{I}_i in D .

For two adjacent branch sets B_o and B_e , the points $p_{o,e}$ and $p_{e,o}$ coincide at (o, e) . Since $B_{o,e} \cup B_{e,o}$ forms a clique in G (Lemma 4.1), we need to show that the corresponding paths share a common point in D . The point $p_{o,e}$ (respectively $p_{e,o}$) is the common point of the intervals corresponding to the vertices in $B_{o,e}$ (respectively $B_{e,o}$). Since both these two points fall at location (o, e) in the plane, all the paths corresponding to vertices in $B_{o,e} \cup B_{e,o}$ have a common intersection. If two vertices $x \in B_o$ and $y \in B_e$ are non-adjacent, then the intervals $I_x \in \mathcal{I}_o, I_y \in \mathcal{I}_e$ satisfy either $p_{e,o} \notin I_x$ or $p_{o,e} \notin I_y$ or both. Thus I_x and I_y do not touch each other in D .

In D , if we consider each \mathcal{I}_i as a single path, then we can use the adjacency property to verify that we get a B_0 -VPG representation of R_G . Hence the non-adjacencies of the two vertices belonging to two non-adjacent branch sets of R_G are maintained in D .

This completes the proof of Theorem 1 since we were able to construct a B_0 -VPG representation D for the cocomparability graph G satisfying the conditions of Theorem 1.