# Finding linearly generated subsequences

Claude Gravel<sup>1</sup>, Daniel Panario<sup>2</sup>, and Bastien Rigault<sup>3</sup>

<sup>1</sup> National Institute of Informatics Tokyo, Japan claudegravel1980@gmail.com Currently working at EAGLYS (Tokyo, Japan)

> <sup>2</sup>School of Mathematics and Statistics Carleton University, Canada daniel@math.carleton.ca

<sup>3</sup>National Institute of Informatics Tokyo, Japan rgaultb@gmail.com

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#### Abstract

We develop a new algorithm to compute determinants of all possible Hankel matrices made up from a given finite length sequence over a finite field. Our algorithm fits within the dynamic programming paradigm by exploiting new recursive relations on the determinants of Hankel matrices together with new observations concerning the distribution of zero determinants among the possible matrix sizes allowed by the length of the original sequence. The algorithm can be used to isolate very efficiently linear shift feedback registers hidden in strings with random prefix and random postfix for instance and, therefore, recovering the shortest generating vector. Our new mathematical identities can be used also in any other situations involving determinants of Hankel matrices. We also implement a parallel version of our algorithm. We compare our results empirically with the trivial algorithm which consists of computing determinants for each possible Hankel matrices made up from a given finite length sequence. Our new accelerated approach on a single processor is faster than the trivial algorithm on 160 processors for input sequences of length 16384 for instance.

**Keywords:** generating polynomial, linear algebra over finite fields, quotient-difference tables, Hankel matrices, linear shift feedback registers, pattern substrings, Berlekamp-Massey algorithm

#### 1 Notation, facts and definitions

Let q be a prime power, n > 0, and  $\mathbf{x} = (x_i)_{i=0}^{n-1} \in \mathbb{F}_q^n$ . For integers  $1 \le j \le \left\lceil \frac{n}{2} \right\rceil$  and  $j-1 \le i < n-j+1$ , define the matrix  $\mathbf{X}_{i,j}$  by

$$\mathbf{X}_{i,j} = \begin{pmatrix} x_i & \dots & x_{i+j-2} & x_{i+j-1} \\ x_{i-1} & \dots & x_{i+j-3} & x_{i+j-2} \\ \vdots & \ddots & \vdots & \vdots \\ x_{i-j+1} & \dots & x_{i-1} & x_i \end{pmatrix}.$$
 (1)

By convention, we let  $\mathbf{X}_{i,0} = 1$  for  $0 \le i < n$ . Every matrix  $\mathbf{X}_{i,j}$  is a Hankel matrix.

Hankel matrices have a large number of applications in applied mathematics. In this paper we are interested in Hankel matrices over finite fields. We explore the well known connection of Hankel matrices and sequences over finite fields; for an introductory explanation see Section 8.6 in [13]. Hankel matrices are strongly connected to coprime polynomials over finite fields. Indeed, the probability of two monic polynomials of positive degree n over  $\mathbb{F}_q$  to be relatively prime is the same as the uniform probability for an  $n \times n$  Hankel matrix over  $\mathbb{F}_q$  be nonsingular [6, 7]. Elkies [5] studies the probability of Hankel matrices over finite fields be nonsingular when independent biased entries are used for the matrix. An algorithm to generate a class of Hankel matrices called superregular (that are related to MDS codes) is given in [16]. Finally, we point out that several results and applications of Hankel matrices over finite fields are given in the Handbook of Finite Fields [14]. In particular, see Section 13.2 for enumeration and classical results, Section 14.8 for connections to (t, m, s)-nets, and Section 16.7 for hardware arithmetic for matrices over finite fields. In this paper, we give a new algorithm to compute determinants of all possible Hankel matrices made up from a given finite length sequence over a finite field.

We denote by  $d_{i,j}$  the determinant of  $\mathbf{X}_{i,j}$ . By definition of  $\mathbf{X}_{i,j}$ , we have for all i

$$d_{i,0} = 1, \quad d_{i,1} = x_i, \quad d_{i,j} = \det \mathbf{X}_{i,j}.$$

For convenience, let  $h = \lfloor n/2 \rfloor$ . We use the determinants to form a quotientdifference table [10, 18]. If h is odd, the determinants form a triangle:

0	:	1	1		1	1	1		1	1
1	:	$x_0$	$x_1$			$x_h$			$x_{n-2}$	$x_{n-1}$
2	:		$d_{2,1}$	$d_{2,2}$				$d_{2,n-3}$	$d_{2,n-2}$	
3	:			$d_{3,2}$				$d_{3,n-3}$		
4	:				·	÷	. · ·			
÷					۰.	÷	<sup>.</sup>			
h	:					$d_{h,h}$				

If n is even, then the triangle is truncated at the hth level where there are two elements  $d_{h,h}$  and  $d_{h,h+1}$ . We observe that i refers to columns and j refers to rows.

For integers  $i_0$ ,  $i_1$ ,  $j_0$ ,  $j_1$  such that  $i_1 > i_0$ ,  $h \ge j_1 - j_0 > 0$ , consider the set  $S(i_0, i_1, j_0, j_1) = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i_0 \le i \le i_1, \quad j_0 \le j \le j_1, \quad j_1 - 1 \le i < n - j_1 + 1\}$ . We observe that S is nonempty and may have a k-side polygonal shape with  $3 \le k \le 6$ . Section 4 gives two examples, one with n = 32, k = 6, and one with n = 81, k = 4. We see that the tip of the triangular table from the example with n = 32 has length two, and therefore it yields to an hexagonal case. For a detailed explanation, see Section 4. If S falls entirely inside the table, then k = 4 necessarily, that is, we have a square of zeros. If S overlaps with the edges of the triangular table, then k may be different than 4. We use  $\partial S$  to denote the boundary of S. We prove in this work that zeros in a difference table are *always* distributed or grouped according to S.

Our goal is to design a dynamic programming algorithm to fill the table that requires the least number of determinant evaluations. More precisely, if we know the first j - 1 rows of the table, then we want to compute determinants for the *j*th row by using the least possible number of rows above the *j*th. In Section 2, we establish relations among determinants  $d_{i,j}$ 's no matter how **x** is generated. Our results amplify any linear patterns that could be used to generate the coordinates of **x**. We show that any run of zeros in the table automatically implies a run of zeros exactly below the former so that we obtain a square of zeros. Moreover, we prove identities, that we call *cross shape identities*, relating determinants  $d_{i,j}$ 's located on a cross as explained later; those identities are based on Sylvester's identities, generalized by Bareiss [2], as well as Dogson's identity [1, 11].

It would be possible to avoid the evaluations of determinants of matrices by generalizing determinantal identities given in Conjecture 1 from Section 2. More precisely, in a true random sequence of length n, the expected length of the maximum run of zeros is  $O(\log_2 n)$ . Therefore using the recursive nature of determinants, and especially determinants of Hankel matrices, we conjecture that the evaluations of determinants of matrices larger than about  $O(\log_2 n)$ are not required to complete the table above which would lead to a linear time algorithm to locate the linear subsequence. Our algorithm may also be used as a statistical test to determine linearity in a pseudo-random sequence.

In Section 3, we apply results from Section 2 to the case of a sequence  $\mathbf{x} = (x_i)_{i=1}^n$  that contains a linearly shifted and fed back subsequence.

**Definition 1.** Using  $\mathbf{x} = (x_i)_{i=0}^{n-1}$  as above, let  $c = (c_0, \ldots, c_{d-1}) \in \mathbb{F}_q^d$  with  $c_{d-1} = 1$  and d < n-1. The sequence  $\mathbf{x}$  contains a *linear subsequence* if there are integers s and t with  $d \leq s \leq t < n$  such that for all  $s \leq \ell \leq t$  we have

$$\sum_{i=0}^{d-1} c_i x_{\ell-d+i} = 0.$$

Indeed one of our motivations is to identify the indices s and t as well as to find the generating vector c. This relates our work to the Berlekamp-Massey algorithm. As shown later our method does not assume any upper bound on the length of c or equivalently on the degree of the generating polynomial in the framework of Berlekamp-Massey. Given a prime power q, d > 0, and a sequence  $\mathbf{x} = (x_i)_{i=0}^{\infty}$  with  $x_i \in \mathbb{F}_q$ , the Berlekamp-Massey algorithm is an iterative algorithm that finds the shortest linear feedback shift register (LFSR) that generates  $\mathbf{x}$ . A register of size d over  $\mathbb{F}_q$  is an element of  $\mathbb{F}_q^d$ . More precisely, an LFSR consists of an initial register  $(x_0, x_1, \ldots, x_{d-1}) \in \mathbb{F}_q^d$ , a non zero vector  $c = (c_0, \ldots, c_{d-1}) \in \mathbb{F}_q^d$  such that for  $i \ge 0$ 

$$(x_i, x_{i+1}, \dots, x_{i+d-2}, x_{i+d-1}) \longrightarrow (x_{i+1}, x_{i+2}, \dots, x_{i+d-1}, \sum_{j=0}^{d-1} c_j x_{i+j}).$$
 (2)

The arrow in Equation (2) expresses the transition. The state of a system at a point in time is the content of the register. In Equation (2), the system transits from the state  $(x_i, x_{i+1}, \ldots, x_{i+d-2}, x_{i+d-1})$  to  $(x_{i+1}, x_{i+2}, \ldots, x_{i+d-1}, x_{i+d})$  where  $x_{i+d}$  is a given as linear combination of the previous  $x_i$ 's. The content of the register at time i + d - 1 is being fed back into the right end of it through the linear combination  $\sum_{j=0}^{d-1} c_j x_{i+j}$ . At time i + d, the register is updated to  $(x_{i+1}, x_{i+2}, \ldots, x_{i+d-1}, x_{i+d})$  where  $x_{i+d} = c_0 x_i + c_1 x_{i+1} + \cdots + c_{d-1} x_{i+d-1}$ .

For more information on the Berlekamp-Massey algorithm, see [3] where interesting connections between this algorithm and the extended Euclidean algorithm are given. LaMacchia and Odlyzko [12] also review how Berlekamp-Massey algorithm is used in the Wiedemann algorithm to find linear recurrences over finite fields and also show interesting connections to determinants of Hankel matrices. For more information on LFSR sequences, see [8, 9].

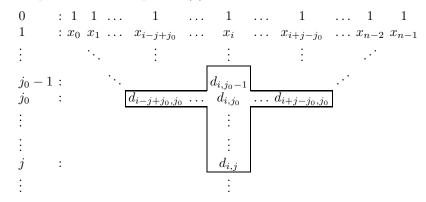
We conclude the section giving the structure of the paper. Section 2 gives several theoretical relations among Hankel determinants that are crucial in this paper. Those relations are used in Section 3 where we provide our algorithm to compute all determinants from Hankel matrices. Illustrative examples over  $\mathbb{F}_2$  are given in Section 4. Due to the lack of space, experimental runs of our algorithm against a standard method of computation are given in Appendix 5. We compare our results empirically with the trivial algorithm which consists of computing determinants for each possible Hankel matrices made up from a given finite length sequence. Our new accelerated approach on a single processor is faster than the trivial algorithm on 160 processors for input sequences of length 16384 for instance as shown in Section 5.

### 2 Relations among Hankel determinants

In this section, we derive useful results to allow the computations of  $d_{i,j}$  without actually computing explicitly or directly determinants of size j and instead using determinants  $d_{i,j'}$  with j < j'. Then in Section 3, we fill the triangular table using a dynamic programming approach. Before that, let us recall one of the results from [2] applied to Hankel matrices and adapted to our notation. If i, jare such that  $i_0 < i < i_1, j_0 \le j \le j_0 + (i_1 - i_0 - 1)$ , and with the convention that  $d_{i,0} = 1, d_{i,1} = x_i$ , then

$$d_{i,j}d_{i,j_0-1}^{j-j_0} = \det \begin{pmatrix} d_{i,j_0} & \dots & d_{i+j-j_0,j_0} \\ \vdots & \ddots & \vdots \\ d_{i-(j-j_0),j_0} & \dots & d_{i,j_0} \end{pmatrix}.$$
 (3)

Equation (3) is called a *j*th-step integer preserving identity in [2]. We call an identity like in Equation (3) a cross shape identity because  $d_{i,j}$ ,  $d_{i,j_0}$  and  $d_{i,j_0-1}$  are located on the vertical part of a cross, and the other non-diagonal elements of the matrix are located on the horizontal part of the aforementioned cross. A visual representation of Equation (3) is as follow:



We come back to Equation (3) at the end of this section with a brief explanation of its proof. Equation (3) remains valid even if  $d_{i,j_0-1} = 0$  as pointed in [2].

**Theorem 1.** Let  $i_0 < i < i_1$ , and  $j_0$  be such that  $d_{i,j_0} \neq 0$ ,  $d_{i,j_0+1} = 0$ ,  $d_{i_0,j_0+1} \neq 0$ ,  $d_{i_1,j_0+1} \neq 0$ . Then with  $j_1 = j_0 + (i_1 - i_0)$  and  $S(i_0, i_1, j_0, j_1)$  non-empty, we have

$$\begin{aligned} d_{i,j} &= 0 \quad for \; all \; (i,j) \in S(i_0,i_1,j_0,j_1), \\ d_{i,j} &\neq 0 \quad for \; all \; (i,j) \in \partial S(i_0,i_1,j_0,j_1). \end{aligned}$$

*Proof.* Without loss of generality, assume that S falls entirely inside the table with left and right boundaries at  $(i_0, j_0)$  and  $(i_1, j_0)$ , respectively, and with upper and lower boundaries at  $(i_0, j_0)$  and  $(i_0, j_1)$ , respectively. To fall entirely inside the table, one must have  $2(i_0 + 1) - i_1 \ge 0$  so that the Hankel matrix  $\mathbf{X}_{i_0,i_1-i_0-1}$  is properly defined; the number of consecutive zeros on level  $j_0$  that occur between  $i_0$  and  $i_1$  is  $i_1 - i_0 - 1$ .

Fix *i* such that  $i_0 < i < i_1$  and let  $j_0 \le w \le j_0 + (i_1 - i_0 - 1)$ . Then using Equation (3) with  $w = j_0 + 1$  as the basis for induction, we obtain that  $d_{i,j_0+1} = 0$ , that is, we obtain the second row of zeros below the first one. For the inductive step, assume that  $d_{i,w'} = 0$  for  $j_0 \le w' < w$ , and rewrite Equation (3) as

$$d_{i,w} = d_{i,j_0-1}^{j_0-w} \det \begin{pmatrix} d_{i,j_0} & \dots & d_{i+w-j_0,j_0} \\ \vdots & \ddots & \vdots \\ d_{i+j_0-w,j_0} & \dots & d_{i,j_0} \end{pmatrix}$$

Therefore at least one row of the previous matrix is made only of zeros which implies the desired result.  $\hfill \Box$ 

We remark that Theorem 1 does not depend on the input sequence, and it is solely a property of determinants for Hankel matrices. If for instance the input sequence is chosen entirely at random with independent identically unbiased distributed Bernoulli random variables, then the biggest squares have average side length  $O(\log_2 n)$  which is the expected length of the longest run of zeros in a random sequence of Bernoulli random variables with length n.

Given a square matrix **X** of size  $\ell \times \ell$ , we consider its sub-matrix **C** of size  $(\ell - 2) \times (\ell - 2)$  located in the center **X**, and its 4 sub-matrices **N**, **S**, **E** and **W** of size  $(\ell - 1) \times (\ell - 1)$  located in the top left, bottom right, top right and bottom left of **X**, respectively. In other words let

$$\mathbf{X} = \begin{pmatrix} \begin{array}{c|c} x_{1,1} & \dots & x_{1,\ell} \\ \vdots & \mathbf{C} & \vdots \\ \hline x_{\ell,1} & \dots & x_{\ell,\ell} \end{pmatrix}$$
$$= \begin{pmatrix} \begin{array}{c|c} \mathbf{N} & x_{1,\ell} \\ \vdots & x_{\ell,1} & \dots & x_{\ell,\ell} \end{pmatrix} = \begin{pmatrix} \begin{array}{c|c} x_{1,1} & \dots & x_{1,\ell} \\ \vdots & \mathbf{S} \end{pmatrix} \\ = \begin{pmatrix} \begin{array}{c|c} x_{1,1} & \dots & x_{1,\ell} \\ \hline \mathbf{W} & \vdots \\ x_{\ell,\ell} \end{pmatrix} = \begin{pmatrix} \begin{array}{c|c} x_{1,1} & \dots & x_{1,\ell} \\ \vdots & \mathbf{S} \\ \hline x_{\ell,1} & \dots & x_{\ell,\ell} \end{pmatrix} .$$

Then we have Dodgson's identity (see [1], or page 29 of [11]):

$$det(\mathbf{X}) det(\mathbf{C}) = det(\mathbf{N}) det(\mathbf{S}) - det(\mathbf{E}) det(\mathbf{W}).$$
(4)

If the entry  $x_{\ell,\ell}$  is an unknown and all other elements of **X** are known, then, for some  $\alpha, \beta \in \mathbb{F}_q$ , we have that

$$(x_{\ell,\ell}\det(\mathbf{N}) + \alpha)\det(\mathbf{C}) = \det(\mathbf{N})(x_{\ell,\ell}\det(\mathbf{C}) + \beta) - \det(\mathbf{E})\det(\mathbf{W}).$$
(5)

Equation (5) implies that  $x_{\ell,\ell}$  cannot be determined if  $\det(\mathbf{N}) = 0$  or  $\det(\mathbf{C}) = 0$ . This simply implies that  $x_{\ell,\ell}$  cannot be determined from a determinantal equation of the type obtained by Dodgson's identity.

We now derive a useful identity using Equation (4) which can also be proved using results from [2].

**Proposition 1** (North-South-East-West). For all (i, j) such that  $i - j + 1 \ge 0$ , and  $2 \le j \le \lceil n/2 \rceil$  the following identity is true:

$$d_{i,j}d_{i,j-2} = d_{i,j-1}^2 - d_{i+1,j-1}d_{i-1,j-1}.$$

*Proof.* Apply Equation (4) on the matrix  $\mathbf{X}_{i,j}$  given from (1) where det( $\mathbf{W}$ ) =  $d_{i-1,j-1}$ , det( $\mathbf{E}$ ) =  $d_{i+1,j-1}$ , det( $\mathbf{N}$ ) =  $d_{i,j-1}$ , det( $\mathbf{S}$ ) =  $d_{i,j-1}$ , and det( $\mathbf{C}$ ) =  $d_{i,j-2}$ .

We observe that Proposition 1 is reminiscent to the North-South-East-West identity [19] for quotient-difference table. Proposition 1 is similar to the 1st-order step integer preserving relation from [2] with a much easier proof. The condition  $d_{i,j-2} \neq 0$  is not required as explained in [2], or as it follows directly from Equation (4), but it matters for our dynamic programming method since we cannot determine  $d_{i,j}$  if  $d_{i,j-2} = 0$  using the table information from the (j-1)th and (j-2)th rows.

In order to accelerate the computation of determinants within a dynamical programming approach, we must ensure that  $d_{i,j_0-1} \neq 0$  from Equation (3). For that we have the next theorem.

**Theorem 2.** For all (i, j) such that  $i - j + 1 \ge 0$ , and  $j_0 \le j \le \lfloor n/2 \rfloor$ , if

$$d_{i,j_0-1} \neq 0, \quad d_{i,k} = 0 \text{ for } j_0 \leq k \leq j-1,$$

then

$$d_{i,j} = d_{i,j_0-1}^{j_0-j} \det \begin{pmatrix} d_{i,j_0} & \dots & d_{i+j-j_0,j_0} \\ \vdots & \ddots & \vdots \\ d_{i-(j-j_0),j_0} & \dots & d_{i,j_0} \end{pmatrix}.$$

*Proof.* Before starting, for a fixed position i and for any size  $j_0$ , we observe that the value  $j - j_0$  expresses the depth of singularity, that is, the number of zeros below the non-zero cell indexed by  $(i, j_0-1)$ . The depth of singularity also relates to the concentration of zeros aligned horizontally around the cell  $(i, j_0 - 1)$ . By concentration of zeros, we mean the length of a run of consecutive zeros.

 $(j = j_0 + 1)$ th step: Suppose that  $d_{i,j_0-1} \neq 0$  and  $d_{i,j_0} = 0$ . If  $d_{i,j_0+1} = 0$ , then there is at least one zero to the left or to the right of  $(i, j_0)$  or both. Indeed Proposition 1 entails that  $d_{i,j_0+1}d_{i,j_0-1} = d_{i,j_0}^2 - d_{i+1,j_0}d_{i-1,j_0}$  which, in this case, is equivalent to  $d_{i,j_0+1}d_{i,j_0-1} = -d_{i+1,j_0}d_{i-1,j_0}$  from which we infer that either  $d_{i-1,j_0} = 0$  or  $d_{i+1,j_0} = 0$  whenever  $d_{i,j_0+1} = 0$ . So there is qualitatively speaking a small concentration of zeros aligned horizontally around the cell  $(i, j_0)$ .

 $(j = j_0 + 2)$ th step: Now suppose that  $d_{i,j_0-1} \neq 0$ ,  $d_{i,j_0} = d_{i,j_0+1} = 0$  and write

$$0 = d_{i,j_0+1}d_{i,j_0-1}^2 = \det \begin{pmatrix} 0 & d_{i+1,j_0} & d_{i+2,j_0} \\ d_{i-1,j_0} & 0 & d_{i+1,j_0} \\ d_{i-2,j_0} & d_{i-1,j_0} & 0 \end{pmatrix}.$$

By the  $(j = j_0 + 1)$ th-step, if  $d_{i+1,j_0} = 0$ , then  $0 = d_{i,j_0+1}d_{i,j_0-1}^2 = d_{i-1,j_0}^2d_{i+2,j_0}$ from which either  $d_{i-1,j_0} = 0$  or  $d_{i+2,j_0} = 0$ ; if  $d_{i-1,j_0} = 0$ , then  $0 = d_{i,j_0+1}d_{i,j_0-1}^2$  $= d_{i+1,j_0}^2d_{i-2,j_0}$  from which either  $d_{i+1,j_0} = 0$  or  $d_{i-2,j_0} = 0$ . Therefore we conclude that  $d_{i-1,j_0} = d_{i+1,j_0} = 0$  as well. The horizontal part of the cross contains therefore a higher concentration of zeros around  $d_{i,j_0}$  with respect to the previous step. We cannot conclude at this moment that  $d_{i+1,j_0} = 0 = d_{i+2,j_0}$  or  $d_{i-1,j_0} = 0 = d_{i-2,j_0}$  without further adding deeper singularities.  $(j = j_0 + 3)$ th step: Now suppose that  $d_{i,j_0-1} \neq 0$  and  $d_{i,j_0} = d_{i,j_0+1} = d_{i,j_0+2} = 0$  and write

$$0 = d_{i,j_0+2} d_{i,j_0-1}^3 = \det \begin{pmatrix} 0 & d_{i+1,j_0} & d_{i+2,j_0} & d_{i+3,j_0} \\ d_{i-1,j_0} & 0 & d_{i+1,j_0} & d_{i+2,j_0} \\ d_{i-2,j_0} & d_{i-1,j_0} & 0 & d_{i+1,j_0} \\ d_{i-3,j_0} & d_{i-2,j_0} & d_{i-1,j_0} & 0 \end{pmatrix}.$$
 (6)

From the  $(j = j_0 + 2)$ th step,  $d_{i-1,j_0} = 0 = d_{i+1,j_0}$ , and Equation (6) is equivalent to

$$0 = d_{i,j_0+2}d_{i,j_0-1}^3 = \det \begin{pmatrix} 0 & 0 & d_{i+2,j_0} & d_{i+3,j_0} \\ 0 & 0 & 0 & d_{i+2,j_0} \\ d_{i-2,j_0} & 0 & 0 & 0 \\ d_{i-3,j_0} & d_{i-2,j_0} & 0 & 0 \end{pmatrix}$$
$$= d_{i+2,j_0}^2 d_{i-2,j_0}^2,$$

so that either  $d_{i-2,j_0} = 0$  or  $d_{i+2,j_0} = 0$ . With the knowledge of the  $(j_0 + 2)$ thstep, we can safely conclude that either  $d_{i+1,j_0} = 0 = d_{i+2,j_0}$  or  $d_{i-2,j_0} = 0 = d_{i-1,j_0}$ . Hence the concentration of zeros increases on the  $j_0$ th row with respect to the previous steps. We observe that the previous determinant has at least one row with 3 consecutive zeros. The position of a run of zeros from one row to the following is shifted cyclically by one position.

The process stops when we can no longer add deeper singularity, that is, we stop for the smallest index  $j > j_0$  such that  $d_{i,j} \neq 0$ . When such index j is found, then we can no longer deduce zero determinants on the horizontal part of the cross.

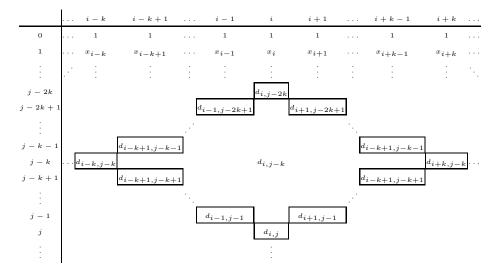
*j***th step:** Assume that  $d_{i,j_0-1} \neq 0$  and  $d_{i,k} = 0$  for  $j_0 \leq k \leq j-1$ , and now assume  $d_{i,j} \neq 0$ . The matrix to consider at this step has size  $(j+1) \times (j+1)$ . Thus at this current *j*th-step, we can find  $d_{i,j}d_{i,j_0-1}^{j-j_0} \neq 0$ . We observe that we hit the boundaries of a square of zeros. At the following (j+1)th step, all rows would contain at least 2 non-zero elements or equivalently there would not be a row with at least j+1 consecutive zeros. Thus  $d_{i,j+1}d_{i,j_0-1}^{j+1-j_0}$  would be the sum of at least two products and it would become impossible to correctly deduce the values of the determinants.

Now we work for our next result, a partially proved conjecture. Given valid indices i, j for the column and the row of the triangular table of determinants, and  $2k \leq j$  let  $\mathbf{G}_{i,j,k}$  be a matrix of size  $(k+1) \times (k+1)$  defined as

$$\mathbf{G}_{i,j,k} = \begin{pmatrix} d_{i,j-2k} & d_{i+1,j-2k+1} & \dots & d_{i+k,j-k} \\ d_{i-1,j-2k+1} & d_{i,j-2k+2} & \dots & d_{i+k-1,j-k+1} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i-k,j-k} & d_{i-k+1,j-k+1} & \dots & d_{i,j} \end{pmatrix}.$$
 (7)

In other words, for  $0 \le r, c \le k$ , the entry of  $\mathbf{G}_{i,j,k}$  located on the *r*th row and *c*th column is given by  $d_{i-r+c,j-2k+r+c}$ . The pair (i-r+c,j-2k+r+c) indexing an

element of  $\mathbf{G}_{i,j,k}$  is the intersection of two perpendicular lines. The intersection of a group of k parallel lines intersecting perpendicularly another group of k parallel lines as it might be easier to see with the following representation by drawing k lines with slope  $\pi/4$  and separated at distance  $\sqrt{2}$  intersecting k other lines with slope  $3\pi/4$  also at distance  $\sqrt{2}$  of each other:



If the information about the determinants  $d_{i,j'}$ ,  $0 \leq j' \leq j-1$ , is known, then we might hope to solve a determinantal equation like  $\det(\mathbf{G}_{i,j,k}) = g$  for some  $g \in \mathbb{F}_q$  for the unknown  $d_{i,j}$  located in the bottom right corner of  $\mathbf{G}_{i,j,k}$ .

We may sometimes abuse the language to denote the index (i, j) or the value indexed by (i, j) which is  $d_{i,j}$ . It is very convenient to refer to k as a radius of an  $\ell_1$ -ball centered around  $d_{i,j-k}$  or more precisely around the index (i, j - k). An  $\ell_1$ -ball is a square grid. The grid can be seen as the intersection of the two families of parallel lines and each family perpendicular to each other as mentioned previously. The indices obtained by the intersection of the two families are used to define  $\mathbf{G}_{i,j,k}$ . We refer the neighbourhood around (i, j - k), which is the center of the grid, as the  $\ell_1$ -ball of radius k. If k is even, then the center (i, j - k) is deleted. If k is odd, then the center is part of the ball. If  $\det(\mathbf{G}_{i,j,k}) = 0$ , then there is a local linear dependency around (i, j - k). From Dodgson's identity,  $\mathbf{G}_{i,j,k-2}$  plays the role of the center which can be seen as the interior of the neighbourhood of (i, j - k).

We postulate the following conjecture about the local linear dependency or more precisely about the minimum amount of information required to determine  $d_{i,j}$  assuming the table is known up to the (j-1)th level, inclusively.

**Conjecture 1.** For  $2 \leq k \leq 6$ ,  $j \geq 2k$ , and  $j-1 \leq i \leq n-j$ , where n is the length of the sequence, the smallest radius k for which det  $\mathbf{G}_{i,j,k} = 0$  is the smallest value k for which det  $\mathbf{G}_{i,j,k-1} \neq 0$ . Assuming det  $\mathbf{G}_{i,j,k'} = 0$  for  $2 \leq k' \leq 6$ , then we have det  $\mathbf{G}_{i,j,7} = 0$  if and only det  $\mathbf{G}_{i,j,6} \neq 0$  and  $d_{i,j-7} = 0$ .

We recall from linear algebra that the condition  $\det(\mathbf{G}_{i,j,k-1}) \neq 0$  is necessary and sufficient for the uniqueness to the solution of the linear equation  $\det(\mathbf{G}_{i,j,k}) = 0$  with  $d_{i,j}$  as unknown for all value of k and  $j \geq 2k$ .

We verified the previous conjecture by comparing  $d_{i,j}$  obtained from solving the corresponding determinantal equation with the value obtained from the trivial algorithm. Given that we never found any counter-example to Conjecture 1, we decided to include its algorithmic flavour, that is Algorithm 1, in our dynamic method given in Algorithm 2.

**Algorithm 1** Growing an  $\ell_1$ -metric ball and solving for the unknown

**Input:** Integer  $n > 0, 2 \le k \le 7, j \ge 2k$ , and  $j - 1 \le i \le n - j$ 1: if  $k \leq 6$  then for k' = 2 to k do 2: if  $det(\mathbf{G}_{i,i,k'-1}) \neq 0$  then 3: Solve  $det(\mathbf{G}_{i,j,k'}) = 0$  with  $d_{i,j}$  as unknown 4: **Return**  $d_{i,j}$ 5:end if 6: end for 7:else if k = 7,  $det(\mathbf{G}_{i,j,6}) \neq 0$ ,  $d_{i,j-7} = 0$  then 8: Solve  $det(\mathbf{G}_{i,j,7}) = 0$  with  $d_{i,j}$  as unknown 9: 10: **Return**  $d_{i,i}$ 11: else // We need more research for larger radius k. 12:k out of range 13: end if

We finish by briefly explaining Equation (3) as we promised at the beginning of this section. Given a square matrix A of size  $h \times h$ , with  $h = \lceil n/2 \rceil$ , we introduce the notation from  $\lceil 2 \rceil$ 

$$a_{r,c}^{(k)} = \det \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,k-1} & a_{0,c} \\ a_{1,0} & a_{1,1} & \dots & a_{1,k-1} & a_{1,c} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-1,0} & a_{k-1,1} & \dots & a_{k-1,k-1} & a_{k-1,c} \\ a_{r,0} & a_{r,1} & \dots & a_{r,k-1} & a_{r,c} \end{pmatrix} \quad \text{for } k \le r, c \le h.$$

Clearly  $a_{r,c}^{(k)}$  is the determinant of a  $(k+1) \times (k+1)$  matrix. We observe that the principal minors of A are  $a_{k,k}^{(k)}$ . In [2], it is shown that

$$a_{r,c}^{(k)} = \frac{1}{\left(a_{\ell,\ell}^{(\ell-1)}\right)^{k-\ell}} \det \begin{pmatrix} a_{\ell,\ell}^{(\ell)} & \dots & a_{\ell,k-1}^{(\ell)} & a_{\ell,c}^{(\ell)} \\ \vdots & \ddots & \vdots & \vdots \\ a_{k-1,\ell}^{(\ell)} & \dots & a_{k-1,k-1}^{(\ell)} & a_{k-1,c}^{(\ell)} \\ a_{r,\ell}^{(\ell)} & \dots & a_{r,k-1}^{(\ell)} & a_{r,c}^{(\ell)} \end{pmatrix}$$
for  $0 < \ell < k$ . (8)

So  $a_{r,c}^{(k)}$  is also the determinant of a  $(k-\ell+1)\times(k-\ell+1)$  matrix of determinants. The left side of Equation (8) does not depend on  $\ell$ . Let us concentrate on the principal minors when r = c = k, and substitute  $\ell = j_0$  and k = j in Equation (8) to get that

$$a_{j,j}^{(j)} = \frac{1}{\left(a_{j_0,j_0}^{(j_0-1)}\right)^{j-j_0}} \det \begin{pmatrix} a_{j_0,j_0}^{(j_0)} & \dots & a_{j_0,j-1}^{(j_0)} & a_{j_0,j}^{(j_0)} \\ \vdots & \ddots & \vdots & \vdots \\ a_{j-1,j_0}^{(j_0)} & \dots & a_{j-1,j-1}^{(j_0)} & a_{j-1,j}^{(j_0)} \\ a_{j,j_0}^{(j_0)} & \dots & a_{j,j-1}^{(j_0)} & a_{j,j}^{(j_0)} \end{pmatrix} \quad \text{for } 0 < j_0 < j$$

Entries (determinants) inside of the previous matrix are those  $d_{j-j_0,j_0}$ 's introduced at the beginning of this section. We observe that we can shift  $j - j_0$  by any quantity modulo n and therefore get Equation (3).

## 3 Algorithm to compute determinants of Hankel matrices over finite fields

In the following algorithm, the symbol j indexes the rows of the table and j is the size of the Hankel matrix  $\mathbf{X}_{i,j}$  as in the introduction. The symbol i indexes the columns and is related to the position in the input vector  $\mathbf{x}$  from where we build the Hankel matrix  $\mathbf{X}_{i,j}$ . We use the symbol M to denote the dynamic table under consideration, and M[j][i] stands for  $d_{i,j} = \det \mathbf{X}_{i,j}$ . The table grows from top to bottom by considering the smallest possible Hankel matrices to the largest one if n is odd or the largest two if n is even. For a given matrix of size j or equivalently a given row j of M, the algorithm sweeps from the left to the right using the input vector  $\mathbf{x}$  in order to consider all possible Hankel matrices of size j. We use also  $M[j][\cdot]$  to refer to the jth row of the table M.

**Algorithm 2** Computing determinants for all possible Hankel matrices made up from a sequence  $\mathbf{x} \in \mathbb{F}_{q}^{n}$ 

**Input:** Integer n > 0 and vector  $\mathbf{x} \in \mathbb{F}_{q}^{n}$ . **Output:** Triangular table M. 1:  $h \leftarrow \lceil n/2 \rceil$  $2: \ M \leftarrow \emptyset$ // Allocate space for M with base h and width n. 3: for i = 0 to n - 1 do // Initialize first two rows M.  $M[0][i] \leftarrow 1$ 4:  $M[1][i] \leftarrow x_i$ 5: 6: end for 7: for j = 2 to h do Find new squares of zeros // Use Theorem 1 with the knowledge 8: of rows  $M[j-1][\cdot]$  and  $M[j-2][\cdot]$ . for i = j - 1 to n - j do // Loop is parallelized. 9: if M[j][i] has not been yet evaluated then 10: if  $M[j-2][i] \neq 0$  then 11: Compute M[j][i] using Proposition 1 12: else if Conditions for Conjecture 1 then 13:

14:	Compute $M[j][i]$ accordingly with Algorithm 1
15:	else if Conditions for Theorem 2 then
16:	Compute $M[j][i]$ accordingly
17:	else
18:	Compute $M[j][i]$ explicitly from its definition
19:	end if
20:	end if
21:	end for
22: <b>e</b> i	nd for

Based on the results from Section 2, the algorithm correctly terminates. We note that an auxiliary table can be maintained to flag entries of M that were computed or not. Given j from line (7), to find squares of zeros using  $M[j-1][\cdot]$ and  $M[j-2][\cdot]$ , we look for consecutive non zero elements between two indices, say  $i_0$  and  $i_1$  (including  $M[j-2][i_0] \neq 0$  and  $M[j-2][i_1] \neq 0$ ) at level j-2, then check for  $M[j-1][i_0] \neq 0$  followed by zero elements until  $M[j-1][i_1] \neq 0$ ; the procedure begins with  $i_0 = j - 1$  and if  $i_1$  is found to be the right upper corner, then a square is filled, and the procedure continues from  $i_1$  until reaching n-j. Once the squares are filled, the remaining elements on a given row must be evaluated. The goal is to use as few as possible knowledge from the previous rows by using Proposition 1, Theorem 2, and Conjecture 1. If none of the previous results applied, then we revert to the trivial and expensive evaluation.

We end this section by explaining briefly how to find the generating vector of a linear subsequence. Once an unusual long run of zeros is found on a row of the table, we stop the computations of determinants since actually there is no need to further complete the table. Indeed, all the knowledge we need to build the adjugate, in order to invert a Hankel matrix connecting the generating vector to a part of the original sequence, is located on the previous rows that had been already computed.

### 4 Illustrative visual examples

In this section, we give two examples illustrating our new results over  $\mathbb{F}_2$ . For the first example, we generate a sequence of length 32 indexed from left to right starting with index 0, ending with index 31, and which is given by

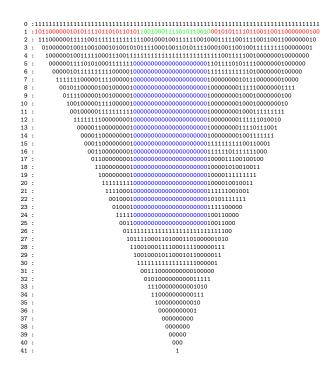
#### 01010110100111010011101011101110

Red color represents the prefix and the postfix that are generated randomly. Green color represents the middle linear substring. The big square of zeros due to the linearity of the middle string is in blue color. The sequence is used to initialize the table so it is identical to the row indexed by 1 below. Row 0 contains only unit elements. The generating vector is  $(1, 0, 1, 1) = (c_0, c_1, c_2, c_3)$ . We note that as mentioned previously,  $c_3 = 1$  to ensure the vector is not trivial. The leftmost index of linear subsequence is 8, that is  $i_0 = 8$ , and so the generated random prefix is  $01010110 = x_0x_1 \cdots x_7$ . The rightmost index of the linear subsequence is 24, and the generated random postfix is  $11101110 = x_{24} \cdots x_{31}$ .

The middle linearly substring is given by  $1001110100111010 = x_8 x_9 \cdots x_{23}$ , and is generated linearly from the prefix string:  $c_3 x_8 + c_2 x_7 + c_1 x_6 + c_0 x_5 = 0$ implies  $x_8 = c_2 x_7 + c_1 x_6 + c_0 x_5 = x_7 + x_5 = 0 + 1 = 1$ ,  $x_9 = x_8 + x_6 = 1 + 1 = 0$ ,  $x_{10} = x_9 + x_7 = 0 + 0 = 0$ , and so on.

Since the generating vector has length 4, then the row at which appears a long run of zeros is on the row indexed by 4. The shape of S is hexagonal, and the values of  $j_1$  varies with those of the positional indices *i*. The value  $j_0 = 4$ .

0	:	111111111111111111111111111111111111111
1	:	01010110100111010011101011101110
2	:	111111110010111001011111011101
3	:	001001111111111111111111111111111111111
4	:	0100100000000000011101111
5	:	11110000000000001111100
6	:	011000000000000100110
7	:	1100000000000010011
8	:	1000000000001111
9	:	00000000000100
10	:	000000000010
11	:	0000000001
12	:	000000000
13	:	0000000
14	:	000000
15	:	0000
16	:	00



In this case the generating vector is  $(1, 0, 0, 1, 1) = (c_0, c_1, c_2, c_3, c_4)$ . The leftmost index of the linear subsequence is 30, and the rightmost index of linear subsequence is 50. The shape of S is a square and  $i_0 = 30$ ,  $i_1 = 50$ ,  $j_0 = 5$ , and  $j_1 = j_0 + (i_1 - i_0) = 25$  as it can be seen as well from the visual aid. The prefix random string is given by  $x_0x_1\cdots x_{29} = 101100000010101111011010110101$ . The postfix random string is  $x_{50}x_{51}\cdots x_{80} = 001010111101100110000000100$ . The middle linear substring is given by  $x_{30}\cdots x_{49} = 10010001111011011001$ .

### 5 Conclusion and further work

We believe that there are still more relations to be found and to be coded in order to avoid the computation of determinants, and this is currently under study. An ultimate goal is to get rid entirely of the evaluations of large determinants by proving and generalizing Conjecture 1. How would the linear dependency vanish as the radius gets larger or synonymously how far does it propagate around the center? Can we further enlarge the radius by adding new conditions for k > 7?

It would be interesting to adapt our algorithm to output the generating vector and compare it to efficient implementations of the Berlekamp-Massey algorithm. We would need to stop at the level containing a long run of zeros and use the information of the row preceding this one to solve efficiently the linear system for the generating vector using adjugate matrices.

It is known that Berlekamp-Massey algorithm is virtually the same as the extended Euclidean algorithm for polynomials over finite field. Could we find a similar equivalence to our algorithm for problems involving Bezout identities that express linear dependencies among elements in fields?

Our dynamic approach can be easily adapted to multiple and combined linear feedback shift registers. Further research also includes to analyze the case of non-linear feedback shift register by linearizing the generator; more precisely, linearizing a non-linear boolean feedback function pertains to add constraints which are reflected in the determinant identities.

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#### Appendix: Run times and distribution of counts

In order to compare in practice the running times between the trivial method and our new method, we generate sequences of length n that we linearly filled. In order to compute determinants of large Hankel matrices whenever necessary, we do not use the Levinson-Durbin algorithm [4], [15] that can be adapted to Hankel matrices instead of Toeplitz matrices. We created an extremely fast C/C++ low-level module to compute determinants over  $\mathbb{F}_2$  in order to do not rely on any external libraries. Our module to compute determinants over  $\mathbf{F}_2$  is quite faster than NTL; it however only applies to binary matrices. We recall that one of our future goals is to avoid such computation of determinants of large matrices, and only use local information. We look at typical worst-case instances when the linear subsequence is "buried" between two long random sequences serving as a prefix and a postfix. The prefix random string together with the generator vector are used to built the linear subsequence in the middle. The prefix random string must be at least as long as the length of the generating vector to be used as initial data. We also consider typical easy-case instances when there is no random postfix sequence and when the length of the random prefix sequence is the same as the generating vector.

For our accelerated dynamic algorithm, we give the distribution of counts of the number of times, with respect to the number of entries in the table, that we branch to Proposition 1, Theorems 1 or 2, Conjecture 1 or to an explicit computation (where Levinson-Durbin could be used for instance). The time to verify that the tables obtained from the trivial and our accelerated methods coincide is not taken into account; we must check this because at this time we cannot prove the validity of Conjecture 1 and/or further enhanced it. We also parallelize both the naive and accelerated algorithms. We notice that our accelerated algorithm on a single core is faster than the trivial algorithm on 160 cores for instance as shown here in the following tables. For  $n = 2^{14} = 16384$ , the time to run the trivial algorithm is prohibitive and we did not run the trivial algorithm for length  $n = 2^{14}$ . The meanings of the abbreviations in the following tables are: Tri. S.T. for trivial algorithm single threaded, Tri. M.T. for trivial algorithm multi threaded, Acc. S.T. for accelerated algorithm single threaded, and Acc. M.T. for accelerated algorithm multi threaded. Roughly speaking, a thread is a core. All threads share a unique space in memory.

EXCERPT OF RUNNING TIMES FOR $n = 4096$ (in milliseconds)					
Tri. S.T.	Tri. M.T.	Acc. S.T.	Acc. M.T.		
15931964.434875	793513.102411	209495.875914	79493.525290		
15930582.584026	793131.331561	212010.715863	79784.405378		

Continued on next page

Tri. S.T.	Tri. M.T.	Acc. S.T.	Acc. M.T.
15931070.671137	793029.664463	211231.649798	79538.533773
$\approx 4h$ 30min	$\approx 14 \mathrm{min}$	$\approx 4 \mathrm{min}$	$\approx 1 \text{min } 33 \text{sec}$

EXCERPT OF RUNNING TIMES FOR $n = 16384$ (in milliseconds)				
Acc. S.T.	Acc. M.T.			
95397251.744779	18823959.088013			
92977447.400304	18820690.826587			
93372757.004548	18817978.092718			
$\approx 26h$ 30min	$\approx 5h$ 36min			

The hardware specification for the computer we used is: Intel(R) Core(TM) i7-8700 CPU @ 3.20Ghz, 160 cores, 1TB RAM.

We coded Algorithm 2 over  $\mathbb{F}_2$ , and a compile switch can be enable to avoid using the library NTL or to use it. Our code is available at https://github.com/63EA13D5/. For the worst-case instances, we generate the sequences using the following parameters:

- 1. Elements indexed from 0 to 7n/16 inclusively are generated randomly.
- 2. Elements indexed from 7n/16 + 1 to 9n/16 inclusively are linearly filled using a non-trivial generating vector of length d = n/8. The generating vector is randomly created and the rightmost coordinate is set to the unit element in base field.
- 3. Elements indexed from 9n/16 to n inclusively are generated randomly.

The ratio of the number of entries in the big square over the number of entries for the table of a given instance is about 1/16 up to a few decimals. For each value of n, a sample of sequences is used to estimate the running time by evaluating the averages over the sample, one average for the trivial and one average for our method. For comparison, each method is applied to a sequence from the sample. The ratios of the averages of the new method by the trivial are given. We generate a sample of 1000 linearly filled vectors as described above for each value of n. Zero counts are not shown in the tables.

Table 3: Time complexity and distribution of counts-hard instances.

Sample size	10140
Sequence length	4096
Generating vector length	256
Subsequence leftmost index	1792
Subsequence rightmost index	2304
Number of entries	4192256

Continued on next page

Average time for accelerate	ed method (ms)	482211.336405 (i)			
Average time for trivial me		39627789.122209 (ii)			
Ratio i/ii		0.012169			
Average counts NSEW		1720642.408481			
Average counts square filli	ng	925814.212032			
Average counts direct	0	60875.169231			
Average counts $2 \times 2$ grid		467181.981164	467181.981164		
Average counts $3 \times 3$ grid		382111.856114			
Average counts $4 \times 4$ grid		265305.747436			
Average counts $5 \times 5$ grid		169767.046746			
Average counts $6 \times 6$ grid		94158.297732			
Average counts $7 \times 7$ grid		51134.827416			
Average counts 2-cross	3772.531657	Average counts 3-cross	2480.830473		
Average counts 4-cross	1473.981558	Average counts 5-cross	821.684813		
Average counts 6-cross	4415.502071	Average counts 7-cross	4449.331460		
Average counts 8-cross	16683.110947	Average counts 9-cross	9334.250197		
Average counts 10-cross	5165.383136	Average counts 11-cross	2828.874063		
Average counts 12-cross	1536.936785	Average counts 13-cross	828.868540		
Average counts 14-cross	444.807101	Average counts 15-cross	237.420710		
Average counts 16-cross	125.906312	Average counts 17-cross	67.308679		
Average counts 18-cross	35.672189	Average counts 19-cross	18.771893		
Average counts 20-cross	9.915779	Average counts 21-cross	5.115089		
Average counts 22-cross	2.713708	Average counts 23-cross	1.373570		
Average counts 24-cross	0.696746	Average counts 25-cross	0.388955		
Average counts 26-cross	0.246154	Average counts 27-cross	0.098422		
Average counts 28-cross	0.054734	Average counts 29-cross	0.043590		
Average counts 30-cross	0.020809	Average counts 31-cross	0.009369		
Average counts 32-cross	0.006312	Average counts 256-cross	21.695464		
Average counts 257-cross	53.271203	Average counts 258-cross	70.017061		
Average counts 259-cross	74.697732	Average counts 260-cross	67.246943		
Average counts 261-cross	56.079487	Average counts 262-cross	45.345661		
Average counts 263-cross	35.773866	Average counts 264-cross	26.859369		
Average counts 265-cross	20.313807	Average counts 266-cross	14.738166		
Average counts 267-cross	9.678205	Average counts 268-cross	7.546055		
Average counts 269-cross	5.817850	Average counts 270-cross	3.680079		
Average counts 271-cross	2.569625	Average counts 272-cross	1.961440		
Average counts 273-cross	1.428895	Average counts 274-cross	1.082840		
Average counts 275-cross	0.923471	Average counts 276-cross	0.490335		
Average counts 277-cross	0.383432	Average counts 278-cross	0.439250		
Average counts 279-cross	0.247929	Average counts 280-cross	0.082840		
Average counts 281-cross	0.027811	Average counts 282-cross	0.027811		
Average counts 285-cross	0.028205	Average counts 286-cross	0.028205		
Average counts 287-cross	0.028402	Average counts 288-cross	0.028402		
Average counts 293-cross	0.028994	Average counts 294-cross	0.028994		
Sum over all	average counts	4192256	•		
	0	I			

Table 4: Time complexity and distribution of counts-easy instances

Sample size	10140
Sequence length	4096

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Subsequence rightmost indexNumber of entriesAverage time for accelerated method (ms)Average time for trivial method (ms)Ratio i/iiAverage counts NSEWAverage counts square fillingAverage counts directAverage counts 3 × 3 gridAverage counts 5 × 5 gridAverage counts 6 × 6 gridAverage counts 7 × 7 gridAverage counts 2-cross3772.586193	256 4096 4192256 4331.896007 (i) 25765730.530170 (ii) 0.000168 407852.065385 3419936.066568 19444.707988 106227.368146 86949.429093 60390.460256			
Subsequence rightmost indexNumber of entriesAverage time for accelerated method (ms)Average time for trivial method (ms)Ratio i/iiAverage counts NSEWAverage counts square fillingAverage counts directAverage counts $3 \times 3$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross	4192256         4331.896007 (i)         25765730.530170 (ii)         0.000168         407852.065385         3419936.066568         19444.707988         106227.368146         86949.429093			
Number of entriesAverage time for accelerated method (ms)Average time for trivial method (ms)Ratio i/iiAverage counts NSEWAverage counts square fillingAverage counts directAverage counts $3 \times 3$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ grid	4331.896007       (i)         25765730.530170       (ii)         0.000168       (ii)         407852.065385       (iii)         3419936.066568       (iiii)         19444.707988       (iiii)         106227.368146       (iiii)         86949.429093       (iii)			
Average time for trivial method (ms)         Ratio i/ii         Average counts NSEW         Average counts square filling         Average counts direct         Average counts 3 × 3 grid         Average counts 5 × 5 grid         Average counts 6 × 6 grid         Average counts 7 × 7 grid	25765730.530170 (ii) 0.000168 407852.065385 3419936.066568 19444.707988 106227.368146 86949.429093			
Ratio i/iiAverage counts NSEWAverage counts square fillingAverage counts directAverage counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross	0.000168 407852.065385 3419936.066568 19444.707988 106227.368146 86949.429093			
Average counts NSEWAverage counts square fillingAverage counts directAverage counts $2 \times 2$ gridAverage counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross	407852.065385 3419936.066568 19444.707988 106227.368146 86949.429093			
Average counts square fillingAverage counts directAverage counts $2 \times 2$ gridAverage counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross	3419936.066568 19444.707988 106227.368146 86949.429093			
Average counts square fillingAverage counts directAverage counts $2 \times 2$ gridAverage counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross	19444.707988           106227.368146           86949.429093			
Average counts $2 \times 2$ gridAverage counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross $3772.586193$	106227.368146           86949.429093			
Average counts $3 \times 3$ gridAverage counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts $2$ -cross $3772.586193$	86949.429093			
Average counts $4 \times 4$ gridAverage counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts 2-cross $3772.586193$		106227.368146		
Average counts $5 \times 5$ gridAverage counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts 2-cross $3772.586193$	60390.460256			
Average counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts 2-cross $3772.586193$				
Average counts $6 \times 6$ gridAverage counts $7 \times 7$ gridAverage counts 2-cross $3772.586193$	38652.743195			
Average counts 7 × 7 grid       Average counts 2-cross       3772.586193	21438.953156			
÷	11646.030572			
	Average counts 3-cross	2479.999310		
Average counts 4-cross 1473.537081	Average counts 5-cross	821.756805		
Average counts 6-cross 1341.515483	Average counts 7-cross	1185.020414		
Average counts 8-cross 3878.167554	Average counts 9-cross	2161.438363		
Average counts 10-cross 1191.903748	Average counts 11-cross	651.711736		
Average counts 12-cross 351.648225	Average counts 13-cross	189.478107		
	Average counts 15-cross	54.392998		
Average counts 16-cross 28.796746	Average counts 17-cross	15.180178		
Average counts 18-cross 8.078008	Average counts 19-cross	4.166469		
Average counts 20-cross 2.130868	Average counts 21-cross	1.140039		
	Average counts 23-cross	0.254734		
	Average counts 25-cross	0.104832		
	Average counts 27-cross	0.035207		
	Average counts 29-cross	0.002860		
	Average counts 1793-cross	0.501775		
0	Average counts 1795-cross	0.212623		
0	Average counts 1797-cross	0.073373		
Average counts 1798-cross 0.039645	Average counts 1799-cross	0.022091		
0	Average counts 1801-cross	0.007298		
	Average counts 1803-cross	0.001578		
0	Average counts 1805-cross	0.000394		
Average counts 1806-cross0.000197		0.000197		
Sum over all average counts	Average counts 1812-cross	0.000191		

We observe that we are about 83 times faster on typical hard instances and about 5947 times faster on easy ones. In practice, to detect linearity or to solve backward for the generating vector, we only need to stop at the first level that contains a long run of zeros.