

Rectilinear Planarity Testing of Plane Series-Parallel Graphs in Linear Time [★]

Walter Didimo¹, Michael Kaufmann², Giuseppe Liotta¹, Giacomo Ortali¹✉

¹ Università degli Studi di Perugia, Italy
{walter.didimo,giuseppe.liotta}@unipg.it,
giacomo.ortali@studenti.unipg.it

² University of Tübingen, Germany
mk@informatik.uni-tuebingen.de

Abstract. A plane graph is *rectilinear planar* if it admits an embedding-preserving straight-line drawing where each edge is either horizontal or vertical. We prove that rectilinear planarity testing can be solved in optimal $O(n)$ time for any plane series-parallel graph G with n vertices. If G is rectilinear planar, an embedding-preserving rectilinear planar drawing of G can be constructed in $O(n)$ time. Our result is based on a characterization of rectilinear planar series-parallel graphs in terms of intervals of orthogonal spirality that their components can have, and it leads to an algorithm that can be easily implemented.

Keywords: Orthogonal drawings · Rectilinear planarity testing · Series-parallel graphs.

1 Introduction

A *planar orthogonal drawing* Γ of a planar graph G is a crossing-free drawing of G that maps each vertex to a distinct point of the plane and each edge to a sequence of horizontal and vertical segments between its end-points [1,9,14]. A graph is *rectilinear planar* if it admits a planar orthogonal drawing without bends.

Testing whether a graph is rectilinear planar is a fundamental question in graph drawing. The problem can be either studied for *plane* graphs, that is graphs that come with a fixed embedding, or in the variable embedding setting, where the algorithm can choose one of the planar embeddings of the input graph. Besides being an interesting topic on its own right, rectilinear planarity testing is at the core of efficient algorithms that compute orthogonal drawings with minimum number of bends. For example, Rahman et al. [17] characterize the rectilinear plane 3-graphs (i.e., graphs with vertex degree at most three) and then use this characterization to design linear time bend-minimization algorithms for these graphs in the fixed embedding setting [15,16]. On the other hand, Garg

[★] Work partially supported by: (i) MIUR, grant 20174LF3T8 “AHeAD: efficient Algorithms for HArnessing networked Data”, (ii) Engineering Dep., Univ. Perugia, grant RICBA19FM: “Modelli, algoritmi e sistemi per la visualizzazione di grafi e reti”.

and Tamassia [11] prove that rectilinear planarity testing is NP-complete for planar 4-graphs in the variable embedding setting. Remarkably, the study of rectilinear plane 3-graphs has turned out to be an essential tool to design linear-time rectilinear planarity testing and bend-minimization algorithms for planar 3-graphs in the variable embedding setting [8,13].

In this paper we study rectilinear planarity testing in the fixed embedding setting. A seminal paper of Tamassia [18] implies that in this setting the problem can be solved in $O(n^2 \log n)$, where n is the number of vertices of the input graph; its approach is based on solving a min-cost flow network problem to compute a bend-minimum orthogonal drawing of the input graph. Since its time of publication, establishing a lower bound on the time complexity of computing bend-minimum orthogonal drawings of plane graphs has remained a fascinating open problem (see, e.g., [3,1,7]). Garg and Tamassia [12] improve the complexity to $O(n^{\frac{7}{4}} \sqrt{\log n})$ and then Cornelsen and Karrenbauer [4] further improve the upper bound to $O(n^{1.5})$. For rectilinear planarity testing, the approach in [18] reduces to compute a maximum flow in an n -vertex planar network with multiple sources and sinks; Borradaile et al. [2] prove that this problem can be solved in $O(n \log^3 n)$ time. Since, as already mentioned, an $O(n)$ -time algorithm for rectilinear planarity testing is known when the input is a plane 3-graph, the challenge is to understand whether an $O(n)$ -time bound exists for plane 4-graphs.

This paper sheds some light on this question by answering it for series-parallel graphs. An essential aspect of our approach is to tackle the problem without using any network-flow computation. Our results are as follows:

- (i) We give a characterization of those plane series-parallel graphs (with two terminals s and t) that are rectilinear planar. This characterization is expressed in terms of values of *spirality* that each series or parallel component can have in a rectilinear drawing. Intuitively, the spirality of a component measures how much it can be “rolled-up” in a rectilinear drawing of the graph.
- (ii) While the possible values of spirality for each component may be linear, we can encode them in constant space. This makes it possible to design a linear-time rectilinear planarity testing algorithm for a two-terminal series-parallel graph G based on a bottom-up visit of its decomposition tree T . If the test is positive, we compute in linear time a rectilinear drawing of G through a top-down visit of T . The algorithm is easy to implement.

The paper is organized as follows. Section 2 recalls basic concepts. Section 3 gives our characterization of rectilinear planar series-parallel graphs in terms of their orthogonal spirality. Section 4 describes the linear-time testing and drawing algorithm. Section 5 lists some open problems. In the main text of the paper some proofs are sketched or omitted, and can be found in the Appendix.

Together with our submission to GD 2020, another paper by Frati [10] was accepted to the same conference. The work of Frati is based on a different technique and it presents an $O(n)$ -time algorithm for rectilinear planarity testing of outerplanar graphs. While the result of [10] does not apply to the family of graphs that are studied in this paper, it covers the variable embedding setting and the case of 1-connected outerplanar graphs.

2 Preliminaries

Orthogonal Representations. We focus on *orthogonal representations* rather than orthogonal drawings. An orthogonal representation H describes the shape of a class of orthogonal drawings in terms of sequences of bends along the edges and angles at the vertices. An (orthogonal) drawing Γ of H can be computed in linear time [18]. If H has no bend, it is a *rectilinear representation* (see Fig. 1(b)). The *degree* $\deg(v)$ of a vertex v denotes the number of edges incident to v .

Series-Parallel graphs and Decomposition Trees. A *two-terminal series-parallel* graph, also called *series-parallel graph* in the rest of the paper, has two distinct vertices s and t , called its *source* and its *sink*, respectively, and it is inductively defined as follows: (i) A single edge (s, t) is a series-parallel graph with source s and sink t . (ii) Given $p \geq 2$ series-parallel graphs G_1, \dots, G_p , each G_i with source s_i and sink t_i ($i = 1, \dots, p$), a new series-parallel graph G can be obtained with any of these two operations: *Series composition* – It identifies t_i with s_{i+1} ($i = 1, \dots, p-1$); G has source $s = s_1$ and sink $t = t_p$. *Parallel composition* – It identifies all sources s_i together and all sinks t_i together; G has source $s = s_i$ and $t = t_i$ ($i = 1, \dots, p$).

A series-parallel graph G is naturally associated with a *decomposition tree* T , which describes the series and parallel compositions that build G . Tree T has three types of nodes: S-, P-, and Q*-nodes. If G is the series composition of $p \geq 2$ graphs G_i that are not all single edges, the root of T is an S-node whose subtrees are the decomposition trees T_i of G_i . If G is the parallel composition of $p \geq 2$ graphs G_i , the root of T is a P-node whose subtrees are the decomposition trees T_i of G_i . If G is a series composition of $\ell \geq 1$ edges, its decomposition tree is a single Q*-node and for brevity we say that ℓ is the *length* of this node.

For a node ν of T , the *pertinent graph* G_ν of ν is the series-parallel subgraph of G formed by all edges associated with the Q*-nodes in the subtree rooted at ν . We also call G_ν a *component* of G . If u and v are the source and the sink of G_ν , respectively, we say that $\{u, v\}$ are the *poles* of G_ν and of ν : u is the *source pole* and v is the *sink pole*. If G is a biconnected plane series-parallel graph, for any edge $e = (s, t)$ on the external face of G , we can associate with G a decomposition tree T where the root is a P-node representing the parallel composition between e and the rest of the graph. Thus, the root of T is always a P-node with two children, one of which is a Q*-node corresponding to e . It will be called the (unique) P^r-node of T , to distinguish it by the other P-nodes. Edge e is the *reference edge* of T and T is the SPQ*-tree of G with respect to e . Also, it is always possible to make T such that each P-node (distinct from the root) has no P-node child and each S-node has no S-node child. Since we only deal with graphs of vertex-degree at most four, a P-node has either two or three children. From now on we assume that T always satisfies the properties above for a biconnected series-parallel graph. Observe that the number of nodes of T is $O(n)$, where n is the number of vertices of G . Figure 1 shows a biconnected series-parallel graph G , a rectilinear planar representation H of G , and the SPQ*-tree T of G with respect to the reference edge $(1, 33)$.

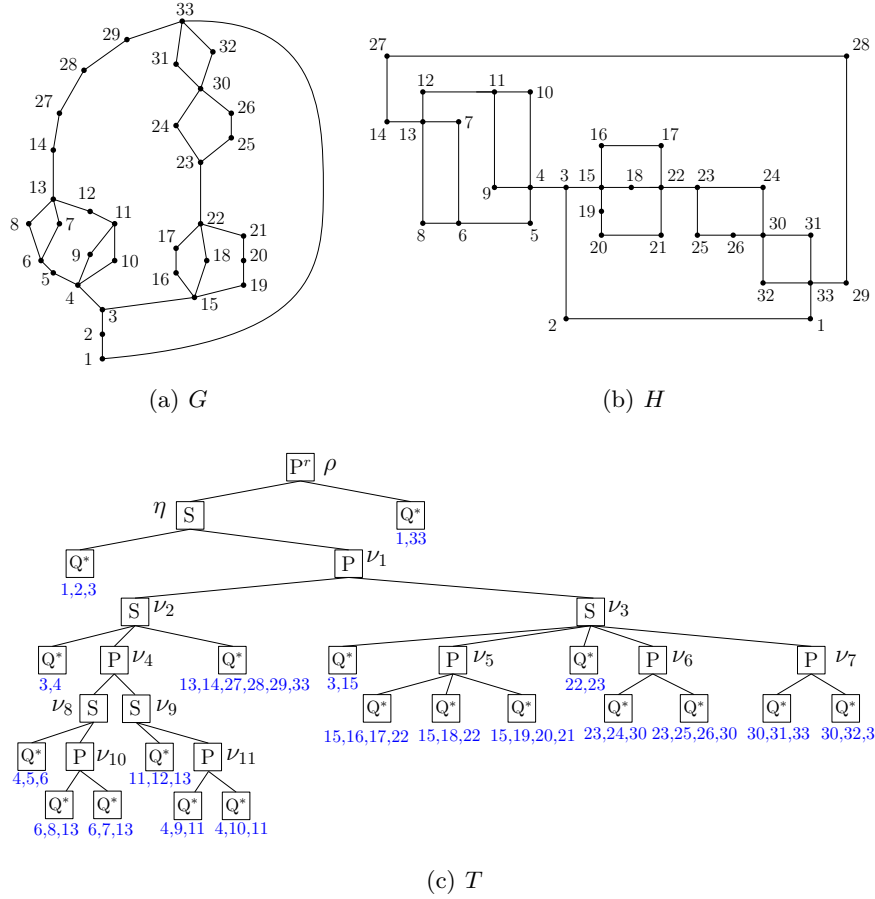


Fig. 1. (a) A biconnected series-parallel graph G . (b) A rectilinear planar representation H of G . (c) The SPQ*-tree T of G with reference edge $(1, 33)$.

3 Characterizing Rectilinear Plane Series-Parallel Graphs

Let G be a plane series-parallel graph. If G is biconnected let $e = (s, t)$ be any edge on the external face of G ; otherwise, by definition of two-terminal series-parallel graph, we can add a dummy edge e on the external face of G to make it biconnected. We assume that the external face of G is to the right of e while moving from s to t (as in Fig. 1(a)). Let T be an SPQ*-tree of G with respect to e . An overview of our algorithm is as follows. It visits T in post-order (a node is visited after its children). When the algorithm visits a node ν , it tests whether G_ν admits a planar rectilinear representation by checking whether a certain condition, which we call *representability condition*, is verified: In the negative case, the algorithm halts and rejects the instance;

else it stores in ν its *representability interval* I_ν . Such an interval is a compact representation of the possible values of *orthogonal spirality* that the pertinent graph G_ν of ν may have in a rectilinear representation of G . Informally speaking, the orthogonal spirality is a measure of how much a rectilinear representation of pertinent graph G_ν is “rolled-up” in a rectilinear planar representation of G . As we shall see, the representability interval is such that for every value $k \in I_\nu$ graph G_ν admits a planar rectilinear representation with spirality k , while it does not for any value outside I_ν . If the testing algorithm does not halt and it reaches the root, two cases are considered: If e is a real edge of G , then the algorithm executes a final test to check whether a rectilinear planar representation of G can be obtained by merging a straight-line representation of e with a rectilinear representation of the child component of the root other than e . If e is a dummy edge added to make G biconnected this check is not required, because e is not present in the final representation and can arbitrarily bend.

We now present the characterization of the rectilinear planar components in terms of representability conditions and intervals that is at the base of the testing algorithm. We start in Section 3.1 with a formal definition of spirality. We characterize Q^{*}-, S-, and P-components with three children in Section 3.2, and P-components with two children in Section 3.3. We summarize in Section 3.4.

3.1 Spirality of Series-Parallel Graphs

Let T be an SPQ^{*}-tree of a biconnected plane series-parallel graph G for a given reference edge $e = (s, t)$. Let H be an embedding-preserving orthogonal representation of G . Also, let ν be a node of T with poles $\{u, v\}$, and let H_ν be the restriction of H to the pertinent graph G_ν of ν . We also say that H_ν is a *component* of H . For each pole $w \in \{u, v\}$, let $\text{indeg}_\nu(w)$ and $\text{outdeg}_\nu(w)$ be the degree of w inside and outside H_ν , respectively. Define two (possibly coincident) *alias vertices* of w , denoted by w' and w'' , as follows: (i) if $\text{indeg}_\nu(w) = 1$, then $w' = w'' = w$; (ii) if $\text{indeg}_\nu(w) = \text{outdeg}_\nu(w) = 2$, then w' and w'' are dummy vertices, each splitting one of the two distinct edge segments incident to w outside H_ν ; (iii) if $\text{indeg}_\nu(w) > 1$ and $\text{outdeg}_\nu(w) = 1$, then $w' = w''$ is a dummy vertex that splits the edge segment incident to w outside H_ν .

Let A^w be the set of distinct alias vertices of a pole w . Let P^{uv} be any simple path from u to v inside H_ν and let $u' \in A^u$ and $v' \in A^v$. The path $S^{u'v'}$ obtained concatenating (u', u) , P^{uv} , and (v, v') is called a *spine* of H_ν . Denote by $n(S^{u'v'})$ the number of right turns minus the number of left turns encountered along $S^{u'v'}$ while moving from u' to v' . The *spirality* $\sigma(H_\nu)$ of H_ν is defined based on the following cases: (a) $A^u = \{u'\}$ and $A^v = \{v'\}$. Then $\sigma(H_\nu) = n(S^{u'v'})$. (b) $A^u = \{u'\}$ and $A^v = \{v', v''\}$. Then $\sigma(H_\nu) = \frac{n(S^{u'v'}) + n(S^{u'v''})}{2}$. (c) $A^u = \{u', u''\}$ and $A^v = \{v'\}$. Then $\sigma(H_\nu) = \frac{n(S^{u'v'}) + n(S^{u''v'})}{2}$. (d) $A^u = \{u', u''\}$ and $A^v = \{v', v''\}$. Without loss of generality, assume that (u, u') precedes (u, u'') counterclockwise around u and that (v, v') precedes (v, v'') clockwise around v . Then $\sigma(H_\nu) = \frac{n(S^{u'v'}) + n(S^{u''v''})}{2}$. Notice that, by definition, the spirality of H_ν also depends on the angles at the poles of H_ν , not only on the shape of H_ν .

Di Battista et al. [5] showed that the spirality of H_ν does not vary with the choice of path P^{uv} and that two distinct representations of G_ν with the same spirality are interchangeable. Fig. 2 reports the spiralities of some P- and S-components in the representation H of Fig. 1(b). For brevity, we shall denote by σ_ν the spirality of an orthogonal representation of G_ν .

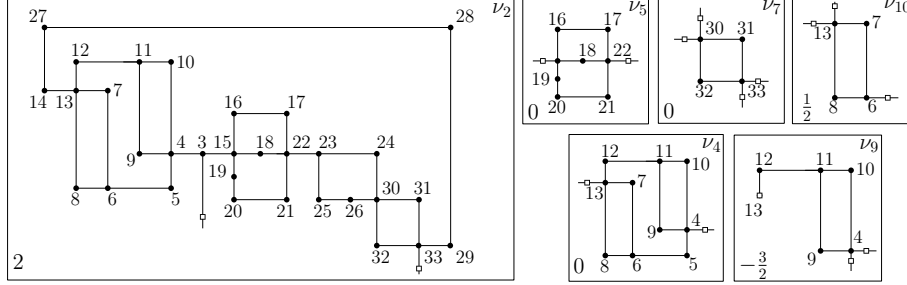


Fig. 2. Spiralities (left-bottom corners) of some components in the representation H of Fig. 1(b). Small squares indicate alias vertices.

Lemma 1 ([5]). *Let ν be an S-node of T with children μ_1, \dots, μ_h . The following relationship holds: $\sigma_\nu = \sum_{i=1}^h \sigma_{\mu_i}$.*

If ν is a P-node with two children, we denote by μ_l and μ_r the left child and the right child of ν , respectively. If ν is a P-node with three children, we denote by μ_l , μ_c , and μ_r , the three children of ν from left to right. Also, for each pole $w \in \{u, v\}$ of ν , the *leftmost angle* at w in H is the angle formed by the leftmost external edge and the leftmost internal edge of H_ν incident to w . The *rightmost angle* at w in H is defined symmetrically. We define two binary variables α_w^l and α_w^r as follows: $\alpha_w^l = 0$ ($\alpha_w^r = 0$) if the leftmost (rightmost) angle at w in H is of 180° , while $\alpha_w^l = 1$ ($\alpha_w^r = 1$) if this angle is of 90° . Observe that if $\deg(w) = 4$, then $\alpha_w^l = \alpha_w^r = 1$. Also, if ν has two children, define two additional variables k_w^l and k_w^r as follows: $k_w^d = 1$ if $\text{indeg}_{\mu_d}(w) = \text{outdeg}_\nu(w) = 1$, while $k_w^d = 1/2$ otherwise, for $d \in \{l, r\}$.

For example, in Fig. 2 the P-component of ν_4 has poles $u = 4$ and $v = 13$, and we have $k_u^l = k_v^r = 1$, $k_u^r = k_v^l = \frac{1}{2}$, and $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$. The P-component of ν_{10} has poles $u = 6$ and $v = 13$, and we have $k_u^l = k_v^r = 1$, $k_u^r = k_v^l = \frac{1}{2}$, $\alpha_u^l = 0$, and $\alpha_u^r = \alpha_v^l = \alpha_v^r = 1$. Fig. 3 reports all the values of k_w^d for the possible types of P-nodes with two children.

Lemma 2 ([5]). *Let ν be a P-node of T with two children μ_l and μ_r . The following relationships hold: $\sigma_\nu = \sigma_{\mu_l} - k_u^l \alpha_u^l - k_v^l \alpha_v^l = \sigma_{\mu_r} + k_u^r \alpha_u^r + k_v^r \alpha_v^r$.*

Lemma 3 ([5]). *Let ν be a P-node of T with three children μ_l , μ_c , and μ_r . The following relationships hold: $\sigma_\nu = \sigma_{\mu_l} - 2 = \sigma_{\mu_c} = \sigma_{\mu_r} + 2$.*

About the values of spirality σ_ν that a component H_ν can take, if ν is a Q*-node or a P-node with three children, σ_ν is always an integer. If ν is an S-node or a P-node with two children, σ_ν is either integer or semi-integer depending on whether the total number of alias vertices for the poles of ν is even or odd.

3.2 Q*-nodes, S-nodes, and P-nodes with three children

From now on, when we say that the spirality σ_ν of an orthogonal planar representation of G_ν can take *all* values in an interval $[a, b]$, we mean that such values are either all the integer numbers or all the semi-integer numbers in $[a, b]$, depending on the cases described above for ν .

Lemma 4. *Let ν be a Q*-node of length ℓ . Graph G_ν is always rectilinear planar (i.e., its representability condition is always true) and its representability interval is $I_\nu = [-\ell + 1, \ell - 1]$.*

Proof. G_ν is a path with $\ell - 1$ degree-2 vertices. For any integer $k \in [-\ell + 1, 0]$, a rectilinear planar representation H_ν of G_ν with spirality k is obtained by making a left turn at k degree-2 vertices of G_ν (going from the source to the sink pole), and no turn at any remaining vertex of G_ν . Symmetrically, for any $k \in (0, \ell - 1]$, we realize H_ν with spirality k by making a right turn at exactly k degree-2 vertices of G_ν . It is clear that no values of spirality out of I_ν can be achieved. \square

Lemma 5. *Let ν be an S-node with h children μ_1, \dots, μ_h . Suppose that, for every $i \in [1, h]$, the representability interval of G_{μ_i} is $I_{\mu_i} = [m_i, M_i]$. Graph G_ν is always rectilinear planar (i.e., its representability condition is always true) and its representability interval is $I_\nu = [\sum_{i=1}^h m_i, \sum_{i=1}^h M_i]$.*

Proof. We use induction on the number of children of ν . In the base case $h = 2$. By hypothesis $I_{\mu_1} = [m_1, M_1]$ and $I_{\mu_2} = [m_2, M_2]$. By Lemma 1, a series composition of a rectilinear representation of G_{μ_1} with spirality σ_{μ_1} and of a rectilinear representation of G_{μ_2} with spirality σ_{μ_2} results in a rectilinear representation of G_ν with spirality $\sigma_\nu = \sigma_{\mu_1} + \sigma_{\mu_2}$. Hence, if $M_1 = m_1 + r_1$ and $M_2 = m_2 + r_2$, for two non-negative integers r_1 and r_2 , then the possible values for σ_ν are exactly $m_1 + m_2, m_1 + 1 + m_2, \dots, m_1 + r_1 + m_2, \dots, m_1 + r_1 + m_2 + 1, \dots, m_1 + r_1 + m_2 + r_2$, i.e., all values in the interval $[m_1 + m_2, M_1 + M_2]$. In the inductive case $h \geq 3$; consider the series composition G'_1 of $G_{\mu_1}, \dots, G_{\mu_{h-1}}$. Graph G_ν is the series composition of G'_1 and G_{μ_h} . By inductive hypothesis the representability interval of G'_1 is $[\sum_{i=1}^{h-1} m_i, \sum_{i=1}^{h-1} M_i]$ and by Lemma 1 applied to G'_1 and G_{μ_h} we have $I_\nu = [\sum_{i=1}^h m_i, \sum_{i=1}^h M_i]$, using the same reasoning as for the base case. \square

Lemma 6. *Let ν be a P-node with three children μ_l, μ_c , and μ_r . Suppose that G_{μ_l}, G_{μ_c} , and G_{μ_r} are rectilinear planar and that their representability intervals are $I_{\mu_l} = [m_l, M_l]$, $I_{\mu_c} = [m_c, M_c]$, and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] \neq \emptyset$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$.*

Proof. Representability condition. Suppose first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν . By Lemma 3, the spiralities σ_{μ_l} , σ_{μ_c} , and σ_{μ_r} for the representations of G_{μ_l} , G_{μ_c} , and G_{μ_r} in H_ν are such that $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$. Since $\sigma_{\mu_l} \in [m_l, M_l]$, $\sigma_{\mu_c} \in [m_c, M_c]$, $\sigma_{\mu_r} \in [m_r, M_r]$, we have $\sigma_\nu \in [m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2]$. Suppose vice versa that $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] \neq \emptyset$, and let k be any value in such intersection. Setting $\sigma_{\mu_l} = k + 2$, $\sigma_{\mu_c} = k$, and $\sigma_{\mu_r} = k - 2$ we have $\sigma_{\mu_l} \in [m_l, M_l]$, $\sigma_{\mu_c} \in [m_c, M_c]$, and $\sigma_{\mu_r} \in [m_r, M_r]$. By Lemma 3, G_ν is rectilinear planar for a value of spirality $\sigma_\nu = k$.

Representability interval. Assume that G_ν is rectilinear planar. Clearly $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}] = [m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2]$, and by the truth of the feasibility condition we have $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}] \neq \emptyset$. Similarly to the first part of the proof of the representability condition, any rectilinear planar representation of G_ν has a value of spirality in the interval $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$. On the other hand, let $k \in [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$. Analogously to the second part of the proof of the representability condition, we can construct a rectilinear planar representation of G_ν with spirality $\sigma_\nu = k$, by combining in parallel rectilinear planar representations of G_{μ_l} , G_{μ_c} , and G_{μ_r} with spiralities $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$, respectively. \square

3.3 P-nodes with two children

For a P-node ν with two children μ_l and μ_r , the representability condition and interval depend on the indegree and outdegree of the poles of ν in G_ν , G_{μ_l} , and G_{μ_r} . We define the *type* of ν and of G_ν as follows (refer to Fig. 3):

- $\text{I}_2\text{O}_{\alpha\beta}$: Both poles of ν have indegree two in G_ν ; also one pole has outdegree α in G_ν and the other pole has outdegree β in G_ν , for $1 \leq \alpha \leq \beta \leq 2$. This gives rise to the specific types I_2O_{11} , I_2O_{12} , and I_2O_{22} .
- $\text{I}_{3d}\text{O}_{\alpha\beta}$: One pole of ν has indegree two in G_ν , while the other pole has indegree three in G_ν and indegree two in G_{μ_d} for $d \in \{l, r\}$; also one pole has outdegree α in G_ν and the other has outdegree β in G_ν , for $1 \leq \alpha \leq \beta \leq 2$, where $\alpha = \beta = 2$ is not possible. This gives rise to the specific types $\text{I}_{3l}\text{O}_{11}$, $\text{I}_{3r}\text{O}_{11}$, $\text{I}_{3l}\text{O}_{12}$, $\text{I}_{3r}\text{O}_{12}$.
- $\text{I}_{3dd'}$: Both poles of ν have indegree three in G_ν ; one of the two poles has indegree two in G_{μ_d} and the other has indegree two in $G_{\mu_{d'}}$, for $dd' \in \{ll, lr, rr\}$ (both poles have outdegree one in G_ν). Hence, the specific types are I_{3ll} , I_{3lr} , I_{3rr} . To characterize P-nodes of type $\text{I}_2\text{O}_{\alpha\beta}$ we start with the following result.

Lemma 7. *Let G_ν be a P-node of type $\text{I}_2\text{O}_{\alpha\beta}$ with children μ_l and μ_r . G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4 - \gamma]$, where $\gamma = \alpha + \beta - 2$.*

Sketch of proof. We only give the proof for $\alpha = \beta = 2$. The other cases are treated similarly (see the appendix). In this case G_ν is of type I_2O_{22} and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear

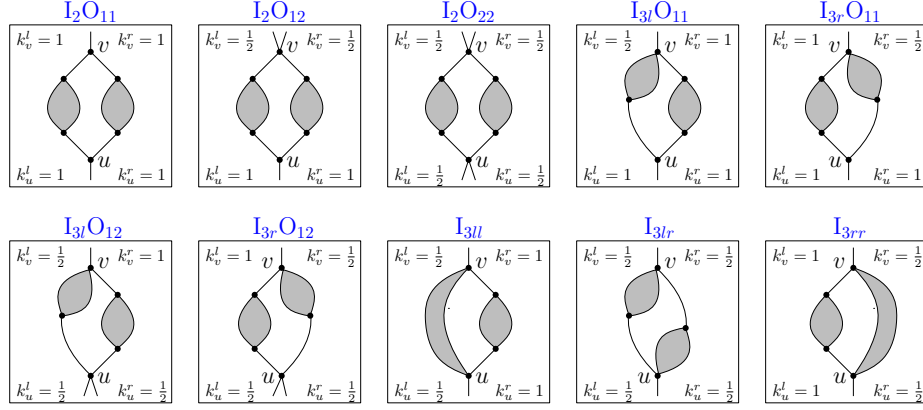


Fig. 3. Schematic illustration of the different types of P-nodes with two children.

planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. We have $k_u^l = k_u^r = \frac{1}{2}$. If G_ν is rectilinear planar, we have that $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 2, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$; hence $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$.

Suppose vice versa that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. We show that G_ν admits a rectilinear planar representation H_ν . We obtain H_ν by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and by suitably setting α_u^d and α_v^d ($d \in \{l, r\}$). For any cycle C through u and v , the number of 90° angles minus the number of 270° angles in the interior of C can be expressed by $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 1 + 1$ (both the angles at u and v inside C is always of 90° degrees). We then set $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$, which guarantees $a_c = 4$. Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} or G_{μ_r} . \square

Lemma 8. *Let ν be a P-node of type $I_2O_{\alpha\beta}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$, where $\gamma = \alpha + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 2, m_r\} + \frac{\gamma}{2}, \min\{M_l, M_r + 2\} - \frac{\gamma}{2}]$.*

Sketch of proof. We consider the case $\alpha = \beta = 2$. The other cases are treated similarly (see the appendix). In this case G_ν is of type I_2O_{22} and we prove that $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$.

Assume first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν . Let H_{μ_l} and H_{μ_r} be the rectilinear planar representations of G_{μ_l} and G_{μ_r} contained in H_ν , and let σ_{μ_l} and σ_{μ_r} their spiralities. Since both u and v have outdegree two in G_ν we have that $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 2, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$. By the representability condition $\sigma_{\mu_r} = \sigma_{\mu_l} - 2$. Hence $\sigma_{\mu_r} \geq m_l - 2$ and $\sigma_{\mu_r} \geq \max\{m_l - 2, m_r\}$. Also by $\sigma_\nu = \sigma_{\mu_r} + 1$, $\sigma_\nu \geq \max\{m_l - 2, m_r\} + 1$. Similarly, by the representability

condition $\sigma_{\mu_l} = \sigma_{\mu_r} + 2$. Hence $\sigma_{\mu_l} \leq M_r + 2$ and $\sigma_{\mu_l} \leq \max\{M_l, M_r + 2\}$. Since $\sigma_{\mu_l} = \sigma_\nu + 1$ we have $\sigma_\nu \leq \max\{M_l, M_r + 2\} - 1$.

Assume vice versa that k is an integer in the interval $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$. We show that there exists a rectilinear planar representation of G_ν with spirality $\sigma_\nu = k$. We have $k + 1 \in [\max\{m_l, m_r + 2\}, \min\{M_l, M_r + 2\}]$ and therefore $k + 1 \in [m_l, M_l]$. Hence there is a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k + 1$. Similarly, $k - 1 \in [\max\{m_l - 2, m_r\}, \min\{M_l - 2, M_r\}]$ and therefore $k - 1 \in [m_r, M_r]$. Hence there is a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = k - 1$. By the representability condition, G_ν has a rectilinear planar representation H_ν ; with the same construction as in Lemma 7, the spirality of H_ν is $\sigma_\nu = k$. \square

The proofs of the next lemmas are similar to Lemma 8 (see the appendix).

Lemma 9. *Let ν be a P-node of type $I_{3d}O_{\alpha\beta}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$, where $\gamma = \alpha + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma - \rho(d)}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma + \rho(d)}{2}]$, where $\rho(\cdot)$ is a function such that $\rho(r) = 1$ and $\rho(l) = 0$.*

Lemma 10. *Let ν be a P-node of type $I_{3dd'}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $3 \in [m_l - M_r, M_l - m_r]$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 1, m_r + 2\} - \frac{\rho(d) + \rho(d')}{2}, \min\{M_l - 1, M_r + 2\} - \frac{\rho(d) + \rho(d')}{2}]$, where $\rho(\cdot)$ is a function such that $\rho(r) = 1$ and $\rho(l) = 0$.*

3.4 Characterization

Lemmas 4, 5, 6, 8, 9, and 10 give rise to the following characterization.

Theorem 1. *Let G be a plane series-parallel graph and let T be an SPQ*-tree of G . Let ν be any non-root node of T . The plane graph G_ν is rectilinear planar if and only if it satisfies the representability condition given in Table 1. Also, if such condition is satisfied, G_ν admits a rectilinear planar representation for all and only the values of spirality in the representability interval given in Table 1.*

To finally achieve a characterization of rectilinear series-parallel graphs we need to consider the representability condition that must be verified at the level of the root, when the reference edge is not a dummy edge. Denote by $e = (u, v)$ be the reference edge of G and let ρ be the root of T with respect to e . Let η be the child of ρ that does not correspond to e , and let u' and v' be the alias vertices associated with the poles u and v of G_η . Suppose that G_η is rectilinear planar with representability interval I_η .

We say that G satisfies the *root condition* if $I_\eta \cap \Delta_\rho \neq \emptyset$, where Δ_ρ is defined as follows: (i) $\Delta_\rho = [2, 6]$ if u' coincides with u and v' coincides with v ; (ii) $\Delta_\rho = [3, 5]$ if exactly one of u' and v' coincides with u and v , respectively; (iii) $\Delta_\rho = 4$ if none of u' and v' coincides with u and v .

Lemma 11. *Let $e = (u, v)$ be the reference edge of G and let ρ be the root of T with respect to e . Let η be the child of ρ that does not correspond to e . Suppose that G_η is rectilinear planar with representability interval I_η . G is rectilinear planar if and only if it satisfies the root condition. Also, if G satisfies the root condition, it admits a rectilinear planar representation H for any value of spirality σ_η of H_η such that $\sigma_\eta \in I_\eta \cap \Delta_\rho$, where H_η is the restriction of H to G_η .*

Proof. Let f_{int} be the internal face of G incident to e . Observe that u and v are the poles of G_η . Let u' be the alias vertex associated with u and let v' be the alias vertex associated with v . H is a rectilinear planar representation of G if and only if the following two conditions hold: The restriction H_η of H to G_η is a rectilinear planar representation; the number A of right turns minus left turns of any simple cycle of G in H containing e and traversed clockwise in H is equal to 4. We have $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'}$, where: σ_η is the spirality of H_η ; for $w \in \{u', v'\}$, $\alpha_w = 1$, $\alpha_w = 0$, and $\alpha_w = -1$ if the angle formed by w in f_{int} is equal to 90° , 180° , or 270° , respectively.

According to the definition of root condition, there are three cases to consider: (i) $\Delta_\rho = [2, 6]$, (ii) $\Delta_\rho = [3, 5]$, and (iii) $\Delta_\rho = 4$. Consider Case (i). Since in this case the alias vertices coincide with the poles, we have $\alpha_{u'} \in [-1, 1]$, $\alpha_{v'} \in [-1, 1]$, and hence $\alpha_{u'} + \alpha_{v'} \in [-2, 2]$. If G is rectilinear planar, we have that $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'} = 4$ for some $\sigma_\eta \in I_\eta$ and for $\alpha_{u'} + \alpha_{v'} \in [-2, 2]$. Hence, $\sigma_\eta = 4 - \alpha_{u'} - \alpha_{v'} \in [2, 6]$, i.e., the root condition $I_\eta \cap \Delta_\rho \neq \emptyset$ holds.

Suppose vice versa that the root condition $I_\eta \cap \Delta_\rho \neq \emptyset$ holds. For any value $\sigma_\eta \in I_\eta \cap \Delta_\rho$ there exists a rectilinear planar representation of H_η of G_η with spirality σ_η . Also, since $\Delta_\rho = [2, 6]$, we have that $4 - \sigma_\eta \in [-2, 2]$, and therefore, for any possible choice of $\sigma_\eta \in I_\eta \cap \Delta_\rho$, we can suitably choose $\alpha_{u'}$ and $\alpha_{v'}$ such that $\alpha_{u'} + \alpha_{v'} = 4 - \sigma_\eta$, i.e., $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'} = 4$. It follows that G is rectilinear planar and it admits a rectilinear planar representation for any value $\sigma_\eta \in I_\eta \cap \Delta_\rho$.

Cases (ii) and (iii) can be proved analogously, observing that in Case (ii) $\alpha_{u'} + \alpha_{v'} \in [-1, 1]$ and in Case (iii) $\alpha_{u'} + \alpha_{v'} = 0$. \square

The next theorem is an immediate consequence of Lemma 11 and Theorem 1, and it provides a characterization of rectilinear plane series-parallel graphs.

Theorem 2. *Let G be a plane series-parallel graph and let T be an SPQ*-tree of G . Let η be the root child of T . Graph G is rectilinear planar if and only if: (i) G_η is rectilinear planar; (ii) G satisfies the root condition.*

Table 1. Representability conditions and intervals for the different types of nodes. In the formulas $\gamma = \alpha + \beta - 2$ and $\rho(\cdot)$ is such that $\rho(r) = 1$ and $\rho(l) = 0$.

Q*-node of length ℓ – Lemma 4	
Representability Condition	true
Representability Interval	$[-\ell + 1, \ell - 1]$
S-node with h children – Lemma 5	
Representability Condition	true
Representability Interval	$[\sum_{i=1}^h m_i, \sum_{i=1}^h M_i]$
P-node with three children – Lemma 6	
Representability Condition	$[m_l - 2, M_l - 2] \cap [m_e, M_e] \cap [m_r + 2, M_r + 2] \neq \emptyset$
Representability Interval	$[\max\{m_l - 2, m_e, m_r + 2\}, \min\{M_l - 2, M_e, M_r + 2\}]$
P-node with two children – $I_2O_{\alpha\beta}$ – Lemma 8	
Representability Condition	$[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$
Representability Interval	$[\max\{m_l - 2, m_r\} + \frac{\gamma}{2}, \min\{M_l, M_r + 2\} - \frac{\gamma}{2}]$
P-node with two children – $I_{3d}O_{\alpha\beta}$ – Lemma 9	
Representability Condition	$[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$
Representability Interval	$[\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma - \rho(d)}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma + \rho(d)}{2}]$
P-node with two children – $I_{3dd'}$ – Lemma 10	
Representability Condition	$3 \in [m_l - M_r, M_l - m_r]$
Representability Interval	$[\max\{m_l - 1, m_r + 2\} - \frac{\rho(d) + \rho(d')}{2}, \min\{M_l - 1, M_r + 2\} - \frac{\rho(d) + \rho(d')}{2}]$

4 Rectilinear Planarity Testing Algorithm

Theorem 3. *Let G be an n -vertex plane series-parallel graph. There exists an $O(n)$ -time algorithm that tests whether G admits a planar rectilinear representation and that constructs one in the positive case.*

Proof. If G is biconnected let e be an edge of G on the external face; otherwise, let e be a dummy edge added on the external face to make G biconnected. Let T be an SPQ*-tree of G with respect to e . We first show how to perform the test in linear time. If the test is positive, we show how to efficiently construct a rectilinear planar representation of G .

Testing Algorithm. Based on Theorem 1, the algorithm visits T in post-order and, for each non-root node ν of T , it checks the representability condition of ν and computes interval I_ν if the condition is positive. If the representability condition is violated for some node, the algorithm halts and returns a negative answer. Otherwise, the algorithm reaches the root ρ of T . If e is a dummy edge, the algorithm halts and returns a positive answer (since e will not appear in the representation, the algorithm does not need to check anything else). If e is real, let η be the child of ρ other than the child associated with e (see Fig. 1(c)). Based on Theorem 2, to complete the test, the algorithm must check the root condition, i.e., it must check whether $I_\eta \cap \Delta_\rho \neq \emptyset$.

We now analyze the time complexity of the testing algorithm. T can be computed in $O(n)$ time and it consists of $O(n)$ nodes [1]. For a node ν of T that

is not a Q^* -node, denote by n_ν the number of children of ν . In the bottom-up visit, each node of T is visited exactly once. By Theorem 1, for a non-root node ν of T we have the following: If ν is a Q^* -node, its representability interval can be computed in $O(1)$ time, assuming that the length ℓ of the chain of edges represented by ν is stored at ν during the construction of T . If ν is an S-node, its representability interval can be computed in $O(n_\nu)$ time. If ν is a P-node, its representability interval can be computed in $O(1)$ time. Finally, the root condition is easily checked in $O(1)$ time by Lemma 11. It follows that the whole test takes $O(n)$ time.

Construction Algorithm. Suppose that the test is positive. By Theorem 2, the root condition holds, and by Lemma 11, for $\sigma_\eta \in I_\eta \cap \Delta_\rho$, G admits a rectilinear planar representation H such that its restriction H_η to G_η has spirality σ_η . Hence, the algorithm starts by arbitrarily choosing a value $\sigma_\eta \in I_\eta \cap \Delta_\rho$ (if e is a dummy edge, it can choose σ_η as any value in I_η). Then, to construct a rectilinear planar representation H of G , the algorithm visits T top-down and determine the right value of spirality required by the component associated with each node of T distinct from η . Once the spiralities for all nodes of T are determined, H is easily defined by fixing the vertex angles in each component as described in the proofs of Lemmas 4–6, 8–10. To compute the spiralities for the children of η we distinguish the following cases:

Case 1: η is an S-node, with children μ_1, \dots, μ_h ($i \in \{1, \dots, h\}$). Let $I_{\mu_i} = [m_i, M_i]$ be the representability interval of μ_i . We must find a value $\sigma_{\mu_i} \in [m_i, M_i]$ for each $i = 1, \dots, h$ such that $\sum_{i=1}^h \sigma_{\mu_i} = \sigma_\eta$. To this aim, initially set $\sigma_{\mu_i} = M_i$ for each $i = 1, \dots, h$ and consider $s = (\sum_{i=1}^h \sigma_{\mu_i}) - \sigma_\eta$. By Lemma 1, $s \geq 0$. If $s = 0$ we are done. Otherwise, iterate over all $i = 1, \dots, h$ and for each i decrease both σ_{μ_i} and s by the value $\min\{s, M_i - m_i\}$, until $s = 0$.

Case 2: η is a P-node with three children, μ_l , μ_c , and μ_r . By Lemma 3, it suffices to set $\sigma_{\mu_l} = \sigma_\eta + 2$, $\sigma_{\mu_c} = \sigma_\eta$, and $\sigma_{\mu_r} = \sigma_\eta - 2$.

Case 3: η is a P-node with two children, μ_l and μ_r . Let u and v be the poles of η . By Lemma 2, σ_{μ_l} and σ_{μ_r} must be fixed in such a way that $\sigma_{\mu_l} = \sigma_\eta + k_u^l \alpha_u^l + k_v^l \alpha_v^l$ and $\sigma_{\mu_r} = \sigma_\eta - k_u^r \alpha_u^r - k_v^r \alpha_v^r$. The values of k_u^l , k_v^l , k_u^r , and k_v^r are fixed by the indegree and outdegree of u and v . Hence, it suffices to choose the values of α_u^l , α_v^l , α_u^r , α_v^r such that they are consistent with the type of η and they yield $\sigma_{\mu_l} \in I_{\mu_l}$ and $\sigma_{\mu_r} \in I_{\mu_r}$. Since each α_w^d ($w \in \{u, v\}$, $d \in \{l, r\}$) is either 0 or 1 there are at most four possible combinations of values to consider.

Once the spiralities for the children of η are computed, the algorithm continues its top-down visit, and for each node ν for which a spirality σ_ν has been fixed, it computes the spiralities of the children of ν with same procedure as for η . Concerning the time complexity, the procedure in Case 1 takes linear time in the number of children of the S-node, while the procedures in Case 2 and Case 3 take constant time. Therefore the whole visit requires $O(n)$ time. \square

Table 2 shows a running example based on Fig. 1. For each P- and S-component it reports the representability interval computed in the bottom-up visit of the tree and the spirality fixed in the top-down visit (see also Fig. 2).

Table 2. Running Example based on Figure 1.

NODE LABEL	NODE TYPE	REPRES. INTERVAL	SPIRALITY IN H
η	S-node	$[-3, 3]$	3
ν_1	P-node (2 children) – $I_{3r}O_{11}$	$[-2, 2]$	2
ν_2	S-node	$[-4, 4]$	4
ν_3	S-node	$[-\frac{5}{2}, \frac{1}{2}]$	$\frac{1}{2}$
ν_4	P-node (2 children) – I_{3lr}	$[0, 0]$	0
ν_5	P-node (3 children)	$[-1, 0]$	0
ν_6	P-node (2 children) – I_2O_{12}	$[-\frac{3}{2}, \frac{1}{2}]$	$\frac{1}{2}$
ν_7	P-node (2 children) – I_2O_{22}	$[0, 0]$	0
ν_8	S-node	$[-\frac{3}{2}, \frac{3}{2}]$	$\frac{3}{2}$
ν_9	S-node	$[-\frac{3}{2}, \frac{3}{2}]$	$-\frac{3}{2}$
ν_{10}	P-node (2 children) – I_2O_{12}	$[-\frac{1}{2}, \frac{1}{2}]$	$\frac{1}{2}$
ν_{11}	P-node (2 children) – I_2O_{12}	$[-\frac{1}{2}, \frac{1}{2}]$	$-\frac{1}{2}$

5 Conclusions and Open Problems

We proved that rectilinear planarity testing can be solved in linear time for series-parallel graphs with two terminals. Several open problems can be studied:

OP1. Can we extend Theorem 3 to 1-connected plane 4-graphs whose biconnected components are two-terminal series-parallel graphs (i.e., partial 2-trees)? The work in [10] solves the problem for 1-connected outerplanar graphs.

OP2. What is the time complexity of rectilinear planarity testing for general plane 4-graphs? The question is interesting even for triconnected plane 4-graphs. A linear-time solution exists for plane 3-graphs [15,16].

OP3. Testing rectilinear planarity is NP-complete in the variable embedding setting but it can be solved in $O(n^3 \log n)$ -time for series-parallel graphs [6]. It is interesting to determine whether this complexity bound can be improved.

References

1. G. D. Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs*. Prentice-Hall, 1999.
2. G. Borradaile, P. N. Klein, S. Mozes, Y. Nussbaum, and C. Wulff-Nilsen. Multiple-source multiple-sink maximum flow in directed planar graphs in near-linear time. *SIAM J. Comput.*, 46(4):1280–1303, 2017.
3. F. Brandenburg, D. Eppstein, M. T. Goodrich, S. G. Kobourov, G. Liotta, and P. Mutzel. Selected open problems in graph drawing. In G. Liotta, editor, *Graph*

- Drawing, 11th International Symposium, GD 2003, Perugia, Italy, September 21-24, 2003, Revised Papers*, volume 2912 of *Lecture Notes in Computer Science*, pages 515–539. Springer, 2003. URL: https://doi.org/10.1007/978-3-540-24595-7_55, doi:10.1007/978-3-540-24595-7_55.
4. S. Cornelsen and A. Karrenbauer. Accelerated bend minimization. *J. Graph Algorithms Appl.*, 16(3):635–650, 2012. URL: <https://doi.org/10.7155/jgaa.00265>, doi:10.7155/jgaa.00265.
 5. G. Di Battista, G. Liotta, and F. Vargiu. Spirality and optimal orthogonal drawings. *SIAM J. Comput.*, 27(6):1764–1811, 1998. URL: <https://doi.org/10.1137/S0097539794262847>, doi:10.1137/S0097539794262847.
 6. E. Di Giacomo, G. Liotta, and F. Montecchiani. Sketched representations and orthogonal planarity of bounded treewidth graphs. *CoRR*, abs/1908.05015, 2019.
 7. E. Di Giacomo, G. Liotta, and R. Tamassia. Graph drawing. In J. E. Goodman, J. O’Rourke, and C. D. Tóth, editors, *Handbook of Discrete and Computational Geometry, Third Edition*, chapter 55, pages 1451–1478. Chapman and Hall/CRC, 2017.
 8. W. Didimo, G. Liotta, G. Ortali, and M. Patrignani. Optimal orthogonal drawings of planar 3-graphs in linear time. In S. Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 806–825. SIAM, 2020. URL: <https://doi.org/10.1137/1.9781611975994.49>, doi:10.1137/1.9781611975994.49.
 9. C. A. Duncan and M. T. Goodrich. Planar orthogonal and polyline drawing algorithms. In R. Tamassia, editor, *Handbook on Graph Drawing and Visualization.*, pages 223–246. Chapman and Hall/CRC, 2013. URL: <https://www.crcpress.com/Handbook-of-Graph-Drawing-and-Visualization/Tamassia/9781584884125>.
 10. F. Frati. Planar rectilinear drawings of outerplanar graphs in linear time. In D. Auber and P. Valtr, editors, *Graph Drawing, Symposium on Graph Drawing and Network Visualization, GD ’20, September 16-18, Proceedings*, Lecture Notes in Computer Science. Springer, 2020.
 11. A. Garg and G. Liotta. Almost bend-optimal planar orthogonal drawings of biconnected degree-3 planar graphs in quadratic time. In J. Kratochvíl, editor, *Graph Drawing, 7th International Symposium, GD’99, Střirín Castle, Czech Republic, September 1999, Proceedings*, volume 1731 of *Lecture Notes in Computer Science*, pages 38–48. Springer, 1999. URL: https://doi.org/10.1007/3-540-46648-7_4, doi:10.1007/3-540-46648-7_4.
 12. A. Garg and R. Tamassia. A new minimum cost flow algorithm with applications to graph drawing. In S. C. North, editor, *Graph Drawing, Symposium on Graph Drawing, GD ’96, Berkeley, California, USA, September 18-20, Proceedings*, volume 1190 of *Lecture Notes in Computer Science*, pages 201–216. Springer, 1996. URL: https://doi.org/10.1007/3-540-62495-3_49, doi:10.1007/3-540-62495-3_49.
 13. M. M. Hasan and M. S. Rahman. No-bend orthogonal drawings and no-bend orthogonally convex drawings of planar graphs (extended abstract). In D. Du, Z. Duan, and C. Tian, editors, *Computing and Combinatorics - 25th International Conference, COCOON 2019, Xi’an, China, July 29-31, 2019, Proceedings*, volume 11653 of *Lecture Notes in Computer Science*, pages 254–265. Springer, 2019. URL: https://doi.org/10.1007/978-3-030-26176-4_21, doi:10.1007/978-3-030-26176-4_21.

14. T. Nishizeki and M. S. Rahman. *Planar Graph Drawing*, volume 12 of *Lecture Notes Series on Computing*. World Scientific, 2004. URL: <https://doi.org/10.1142/5648>, doi:10.1142/5648.
15. M. S. Rahman, S. Nakano, and T. Nishizeki. A linear algorithm for bend-optimal orthogonal drawings of triconnected cubic plane graphs. *J. Graph Algorithms Appl.*, 3(4):31–62, 1999. URL: <http://www.cs.brown.edu/publications/jgaa/accepted/99/SaidurNakanoNishizeki99.3.4.pdf>.
16. M. S. Rahman and T. Nishizeki. Bend-minimum orthogonal drawings of plane 3-graphs. In L. Kucera, editor, *Graph-Theoretic Concepts in Computer Science, 28th International Workshop, WG 2002, Cesky Krumlov, Czech Republic, June 13-15, 2002, Revised Papers*, volume 2573 of *Lecture Notes in Computer Science*, pages 367–378. Springer, 2002. URL: https://doi.org/10.1007/3-540-36379-3_32, doi:10.1007/3-540-36379-3_32.
17. M. S. Rahman, T. Nishizeki, and M. Naznin. Orthogonal drawings of plane graphs without bends. *J. Graph Algorithms Appl.*, 7(4):335–362, 2003. URL: <http://jgaa.info/accepted/2003/Rahman+2003.7.4.pdf>.
18. R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–444, 1987. URL: <https://doi.org/10.1137/0216030>, doi:10.1137/0216030.

A Additional Material for Section 3

Proof of Lemma 8

We first prove the following result.

Lemma 7. *Let G_ν be a P -node of type $\mathbf{I_2O_{\alpha\beta}}$ with children μ_l and μ_r . G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4 - \gamma]$, where $\gamma = \alpha + \beta - 2$.*

Proof. We distinguish three cases, based on the values of α and β .

Case 1: $\alpha = \beta = 1$. In this case G_ν is of type $\mathbf{I_2O_{11}}$ and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4]$. For a $\mathbf{I_2O_{11}}$ component we have $k_u^l = k_v^l = k_u^r = k_v^r = 1$.

If G_ν is rectilinear planar, we have $1 \leq \alpha_u^l + \alpha_u^r \leq 2$ and $1 \leq \alpha_v^l + \alpha_v^r \leq 2$ in any rectilinear planar representation of G_ν . Hence, by Lemma 2, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \alpha_v^l + \alpha_u^r + \alpha_v^r \in [2, 4]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4]$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and suitably assign the values of α_u^d and α_v^d ($d \in \{l, r\}$), depending on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$. This assignment is such that for any cycle C of G_ν through u and v , the number of 90° angles minus the number of 270° angles in the interior of C is equal to four. Poles u and v split C into two paths π_l and π_r . The spirality σ_{μ_l} equals the number of right turns minus the number of left turns along π_l while going from u to v , which in turns corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l . Similarly, $-\sigma_{\mu_r}$ equals the number of right turns minus the number of left turns along π_r while going from v to u , which in turns corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_r . By also taking into account the angles at u and v inside C , the number of 90° angles minus the number of 270° angles in the interior of C can be expressed as $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 4 - \alpha_u^l - \alpha_u^r - \alpha_v^l - \alpha_v^r$. We distinguish the following three cases: (i) If $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$, then for every pole $w \in \{u, v\}$ we set α_w^l and α_w^r such that $\alpha_w^l + \alpha_w^r = 1$. (ii) If $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, then for one pole $w \in \{u, v\}$ we set α_w^l and α_w^r such that $\alpha_w^l + \alpha_w^r = 1$, and for the other pole $w' \in \{u, v\}$ we set $\alpha_{w'}^l = \alpha_{w'}^r = 1$. (iii) If $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$, then for every pole $w \in \{u, v\}$ we set $\alpha_w^l = \alpha_w^r = 1$. In all the cases above, we have that $a_c = 4$. Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $\alpha = 1, \beta = 2$. In this case G_ν is of type I_2O_{12} and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$. Suppose, w.l.o.g., that $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$. We have $k_u^l = k_u^r = 1$ and $k_v^l = k_v^r = \frac{1}{2}$.

If G_ν is rectilinear planar, we have $\alpha_v^l + \alpha_v^r = 2$ and $\alpha_u^l + \alpha_u^r \in [1, 2]$. By Lemma 2 we have $\sigma_{\mu_l} - \sigma_{\mu_r} = k_u^l \alpha_u^l + k_v^r \alpha_v^l + k_u^r \alpha_u^r + k_v^r \alpha_v^r$, and hence $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2} \alpha_v^l + \alpha_u^r + \frac{1}{2} \alpha_v^r \in [2, 3]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} , such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and suitably set α_u^d and α_v^d ($d \in \{l, r\}$). Namely, we set $\alpha_v^l = \alpha_v^r = 1$. The values of α_u^l and α_u^r are set as follows: (i) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$, we set α_u^l and α_u^r such that $\alpha_u^l + \alpha_u^r = 1$; (ii) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, we set $\alpha_u^l = \alpha_u^r = 1$. With an argument similar to the previous case, for any cycle C through u and v , the number of 90° angles minus the number of 270° angles in the interior of C can be expressed in this case by $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 4 - \alpha_u^l - \alpha_u^r - 1$ (the angle at v inside C is always of 90° degrees). In case (i) we have $a_c = 2 + 4 - 1 - 1 = 4$; in case (ii) we have $a_c = 3 + 4 - 2 - 1 = 4$. Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} or G_{μ_r} .

Case 3: $\alpha = \beta = 2$. In this case G_ν is of type I_2O_{22} and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. We have $k_u^l = k_u^r = \frac{1}{2}$.

If G_ν is rectilinear planar, we have that $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 2, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$; hence $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$.

Suppose vice versa that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. We show that G_ν admits a rectilinear planar representation H_ν . Again, we obtain H_ν by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and by suitably setting α_u^d and α_v^d ($d \in \{l, r\}$). In this case, for any cycle C through u and v , the number of 90° angles minus the number of 270° angles in the interior of C can be expressed by $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 1 + 1$ (both the angles at u and v inside C is always of 90° degrees). We then set $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, which guarantees $a_c = 4$. Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} or G_{μ_r} . \square

Lemma 8. *Let ν be a P-node of type $I_2O_{\alpha\beta}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$, where $\gamma = \alpha + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 2, m_r\} + \frac{\gamma}{2}, \min\{M_l, M_r + 2\} - \frac{\gamma}{2}]$.*

Proof. We first prove the correctness of the representability condition and then the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 7, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spiralities σ_{μ_l} and σ_{μ_r} , respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4 - \gamma]$. Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 4 - \gamma$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq 2$, i.e., $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$.

Suppose, vice versa that $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every integer value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that for every integer value k in the interval $[m_l - M_r, M_l - m_r]$, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k$. Since by hypothesis there exists a value $k \in [m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma]$, there must be two values of spiralities σ_{μ_l} and σ_{μ_r} for the representations of G_{μ_l} and G_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k \in [2, 4 - \gamma]$. Hence, by Lemma 7 G_ν is rectilinear planar.

Representability interval. We analyze three cases, based on the values of α and β .

Case 1: $\alpha = \beta = 1$. In this case G_ν is of type I_2O_{11} and we prove that $I_\nu = [\max\{m_l - 2, m_r\}, \min\{M_l, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear representation of G_ν . By Lemma 2, we have $\sigma_\nu \in [m_l - 2, M_r + 2]$. Also, since for a I_2O_{11} component we have $k_u^l = k_v^l = k_u^r = k_v^r = 1$, we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$, which implies $\sigma_\nu \geq m_r$. Analogously, $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \alpha_v^l \leq M_l$. Hence, $\sigma_\nu \in I_\nu = [\max\{m_l - 2, m_r\}, \min\{M_l, M_r + 2\}]$.

Assume vice versa that k is any integer in the interval $I_\nu = [\max\{m_l - 2, m_r\}, \min\{M_l, M_r + 2\}]$. We show that G_ν admits a rectilinear planar representation with spirality $\sigma_\nu = k$. By hypothesis $k \leq \min\{M_l, M_r + 2\} \leq M_l$; also, $k \geq \max\{m_l - 2, m_r\} \geq m_l - 2$, i.e., $k + 2 \geq m_l$. Hence $[k, k + 2] \cap [m_l, M_l] \neq \emptyset$. Analogously, $k \leq \min\{M_l, M_r + 2\} \leq M_r + 2$, i.e., $k - 2 \leq M_r$; also, $k \geq \max\{m_l - 2, m_r\} \geq m_r$. Hence $[k - 2, k] \cap [m_r, M_r] \neq \emptyset$. We now distinguish the following sub-cases:

- **Case 1.1:** $k \leq M_l - 2$. Consider any two rectilinear planar representations H_{μ_l} of G_{μ_l} and H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_l} = k + 2$ and $\sigma_{\mu_r} \in [k - 2, k] \cap [m_r, M_r] \neq \emptyset$, respectively. Notice that, as already observed, $k + 2 \geq m_l$ and by hypothesis $k + 2 \leq M_l$; hence $\sigma_{\mu_l} \in [m_l, M_l]$. With this choice we have $2 \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 4$, and we can combine H_{μ_l} and H_{μ_r} in parallel as in the proof of Lemma 7 to obtain a rectilinear planar representation H_ν of G_ν . By Lemma 2 the spirality of H_ν equals $\sigma_{\mu_l} - \alpha_l^u - \alpha_l^v = k + 2 - \alpha_l^u - \alpha_l^v$ and it suffices to set $\alpha_l^u = \alpha_l^v = 1$ (which is always possible, as these two values correspond to 90° angles) to get $\sigma_\nu = k$.
- **Case 1.2:** $k = M_l - 1$. Consider any rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k + 1 = M_l$. To suitably choose the spirality of a rectilinear planar representation H_{μ_r} of G_{μ_r} , observe that by the representability condition $M_l - 2 \geq m_r$ and, as already proved, $M_r \geq k - 2$, i.e., $M_r \geq M_l - 3$. It follows that $[M_l - 3, M_l - 2] \cap [m_r, M_r] \neq \emptyset$. Hence, either $M_l - 3 \in [m_r, M_r]$ (possibly $m_r = M_r = M_l - 3$) or $M_l - 2 \in [m_r, M_r]$ (possibly $m_r = M_r = M_l - 2$). In the first case, choose any representation H_{μ_r} with spirality $\sigma_r = M_l - 3$, which implies $\sigma_{\mu_l} - \sigma_{\mu_r} = 3 \in [2, 4]$. In the second case, choose H_{μ_r} with spirality $\sigma_r = M_l - 2$, which implies $\sigma_{\mu_l} - \sigma_{\mu_r} = 2 \in [2, 4]$.

The two representations H_{μ_l} and H_{μ_r} can be combined in parallel to get a representation of G_ν with spirality $\sigma_\nu = k$. Namely, by Lemma 2 we can set $\alpha_u^l = 0$ and $\alpha_v^l = 1$ (or vice versa); also, if $\sigma_{\mu_r} = M_l - 2$ we set $\alpha_u^r = 0$ and $\alpha_v^r = 1$ (or vice versa), while if $\sigma_{\mu_r} = M_l - 3$ we set $\alpha_u^r = \alpha_v^r = 1$.

- **Case 1.3:** $k = M_l$. In this case, we can combine in parallel a representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k = M_l$ and a representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = k - 2 = M_l - 2$, which implies that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. By the representability condition we have $M_l - 2 \geq m_r$, i.e., $\sigma_{\mu_r} \geq m_r$; also, $k \leq \min\{M_l, M_r + 2\} \leq M_r + 2$, i.e., $\sigma_{\mu_r} \leq M_r$. Hence, $\sigma_{\mu_r} \in [m_r, M_r]$. By Lemma 2 we can set $\alpha_u^l = \alpha_v^l = 0$ and $\alpha_u^r = \alpha_v^r = 1$ to get a representation of G_ν with spirality $\sigma_\nu = k$.

Case 2: $\alpha = 1, \beta = 2$. In this case G_ν is of type I_2O_{12} and we prove that $I_\nu = [\max\{m_l - 2, m_r\} + \frac{1}{2}, \min\{M_l, M_r + 2\} - \frac{1}{2}]$.

Assume first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν . Let H_{μ_l} and H_{μ_r} be the rectilinear planar representations of G_{μ_l} and G_{μ_r} contained in H_ν , and let σ_{μ_l} and σ_{μ_r} be their corresponding spiralities. By Lemma 7, $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$, i.e., $\sigma_{\mu_l} \in [2 + \sigma_{\mu_r}, 3 + \sigma_{\mu_r}]$. Since $\sigma_{\mu_l} \in [m_l, M_l]$ and $\sigma_{\mu_r} \in [m_r, M_r]$, we have $\sigma_{\mu_l} \geq \max\{m_l, m_r + 2\}$.

Suppose, w.l.o.g., that $\text{outdeg}_\nu(v) = 2$ and $\text{outdeg}_\nu(u) = 1$. We have $k_u^r = k_u^l = 1$, $k_v^r = k_v^l = \frac{1}{2}$, $\alpha_u^l \in [0, 1]$, $\alpha_v^l \in [0, 1]$, and $\alpha_v^r = \alpha_u^r = 1$. By Lemma 2 we have $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $-\alpha_u^l - \frac{1}{2}\alpha_v^l \geq -\frac{3}{2}$, we have $\sigma_\nu \geq \max\{m_l, m_r + 2\} - \frac{3}{2}$. It follows that $\sigma_\nu \geq \max\{m_l - 2, m_r\} + \frac{1}{2}$. Analogously, since $\sigma_{\mu_r} \in [\sigma_{\mu_l} - 3, \sigma_{\mu_l} - 2]$, we have $\sigma_{\mu_r} \leq \min\{M_l - 2, M_r\}$. By Lemma 2 we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \frac{1}{2}\alpha_v^r$. Since $\alpha_u^r + \frac{1}{2}\alpha_v^r \leq \frac{3}{2}$, we have $\sigma_\nu \leq \min\{M_l - 2, M_r\} + \frac{3}{2}$. It follows that $\sigma_\nu \leq \min\{M_l, M_r + 2\} - \frac{1}{2}$. Therefore, $\sigma_\nu \in I_\nu$.

Assume vice versa that k is a semi-integer in the interval $I_\nu = [\max\{m_l - 2, m_r\} + \frac{1}{2}, \min\{M_l, M_r + 2\} - \frac{1}{2}]$. We show that G_ν has a rectilinear planar representation with spirality $\sigma_\nu = k$. Since $k \in [m_l - \frac{3}{2}, M_l - \frac{1}{2}]$ we have $k + \frac{1}{2} \leq M_l$ and $k + \frac{3}{2} \geq m_l$, i.e., $[k + \frac{1}{2}, k + \frac{3}{2}] \cap [m_l, M_l] \neq \emptyset$. Also, since m_l and M_l are both integer numbers while k is semi-integer, it is impossible to have $k + 1 = m_l = M_l$. It follows that $k + \frac{1}{2} \in [m_l, M_l]$ or $k + \frac{3}{2} \in [m_l, M_l]$. With the same reasoning, we have $k \in [m_r + \frac{1}{2}, M_r + \frac{3}{2}]$ and $[k - \frac{3}{2}, k - \frac{1}{2}] \cap [m_r, M_r] \neq \emptyset$. Hence, $k - \frac{3}{2} \in [m_r, M_r]$ or $k - \frac{1}{2} \in [m_r, M_r]$. We now prove that $k + \frac{3}{2} \in [m_l, M_l]$ or $k - \frac{3}{2} \in [m_r, M_r]$. Suppose by contradiction that $k + \frac{3}{2} \notin [m_l, M_l]$ and $k - \frac{3}{2} \notin [m_r, M_r]$. In that case $k + \frac{1}{2} \in [m_l, M_l]$ and $k - \frac{1}{2} \in [m_r, M_r]$. Consequently, $k + \frac{1}{2} = M_l$ and $k - \frac{1}{2} = m_r$. Hence, $M_l - m_r = 1$ and, by the representability condition, G_ν is not rectilinear planar, a contradiction. As in the previous case, a rectilinear representation of G_ν with spirality k is obtained by combining in parallel a representations H_{μ_l} of G_{μ_l} with spirality σ_{μ_l} and a representation H_{μ_r} of G_{μ_r} with spirality σ_{μ_r} , for two suitable values σ_{μ_l} and σ_{μ_r} . Based on the aforementioned analysis, we distinguish the following sub-cases:

- **Case 2.1:** $k + \frac{3}{2} \notin [m_l, M_l]$. This implies that $k + \frac{1}{2} \in [m_l, M_l]$ and $k - \frac{3}{2} \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = k + \frac{1}{2}$ and $\sigma_{\mu_r} = k - \frac{3}{2}$.
- **Case 2.2:** $k - \frac{3}{2} \notin [m_r, M_r]$. This implies that $k + \frac{3}{2} \in [m_l, M_l]$ and $k - \frac{1}{2} \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = k + \frac{3}{2}$ and $\sigma_{\mu_r} = k - \frac{1}{2}$.

– **Case 2.3:** $k + \frac{3}{2} \in [m_l, M_l]$ and $k - \frac{3}{2} \in [m_r, M_r]$. We set $\sigma_{\mu_l} = k + \frac{3}{2}$ and $\sigma_{\mu_r} = k - \frac{3}{2}$.

Notice that in all the three sub-cases we have $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$, hence by Lemma 7 there exists a rectilinear planar representation H_ν of G_ν that contains H_{μ_l} and H_{μ_r} . It remains to prove that the spirality σ_ν of H_ν is equal to k . Suppose, w.l.o.g, that $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$. We have $k_u^r = k_u^l = 1$ and $k_v^r = k_v^l = \frac{1}{2}$. Since G_ν is rectilinear planar, we have $\alpha_u^l \in [0, 1]$ and $\alpha_v^l = 1$. By Lemma 2 we have $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ_ν is the spirality of the representation H_ν of G_ν . In Case 2.1 we have $\sigma_\nu = k + \frac{1}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. By choosing $\alpha_u^l = 0$ and $\alpha_v^l = 1$ we have $\sigma_\nu = k$. In Case 2.2 and in Case 2.3 we have $\sigma_\nu = k + \frac{3}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. By choosing $\alpha_u^l = 1$ and $\alpha_v^l = 1$ we have $\sigma_\nu = k$.

Case 3: $\alpha = \beta = 2$. In this case G_ν is of type I_2O_{22} and we prove that $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$.

Assume first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν . Let H_{μ_l} and H_{μ_r} be the rectilinear planar representations of G_{μ_l} and G_{μ_r} contained in H_ν , and let σ_{μ_l} and σ_{μ_r} their spiralities. Since both u and v have outdegree two in G_ν we have that $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 2, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$. By the representability condition $\sigma_{\mu_r} = \sigma_{\mu_l} - 2$. Hence $\sigma_{\mu_r} \geq m_l - 2$ and $\sigma_{\mu_r} \geq \max\{m_l - 2, m_r\}$. Also by $\sigma_\nu = \sigma_{\mu_r} + 1$, $\sigma_\nu \geq \max\{m_l - 2, m_r\} + 1$. Similarly, by the representability condition $\sigma_{\mu_l} = \sigma_{\mu_r} + 2$. Hence $\sigma_{\mu_l} \leq M_r + 2$ and $\sigma_{\mu_l} \leq \max\{M_l, M_r + 2\}$. Since $\sigma_{\mu_l} = \sigma_\nu + 1$ we have $\sigma_\nu \leq \max\{M_l, M_r + 2\} - 1$.

Assume vice versa that k is an integer in the interval $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$. We show that there exists a rectilinear planar representation of G_ν with spirality $\sigma_\nu = k$. We have $k + 1 \in [\max\{m_l, m_r + 2\}, \min\{M_l, M_r + 2\}]$ and therefore $k + 1 \in [m_l, M_l]$. Hence there exists a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k + 1$. Similarly, we have $k - 1 \in [\max\{m_l - 2, m_r\}, \min\{M_l - 2, M_r\}]$ and therefore $k - 1 \in [m_r, M_r]$. Hence there exists a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = k - 1$. By the representability condition G_ν has a rectilinear planar representation H_ν ; also, following the same construction as in the proof of Lemma 7, the spirality of H_ν is $\sigma_\nu = k$. \square

Proof of Lemma 9

We first prove the following result.

Lemma 12. *Let G_ν be a P-node of type $I_{3d}O_{\alpha\beta}$ and let μ_l and μ_r be its two children. G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} , respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2} - \gamma]$, where $\gamma = \alpha + \beta - 2$.*

Proof. We distinguish four cases, based on the values of α , β , and d .

Case 1: $\alpha = \beta = 1$, $d = l$. In this case G_ν is of type $I_{3l}O_{11}$ and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values

of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$. For a $\text{I}_{3l}\text{O}_{11}$ component we have $k_u^l = k_u^r = k_v^r = 1$ and $k_v^l = \frac{1}{2}$.

If G_ν is rectilinear planar, we have $1 \leq \alpha_u^l + \alpha_u^r \leq 2$ and $\alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν . Hence, by Lemma 2, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2}\alpha_v^l + \alpha_u^r + \alpha_v^r \in [\frac{5}{2}, \frac{7}{2}]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and suitably assign the values of α_u^l and α_u^r , depending on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$.

Let v' be the alias vertex of G_{μ_l} that is in G_ν . Any cycle C that goes through u and v also passes through v' . We show that the number of 90° angles minus the number of 270° angles in the interior of C is equal to four.

Vertices u and v' split C into two paths π_l and π_r . Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right turns minus left turns along π_r while going from v' to u equals $-\sigma_{\mu_r}$. Hence, the sum $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l$ corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l . Notice that $\alpha_u^r + \alpha_u^l \in \{1, 2\}$ since u is a vertex of degree 3. If $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ we set $\alpha_u^r + \alpha_u^l = 1$ and we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l = \frac{5}{2} + \frac{1}{2} + 2 - 1 = 4$. Else, if $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{7}{2}$ we set $\alpha_u^r + \alpha_u^l = 2$ and we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l = \frac{7}{2} + \frac{1}{2} + 2 - 2 = 4$.

Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $\alpha = 1, \beta = 2, d = l$. In this case G_ν is of type $\text{I}_{3l}\text{O}_{12}$ and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ (note that this corresponds to the interval $[\frac{5}{2}, \frac{7}{2} - \gamma]$ claimed in the lemma). For a $\text{I}_{3l}\text{O}_{12}$ component we have $k_u^l = k_u^r = k_v^l = \frac{1}{2}$ and $k_v^r = 1$.

If G_ν is rectilinear planar, we have $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν . Hence, by Lemma 2, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{1}{2}\alpha_u^l + \frac{1}{2}\alpha_v^l + \frac{1}{2}\alpha_u^r + \alpha_v^r = \frac{5}{2}$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and assign values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$. Let v' be the alias vertex of G_{μ_l} that is in G_ν . Any cycle C that goes through u and v also passes through v' . We show that the number of 90° angles minus the number of 270° angles in the interior of C is equal to four.

Vertices u and v' split C into two paths π_l and π_r . Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right turns minus left turns along π_r while going from v' to u equals $-\sigma_{\mu_r}$. Also, pole u forms a 90° angle inside C . Hence, the sum $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 1$ corresponds to the number of 90° angles minus the number

of 270° angles in the interior of C at the vertices of π_l . Since $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 1 = \frac{5}{2} + \frac{1}{2} + 1 = 4$.

Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}).

Case 3: $\alpha = \beta = 1$, $d = r$. This case is symmetric to Case 1.

Case 4: $\alpha = 1$, $\beta = 2$, $d = r$. This case is symmetric to Case 2. \square

Lemma 9. *Let ν be a P -node of type $I_{3d}O_{\alpha\beta}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$, where $\gamma = \alpha + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma - \rho(d)}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma + \rho(d)}{2}]$, where $\rho(\cdot)$ is a function such that $\rho(r) = 1$ and $\rho(l) = 0$.*

Proof. We first prove the correctness of the representability condition and then the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 12, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spiralities σ_{μ_l} and σ_{μ_r} , respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2} - \gamma]$, where $\gamma = \alpha + \beta - 2$. Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq \frac{7}{2} - \gamma$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq \frac{5}{2}$, i.e., $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$.

Suppose, vice versa that $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that for every semi-integer value k in the interval $[m_l - M_r, M_l - m_r]$, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k$. Since by hypothesis there exists a value $k \in [m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma]$, there must be two values of spiralities σ_{μ_l} and σ_{μ_r} for the representations of G_{μ_l} and G_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k \in [\frac{5}{2}, \frac{7}{2} - \gamma]$. Hence, by Lemma 12 G_ν is rectilinear planar.

Representability interval. We analyze four cases, based on the values of α , β , and d and we assume, w.l.o.g., that ν is the pole of degree four.

Case 1: $\alpha = \beta = 1$, $d = l$. In this case G_ν is of type $I_{3l}O_{11}$ and we prove that $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\}, \min\{M_l - \frac{1}{2}, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν . Since for a $I_{3l}O_{11}$ component we have $k_u^l = k_u^r = k_v^r = 1$ and $k_v^l = \frac{1}{2}$, by Lemma 2 we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l + \alpha_u^r \in \{1, 2\}$ and $\alpha_v^l = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + 1$, $\sigma_\nu \leq M_r + 2$, $\sigma_\nu \geq m_l - \frac{3}{2}$, and $\sigma_\nu \leq M_l - \frac{1}{2}$.

We now show that, if $\sigma_\nu \in [\max\{m_l - \frac{3}{2}, m_r + 1\}, \min\{M_l - \frac{1}{2}, M_r + 2\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν . We have $\sigma_\nu \in [m_l - \frac{3}{2}, M_l - \frac{1}{2}]$. Hence, $\sigma_\nu + \frac{1}{2} \leq M_l$ and $\sigma_\nu + \frac{3}{2} \geq m_l$, i.e., $[\sigma_\nu + \frac{1}{2}, \sigma_\nu + \frac{3}{2}] \cap [m_l, M_l] \neq \emptyset$. Also, since m_l and M_l are both semi-integer numbers while σ_ν is integer, it is impossible to have $\sigma_\nu + 1 = m_l = M_l$. It follows that $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ or $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$. With the same reasoning, we have $\sigma_\nu \in [m_r + 1, M_r + 2]$

and $[\sigma_\nu - 2, \sigma_\nu - 1] \cap [m_r, M_r] \neq \emptyset$. Hence, $\sigma_\nu - 2 \in [m_r, M_r]$ or $\sigma_\nu - 1 \in [m_r, M_r]$. We now prove the following.

Claim. Either $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ or $\sigma_\nu - 2 \in [m_r, M_r]$.

Proof. Suppose by contradiction that $\sigma_\nu + \frac{3}{2} \notin [m_l, M_l]$ and $\sigma_\nu - 2 \notin [m_r, M_r]$. In that case $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ and $\sigma_\nu - 1 \in [m_r, M_r]$. Consequently, $\sigma_\nu + \frac{1}{2} = M_l$ and $\sigma_\nu - 1 = m_r$. Hence, $M_l - m_r = \frac{3}{2}$ and, by the representability condition, G_ν is not rectilinear planar. This is a contradiction. Hence, either $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ or $\sigma_\nu - 2 \in [m_r, M_r]$. \square

We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality σ_{μ_l} and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality σ_{μ_r} , based on the following cases:

- **Case (a):** $\sigma_\nu + \frac{3}{2} \notin [m_l, M_l]$. This implies that $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = \sigma_\nu + \frac{1}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 2$.
- **Case (b):** $\sigma_\nu - 2 \notin [m_r, M_r]$. This implies that $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - 1 \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 1$.
- **Case (c):** $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$. We set $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 2$.

By the claim proved above either the condition of Case (a), Case (b), or Case (c) is verified. Notice that in all the three cases we have $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r} , described in the three cases. We have to prove that in the three cases the spirality of H_ν is σ_ν . By Lemma 2 we have $\sigma'_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l} , α_u^l , and α_v^l . In Case (a) we have $\sigma'_\nu = \sigma_\nu + \frac{1}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. By choosing $\alpha_u^l = 0$ and $\alpha_v^l = 1$ we have $\sigma'_\nu = \sigma_\nu$. In Case (b) and in Case (c) we have $\sigma'_\nu = \sigma_\nu + \frac{3}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. By choosing $\alpha_u^l = 1$ and $\alpha_v^l = 1$ we have $\sigma'_\nu = \sigma_\nu$.

Case 2: $\alpha = 1$, $\beta = 2$, $d = l$. In this case G_ν is of type $I_{3l}O_{12}$ and we prove that $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{1}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{1}{2}] = [\max\{m_l - 1, m_r + \frac{3}{2}\}, \min\{M_l - 1, M_r + \frac{3}{2}\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν . Since for a $I_{3l}O_{12}$ component we have $k_u^l = k_u^r = k_v^l = \frac{1}{2}$ and $k_v^r = 1$, by Lemma 2 we have $\sigma_\nu = \sigma_{\mu_r} + \frac{1}{2}\alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + \frac{3}{2}$, $\sigma_\nu \leq M_r + \frac{3}{2}$, $\sigma_\nu \geq m_l - 1$, and $\sigma_\nu \leq M_l - 1$.

We now show that, if $\sigma_\nu \in [\max\{m_l - 1, m_r + \frac{3}{2}\}, \min\{M_l - 1, M_r + \frac{3}{2}\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν . We have $\sigma_\nu \in [m_l - 1, M_l - 1]$ and $\sigma_\nu \in [m_r + \frac{3}{2}, M_r + \frac{3}{2}]$. Hence, $\sigma_\nu + 1 \in [m_l, M_l]$ and $\sigma_\nu - \frac{3}{2} \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + 1$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - \frac{3}{2}$. Notice that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r} . We have to prove that the spirality of H_ν is σ_ν .

By Lemma 2 we have $\sigma'_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l} , α_v^l , and α_v^r . Since $\sigma_\nu = \sigma_{\mu_l} - 1$, $\alpha_v^l = 1$, and $\alpha_v^r = 1$, we have $\sigma'_\nu = \sigma_\nu + 1 - 1 = \sigma_\nu$.

Case 3: $\alpha = \beta = 1$, $d = r$. This case is symmetric to Case 1.

Case 4: $\alpha = 1$, $\beta = 2$, $d = r$. This case is symmetric to Case 2.

□

Proof of Lemma 10

We first prove the following result.

Lemma 13. *Let G_ν be a P -node of type $\mathbf{I}_{3dd'}$ and let μ_l and μ_r be its two children. G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} , respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$.*

Proof. We distinguish three cases, based on the values of d and d' . Note that the proof for \mathbf{I}_{3rl} is symmetric to the proof for \mathbf{I}_{3lr} .

Case 1: $d = d' = l$. In this case G_ν is of type \mathbf{I}_{3ll} and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of spiralities σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. For a \mathbf{I}_{3ll} component we have $k_u^l = k_v^l = \frac{1}{2}$ and $k_u^r = k_v^r = 1$.

If G_ν is rectilinear planar, we have $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν . Hence, by Lemma 2, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{1}{2}\alpha_u^l + \frac{1}{2}\alpha_v^l + \alpha_u^r + \alpha_v^r = 3$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and assign values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$. Let u' and v' be the alias vertices of G_{μ_l} that are in G_ν . Any cycle C that goes through u and v also passes through u' and v' . We show that the number of 90° angles minus the number of 270° angles in the interior of C is equal to four.

Vertices u' and v' split C into two paths π_l and π_r . Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u' to v' equals $\sigma_{\mu_l} + 1$. The number of right turns minus left turns along π_r while going from v' to u' equals $-\sigma_{\mu_r}$. The sum of these two values corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l . Hence, $\sigma_{\mu_l} + 1 - \sigma_{\mu_r} = 3 + 1 = 4$.

Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $d = d' = r$. This case is symmetric to Case 1, observing that $k_u^r = k_v^r = \frac{1}{2}$ and $k_u^l = k_v^l = 1$.

Case 3: $d = l$, $d' = r$. In this case G_ν is of type \mathbf{I}_{3lr} and we prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for values of

spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. For a I_{3lr} component we have $k_u^r = k_v^l = \frac{1}{2}$ and $k_u^l = k_v^r = 1$.

If G_ν is rectilinear planar, we have $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν . Hence, by Lemma 2, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2}\alpha_v^l + \frac{1}{2}\alpha_u^r + \alpha_v^r = 3$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. We show that G_ν admits a rectilinear planar representation H_ν . To define H_ν , we combine in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and assign values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$. Let v' be the alias vertex of the pole v of G_{μ_l} such that v' is along an edge of G_ν . Similarly, let u' be the alias vertex of the pole u of G_{μ_r} such that u' is along an edge of G_ν . Any cycle C that goes through u and v also passes through u' and v' . We show that the number of 90° angles minus the number of 270° angles in the interior of C is equal to four.

Vertices u' and v' split C into two paths π_l and π_r . Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u' to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right turns minus left turns along π_r while going from v' to u' equals $-\sigma_{\mu_r} + \frac{1}{2}$. The sum of these two values corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l . Hence, $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + \frac{1}{2} = 3 + \frac{1}{2} + \frac{1}{2} = 4$.

Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}). \square

Lemma 10. *Let ν be a P -node of type $\text{I}_{3dd'}$ with children μ_l and μ_r . Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $3 \in [m_l - M_r, M_l - m_r]$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 1, m_r + 2\} - \frac{\rho(d) + \rho(d')}{2}, \min\{M_l - 1, M_r + 2\} - \frac{\rho(d) + \rho(d')}{2}]$, where $\rho(\cdot)$ is a function such that $\rho(r) = 1$ and $\rho(l) = 0$.*

Proof. We first prove the correctness of the representability condition and then the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 13, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spirality σ_{μ_l} and σ_{μ_r} , respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 3$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq 3$, i.e., $3 \in [m_l - M_r, M_l - m_r]$.

Suppose, vice versa that $3 \in [m_l - M_r, M_l - m_r]$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spirality $\sigma_{\mu_l} \in [m_l, M_l]$ and $\sigma_{\mu_r} \in [m_r, M_r]$ such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. Hence, by Lemma 13 G_ν is rectilinear planar.

Representability interval. We distinguish three cases, based on the values of d and d' . Note that a possible forth case for I_{3rl} is symmetric to the case I_{3lr} .

Case 1: $d = d' = l$. In this case G_ν is of type I_{3ll} and we prove that $I_\nu = [\max\{m_l - 1, m_r + 2\}, \min\{M_l - 1, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν . Since for a \mathbf{I}_{3ll} component we have $k_u^l = k_v^l = \frac{1}{2}$ and $k_u^r = k_v^r = 1$, by Lemma 2 we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + 2$, $\sigma_\nu \leq M_r + 2$, $\sigma_\nu \geq m_l - 1$, and $\sigma_\nu \leq M_l - 1$.

We now show that, if $\sigma_\nu \in [\max\{m_l - 1, m_r + 2\}, \min\{M_l - 1, M_r + 2\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν . We have $\sigma_\nu \in [m_l - 1, M_l - 1]$ and $\sigma_\nu \in [m_r + 2, M_r + 2]$. Hence, $\sigma_\nu + 1 \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + 1$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - 2$. Notice that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r} . We have to prove that the spirality of H_ν is σ_ν . By Lemma 2 we have $\sigma'_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l} , α_u^l , and α_v^l . Since $\sigma_\nu = \sigma_{\mu_l} - 1$, $\alpha_u^l = 1$, and $\alpha_v^l = 1$, we have $\sigma'_\nu = \sigma_\nu + 1 - \frac{1}{2} - \frac{1}{2} = \sigma_\nu$.

Case 2: $d = d' = r$. This case is symmetric to Case 1.

Case 3: $d = l$, $d' = r$. In this case G_ν is of type \mathbf{I}_{3lr} and we prove that $I_\nu = [\max\{m_l - 1, m_r + 2\} - \frac{1}{2}, \min\{M_l - 1, M_r + 2\} - \frac{1}{2}] = [\max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\}, \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν . Since for a \mathbf{I}_{3lr} component we have $k_u^r = k_v^l = \frac{1}{2}$ and $k_u^l = k_v^r = 1$, by Lemma 2 we have $\sigma_\nu = \sigma_{\mu_r} + \frac{1}{2}\alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + \frac{3}{2}$, $\sigma_\nu \leq M_r + \frac{3}{2}$, $\sigma_\nu \geq m_l - \frac{3}{2}$ and $\sigma_\nu \leq M_l - \frac{3}{2}$.

We now show that, if $\sigma_\nu \in [\max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\}, \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν . We have $\sigma_\nu \in [m_l - \frac{3}{2}, M_l - \frac{3}{2}]$ and $\sigma_\nu \in [m_r + \frac{3}{2}, M_r + \frac{3}{2}]$. Hence, $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - \frac{3}{2} \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - \frac{3}{2}$. Notice that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r} . We have to prove that the spirality of H_ν is σ_ν . By Lemma 2 we have $\sigma'_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l} , α_u^l , and α_v^l . Since $\sigma_\nu = \sigma_{\mu_l} - \frac{3}{2}$, $\alpha_u^l = 1$, and $\alpha_v^l = 1$, we have $\sigma'_\nu = \sigma_\nu + \frac{3}{2} - 1 - \frac{1}{2} = \sigma_\nu$. \square