The Persistence of False Memory: Brain in a Vat Despite Perfect Clocks

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Abstract. Recently, a detailed epistemic reasoning framework for multiagent systems with byzantine faulty asynchronous agents and possibly unreliable communication was introduced. We have developed a modular extension framework implemented on top of it, which allows to encode and safely combine additional system assumptions commonly used in the modeling and analysis of fault-tolerant distributed systems, like reliable communication, time-bounded communication, multicasting, synchronous and lock-step synchronous agents and even agents with coordinated actions. We use this extension framework for analyzing basic properties of synchronous and lock-step synchronous agents, such as the agents' local and global fault detection abilities. Moreover, we show that even the perfectly synchronized clocks available in lock-step synchronous systems cannot be used to avoid "brain-in-a-vat" scenarios.

1 Introduction

Epistemic reasoning is a powerful technique for modeling and analysis of distributed systems [5,9], which has proved its utility also for fault-tolerant systems: Benign faults, i.e., nodes (termed agents subsequently) that may crash and/or drop messages, have been studied right from the beginning [16,17,4]. Recently, a comprehensive epistemic reasoning framework for agents that may even behave arbitrarily ("byzantine" [14]) faulty have been introduced in [12,11]. Whereas it fully captures byzantine asynchronous systems, it is currently not suitable for modeling and analysis of the wealth of other distributed systems, most notably, synchronous agents and reliable multicast communication.

As we extend the framework [12] in this paper, we briefly summarize its basic notation. There is a finite set $\mathcal{A} = \{1, \ldots, n\}$ (for $n \geq 2$) of **agents**, who do not have access to a global clock and execute a possibly non-deterministic joint **protocol**. In such a protocol, agents can perform **actions**, e.g., send **messages** $\mu \in Msgs$, and witness **events**, in particular, message deliveries: the action of sending a copy (numbered k) of a message $\mu \in Msgs$ to an agent $j \in \mathcal{A}$ in a protocol is denoted by $send(j, \mu_k)$, whereas a receipt of such a message from $i \in \mathcal{A}$ is recorded locally as $recv(i, \mu)$. The set of all **actions** (**events**) available to an agent $i \in \mathcal{A}$ is denoted by $Actions_i$ ($Events_i$), subsumed as **haps** $Haps_i :=$ $Actions_i \sqcup Events_i$, with $Actions := \bigcup_{i \in \mathcal{A}} Actions_i$, $Events := \bigcup_{i \in \mathcal{A}} Events_i$, and $Haps := Actions \sqcup Events$.

The other main player in [12] is the **environment** ϵ , which takes care of scheduling haps, failing agents, and resolving non-deterministic choices in the joint protocol. Since the notation above only describes the local view of agents, there is also a **global** syntactic representation of each hap, which is only available to the environment and contains additional information (regarding the time of a hap, a distinction whether a hap occurred in a correct or byzantine way, etc.) that will be detailed in Sect. 2.

The model utilizes a discrete time model, of arbitrarily fine resolution, with time domain $t \in \mathbb{T} := \mathbb{N} = \{0, 1, ...\}$. All haps taking place after a **timestamp** $t \in \mathbb{T}$ and no later than t+1 are grouped into a **round** denoted $t\frac{1}{2}$ and treated as happening simultaneously. In order to prevent agents from inferring the global time by counting rounds, agents are generally unaware of a round, unless they perceive an event or are prompted to act by the environment. The latter is accomplished by special system events go(i), which are complemented by two more system events for faulty agents: sleep(i) and hibernate(i) signify a failure to activate the agent's protocol and differ in that the latter does not even wake up the agent. None of the **system events** $SysEvents_i := \{go(i), sleep(i), hibernate(i)\}$ is directly observable by agents.

Events and actions that can occur in each round, if enabled by go(i), are determined by the protocols for agents and the environment, with non-deterministic choices resolved by the **adversary** that is considered part of the environment. A **run** r is a function mapping a point in time t to an n + 1 tuple, consisting of the environment's history and local histories $r(t) = (r_{\epsilon}(t), r_1(t), \ldots, r_n(t))$ representing the state of the whole system at that time t. The **environment's history** $r_{\epsilon}(t) \in \mathscr{L}_{\epsilon}$ is a sequence of all haps that happened, in contrast to the local histories faithfully recorded in the global format. Accordingly, $r_{\epsilon}(t+1) = X: r_{\epsilon}(t)$ for the set $X \subseteq GHaps$ of all haps from round $t\frac{1}{2}$. The exact updating procedure including the update of the local **agent histories** is the result of a complex state transition consisting of several phases, which are described in Sect. 2. Proving the correctness of a protocol for solving a certain distributed computing problem boils down to studying the set of runs that can be generated.

In its current version, [12] only supports asynchronous agents and communication, where both agents and message transmission may be arbitrarily slow. Notwithstanding the importance of asynchronous distributed systems in general [15], however, it is well-known that adding faults to the picture renders important distributed computing problems like consensus impossible [6]. There is hence a vast body of research that relies on stronger system models that add additional assumptions. One prominent example are **lockstep synchronous systems**, where agents take actions simultaneously at times $t \in \mathbb{N}$, i.e., have access to a perfectly synchronized global clock, and messages sent at time t are received before time t + 1. It is well-known that consensus can be solved in synchronous systems with $n \geq 3f + 1$ nodes, if at most f of those behave byzantine [14].

Related work. Epistemic analysis has been successfully applied to synchronous systems with both fault-free [2] and benign faulty agents [3,8] in the past. In [2], Ben-Zvi and Moses considered the *ordered response* problem in fault-free time-

bounded distributed systems and showed that any correct solution has to establish a certain nested knowledge. They also introduced the syncausality relation, which generalizes Lamport's happens-before relation [13] and formalizes the knowledge gain due to "communication-by-time" in synchronous systems. This work was extended to tightly coordinated responses in [1,7].

Synchronous distributed systems with agents suffering from benign faults such as crashes or message send/receive omissions have already been studied in [16,17], primarily in the context of agreement problems [4,10], which require some form of common knowledge. More recent results are unbeatable consensus algorithms in synchronous systems with crash faults [3], and the discovery of the importance of *silent choirs* [8] for message-optimal protocols in crash-resilient systems. By contrast, we are not aware of any attempt on the epistemic analysis of fault-tolerant distributed systems with byzantine agents.

Main contributions. In the present paper, we extend [12] by a modular *extension framework*, which allows to encode and safely combine additional system assumptions typically used in the modeling and analysis of fault-tolerant distributed systems, like reliable communication, time-bounded communication, multicasting, synchronous and lock-step synchronous agents and even agents with coordinated actions. We therefore establish the first framework that facilitates a rigorous epistemic modeling and analysis of general distributed systems with byzantine faulty agents. We demonstrate its utility by analyzing some basic properties of the synchronous and lock-step synchronous agent extensions, namely, the agents' local and global fault detection abilities. Moreover, we prove that even the perfectly synchronized clocks available in the lock-step synchronous extension cannot prevent a "brain-in-a-vat" scenario.

Paper organization. Additional details of the existing basic modeling framework [12] required for our extensions are provided in Sect. 2. Sect. 3 presents the cornerstones of our synchronous extension and establishes some introspection results and the possibility of brain-in-a-vat scenarios. Sect. 4 provides an overview of our fully-fledged extension framework, the utility of which is demonstrated by investigating these issues in lock-step synchronous systems in Sect. 5. Some conclusions in Sect. 6 round-off our paper. All the material omitted from the main body of the paper due to lack of space is provided in Appendix A.¹

2 The Basic Model

Since this paper extends the framework from [12], we first briefly recall the necessary details and aspects needed for defining our extension framework.

Global haps and faults. As already mentioned in Section 1, there is a global version of every *Haps* that provides additional information that is only accessible

¹ Please note that the comprehensive appendix has been provided solely for the convenience of the reviewers; all material collected there can reasonably be omitted. It is/will of course be available in the existing publications and in an extended report.

to the environment. Among it is the timestamp t of every correct action $a \in Actions_i$, as initiated by agent i in the local format, which is provided by a one-to-one function global(i, t, a). Timestamps are especially crucial for proper message processing with $global(i, t, send(j, \mu_k)) := gsend(i, j, \mu, id(i, j, \mu, k, t))$ for some one-to-one function $id: \mathcal{A} \times \mathcal{A} \times Msgs \times \mathbb{N} \times \mathbb{T} \to \mathbb{N}$ that assigns each sent message a unique **global message identifier** (GMI). These GMIs enable the direct linking of send actions to their corresponding delivery events, most importantly used to ensure that only sent messages can be delivered (causality). The resulting sets $\overline{GActions_i} := \{global(i, t, a) \mid t \in \mathbb{T}, a \in Actions_i\}$ of correct actions in global format are pairwise disjoint due to the injectivity of global. We set $\overline{GActions} := \bigsqcup_{i \in \mathcal{A}} \overline{GActions_i}$.

Unlike correct actions, correct events witnessed by agent i are generated by the environment ϵ , hence are already produced in the global format *GEvents*_i. We define $\overline{GEvents} := \bigsqcup_{i \in \mathcal{A}} \overline{GEvents_i}$ assuming them to be pairwise disjoint and $\overline{GHaps} = \overline{GEvents} \sqcup \overline{GActions}$. A byzantine event is an event that was perceived by an agent despite not taking place. In other words, for each correct event $E \in \overline{GEvents}_i$, we use a faulty counterpart fake (i, E) and will make sure that agent i cannot distinguish between the two. An important type of correct global events is delivery $grecv(j, i, \mu, id) \in GEvents_i$ of message μ with GMI $id \in \mathbb{N}$ sent from agent i to agent j. The GMI must be a part of the global format (especially for ensuring causality) but cannot be part of the local format because it contains information about the time of sending, which should not be accessible to agents. The stripping of this information before updating local histories is achieved by the function *local*: $\overline{GHaps} \longrightarrow Haps$ converting correct haps from the global into the local formats for the respective agents in such a way that *local* reverses global, i.e., local(global(i, t, a)) := a, in particular, $local(qrecv(i, j, \mu, id)) := recv(j, \mu).$

To allow for the most flexibility regarding who is to blame for an erroneous action, faulty actions are modeled as byzantine events of the form $fake (i, A \mapsto A')$ where $A, A' \in \overline{GActions_i} \sqcup \{\mathbf{noop}\}$ for a special **non-action noop** in global format. These byzantine events are controlled by the environment and correspond to an agent violating its protocol by performing the action A (in global format), while recording in its local history that it either performs $a' = local(A') \in Actions_i$ if $A' \in \overline{GActions_i}$ or does nothing if $A' = \mathbf{noop}$ (note that performing $A = \mathbf{noop}$ means not acting). The byzantine inaction fail(i) defined as $fake(i, \mathbf{noop} \mapsto \mathbf{noop})$ can be used to make agent i faulty without performing any actions and without leaving a record in i's local history. The set of all i's byzantine events, corresponding to both faulty events and actions, is denoted by $BEvents_i$, with $BEvents := \bigsqcup_{i \in \mathcal{A}} BEvents_i$. To summarize, $GEvents_i := GEvents_i \sqcup BEvents_i \sqcup SysEvents_i$ with $GEvents := \bigsqcup_{i \in \mathcal{A}} GEvents_i$, $GHaps := GEvents \sqcup \overline{GActions}$. Horizontal bars signify phenomena that are correct, as contrasted by those that may be correct or byzantine.

Protocols, state transitions and runs. The events and actions that occur in each round are determined by protocols (for agents and the environment) and non-determinism (adversary). Agent *i*'s **protocol** $P_i: \mathscr{L}_i \to 2^{2^{Actions_i}} \setminus \{\varnothing\}$ pro-

vides a range $P_i(r_i(t))$ of sets of actions based on *i*'s current local state $r_i(t) \in \mathscr{L}_i$ at time *t* in run *r*, from which the adversary non-deterministically picks one. Similarly the environment provides a range of (correct, byzantine, and system) events via its protocol $P_{\epsilon} : \mathbb{T} \to 2^{2^{GEvents}} \setminus \{\emptyset\}$, which depends on a timestamp $t \in \mathbb{T}$ but **not** on the current state, in order to maintain its impartiality. It is required that all events of round t_2^1 be mutually compatible at time *t*, called *t*-coherent (for details see Appendix, Def. A.7). The set of all global states is denoted by \mathscr{G} .

Agent *i*'s local view of the system after round t_2^1 is recorded in *i*'s local state $r_i(t+1)$, also called *i*'s local history, sometimes denoted h_i , which is agent *i*'s share of the global state $h = r(t) \in \mathcal{G}$. $r_i(0) \in \Sigma_i$ are the initial local states, with $\mathcal{G}(0) := \prod_{i \in \mathcal{A}} \Sigma_i$. If a round contains neither go(i) nor any event to be recorded in *i*'s local history, then the history $r_i(t+1) = r_i(t)$ remains unchanged, denying the agent knowledge that the round just passed. Otherwise, $r_i(t+1) = X : r_i(t)$, for the set $X \subseteq Haps_i$ of all actions and events perceived by *i* in round t_2^1 , where : stands for concatenation. Thus the local history $r_i(t)$ is a sequence of all haps as perceived by *i* in rounds it was active in.

Given the **joint protocol** $P := (P_1, \ldots, P_n)$ and the environment's protocol P_{ϵ} , we focus on $\tau_{P_{\epsilon},P}$ -**transitional runs** r that result from following these protocols and are built according to a **transition relation** $\tau_{P_{\epsilon},P} \subseteq \mathscr{G} \times \mathscr{G}$. Each such transitional run begins in some initial global state $r(0) \in \mathscr{G}(0)$ and progresses, satisfying $(r(t), r(t+1)) \in \tau_{P_{\epsilon},P}$ for each timestamp $t \in \mathbb{T}$. The transition relation $\tau_{P_{\epsilon},P}$ consisting of five consecutive phases is graphically represented in Appendix, Fig. 1 and described in detail below:

In the **protocol phase** a range $P_{\epsilon}(t) \subset 2^{GEvents}$ of *t*-coherent sets of events is determined by the environment's protocol P_{ϵ} . Similarly for each $i \in \mathcal{A}$, a range $P_i(r_i(t)) \subseteq 2^{Actions_i}$ of sets of *i*'s actions is determined by the joint protocol P.

In the **adversary phase**, the adversary non-deterministically chooses a set $X_{\epsilon} \in P_{\epsilon}(t)$ and one set $X_i \in P_i(r_i(t))$ for each $i \in \mathcal{A}$.

In the **labeling phase**, actions in the sets X_i are translated into the global format: $\alpha_i^t(r) := \{global(i, t, a) \mid a \in X_i\} \subseteq \overline{GActions_i}$.

In the **filtering phase**, filter functions remove all unwanted or impossible attempted events from $\alpha_{\epsilon}^{t}(r) := X_{\epsilon}$ and actions from $\alpha_{i}^{t}(r)$. This is done in two stages:

First, $filter_{\epsilon}$ filters out "illegal" events. This filter will vary depending on the concrete system assumptions (in the byzantine asynchronous case, "illegal" constitutes receive events that violate causality). The resulting set of events to actually occur in round t_2^1 is $\beta_{\epsilon}^t(r) := filter_{\epsilon}(r(t), \alpha_{\epsilon}^t(r), \alpha_1^t(r), \ldots, \alpha_n^t(r))$.

Definition 1. The standard action filter $filter_i^B(X_1, \ldots, X_n, X_{\epsilon})$ for $i \in \mathcal{A}$ either removes all actions from X_i when $go(i) \notin X_{\epsilon}$ or else leaves X_i unchanged.

Second, $filter_i^B$ for each *i* returns the sets of actions to be actually performed by agents in round $t\frac{1}{2}$, i.e., $\beta_i^t(r) := filter_i^B(\alpha_1^t(r), \ldots, \alpha_n^t(r), \beta_{\epsilon}^t(r)))$. Note that $\beta_i^t(r) \subseteq \alpha_i^t(r) \subseteq \overline{GActions_i}$ and $\beta_{\epsilon}^t(r) \subseteq \alpha_{\epsilon}^t(r) \subset GEvents$.

In the **updating phase**, the events $\beta_{\epsilon}^{t}(r)$ and actions $\beta_{i}^{t}(r)$ are appended to the global history r(t). For each $i \in \mathcal{A}$, all non-system events from $\beta_{\epsilon_{i}}^{t}(r) :=$ $\beta_{\epsilon}^{t}(r) \cap GEvents_{i}$ and all actions $\beta_{i}^{t}(r)$ as **perceived** by the agent are appended in the local form to the local history $r_{i}(t)$. Note the local history may remain unchanged if no events trigger an update (see Appendix, Def. A.6 for more details).

$$r_{\epsilon}(t+1) := update_{\epsilon}\left(r_{\epsilon}(t), \quad \beta_{\epsilon}^{t}(r), \quad \beta_{1}^{t}(r), \quad \dots, \quad \beta_{n}^{t}(r)\right)$$
(1)

$$r_i(t+1) := update_i\left(r_i(t), \quad \beta_i^t(r), \quad \beta_\epsilon^t(r)\right).$$
⁽²⁾

The operations in the phases 2–5 (adversary, labeling, filtering and updating phase) are grouped into a **transition template** τ that yields a transition relation $\tau_{P_{\epsilon},P}$ for any joint and environment protocol P and P_{ϵ} . Particularly, we denote as τ^B the transition template utilizing $filter_{\epsilon}^B$ and $filter_i^B$ (for all $i \in \mathcal{A}$).

As **liveness properties** cannot be ensured on a round-by-round basis, they are enforced by restricting the allowable set of runs via **admissibility conditions** Ψ , which are subsets of the set R of all **transitional runs**.

A context $\gamma = (P_{\epsilon}, \mathscr{G}(0), \tau, \Psi)$ consists of an environment's protocol P_{ϵ} , a set of global initial states $\mathscr{G}(0)$, a transition template τ , and an admissibility condition Ψ . For a joint protocol P, we call $\chi = (\gamma, P)$ an **agent-context**. A run $r \in R$ is called **weakly** χ -consistent if $r(0) \in \mathscr{G}(0)$ and the run is $\tau_{P_{\epsilon},P^{-}}$ transitional. A weakly χ -consistent run r is called (strongly) χ -consistent if $r \in \Psi$. The set of all χ -consistent runs is denoted R^{χ} . An agent-context χ is called **non-excluding** if any finite prefix of a weakly χ -consistent run can be extended to a χ -consistent run. (For more details see Appendix, Defs. A.8–A.9.)

Epistemics. [12] defines interpreted systems in this framework as Kripke models for multi-agent distributed environments [5]. The states in such a Kripke model are given by global histories $r(t') \in \mathscr{G}$ for runs $r \in R^{\chi}$ given some agent-context χ and timestamps $t' \in \mathbb{T}$. A valuation function $\pi: \operatorname{Prop} \to 2^{\mathscr{G}}$ determines states where an atomic proposition from Prop is true. This determination is arbitrary except for a small set of designated atomic propositions: For $\operatorname{FEvents}_i :=$ $\operatorname{BEvents}_i \sqcup \{\operatorname{sleep}(i), \operatorname{hibernate}(i)\}, i \in \mathcal{A}, o \in \operatorname{Haps}_i, \text{ and } t \in \mathbb{T} \text{ such that } t \leq t',$

- $-correct_{(i,t)}$ is true at r(t') iff no faulty event happened to *i* by timestamp *t*, i.e., no event from *FEvents*_i appears in $r_{\epsilon}(t)$,
- correct_i is true at r(t') iff no faulty event happened to *i* yet, i.e., no event from *FEvents*_i appears in $r_{\epsilon}(t')$,
- $fake_{(i,t)}(o)$ is true at r(t') iff *i* has a **faulty** reason to believe that $o \in Haps_i$ occurred in round $(t-1)\frac{1}{2}$, i.e., $o \in r_i(t)$ because (at least in part) of some $O \in BEvents_i \cap \beta_{\epsilon_i}^{t-1}(r)$,
- $\overline{occurred}_{(i,t)}(o) \text{ is true at } r(t') \text{ iff } i \text{ has a correct reason to believe } o \in Haps_i \\ \text{occurred in round } (t-1)\frac{1}{2}, \text{ i.e., } o \in r_i(t) \text{ because (at least in part) of } O \in (\overline{GEvents}_i \cap \beta_{\epsilon_i}^{t-1}(r)) \sqcup \beta_i^{t-1}(r),$
- $\overline{occurred}_{i}(o) \text{ is true at } r(t') \text{ iff at least one of } \overline{occurred}_{(i,m)}(o) \text{ for } 1 \leq m \leq t'$ is; also $\overline{occurred}(o) := \bigvee_{i \in \mathcal{A}} \overline{occurred}_{i}(o),$
- $occurred_i(o)$ is true at r(t') iff either $occurred_i(o)$ is or at least one of $fake_{(i,m)}(o)$ for $1 \le m \le t'$ is.

The following terms are used to categorize agent faults caused by the environment's protocol P_{ϵ} : agent $i \in \mathcal{A}$ is *fallible* if for any $X \in P_{\epsilon}(t), X \cup \{fail(i)\} \in$ $P_{\epsilon}(t)$; delayable if $X \in P_{\epsilon}(t)$ implies $X \setminus GEvents_i \in P_{\epsilon}(t)$; gullible if $X \in P_{\epsilon}(t)$ implies that, for any $Y \subseteq FEvents_i$, the set $Y \sqcup (X \setminus GEvents_i) \in P_{\epsilon}(t)$ whenever it is t-coherent. Informally, fallible agents can be branded byzantine at any time; delayable agents can always be forced to skip a round completely (which does not make them faulty); gullible agents can exhibit any faults in place of correct events. Common types of faults, e.g., crash or omission failures, can be obtained by restricting allowable sets Y in the definition of gullible agents.

An **interpreted system** is a pair $\mathcal{I} = (R^{\chi}, \pi)$. The following BNF defines the **epistemic language** considered throughout this paper, for $p \in Prop$ and $i \in \mathcal{A}$: $\varphi ::= p \mid \neg \varphi \mid (\varphi \land \varphi) \mid K_i \varphi$ (other Boolean connectives are defined as usual; belief $B_i \varphi := K_i(correct_i \rightarrow \varphi)$ and hope $H_i \varphi := correct_i \rightarrow B_i \varphi$. The interpreted systems semantics is defined as usual with global states r(t) and r'(t') indistinguishable for i iff $r_i(t) = r'_i(t')$ (see Appendix, Defs. A.10–A.11 for the exact details).

Unless stated otherwise, the global history h ranges over \mathscr{G} , and $X_{\epsilon} \subseteq GEvents$ and $X_i \subseteq \overline{GActions_i}$ for each $i \in \mathcal{A}$. The tuple X_1, \ldots, X_n is abbreviated $X_{\mathcal{A}}$. We use $X_{[i,j]}$ for the tuple X_i, \ldots, X_j and $X'_{\mathcal{A}}$ for X'_1, \ldots, X'_n . For instance, $X_{\mathcal{A}} = X_{[1,n]}$. Further, \mathscr{C}_{ϵ} and \mathscr{C} are sets of all environment and joint protocols respectively.

3 Synchronous Agents

Synchronous agents, i.e., agents who have access to a global clock that can be used to synchronize actions, is a common type of distributed systems. All correct agents perform their actions at the same time here, with the time between consecutive actions left arbitrary and bearing no relation with message delays (except for lock-step synchronous agents, see Sect. 5). A natural malfunction for such an agent is losing synch with the global clock, however, so *byzantine synchronous agents* can err by both lagging behind and running ahead of the global clock. We implement this feature by means of **synced rounds**: correct agents act in a round $t\frac{1}{2}$ iff the round is synced, whereas a faulty agent may skip a synced round and/or act in between synced rounds. Note, however, that the agents do not a priori know whether any given (past, current, or future) round is synced.

Definition 2. A round $t_{\mathbb{Z}}^{1}$ is a synced round of a run $r \in R$ iff $\beta_{g_{i}}^{t}(r) \neq \emptyset$, where $\beta_{g_{i}}^{t}(r) = \beta_{\epsilon_{i}}^{t}(r) \cap SysEvents_{i}$, for all $i \in \mathcal{A}$. We denote the number of synced rounds in $h \in \mathcal{G}$ by NSR (h).

In other words, a synced round requires from each agent i either go(i) or one of two sync errors *sleep* (i) or *hibernate* (i). Conversely, the permission go(i) to act correctly should only be given during synced rounds, which we implement via the following event filter function:

Definition 3. The event filter function $filter_{\epsilon}^{S}(h, X_{\epsilon}, X_{A})$ for the synchronous agents extension $outputs X_{\epsilon} \setminus \{go(i) \mid i \in A\}$ if $SysEvents_{j} \cap X_{\epsilon} = \emptyset$ for some $j \in A$, or else leaves X_{ϵ} unchanged.

Since it is important for correct agents to be aware of synced rounds, we require agent protocols to issue the special internal action \diamond whenever activated:

Definition 4. The set of synchronous joint protocols is

 $\mathscr{C}^{S} := \left\{ (P_1, \dots, P_n) \in \mathscr{C} \mid (\forall i \in \mathcal{A}) (\forall h_i \in \mathscr{L}_i) (\forall D \in P_i(h_i)) \stackrel{\circ}{=} \in D \right\}.$

The action $\mathring{}$ enables correct agents to distinguish between an active round requiring no actions and a passive round with no possibility to act.² The choices behind our implementation of synchronicity will become clearer in Sect. 4.

Run modification [12] is a crucial technique for proving agents' ignorance of a fact, by creating an indistinguishable run falsifying this fact. First, we define what it means for an agent to become byzantine. Given a run r and timestamp t, a node $(i, t') \in \mathcal{A} \times \mathbb{T}$ belongs to the set *Failed* (r, t) of **byzantine nodes**, i.e., agent i is byzantine in r by time t', iff the global history r(t') contains at least one event from *FEvents_i*.

Definition 5 (Run modifications). A function $\rho: \mathbb{R}^{\chi} \mapsto 2^{\overline{GActions_i}} \times 2^{GEvents_i}$ is called an *i*-intervention for an agent-context χ and agent $i \in \mathcal{A}$. A joint intervention $B = (\rho_1, \ldots, \rho_n)$ consists of *i*-interventions ρ_i for each agent $i \in \mathcal{A}$. An adjustment $[B_t; \ldots; B_0]$ (with extent t) is a sequence of joint interventions B_0, \ldots, B_t to be performed at rounds $0\frac{1}{2}, \ldots, t\frac{1}{2}$ respectively.

An *i*-intervention $\rho(r) = (X, X_{\epsilon})$ applied to a round $t_{\underline{1}}^{\underline{1}}$ of a given run r is intended to modify the results of this round for i in such a way that $\beta_i^t(r') = X$ and $\beta_{\epsilon_i}^t(r') = X_{\epsilon}$ in the artificially constructed new run r'. We denote $\mathfrak{a}\rho(r) := X$ and $\mathfrak{e}\rho(r) := X_{\epsilon}$. Accordingly, a joint intervention (ρ_1, \ldots, ρ_n) prescribes actions $\beta_i^t(r') = \mathfrak{a}\rho_i(r)$ for each agent i and events $\beta_{\epsilon}^t(r') = \bigsqcup_{i \in \mathcal{A}} \mathfrak{e}\rho_i(r)$ for the round in question. Thus, an adjustment $[B_t; \ldots; B_0]$ fully determines actions and events in the initial t + 1 rounds of the modified run r':

Definition 6. Let $adj = [B_t; \ldots; B_0]$ be an adjustment with $B_m = (\rho_1^m, \ldots, \rho_n^m)$ for each $0 \le m \le t$ and each ρ_i^m be an *i*-intervention for an agent-context $\chi = ((P_{\epsilon}, \mathscr{G}(0), \tau, \Psi), P)$. The set $R(\tau_{P_{\epsilon}, P}, r, adj)$ consists of all runs r' obtained from $r \in R^{\chi}$ by adjustment adj, *i.e.*, runs r' such that (a) r'(0) = r(0),

(b) $r'_i(t'+1) = update_i(r'_i(t'), \mathfrak{a}\rho_i^{t'}(r), \bigsqcup_{i \in \mathcal{A}} \mathfrak{e}\rho_i^{t'}(r))$ for all $i \in \mathcal{A}$ and $t' \leq t$, (c) $r'_{\epsilon}(t'+1) = update_{\epsilon}(r'_{\epsilon}(t'), \bigsqcup_{i \in \mathcal{A}} \mathfrak{e}\rho_i^{t'}(r), \mathfrak{a}\rho_1^{t'}(r), \ldots, \mathfrak{a}\rho_n^{t'}(r))$ for all $t' \leq t$, (d) $r'(T') \tau_{P_{\epsilon},P} r'(T'+1)$ for all T' > t.

The main interventions we use are as follows, where $\beta_{b_i}^t(r) = \beta_{\epsilon_i}^t(r) \cap BEvents_i$:

Definition 7. For $i \in A$ and $r \in R$, the interventions CFreeze $(r) := (\emptyset, \emptyset)$ and BFreeze_i $(r) := (\emptyset, \{fail(i)\})$ freeze agent i with and without fault respectively.

 $\begin{aligned} PFake_{i}^{t}\left(r\right) &:= \left(\varnothing, \quad \beta_{b_{i}}^{t}\left(r\right) \ \cup \ \left\{fake\left(i, E\right) \mid E \in \overline{\beta}_{\epsilon_{i}}^{t}\left(r\right)\right\} \quad \cup \\ \left\{fake\left(i, \textit{noop} \mapsto A\right) \mid A \in \beta_{i}^{t}\left(r\right)\right\} \sqcup \left\{sleep\left(i\right) \mid \beta_{g_{i}}^{t}\left(r\right) \in \left\{\{go(i)\}, \{sleep\left(i\right)\}\}\right\} \sqcup \\ \left\{hibernate\left(i\right) \mid \beta_{g_{i}}^{t}\left(r\right) \notin \left\{\{go(i)\}, \{sleep\left(i\right)\}\}\right\}\right) \end{aligned}$

 $^{^{2}}$ For the formal statement of this distinction, see Appendix, Lemma A.34.

The Persistence of False Memory: Brain in a Vat Despite Perfect Clocks

turns all correct actions and events into indistinguishable byzantine events.

Until the end of this section, we assume that $P_{\epsilon} \in \mathscr{C}_{\epsilon}$ and $P^{S} \in \mathscr{C}^{S}$ are protocols for the environment and synchronous agents, that $\chi = ((P_{\epsilon}, \mathscr{G}(0), \tau^{S}, R), P^{S})$ is an agent-context where τ^{S} uses the synchronous event filter from Def. 3 and the standard action filters from Def. 1, and that $\mathcal{I} = (R^{\chi}, \pi)$ is an interpreted system. We additionally assume that P_{ϵ} makes a fixed agent *i*, called the "brain," gullible and all other agents $j \neq i$ delayable and fallible.

Lemma 8 (Synchronous Brain-in-the-Vat Lemma). Consider the adjustment $adj = [B_{t-1}; \ldots; B_0]$ with $B_m = (\rho_1^m, \ldots, \rho_n^m)$ where $\rho_i^m = PFake_i^m$ for the "brain" i and $\rho_j^m \in \{CFreeze, BFreeze_j\}$ for other $j \neq i$, for $m = 0, \ldots, t-1$. For any run $r \in \mathbb{R}^{\chi}$, all modified runs $r' \in \mathbb{R}(\tau_{P_e,PS}^S, r, adj)$ are $\tau_{P_e,PS}^S$ -transitional and satisfy the following properties:

1. "Brain" agent i cannot distinguish r from $r': r'_i(m) = r_i(m)$ for all $m \leq t$.

2. Other agents $j \neq i$ remain in their initial states: $r'_{j}(m) = r'_{j}(0)$ for all $m \leq t$.

3. Agent i is faulty from the beginning: $(i,m) \in Failed(r',t)$ for all $1 \le m \le t$.

4. Other agents $j \neq i$ are faulty by time t iff $\rho_j^m = BFreeze_j$ for some m.

Proof. The proof is almost identical to that of the case for asynchronous agents from [12]. The only difference is in the proof that r' is transitional. The gullibility/delayability/fallibility assumptions ensure that the protocols can issue the sets of events prescribed by adjustment adj. By Def. 7, none of the interventions $PFake_i^t$, CFreeze, or $BFreeze_j$ prescribes a go before time t. Thus, all actions are filtered out, and $filter_e^S$ from Def. 3 does not remove any prescribed events. \Box

Lemma 9. In the setting of Lemma 8, no hap o occurs correctly in any modified run $r' \in R(\tau_{P_e,P^S}^S, r, adj)$ before time t, in other words, $(\mathcal{I}, r', m) \not\models \overline{occurred}(o)$ for all $m \leq t$.

Proof. Follows directly by unfolding Def. 7 of the interventions $PFake_i^t$, CFreeze, and $BFreeze_i$ and the definition of $\overline{occurred}(o)$ on p. 6.

Theorem 10. If the agent-context χ is non-excluding, the "brain" i cannot know (a) that any hap occurred correctly, i.e., $\mathcal{I} \models \neg K_i \overline{occurred}(o)$ for all haps o;

(b) that it itself is correct, i.e., $\mathcal{I} \models \neg K_i correct_i$;

(c) that another agent $j \neq i$ is faulty, i.e., $\mathcal{I} \models \neg K_i$ faulty_i;

(d) that another agent $j \neq i$ is correct, i.e., $\mathcal{I} \models \neg K_i correct_i$.

Proof. We need to show that all these knowledge statements are false at (\mathcal{I}, r, t) for any $r \in \mathbb{R}^{\chi}$ and any $t \in \mathbb{T}$.

For t > 0, consider adj from Lemma 8 for this t. It is possible to pick one modified run $r' \in R(\tau_{P_e,P^S}^S, r, adj)$ because χ is non-excluding. "Brain" icannot distinguish r(t) from r'(t) by Lemma 8.1. For Statement (a), we have $(\mathcal{I}, r', t) \not\models \overline{occurred}(o)$ by Lemma 9. For Statement (b), $(\mathcal{I}, r', t) \not\models correct_i$ by Lemma 8.3. For Statement (c), we have $(\mathcal{I}, r', t) \not\models faulty_j$ if all $\rho_j^m = CFreeze$. Finally, for Statement (d), we have $(\mathcal{I}, r', t) \not\models correct_j$ if $\rho_j^0 = BFreeze_j$.

It remains to consider the case of t = 0. Here, $(\mathcal{I}, r, 0) \not\models \overline{occurred}(o)$ and $(\mathcal{I}, r, 0) \not\models faulty_j$ trivially for the run r itself, which completes the argument for Statements (a) and (c). However, since all agents are still correct at t = 0, for Statements (b) and (d), we additionally notice "brain" i is delayable because it is gullible. Since delaying i for the first round would prevent it from distinguishing whether t = 0 or t = 1, the case of t = 0 is thereby reduced to the already considered case of t > 0.

Remark 11. By contrast, it is sometimes possible for synchronous agents to learn of their own defectiveness, i.e., $\mathcal{I} \not\models \neg K_i fault y_i$. This may happen, for instance, if there is a mismatch between actions recorded in the agent's local history and actions prescribed by the agent's protocol for the preceding local state.

While the above limitations of knowledge apply to both asynchronous [12] and synchronous agents, as we just showed, synchronous agents do gain awareness of the global clock in the following precise sense.

Definition 12. For all $l \in \mathbb{N}$, we add the following definition of truth for special propositional variables nsr_l in interpreted systems $\mathcal{I}' = (R', \pi)$ (for $R' \subseteq R$): $(\mathcal{I}', r, t) \models nsr_l$ iff NSR(r(t)) = l.

Theorem 13. A synchronous agent k can always infer how many synced rounds elapsed from the beginning of a run under the assumption of its own correctness, i.e., for any run $r \in R^{\chi}$ and timestamp $t \in \mathbb{T}$, we have $(\mathcal{I}, r, t) \models H_k nsr_{NSR(r(t))}$.

Proof. Since $H_k nsr_{NSR(r(t))} = correct_k \rightarrow K_k (correct_k \rightarrow nsr_{NSR(r(t))})$, we need to show that $(\mathcal{I}, r', t') \models nsr_{NSR(r(t))}$ whenever $r_k(t) = r'_k(t')$ and agent k is correct both at r(t) and r'(t'). It is not hard though tedious to prove that, for a correct agent, the number of $\mathring{\otimes}$ actions in its local history is equal to the number of synced rounds elapsed in the run. Thus, NSR(r(t)) is equal to the number of $\mathring{\otimes}$'s in $r_k(t) = r'_k(t')$, which, in turn, equals NSR(r'(t')).

4 The Extension Framework

In this section, we present a glimpse into our modular extension framework, which augments the asynchronous byzantine framework [12] and enables us to implement and combine a variety of system assumptions and, consequently, extend the epistemic analysis akin to that just performed for synchronous agents.

Definition 14 (Extension). Let $\mathscr{E}^{\alpha} := (PP^{\alpha}, IS^{\alpha}, \tau^{\alpha}, \Psi^{\alpha})$ with nonempty sets $PP^{\alpha} \subseteq \mathscr{C}_{\epsilon} \times \mathscr{C}$, $IS^{\alpha} \subseteq 2^{\mathscr{G}(0)}$, and $\Psi^{\alpha} \subseteq R$ and a transition template τ^{α} . An agent-context $\chi = ((P_{\epsilon}, \mathscr{G}_{\chi}(0), \tau, \Psi), P)$ is part of \mathscr{E}^{α} , denoted $\chi \in \mathscr{E}^{\alpha}$, iff $(P_{\epsilon}, P) \in PP^{\alpha}, \mathscr{G}_{\chi}(0) \in IS^{\alpha}, \tau = \tau^{\alpha}, \Psi = \Psi^{\alpha}$, and $R^{\chi} \neq \emptyset$. We call \mathscr{E}^{α} a (framework) extension iff there exists an agent-context χ such that $\chi \in \mathscr{E}^{\alpha}$.

Extension combination. Combining extension requires combining their constituent parts. Since allowable pairs of protocols, runs, and collections of initial states are restricted by an extension, combining two extensions naturally

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means imposing both restrictions, i.e., taking their intersection. Combining the respective transition templates imposed by these extensions, on the other hand, warrants more explanation. Transition templates differ from each other only in the filtering phase. Therefore, combining transition templates, in effect, means combining their respective filter functions, which can be done in various ways. In this section we discuss *filter composition*.

Definition 15 (Basic Filter Property). We call a function $filter_{\epsilon}^{\alpha}$ ($filter_{i}^{\alpha}$ for $i \in \mathcal{A}$) an event (action) filter function iff $filter_{\epsilon}^{\alpha}(h, X_{\epsilon}, X_{\mathcal{A}}) \subseteq X_{\epsilon}$ and $filter_{i}^{\alpha}(X_{\mathcal{A}}, X_{\epsilon}) \subseteq X_{i}$ (for the exact function typification see Appendix, Def. A.1).

Definition 16. Given event (action) filter functions $filter_{\epsilon}^{\alpha}$ and $filter_{\epsilon}^{\beta}$ (filter_i^{α} and $filter_{i}^{\beta}$ for the same $i \in \mathcal{A}$), their filter composition is defined as

$$\begin{aligned} filter_{\epsilon}^{\beta\circ\alpha}\left(h, X_{\epsilon}, X_{\mathcal{A}}\right) &\coloneqq filter_{\epsilon}^{\beta}\left(h, filter_{\epsilon}^{\alpha}\left(h, X_{\epsilon}, X_{\mathcal{A}}\right), X_{\mathcal{A}}\right), \\ filter_{i}^{\beta\circ\alpha}\left(X_{\mathcal{A}}, X_{\epsilon}\right) &\coloneqq filter_{i}^{\beta}\left(X_{[1,i-1]}, filter_{i}^{\alpha}\left(X_{\mathcal{A}}, X_{\epsilon}\right), X_{[i+1,n]}, X_{\epsilon}\right). \end{aligned}$$

Definition 17. For two extensions $\mathscr{E}^{\dagger} = (PP^{\dagger}, IS^{\dagger}, \tau^{\dagger}, \Psi^{\dagger})$ with $\dagger \in \{\alpha, \beta\}$, we define their composition $\mathscr{E}^{\alpha\circ\beta} := (PP^{\alpha} \cap PP^{\beta}, IS^{\alpha} \cap IS^{\beta}, \tau^{\alpha\circ\beta}, \Psi^{\alpha} \cap \Psi^{\beta})$, where in $\tau^{\alpha\circ\beta}$ the filters of τ^{α} and τ^{β} are combined via filter composition (resulting in filter $_{\epsilon}^{\alpha\circ\beta}$ and filter $_{i}^{\alpha\circ\beta}$ for each $i \in \mathcal{A}$).

Since such a combination $\mathscr{E}^{\alpha\circ\beta}$ may not be a valid extension, we introduce the notion of extension compatibility. Informally, extensions are **compatible** if their combination can produce runs (see Appendix, Def. A.12 for the details).

We recall the definition of the conventional (trace-based) safety properties.

Definition 18. Let $PR^{trans} := \{r(t) \mid r \in R, t \in \mathbb{T}\} \subseteq \mathscr{G}$. A nonempty set $S' \subseteq R \sqcup PR^{trans}$ is a safety property if

(I) S' is prefix-closed in that

 $-r(t) \in S'$ implies that $r(t') \in S'$ for $t' \leq t$ and

 $-r' \in S'$ implies that $r'(t'') \in S'$ for all $t'' \in \mathbb{T}$;

(II) S' is **limit-closed**, i.e., $r(t) \in S'$ for all $t \in \mathbb{T}$ implies that $r \in S'$.

For the formal reasoning that any property $P^{\alpha} \subseteq R$ can be written as intersection of a safety and liveness property, see Appendix, Defs. A.16 and A.18 and Lemmas A.17 and A.19. Since safety properties based on traces are inconvenient for reasoning on a round by round basis, we introduce an equivalent safety property representation, better suited for this task. The fact that our alternative safety property definition is indeed equivalent to the trace safety property representation, follows from Appendix, Def. A.22 and Lemma A.33.

Definition 19. An operational safety property *S* is defined as a function $S: PR^{trans} \rightarrow 2^{2^{GEvents} \times 2^{GActions_1} \times \ldots \times 2^{GActions_n}}$, which satisfies the following two conditions, called operational safety property attributes. ([] represents the empty sequence.)

1. $(\exists h \in PR^{trans}) h_{\epsilon} = [] \land S(h) \neq \emptyset;$

2. $(\forall h \in PR^{trans}) h_{\epsilon} \neq [] \rightarrow ((\exists h' \in PR^{trans})(\exists X \in S(h')) h = update(h', X)) \leftrightarrow S(h) \neq \emptyset).$ The set of all operational safety properties is denoted by \emptyset .

Informally, Condition 1 means that there exists at least one safe initial state. Condition 2 means that every non-initial state is safely extendable if and only if it is safely reachable. From this point on, whenever we refer to a safety property, we mean the operational safety property (the trace safety property representation can always be retrieved if desired). We have discovered that **downward closure** of the safety property of an extension greatly improves its composability. Fortunately, it turned out that a few real-life safety properties (e.g., time-bounded communication, at-most-f byzantine agents) are, in fact, downward closed.

Definition 20. A safety property S is downward closed iff for all $h \in PR^{trans}$, $(X_{\epsilon}, X_{\mathcal{A}}) \in S(h), X'_{\epsilon} \subseteq X_{\epsilon}$, and $X'_{i} \subseteq X_{i}$ for $i \in \mathcal{A}$, we have $(X'_{\epsilon}, X'_{\mathcal{A}}) \in S(h)$.

Implementation classes. According to Def. 14, specific system assumptions can be implemented via extensions using a combination of altering the set of environment protocols, set of agent protocols, event/action filter functions, and the admissibility condition. One crucial question arises: *if a particular property could be implemented using different combinations of these mechanisms, which one of them should be favored?* The answer to this question is informed by our goal to construct extensions in the most modular and composable manner. Indeed, while the compatibility of two extensions guarantees their composition to produce runs, these runs may violate the safety property of one of the extensions, thereby defying the purpose of their combination. Here are two examples:

Example 21. A necessary event whose presence is ensured by the protocol restrictions of one extension may be removed by the event filter of the other extension.

Example 22. Consider composing $filter_{\epsilon}^{B}$ from [12], the causal filter that removes any receive event without a matching send (correct or fake) (see (A.1) for the formal definition), with the synchronous agents filter $filter_{\epsilon}^{S}$ (see Def. 3). If $filter_{\epsilon}^{S}$ is applied last, it may remove some go(i) event, preventing agent *i* from sending a message necessary to support the causality of some receive event. Thus, one must first apply $filter_{\epsilon}^{S}$, followed by $filter_{\epsilon}^{B}$.

Therefore, in this section, we provide a classification of extension implementations, which we call **implementation classes**, in order to analyze their composability and answer our posed question.

Definition 23. Implementation classes are sets of extensions presented in Table 1, where the name of the implementation class is stated in the leftmost column and parts manipulated and properties satisfied by this extension are marked by "x" in its row. Note that the last seven classes are subsets of other classes. We consider them separately due to their altered attributes regarding composability. The set of all implementation classes is denoted by \mathscr{I} .³

 $^{^{3}}$ For a detailed definition see Appendix, Def. A.35.

Ţ	admiss. condition	initial states	joint protocols	environ. protocols	arbitrary event filter	standard action filters	arbitrary action filters	downward closed	monotonic filters
Adm	х	х							
JP	х	х	x						
JP – AFB	х	х	x			х			
EnvJP	х	х	x	х					
EnvJP – AFB	х	х	x	х		х			
EvFJP	х	х	x		х				
EvFJP – AFB	х	х	x		х	x			
EvFEnvJP	х	х	x	х	х				
EvFEnvJP – AFB	х	х	x	х	х	х			
Others	х	х	x	х	х		х		
JP _{DC}	х	х	x					x	
EnvJPDC	х	х	x1	x ¹				x	
EvFJPDC	х	х	x ¹		x ¹			x	
EvFEnvJPDC	х	х	x ¹	x ¹	x ¹			x	
OthersDC	х	х	x1	x ¹	x ¹		x ¹	x	
EvFEnvJP _{DC mono}	х	х	x ¹	x ¹	x ^{1,2}			x	x
Others _{DC} mono	x	х	x1	x ¹	$x^{1,2}$		x ^{1,2}	x	x

Table 1. Implementation classes

1) such that the extension's safety property remains downward closed 2) such that the extension's filters are monotonic

To describe implementation class **composability**, we introduce the forth and reverse composability relations.

Definition 24. Two implementation classes $IC^{\alpha}, IC^{\beta} \in \mathscr{I}$ are forth (reverse) composable iff for all extensions $\mathscr{E}^{\alpha} \in IC^{\alpha}$ and $\mathscr{E}^{\beta} \in IC^{\beta}$ compatible with respect to forth (reverse) composition $\alpha \circ \beta$ ($\beta \circ \alpha$), the extension $\mathscr{E}^{\alpha \circ \beta}$ ($\mathscr{E}^{\beta \circ \alpha}$) adheres to the safety property S^{β} of \mathscr{E}^{β} .⁴

Our synchronous agents introduced in Sect. 3 correspond to the following extension from the class EvFJP - AFB:

Definition 25. We denote by $\mathscr{E}^S := (\mathscr{C}_{\epsilon} \times \mathscr{C}^S, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^S, R)$ the synchronous agents extension, where the transition template τ^S uses the synchronous agents event filter and the standard action filters.

The entries in Table 2 are to be read as follows supposing x is the content of the entry, LC is the implementation class to the left and TC is the implementation class on the top:

- -x = c means that *LC* is both forth and reverse composable with *TC*.
- -x = f means that *LC* is forth composable with *TC*.
- -x = r means that *LC* is reverse composable with *TC*.
- An empty entry means that LC can generally not be safely combined with TC (we do not have a positive result stating the opposite).

 $^{^4}$ Table 2 states our composability results for various implementation class combinations.

	Adm	JP	Env JP	EvF JP	EvF Env JP	JP - AFB	Env JP - AFB	EvF JP - AFB	EvF Env JP - AFB	Oth_{ers}	JP DC	Env JP DC	EvF JP DC	EvF Env JP DC	Oth ers DC	EvF Env JP DC mono	Oth ers DC mono
Adm	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с
JP	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с
EnvJP	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с
EvFJP	с	с		r		с		r			с	с	с	f	f	с	с
EvFEnvJP	с	с		r		с		r			с	с	с	f	f	с	с
JP – AFB	с					с	с	с	с		с	с	с	с	f	с	с
EnvJP – AFB	с					с	с	с	с		с	с	с	с	f	с	с
EvFJP – AFB	с					с		r			с	с	с	f	f	с	с
EvFEnvJP – AFB	с					с		r			с	с	с	f	f	с	с
Others	с										с	с	с	f	f	с	с
JP _{DC}	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с
EnvJPDC	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с	с
EvFJPDC	с	с		r		с		r			с	с	с	f	f	с	с
EvFEnvJPDC	с	с		r		с		r			с	с	с	f	f	с	с
OthersDC	с										с	с	с	f	f	с	с
EvFEnvJP _{DC mono}	с	с		r		с		r			с	с	с	f	f	с	с
Others _{DC} mono	с										с	с	с	f	f	с	с

Table 2. Composability matrix of implementation classes

5 Lock-step Synchronous Agents

In lock-step synchronous distributed systems [15], agents act synchronously in *communication-closed* rounds. In each such round, every correct agent sends a message to every agent, which is received in the same round, and finally processes all received messages, which happens simultaneously at all correct agents. Thus, agents are not only synchronous, but additionally their communication is reliable, broadcast, synchronous (and causal). Our *lock-step round extension* combines 5 different extensions corresponding to the aforementioned properties: the (i) byzantine agents [12], (ii) synchronous agents extension from Def. 25, (iii) reliable communication extension ensuring that every sent message is eventually delivered, (iv) synchronous communication extension ensuring that every message is either received instantaneously or not at all, and (v) broadcast communication extension ensuring that every sent to all agents.⁵

Since, by Lemma 8, even synchronous agents can be fooled by their own (faulty) imagination, it is natural to ask whether a brain-in-a-vat scenario is still possible in the more restricted lock-step synchronous setting. The proof of the possibility of the brain-in-a-vat scenario from Lemma 8 in an asynchronous setting provided in [12] suggests this not to be the case. However, by considering extension combinations more closely and in more detail we were able to implement such a scenario despite the additional restrictions. The issue is that the *i*-intervention $PFake_i^t$ from Def. 7 makes it possible for byzantine actions of the dreaming "brain" to affect other agents. This possibility becoming a certainty due to the more reliable communication of lock-step synchronous agents

⁵ Formal definitions for (i), (iii), (iv), (v) can be found in Appendix, starting from Def. A.41.

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is the obstacle preventing the complete isolation of the brain. This effect can be avoided by modifying $PFake_i^t$ to make all byzantine actions entirely imaginary. This new *i*-intervention $BPFake_i^t$ is obtained from Def. 7 by replacing $\beta_{b_i}^t(r)$ in the modified events with:

 $\begin{cases} fake (i, E) \mid fake (i, E) \in \beta_{b_i}^t (r) \end{cases} \quad \cup \\ \{ fake (i, \mathbf{noop} \mapsto A) \mid (\exists A' \in \overline{GActions}_i \sqcup \{\mathbf{noop}\}) fake (i, A' \mapsto A) \in \beta_{b_i}^t (r) \end{cases}.$

(The full reformulation of Lemma 8 for this case with a short proof can be found in Appendix, Lemma A.65.) Therefore, even perfect clocks and communicationclosed rounds do not exclude the "brain-in-a-vat" scenario, with the consequence that most (negative) introspection results for synchronous systems also hold for lock-step synchronous systems:

Theorem 26. Replacing $PFake_i^t$ with $BPFake_i^t(r)$ in Lemma 8 extends the latter's "brain-in-the-vat" properties to lock-step synchronous system.

But besides the fact that our lock-step round extension was instrumental for identifying the subtle improvements in implementing the brain-in-the-vat scenario, it does have positive consequences for the fault-detection abilities of the agents as well: using a weaker epistemic notion of the hope modality H, we have shown that in a lock-step synchronous context it is possible to design agent protocols to detect faults of other agents.

Theorem 27. There exists an agent context $\chi \in \mathscr{E}^{LSS}$, where $\chi = ((P_{\epsilon}^{SC_{A^2}}, \mathscr{G}(0), \tau^{B \circ S}, EDel_{A^2}), \tilde{P}^{SMC_{BCh}})$, and a run $r \in R^{\chi}$, such that for agents $i, j \in \mathcal{A}$, where $i \neq j$, some timestamp $t \in \mathbb{N}$, and a χ -based interpreted system $\mathcal{I} = (R^{\chi}, \pi)$

$$(\mathcal{I}, r, t) \models H_i fault y_j.$$

Proof. See Appendix, Theorem 68.

6 Conclusions

We substantially augmented the epistemic reasoning framework for byzantine distributed systems [12] with extensions, which allow to incorporate additional system assumptions in a modular fashion. By instantiating our extension framework for both synchronous and lock-step synchronous systems, we proved that even adding perfect clocks and communication-closed rounds cannot circumvent the possibility of a brain-in-the-vat scenario and resulting negative introspection results, albeit they do enable some additional fault detection capabilities.

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A Appendix

Definition A.1. We define an event filter function

 $filter_{:}: \mathscr{G} \times 2^{GEvents} \times 2^{\overline{GActions}_1} \times \cdots \times 2^{\overline{GActions}_n} \longrightarrow 2^{GEvents}.$

In addition, we define action filter functions for agents $i \in A$

 $filter_i: 2^{\overline{GActions_1}} \times \cdots \times 2^{\overline{GActions_n}} \times 2^{\overline{GEvents}} \longrightarrow 2^{\overline{GActions_i}}$



Fig. 1. Details of round t_2^1 of a $\tau_{P_{\epsilon},P}$ -transitional run r.

Definition A.2. The causal event filter returns the set of all attempted events that are "causally" possible. For a set $X_{\epsilon} \subseteq GEvents$, sets $X_i \subseteq \overline{GActions_i}$ for each agent $i \in \mathcal{A}$, and a global history $h = (h_{\epsilon}, h_1, \ldots, h_n) \in \mathscr{G}$, we define

$$filter^{B}_{\epsilon}(h, X_{\epsilon}, X_{1}, \dots, X_{n}) := X_{\epsilon} \setminus \left\{ grecv(j, i, \mu, id) \mid gsend(i, j, \mu, id) \notin h_{\epsilon} \land (\forall A \in \{noop\} \sqcup \overline{GActions}_{i}) fake(i, gsend(i, j, \mu, id) \mapsto A) \notin h_{\epsilon} \land (gsend(i, j, \mu, id) \notin X_{i} \lor go(i) \notin X_{\epsilon}) \land (\forall A \in \{noop\} \sqcup \overline{GActions}_{i}) fake(i, gsend(i, j, \mu, id) \mapsto A) \notin X_{\epsilon} \right\}$$
(A.1)

Definition A.3. A history h_i of agent $i \in A$, or its local state, is a nonempty sequence $h_i = [\lambda_m, \ldots, \lambda_1, \lambda_0]$ for some $m \ge 0$ such that $\lambda_0 \in \Sigma_i$ and $\forall k \in [\![1;m]\!]$ we have $\lambda_k \subseteq Haps_i$. In this case m is called the **length of history** h_i and denoted $|h_i|$. We say that a set $\lambda \subseteq Haps_i$ is recorded in the history h_i of agent i and write $\lambda \subseteq h_i$ iff $\lambda = \lambda_k$ for some $k \in [\![1;m]\!]$. We say that $o \in Haps_i$ is recorded in the history h_i and write $o \in h_i$ iff $o \in \lambda$ for some set $\lambda \subseteq h_i$.

Definition A.4. A history h of the system with n agents, or the global state, is a tuple $h := (h_{\epsilon}, h_1, \ldots, h_n)$ where the history of the environment is a sequence $h_{\epsilon} = [\Lambda_m, \ldots, \Lambda_1]$ for some $m \ge 0$ such that $\forall k \in [1; m]$ we have

 $\Lambda_k \subseteq GHaps$ and h_i is a local state of each agent $i \in [\![1;n]\!]$. In this case m is called the **length of history** h and denoted $|h| := |h_{\epsilon}|$, i.e., the environment has the true global clock. We say that a set $\Lambda \subseteq GHaps$ happens in the environment's history h_{ϵ} or in the system history h and write $\Lambda \subseteq h_{\epsilon}$ iff $\Lambda = \Lambda_k$ for some $k \in [\![1;m]\!]$. We say that $O \in GHaps$ happens in the environment's history h_{ϵ} or in the system history h and write $O \in \Lambda$ for some set $\Lambda \subseteq h_{\epsilon}$.

Definition A.5 (Localization function). The function $\sigma: 2^{GHaps} \longrightarrow 2^{Haps}$ is defined as follows

$$\sigma(X) := local\left(\left(X \cap \overline{GHaps}\right) \cup \{E \mid (\exists i) fake (i, E) \in X\} \cup \{A' \neq noop \mid (\exists i) (\exists A) fake (i, A \mapsto A') \in X\}\right).$$

Definition A.6 (State update functions). Given $h = (h_{\epsilon}, h_1, \ldots, h_n) \in \mathscr{G}$, a tuple of performed actions/events $X = (X_{\epsilon}, X_1, \ldots, X_n) \in 2^{GEvents} \times 2^{\overline{GActions_1}} \times \ldots \times 2^{\overline{GActions_n}}$, we use the following abbreviation $X_{\epsilon_i} = X_{\epsilon} \cap GEvents_i$ for each $i \in \mathcal{A}$. Agents *i*'s update function

$$update_i: \mathscr{L}_i \times 2^{\overline{GActions}_i} \times 2^{GEvents} \to \mathscr{L}_i$$

outputs a new local history from \mathscr{L}_i based on i's actions X_i and environmentcontrolled events X_{ϵ} as follows:

$$update_{i}(h_{i}, X_{i}, X_{\epsilon}) := \begin{cases} h_{i} & \text{if } \sigma(X_{\epsilon_{i}}) = \varnothing \text{ and} \\ & X_{\epsilon_{i}} \cap SysEvents_{i} \notin \{\{go(i)\}, \{sleep(i)\}\} \\ \left[\sigma(X_{\epsilon_{i}} \sqcup X_{i})\right] : h_{i} & \text{otherwise} \end{cases}$$
(A.2)

where : represents sequence concatenation. Similarly, the environment's state update function update_{\epsilon}: $\mathscr{L}_{\epsilon} \times \left(2^{GEvents} \times 2^{\overline{GActions_1}} \times \ldots \times 2^{\overline{GActions_n}}\right) \to \mathscr{L}_{\epsilon}$ outputs a new state of the environment based on X:

$$update_{\epsilon}(h_{\epsilon}, X) := (X_{\epsilon} \sqcup X_1 \sqcup \ldots \sqcup X_n) \colon h_{\epsilon}$$

Thus, the global state is modified as follows:

 $update(h, X) \coloneqq (update_{\epsilon}(h_{\epsilon}, X), update_{1}(h_{1}, X_{1}, X_{\epsilon}), \dots, update_{n}(h_{n}, X_{n}, X_{\epsilon}))$

Definition A.7. Let $t \in \mathbb{T}$ be a timestamp. A set $S \subset GE$ vents of events is called *t*-coherent if it satisfies the following conditions:

- 1. for any fake $(i, gsend(i, j, \mu, id) \mapsto A) \in S$, the GMI $id = id(i, j, \mu, k, t)$ for some $k \in \mathbb{N}$;
- 2. for any $i \in A$ at most one event from $SysEvents_i$ is present in S;
- 3. for any $i \in A$ and any locally observable event e at most one of global (i, t, e)and fake (i, global (i, t, e)) is present in S;

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- 4. for any $grecv(i, j, \mu, id_1) \in S$, no event of the form fake $(i, grecv(i, j, \mu, id_2))$ belongs to S for any $id_2 \in \mathbb{N}$;
- 5. for any fake $(i, grecv(i, j, \mu, id_1)) \in S$, no event of the form $grecv(i, j, \mu, id_2)$ belongs to S for any $id_2 \in \mathbb{N}$;

Definition A.8. For a context $\gamma = (P_{\epsilon}, \mathscr{G}(0), \tau, \Psi)$ and a joint protocol P, we define the set of runs **weakly consistent** with P in γ (or weakly consistent with $\chi = (\gamma, P)$), denoted $R^{w\chi} = R^{w(\gamma, P)}$, to be the set of $\tau_{P_{\epsilon}, P}$ -transitional runs that start at some global initial state from $\mathscr{G}(0)$:

 $R^{w(\gamma,P)} := \{ r \in R \mid r(0) \in \mathscr{G}(0) \text{ and } (\forall t \in \mathbb{T}) r(t+1) \in \tau_{P_{\epsilon},P}(r(t)) \}$

A run r is called **strongly consistent**, or simply **consistent**, with P in γ (or with χ) if it is weakly consistent with P in γ and, additionally, satisfies the admissibility condition: $r \in \Psi$. We denote the system of all runs consistent with P in γ by $R^{(\gamma,P)} := R^{w(\gamma,P)} \cap \Psi$.

Definition A.9. An agent-context χ is non-excluding iff

$$R^{\chi} \neq \varnothing \quad and \quad (\forall r \in R^{w\chi})(\forall t \in \mathbb{T})(\exists r' \in R^{\chi})(\forall t' \le t) r'(t') = r(t')$$

Definition A.10. For agent $i \in \mathcal{A} = [\![1;n]\!]$, the *indistinguishability relation* $\sim_i \subseteq \mathscr{G}^2$ is formally defined by $\sim_i := \{(h,h') \mid \pi_{i+1}h = \pi_{i+1}h'\}.$

Definition A.11. Given an interpreted system $\mathcal{I} = (R^{\chi}, \pi)$ an agent $i \in \mathcal{A}$, a run $r \in R^{\chi}$, and a timestamp $t \in \mathbb{T}$:

$$\begin{aligned} (\mathcal{I}, r, t) &\models p & iff & r(t) \in \pi(p) \\ (\mathcal{I}, r, t) &\models \neg \varphi & iff & (\mathcal{I}, r, t) \not\models \varphi \\ (\mathcal{I}, r, t) &\models \varphi \land \varphi' & iff & (\mathcal{I}, r, t) \models \varphi \text{ and } (\mathcal{I}, r, t) \models \varphi' \\ (\mathcal{I}, r, t) &\models K_i \varphi & iff & (\forall r' \in R')(\forall t' \in \mathbb{T}) (r'_i(t') = r_i(t) \Rightarrow (\mathcal{I}, r', t') \models \varphi) \end{aligned}$$

Definition A.12 (Compatibility). For a number of $l \geq 2$ extensions \mathscr{E}^{α_1} , \mathscr{E}^{α_2} , ..., \mathscr{E}^{α_l} we say the extensions \mathscr{E}^{α_1} , \mathscr{E}^{α_2} , ..., \mathscr{E}^{α_l} are compatible w.r.t. to some series of extension combinations $\star_1, \star_2, ..., \star_{l-1}$ ⁶ iff $PP_1^{\alpha} \cap \ldots \cap PP_l^{\alpha} \neq \emptyset$, $IS_1^{\alpha} \cap \ldots \cap IS_l^{\alpha} \neq \emptyset$, $\Psi^{\alpha_1} \cap \ldots \cap \Psi^{\alpha_l} \neq \emptyset$ and $\exists \chi \in \mathscr{E}^{\alpha_1 \star_1 \alpha_2 \star_2 \ldots \star_{l-1} \alpha_l}$.

Iff extensions \mathscr{E}^{α_1} , \mathscr{E}^{α_2} , ..., \mathscr{E}^{α_l} $(l \ge 2)$ are **compatible** w.r.t. the extension combination series $\star_1, \star_2, \ldots, \star_{l-1}$, then $\mathscr{E}^{\alpha_1 \star_1 \alpha_2 \star_2 \ldots \star_{l-1} \alpha_l}$ is also an extension.

Definition A.13. We define $PD_{\epsilon}^{t\text{-coh}}$ as the (downward closed) domain of all tcoherent events: $PD_{\epsilon}^{t\text{-coh}} := \{X_{\epsilon} \in 2^{GEvents} \mid X_{\epsilon} \text{ is t-coherent for some } t \in \mathbb{T}\}.$

Definition A.14. We define a *liveness* property as a subset $L \subseteq R$, where $L \neq \emptyset \land (\forall r \in R) (\forall t \in \mathbb{T}) (\exists r' \in L) r'(t) = r(t)$.

⁶ For some $l' \in \mathbb{N}$ we use $\alpha \star_{l'} \beta$ or just \star to represent either forth composition $(\alpha \circ \beta)$ or reversed composition $(\beta \circ \alpha)$. Note that in our complete framework we distinguish between further types of combinations.

Informally, liveness says that every prefix r(t) of every run r can be extended in L.

Definition A.15. An extension \mathscr{E}^{α} adheres to a safety property S' (resp. liveness property L) iff $\bigcup_{\chi^{\alpha} \in \mathscr{E}^{\alpha}} R^{\chi^{\alpha}} \subseteq S' (\bigcup_{\chi^{\alpha} \in \mathscr{E}^{\alpha}} R^{\chi^{\alpha}} \subseteq L)$.

Definition A.16. For a set $P^{\alpha} \subseteq R$ of transitional runs, where $P^{\alpha} \neq \emptyset$,

$$L^{\prime\alpha} := \{ r \in R \mid (\exists t \in \mathbb{T}) (\forall r' \in P^{\alpha}) (\forall t' \in \mathbb{T}) \ r(t) \neq r'(t') \}$$
(A.3)
$$\overline{L^{\alpha}} := P^{\alpha} + L^{\prime\alpha}$$
(A.4)

$$L^{\alpha} := P^{\alpha} \cup L^{\prime \alpha}. \tag{A.4}$$

Lemma A.17. $\overline{L^{\alpha}}$ is a liveness property.

Proof. Since $P^{\alpha} \neq \emptyset$, $\overline{L^{\alpha}} \neq \emptyset$ as well by (A.4).

Take any finite prefix r(t) of a run $r \in R$ for some timestamp $t \in \mathbb{T}$. If r(t) has an extension in P^{α} , then there exists a run $r' \in P^{\alpha}$, s.t. r(t) = r'(t). Since by Def. A.16 $P^{\alpha} \subseteq \overline{L^{\alpha}}$, $r' \in \overline{L^{\alpha}}$ as well. If r(t) has no extension in P^{α} , then by Def. A.16 $r \in L'^{\alpha}$, thus $r \in \overline{L^{\alpha}}$.

Definition A.18. The smallest trace safety property containing $P \subseteq R$, for $P \neq \emptyset$, is the **prefix and limit closure** of P, formally

$$S'(P) := \{h \in PR^{trans} \mid (\exists r \in P)(\exists t \in \mathbb{T}) \ r(t) = h\} \sqcup \\ \{r \in R \mid (\forall t \in \mathbb{T})(\exists r' \in P) \ r(t) = r'(t)\}.$$

The set of all trace safety properties is denoted by \mathscr{T} .

Lemma A.19. $P^{\alpha} = \overline{L^{\alpha}} \cap S'(P^{\alpha})$, where $S'(P^{\alpha})$ is the prefix and limit closure of P^{α} (see Def. A.18).

Proof. Since $P^{\alpha} \subseteq S'(P^{\alpha})$ and $P^{\alpha} \subseteq \overline{L^{\alpha}}$, it follows that $P^{\alpha} \subseteq \overline{L^{\alpha}} \cap S'(P^{\alpha})$. Hence it remains to show that $\overline{L^{\alpha}} \cap S'(P^{\alpha}) \subseteq P^{\alpha}$. Assume by contradiction that there exists a run $r \in \overline{L^{\alpha}} \cap S'(P^{\alpha})$, but $r \notin P^{\alpha}$, hence $r \in \overline{L^{\alpha}}$ —specifically $r \in L'^{\alpha}$ —and $r \in S'(P^{\alpha})$. Since $r \in S'(P^{\alpha})$ (by prefix closure of $S'(P^{\alpha})$) for all $t' \in \mathbb{T}$, $r(t') \in S'(P^{\alpha})$ as well. This implies (by limit closure of $S'(P^{\alpha})$) that for all $t \in \mathbb{T}$ there must exist a run $r' \in P^{\alpha}$ such that r(t) = r'(t). This however contradicts that $r \in L'^{\alpha}$.

Definition A.20. A construction F' of an operational safety property from a trace safety property $S' \in \mathscr{T}$ is $F(S')(h) := \{\beta^t(r) \mid r \in S' \land t \in \mathbb{T} \land h = r(t)\}.$

Lemma A.21. $F'(S') \in \mathcal{O}$ for any $S' \in \mathcal{T}$.

Proof. Suppose by contradiction there exists some $S'^{\alpha} \in \mathscr{T}$ s.t. $F'(S'^{\alpha}) = S^{\alpha}$, where S^{α} violates the first operational safety property attribute (1). This implies $(\forall h \in PR^{trans}) h_{\epsilon} \neq [] \lor S^{\alpha}(h) = \varnothing$. Since by Def. A.18 $P^{\alpha} \neq \varnothing$, we get that there has to exist a run $r \in S'^{\alpha}$. Further, by prefix closure of S'^{α} , we have $(\forall t \in \mathbb{T}) r(t) \in S'^{\alpha}$, from which by universal instantiation we get that $r(0) \in S'^{\alpha}$. Since $S'^{\alpha} \subseteq R \sqcup PR^{trans}$, r is transitional, hence $r_{\epsilon}(0) = []$, from which by our assumption $S^{\alpha}(r(0)) = \emptyset$ follows. However, by Def. A.20 of construction F', it follows that $\beta^0(r) \in S^{\alpha}(r(0))$, thus $S^{\alpha}(r(0)) \neq \emptyset$.

Next, suppose by contradiction there exists some $S^{\prime \alpha} \in \mathscr{T}$ s.t. $F^{\prime}(S^{\prime \alpha}) = S^{\alpha}$, where S^{α} violates the second operational safety property attribute (2). This implies that there exists some $h \in PR^{trans}$ s.t. $h_{\epsilon} \neq []$ and

$$(((\exists h' \in PR^{trans})(\exists X \in S^{\alpha}(h'))h = update(h', X)) \land S^{\alpha}(h) = \emptyset) \lor$$
 (A.5)

$$(((\forall h'' \in PR^{trans})(\forall X' \in S^{\alpha}(h''))h \neq update(h'', X')) \land S^{\alpha}(h) \neq \varnothing).$$
(A.6)

Suppose (A.5) is true. This implies that there exists some $h' \in PR^{trans}$ and some $X \in S^{\alpha}(h')$ such that h = update(h', X). By Def. A.20 of F' there exists a run $r \in S'^{\alpha}$ and a timestamp $t \in \mathbb{T}$ s.t. r(t) = h' and $X = \beta^t(r)$. By transitionality of r and Def. A.6 of update r(t+1) = h. Again by Def. A.20, we have $\beta^{t+1}(r) \in S^{\alpha}(h)$, hence $S^{\alpha}(h) \neq \emptyset$ and we conclude that (A.5) is false.

Suppose (A.6) is true. This implies by Def. A.20 that there exists a run $r \in S'^{\alpha}$ and timestamp $t \in \mathbb{T} \setminus \{0\}$, where h = r(t), since r is transitional and $h_{\epsilon} \neq []$. Further we get that $\beta^{t-1}(r) \in S^{\alpha}(r(t-1))$. Thus by Def. A.6 of *update*, we have $r(t) = update(r(t-1), \beta^{t-1}(r))$ and we conclude that (A.6) is false as well.

Definition A.22. We define $F: \mathscr{T} \to \mathscr{O}$, where for any $S' \in \mathscr{T}$ we have F(S') := F'(S'), for F' from Def. A.20, which is indeed a mapping from \mathscr{T} to \mathscr{O} by Lemma A.21.

Lemma A.23. F from Def. A.22 is injective.

Proof. Suppose by contradiction that the opposite is true: there are $S^{\prime\alpha}, S^{\prime\beta} \in \mathscr{T}$ s.t. $S^{\prime\alpha} \neq S^{\prime\beta}$, but $F(S^{\prime\alpha}) = F(S^{\prime\beta})$. Since $S^{\prime\alpha} \neq S^{\prime\beta}$, either

 $1\,$ w.l.o.g. there exists some history $h\in S'^{\alpha}$ s.t. $h\notin S'^{\beta}$ or

2 w.l.o.g. there exists some run $r \in S'^{\alpha}$ s.t. $r \notin S'^{\beta}$. We show that this implies 1. Suppose by contradiction that there does not exist some $h \in S'^{\alpha}$ s.t. $h \notin S'^{\beta}$, meaning $(\forall h \in S'^{\alpha})$ $h \in S'^{\beta}$. By limit closure of S'^{β} however it follows that $r \in S'^{\beta}$, hence there has to exist a history $h \in S'^{\alpha}$ such that $h \notin S'^{\beta}$.

Therefore, we can safely assume 1, i.e., w.l.o.g. that there exists some $h \in S'^{\alpha}$ s.t. $h \notin S'^{\beta}$. By Def. A.22 of F, we get that $F(S'^{\beta})(h) = \emptyset$, as otherwise there would exist a run $r' \in S'^{\beta}$ and time $t' \in \mathbb{T}$ s.t. r'(t') = h, from which by prefix closure of S'^{β} it would follow that $h \in S'^{\beta}$. Since S'^{α} is the prefix closure of some non-empty set $P^{\alpha} \subseteq R$ by Def. A.18, we get that there exists some run $r \in S'^{\alpha}$ and time $t \in \mathbb{T}$ s.t. r(t) = h, additionally by Def. A.22 of F, $\beta^t(r) \in F(S'^{\alpha})(h)$. Therefore $F(S'^{\alpha}) \neq F(S'^{\beta})$ and we are done.

Definition A.24. For some arbitrary $S \in \mathcal{O}$, we define

$$\widetilde{S}_0^{\prime S} := R \tag{A.7}$$

$$\widetilde{S}'_{t}^{S} := \widetilde{S'_{t-1}}^{S} \setminus \{ r \in R \mid \beta^{t-1}(r) \notin S(r(t-1)) \}$$
(A.8)

$$\widetilde{S'_{\infty}}^{S} \coloneqq \lim_{t' \to \infty} \widetilde{S'_{t'}}^{S} \tag{A.9}$$

$$\widetilde{S'}^{S} := \widetilde{S'_{\infty}}^{S} \sqcup \{ h \in PR^{trans} \mid (\exists r \in \widetilde{S'_{\infty}}^{S}) (\exists t \in \mathbb{T}) \ h = r(t) \}.$$
(A.10)

Note that the limit in (A.9) exists, as by (A.8) the set $\widetilde{S'_t}^S$ is non-increasing in t.

Lemma A.25. For $\widetilde{S'_m}^{\widetilde{S}}$ (for $m \in \mathbb{T} \setminus \{0\}$ and $\widetilde{S} \in \mathcal{O}$) from Def. A.24, it holds that $\widetilde{S'_m}^{\widetilde{S}} = \{r \in R \mid (\forall t < m) \ \beta^t (r) \in \widetilde{S}(r(t))\}.$

Proof. By induction: **Induction Hypothesis:**

$$\widetilde{S'_m}^{\widetilde{S}} = \{ r \in R \mid (\forall t < m) \ \beta^t (r) \in \widetilde{S}(r(t)) \}.$$
(A.11)

Base Case: For m = 1 by Def. A.24 it follows that

$$\widetilde{S'_1}^S = R \setminus \{ r \in R \mid \beta^0(r) \notin \widetilde{S}(r(0)) \} = \{ r \in R \mid \beta^0(r) \in \widetilde{S}(r(0)) \}.$$

Induction Step: Suppose the induction hypothesis (A.11) holds for m, but by contradiction does not hold for m + 1. There are two cases:

- 1. There exists a run $r' \in \widetilde{S'_{m+1}}^{\widetilde{S}}$ s.t. $r' \notin \{r \in R \mid (\forall t < m+1) \ \beta^t(r) \in \mathbb{C}\}$ $\widetilde{S}(r(t))$. This implies that there exists some timestamp t' < m + 1 s.t. $\beta^{t'}(r') \notin \widetilde{S}(r'(t'))$. We distinguish two cases:
 - (a) t' = m: Then $r' \in \{r \in R \mid \beta^m(r) \notin \widetilde{S}(r(m))\}$. Hence by Def. A.24 of
- (a) to the first of the entry of

 $\widetilde{S'_{m+1}}^{\widetilde{S}}$. We distinguish two cases regarding at which step r has been removed:

- (a) $r' \in \widetilde{S'_m}^{\widetilde{S}}$: Then $r' \in \{r \in R \mid \beta^m(r) \notin \widetilde{S}(r(m))\}$. This implies that $\beta^m(r') \notin \widetilde{S}(r'(m))$ contradicting $r' \in \{r \in R \mid (\forall t < m+1) \ \beta^t(r) \in \widetilde{S'_m}^{\widetilde{S}}$. $S(r(t))\}.$
- (b) $r' \notin \widetilde{S'_m}^{\widetilde{S}}$: This directly contradicts the induction hypothesis (A.11), thus concluding the induction step.

Lemma A.26. For $\widetilde{S'_{\infty}}^{S}$ from Def. A.24 it holds that

$$\widetilde{S'_{\infty}}^{S} = \{ r \in R \mid (\forall t \in \mathbb{T}) \ \beta^{t} (r) \in S(r(t)) \}$$

Proof. Follows from Lemma A.25 and Def. A.24.

Lemma A.27. For $\widetilde{S'}^{S}$, from Def. A.24, where $S \in \mathcal{O}$, it holds that $\widetilde{S'}^{S} \in \mathcal{T}$, *i.e.* $\widetilde{S'}^{S}$ *is a trace safety property.*

Proof. From Def. A.24, particularly (A.9) and (A.10), it follows that $\widetilde{S'}^{S}$ is the prefix and limit closure of $\widetilde{S'_{\infty}}^{S}$.

Lemma A.28. The state update function—update—from Def. A.6 is injective.

Proof. Recall that according to Def. A.6 update and its constituent parts are defined for $h \in \mathcal{G}$, $i \in \mathcal{A}$ and $X \in 2^{GEvents} \times 2^{\overline{GActions}_1} \times \ldots \times 2^{\overline{GActions}_n}$ as

$$update(h, X) := \left(update_{\epsilon}(h_{\epsilon}, X), update_{1}(h_{1}, X_{1}, X_{\epsilon}), \dots, update_{n}(h_{n}, X_{n}, X_{\epsilon})\right) \quad (A.12)$$

$$update_{i}(h_{i}, X_{i}, X_{\epsilon}) := \begin{cases} h_{i} & \text{if } \sigma(X_{\epsilon_{i}}) = \emptyset \text{ and } unaware(i, X_{\epsilon}) \\ \left[\sigma(X_{\epsilon_{i}} \sqcup X_{i})\right] : h_{i} & \text{otherwise} \end{cases}$$
(A.13)

$$update_{\epsilon}(h_{\epsilon}, X) := (X_{\epsilon} \sqcup X_1 \sqcup \ldots \sqcup X_n) \colon h_{\epsilon}.$$
 (A.14)

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Suppose by contradiction that update is not injective, i.e., $update(h^1, X^1) =$ update (h^2, X^2) for some $(h^1, X^1) \neq (h^2, X^2) \in \mathscr{G} \times 2^{GEvents} \times 2^{\overline{GActions_1}} \times \ldots \times$ $2^{\overline{GActions}_n}$. We distinguish the following cases:

- 1. $X^1 \neq X^2$: By (A.14) $X^1 : h_{\epsilon}^1 \neq X^2 : h_{\epsilon}^2$, as irrespective of h^1 and h^2 the two resulting histories have different suffixes (of size one) X^1 and X^2 (recall that by Defs. A.3 and A.4 the environment and the agent histories are sequences of sets).
- 2. $h_{\epsilon}^1 \neq h_{\epsilon}^2$: If $X_1 \neq X_2$ it follows from case (1) that $X^1: h_{\epsilon}^1 \neq X^2: h_{\epsilon}^2$. Else if $X_1 = X_2$: if further $|h_{\epsilon}^1| = |h_{\epsilon}^2|$ then the two resulting histories now have the same suffix X^1 , however still different prefixes h_{ϵ}^1 and h_{ϵ}^2 . Otherwise if w.l.o.g. $|h_{\epsilon}^1| > |h_{\epsilon}^2|$, then $|X^1:h_{\epsilon}^1| > |X^1:h_{\epsilon}^2|$ and we are done.

- 3. $h_i^1 \neq h_i^2$ (for some $i \in \mathcal{A}$): if $-X^1 \neq X^2$: follows from case (1) $-X^1 = X^2$: either $update_i (h_i^1, X_i^1, X_{\epsilon}^1) = h_i^1$ or $update_i (h_i^2, X_i^1, X_{\epsilon}^1) = h_i^2$, since $h_i^1 \neq h_i^2$ we conclude that the resulting (local) histories are different, or

$$update_i \left(h_i^1, X_i^1, X_{\epsilon}^1\right) = \left[\sigma\left(X_{\epsilon_i}^1 \sqcup X_i^1\right)\right] : h_i^1$$
$$update_i \left(h_i^2, X_i^1, X_{\epsilon}^1\right) = \left[\sigma\left(X_{\epsilon_i}^1 \sqcup X_i^1\right)\right] : h_i^2.$$

Suppose that $|h_i^1| = |h_i^2|$. It follows that the two resulting (local) histories have matching suffixes $(\left|\sigma(X_{\epsilon_i}^1 \sqcup X_i^1)\right|)$, however different prefixes. If on the other hand w.l.o.g. $|h_i^1| > |h_i^2|$, it also follows that $|[\sigma(X_{\epsilon_i}^1 \sqcup X_i^1)] : h_i^1| > |[\sigma(X_{\epsilon_i}^1 \sqcup X_i^1)] : h_i^2|$, hence they cannot be the same and we are done.

Lemma A.29. Given an operational safety property $S \in \mathcal{O}$, a transitional run $r \in R$ and timestamps $t, t' \in \mathbb{T}$, where $t' \geq t$, if $S(r(t)) = \emptyset$, then $S(r(t')) = \emptyset$.

Proof. By induction over t' > t.

Induction Hypothesis: For $t' \ge t$ and $S \in \mathcal{O}$ it holds that if $S(r(t)) = \emptyset$, then $S(r(t')) = \emptyset$.

Base Case for t' = t: it trivially follows that $S(r(t')) = S(r(t)) = \emptyset$.

Induction Step for $t' \to t' + 1$: Since the state update function is injective by Lemma A.28 and run r is transitional, the only way to achieve the prefix r(t' + 1) via state update is by $update(r(t'), \beta^{t'}(r))$. However since by the induction hypothesis $S(r(t')) = \emptyset$, it follows by the second operational safety property attribute (2) that $S(r(t' + 1)) = \emptyset$ as well.

Lemma A.30. For any $h \in PR^{trans}$ and operational safety property $S \in \mathcal{O}$ it holds that $(\exists r \in R)(r(|h|) = h) \land ((\forall t \in \mathbb{T}) \ S(r(t)) \neq \emptyset$ whenever $S(h) \neq \emptyset$.

Proof. Assuming that $S(h) \neq \emptyset$ we construct r as follows: Since $h \in PR^{trans}$, h = r'(|h|) for some $r' \in R$. Contraposition of the statement of Lemma A.29 gives for $t \leq |h|$, if $S(r'(|h|)) \neq \emptyset$ then $S(r'(t')) \neq \emptyset$. Hence for $t \leq |h|$ we define r(t) := r'(t).

Next assume an order on the set $Z := 2^{GEvents} \times 2^{\overline{GActions_1}} \times \ldots \times 2^{\overline{GActions_n}}$ and let $\widetilde{X_1}(S)$ be the first element of some subset $S \subseteq Z$ according to this order. For t > |h| we define $r(t) := update(r(t-1), \widetilde{X_1}(S(r(t-1)))))$. It remains to show that S(r(t-1)) can never be empty. The proof is by induction over t > |h|. Induction Hypothesis: $S(r(t-1)) \neq \emptyset$.

Base Case for t = |h| + 1: We get that S(t) = S(|h|), which is not empty by assumption.

Induction Step for $t \to t+1$: Suppose the induction hypothesis holds for t. Since r(t) is defined as $update(r(t-1), \widetilde{X}_1(S(r(t-1))))$ and it holds that $r(t-1) \in PR^{trans}$ and $\widetilde{X}_1(S(r(t-1))) \in S(r(t-1))$, by the second operational safety property attribute (2) and semantics of \iff we get that also $S(r(t)) \neq \emptyset$, thus completing the induction step.

Lemma A.31. For an operational safety property $S \in \mathcal{O}$, transitional run $r \in R$, and timestamp $t \in \mathbb{T} \setminus \{0\}$, $S(r(t)) \neq \emptyset$ implies $\beta^{t-1}(r) \in S(r(t-1))$.

Proof. If $S(r(t)) \neq \emptyset$, by the operational safety property attribute 2, r(t) has to be safely reachable, meaning $(\exists h \in PR^{trans})(\exists X \in S(h)) \ r(t) = update (h, X)$. By injectivity (Lemma A.28) of update (Def. A.6) it only maps to the prefix r(t)for h = r(t-1) and $X = \beta^{t-1}(r)$. Hence $\beta^{t-1}(r) \in S(r(t-1))$.

Lemma A.32. F from Def. A.22 is surjective.

Proof. Suppose by contradiction that F is not surjective. This implies that there exists some $S \in \mathcal{O}$ s.t. for all $S' \in \mathcal{T}$, $F(S') \neq S$.

To arrive at a contradiction, we use the trace safety property $\tilde{S'}^S$ from Def. A.24. This is safe to use, as by Lemma A.27 $\tilde{S'}^S \in \mathcal{T}$. There are two cases causing $F(\tilde{S'}^S) \neq S$:

1. There is a run $r' \in \widetilde{S'}^{S}$ and timestamp $t' \in \mathbb{T}$ s.t. $\beta^{t'}(r') \notin S(r'(t'))$. Hence, $r' \in \{r \in R \mid \beta^{t'}(r) \notin S(r(t'))\}$, such that by (A.8) $r' \notin \widetilde{S'_{t'+1}}^{S}$, from which further by (A.9) and (A.10) $r' \notin \widetilde{S'}^{S}$ follows, providing a contradiction. The Persistence of False Memory: Brain in a Vat Despite Perfect Clocks 25

2. There exists a prefix $h \in PR^{trans}$ and some $X \in S(h)$, but

$$h \notin \widetilde{S'}^S. \tag{A.15}$$

Since $X \in S(h)$ by Lemma A.30 there exists a transitional run $r \in R$ s.t. r(|h|) = h and $(\forall t \in \mathbb{T}) \ S(r(t)) \neq \emptyset$. By Lemma A.31 we further get that for any $t \in \mathbb{T}$, if $S(r(t)) \neq \emptyset$, then $\beta^{t-1}(r) \in S(r(t-1))$. By Lemma A.26 it follows that $r \in \widetilde{S'_{\infty}}^{S}$ and by prefix closure (A.10) we finally get that $h \in \widetilde{S'}^{S}$ contradicting (A.15).

Thus, by definition of our construction (A.7)–(A.9), $F(\widetilde{S'}^S) = S$.

Lemma A.33. F from Def. A.20 is bijective.

Proof. Follows from Lemma A.23 and A.32.

Lemma A.34. For the general asynchronous byzantine framework given two $\tau_{P_e,P}^B$ -transitional runs $r, r' \in R$ and timestamps $t, t' \in \mathbb{T} \setminus \{0\}$, an agent $i \in \mathcal{A}$ cannot distinguish

- a round $t\frac{1}{2}$ in run r, where a nonempty set of events $Q \subseteq \overline{GEvents_i} \sqcup BEvents_i$ was observed by i, but no go(i) occurred \Rightarrow

$$go(i) \notin \beta_{g_i}^t(r), \quad \beta_i^t(r) = \varnothing, \quad \overline{\beta}_{\epsilon_i}^t(r) \sqcup \beta_{b_i}^t(r) = Q$$

- from a round $t'_{\overline{Z}}$ in run r', where the same set of events Q was observed by *i*, go(i) occurred, but the protocol prescribed the empty set $(\emptyset \in P_i(r'(t')))$, which was chosen by the adversary \Rightarrow

$$go(i) \in \beta_{g_i}^{t'}(r'), \quad \beta_i^{t'}(r') = \varnothing, \quad \overline{\beta}_{\epsilon_i}^{t'}(r') \sqcup \beta_{b_i}^{t'}(r') = Q.$$

Proof. This immediately follows from the definition of the update function (A.2), as in this scenario (for $r_i(t+1) = [\lambda_m, \ldots, \lambda_1, \lambda_0]$ and $r'_i(t'+1) = [\lambda'_{m'}, \ldots, \lambda'_1, \lambda'_0]$) $\lambda_m = \lambda'_{m'} = Q.$

Definition A.35. We define the following implementation classes:

- Adm The desired extension property is only implemented via an appropriate admissibility condition $\Psi^{\alpha} \subseteq R$. An extension $\mathscr{E}^{\alpha} \in \operatorname{Adm}$ iff $\mathscr{E}^{\alpha} = (\mathscr{C}_{\epsilon} \times \mathscr{C}, IS^{\alpha}, \tau^{N,N}, \Psi^{\alpha}).$
- **JP** The extension property is implemented via restricting the set of joint protocols \mathscr{C} . An extension $\mathscr{E}^{\alpha} \in \mathbf{JP}$ iff $\mathscr{E}^{\alpha} = (\mathscr{C}_{\epsilon} \times \mathscr{C}^{\alpha}, IS^{\alpha}, \tau^{N,N}, \Psi^{\alpha}).$
- - **EnvJP** The extension property is implemented via restricting the set of environment protocols \mathscr{C}_{ϵ} possibly in conjunction with the set of joint protocols \mathscr{C} . An extension $\mathscr{E}^{\alpha} \in \mathbf{EnvJP}$ iff $\mathscr{E}^{\alpha} = (PP^{\alpha}, IS^{\alpha}, \tau^{N,N}, \Psi^{\alpha})$, where $PP^{\alpha} \subset \mathscr{C}_{\epsilon} \times \mathscr{C}$ and $\mathscr{E}^{\alpha} \notin \mathbf{JP}$.

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- **EnvJP AFB** An extension $\mathscr{E}^{\alpha} \in \mathbf{EnvJP} \mathbf{AFB}$ iff $\mathscr{E}^{\alpha} = (PP^{\alpha}, IS^{\alpha}, \tau^{N,B}, \Psi^{\alpha})$, where in $\tau^{N,B}$ the filter functions filter^N_{\varepsilon} and filter^B_{\varepsilon} (for all $i \in \mathcal{A}$) are used, $PP^{\alpha} \subset \mathscr{C}_{\epsilon} \times \mathscr{C}$ and $\mathscr{E}^{\alpha} \notin \mathbf{JP} - \mathbf{AFB}$.
 - **EvFJP** An extension $\mathscr{E}^{\alpha} \in \mathbf{EvFJP}$ iff $\mathscr{E}^{\alpha} = (\mathscr{C}_{\epsilon} \times \mathscr{C}^{\alpha}, IS^{\alpha}, \tau^{\alpha, N}, \Psi^{\alpha})$, where in $\tau^{\alpha, N}$ the filter functions filter $_{\epsilon}^{\alpha}$ and filter $_{i}^{N}$ (for all $i \in \mathcal{A}$) are used, $\mathscr{C}^{\alpha} \subseteq \mathscr{C}$ and $\mathscr{E}^{\alpha} \notin \mathbf{JP}$.
- $$\begin{split} \mathbf{EvFJP} \mathbf{AFB} \ An \ extension \ \mathscr{E}^{\alpha} &\in \mathbf{EvFJP} \mathbf{AFB} \ iff \ \mathscr{E}^{\alpha} \ = \ (\mathscr{C}_{\epsilon} \times \mathscr{C}^{\alpha}, IS^{\alpha}, \tau^{\alpha, B}, \Psi^{\alpha}), \\ where \ in \ \tau^{\alpha, B} \ the \ filter \ functions \ filter_{\epsilon}^{\alpha} \ and \ filter_{i}^{B} \ (for \ all \ i \in \mathcal{A}) \ are \\ used, \ \mathscr{C}^{\alpha} \subseteq \mathscr{C} \ and \ \mathscr{E}^{\alpha} \notin \mathbf{JP} \mathbf{AFB}. \end{split}$$
 - **EvFEnvJP** An extension $\mathscr{E}^{\alpha} \in \mathbf{EvFEnvJP}$ iff $\mathscr{E}^{\alpha} = (PP^{\alpha}, IS^{\alpha}, \tau^{\alpha, N}, \Psi^{\alpha})$, where in $\tau^{\alpha, N}$ the filter functions $filter_{\epsilon}^{\alpha} \subset filter_{\epsilon}^{N}$ and the neutral action filters $filter_{i}^{N}$ (for all $i \in \mathcal{A}$) are used, $PP^{\alpha} \subset \mathscr{C}_{\epsilon} \times \mathscr{C}$ and $\mathscr{E}^{\alpha} \notin \mathbf{EvFJP}$.
- $$\begin{split} \mathbf{EvFEnvJP} \mathbf{AFB} \ \ An \ \ extension \ \ \mathcal{E}^{\alpha} \ \in \ \mathbf{EvFEnvJP} \mathbf{AFB} \ \ iff \ \ \mathcal{E}^{\alpha} \ = \ (PP^{\alpha}, IS^{\alpha}, \tau^{\alpha, B}, \Psi^{\alpha}), \\ where \ in \ \tau^{\alpha, B} \ \ the \ filter \ functions \ filter_{\epsilon}^{\alpha} \ \subset \ filter_{\epsilon}^{N} \ \ and \ the \ byzantine \ action \ filters \ \ filter_{i}^{B} \ \ (for \ all \ i \ \in \ \mathcal{A}) \ are \ used, \ PP^{\alpha} \ \subset \ \ \mathcal{C}_{\epsilon} \times \ \ \mathcal{C} \ \ and \ \ \ \mathcal{E}^{\alpha} \ \notin \\ \mathbf{EvFJP} \mathbf{AFB}. \end{split}$$
 - **Others** This class contains all remaining extension implementations including restrictions via arbitrary action filters $filter_i$ (for $i \in A$). An extension $\mathscr{E}^{\alpha} \in \mathbf{Others}$ iff it is not in any other class.

We list important subsets of these implementation classes, which we will treat as individual implementation classes in their own right (see listing below):

- $\mathbf{JP_{DC}} := \{ \mathscr{E}^{\alpha} \in \mathbf{JP} \mid S^{\alpha} \text{ is downward closed} \}$
- $\mathbf{EnvJP_{DC}} := \{ \mathscr{E}^{\alpha} \in \mathbf{EnvJP} \mid S^{\alpha} \text{ is downward closed} \}$
- $\mathbf{EvFJP_{DC}} := \{ \mathscr{E}^{\alpha} \in \mathbf{EvFJP} \mid S^{\alpha} \text{ is downward closed} \}$
- $\mathbf{EvFEnvJP_{DC}} := \{ \mathscr{E}^{\alpha} \in \mathbf{EvFEnvJP} \mid S^{\alpha} \text{ is downward closed} \}$
- **Others**_{DC} := { $\mathscr{E}^{\alpha} \in$ **Others** | S^{α} is downward closed}
- **EvFEnvJP**_{DC mono} := { $\mathscr{E}^{\alpha} \in$ **EvFEnvJP**_{DC} | ($\forall i \in \mathcal{A}$) filter^{α} and filter^{α} are monotonic for the domain $PD^{t-coh}_{\epsilon}, 2^{\overline{GActions_1}}, \dots, 2^{\overline{GActions_n}}$ }.
- $\text{ Others}_{\mathbf{DC\,mono}} := \{ \mathscr{E}^{\alpha} \in \text{ Others}_{\mathbf{DC}} \mid (\forall i \in \mathcal{A}) \ filter_{i}^{\alpha} \ and \ filter_{\epsilon}^{\alpha} \ are monotonic \ for \ the \ domain \ PD_{\epsilon}^{t-coh}, 2^{\overline{GActions}_{1}}, \dots, 2^{\overline{GActions}_{n}} \}.$

Lemma A.36. An agent *i* in a synchronous agents context executes its protocol only during synced rounds, i.e., for every $\chi \in \mathscr{E}^S$ and $r \in R^{\chi}$, $go(i) \in \beta_{g_i}^t(r)$ if $t_{\frac{1}{2}}$ is a synced round.

Proof. From Defs. 2 and 3, it immediately follows that in a synchronous agents context go(i) events can only ever occur during a synced round.

Lemma A.37. For a correct agent *i*, a $\tau_{P_{\epsilon},P^S}^S$ -transitional run *r* (where $P^S \in \mathscr{C}^S$), some timestamp $t' \geq 1$, agent *i*'s local history $r_i(t') = h_i = [\lambda_m, \ldots, \lambda_1, \lambda_0]$ (given the global history $h = r(t') \in \mathscr{G}$) and some round $(t-1)\frac{1}{2}(t' \geq t \geq 1)$, there exists some $a \in Actions_i$ such that $a \in \lambda_{k_t}$ where $\lambda_{k_t} = \sigma(\beta_{\epsilon_i}^{t-1}(r) \sqcup \beta_i^{t-1}(r))$ if and only if $(t-1)\frac{1}{2}$ is a synced round.

Proof. From left to right. From Lemma A.36, we know that an agent can only execute its protocol during synced rounds. Therefore, since agent i is assumed

to be correct and $(t-1)\frac{1}{2}$ is a synced round, it follows that $\{go(i)\} = \beta_{g_i}^{t-1}(r)$ (*sleep*(*i*) or *hibernate*(*i*) would make the agent byzantine). By (A.2) (the definition of the update function) and Def. 4 (the definition of the synchronous agents joint protocols, which dictates that at least $\mathring{\circ}$ has to be among the attempted actions, hence the empty set can never be issued) an action $a \in Actions_i$ such that $a \in \lambda_{k_t}$ has to exist.

From right to left. Suppose there exists $a \in Actions_i$ such that $a \in \lambda_{k_i}$. Since agent *i* is assumed to be correct, by Lemma A.36 agents only execute their protocol during synced rounds and by the definition of the update function (A.2) round $(t-1)\frac{1}{2}$ has to be a synced round.

Lemma A.38. For any agent $i \in A$, any run $r \in R^{\chi}$, where $\chi \in \mathscr{E}^S$ and any timestamp $t \in \mathbb{T}$, it holds that $\{go(i)\} = \beta_{g_i}^t(r)$ iff $(\exists A \in \overline{GActions_i}) A \in \beta_i^t(r)$.

Proof. This directly follows from Def. 4 of the synchronous agents joint protocol and the standard action filter function 1. As no synchronous agents protocol can prescribe the empty set, whenever an agent *i* receives a go(i) event during some round $t\frac{1}{2}$, it will perform some action $a \in Actions_i$, as by *t*-coherence of the environment protocol's event sets, there can always only be one system event present for any agent during one round. Similarly, if $(\exists A \in \overline{GActions_i}) A \in \beta_i^t(r)$ by definition of the byzantine action filter, *i* must have gotten a go(i).

Corollary A.39. Lemma A.34 does not hold for runs $r, r' \in R^{\chi}$ for $\chi \in \mathscr{E}^S$.

Definition A.40. For $i \in A$, global history $h \in \mathscr{G}$, we define the **neutral** event and action filters (the weakest filters) as $filter_{\epsilon}^{N}(h, X_{\epsilon}, X_{A}) := X_{\epsilon}$ and $filter_{i}^{N}(X_{A}, X_{\epsilon}) := X_{i}$. The transition template using only the neutral filters is denoted $\tau^{N,N}$ or τ^{N} .

A.1 Asynchronous Byzantine Agents

Definition A.41. We denote by $\mathscr{E}^B := (\mathscr{C}_{\epsilon} \times \mathscr{C}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^B, R)$ the asynchronous byzantine agents extension.

Lemma A.42. $\mathscr{E}^B \in EvFJP - AFB$.

Proof. Follows from Defs. A.41 and 23.

A.2 Reliable Communication

In the **reliable communication** extension agents can behave arbitrarily. However the communication—the transmission of messages by the environment—is reliable for a particular set of (reliable) channels, i.e., a message that was sent through one of these (reliable) channels, is guaranteed to be delivered by the environment in finite time. This also holds for the delivery of messages to and from byzantine agents. Since a byzantine agent can always "choose" to ignore any messages it receives anyway, this does not restrict its byzantine power to exhibit arbitrary behaviour. Formally, we define a set of (reliable) channels as $C \subseteq \mathcal{A}^2$.

The reliable communication property will be ensured by the admissibility condition $EDel_C$, which is a liveness property.

Definition A.43 (Eventual Message Delivery).

$$EDel_{C} = \left\{ r \in R \middle| \left(\left(gsend(i, j, \mu, id) \in r_{\epsilon}(t) \lor (\exists A \in \{ noop \} \sqcup \overline{GActions}_{i}) fake(i, gsend(i, j, \mu, id) \mapsto A) \in r_{\epsilon}(t) \right) \land (A.16) \\ (i, j) \in C \right) \longrightarrow (\exists t' \in \mathbb{N}) greev(j, i, \mu, id) \in r_{\epsilon}(t') \right\}$$

Definition A.44. We define by $\mathscr{E}^{RC_C} := (\mathscr{C}_{\epsilon} \times \mathscr{C}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^N, EDel_C)$ the reliable communication extension.

A.3 Time-bounded Communication

We say that communication is time-bounded if for every channel and for every message there is an upper-bound (possibly infinite) on the transmission time. Since the transmission is not reliable a priori, the **time-bounded communi-cation** extension only specifies the time window during which the delivery of a message can occur. In order to gain flexibility, bounds can be changed depending on the sending time and depending on the message too—for instance a byte of data and picture will not have the same time bound. We encode these bounds in an upper-bound structure defined as follows:

Definition A.45. For the first infinite ordinal number ω , agents $(i, j) \in \mathcal{A}^2$, and the channel $i \mapsto j$, we define the message transmission upper-bound for the channel $i \mapsto j$ as follows $\delta_{i \mapsto j} \colon Msgs \times \mathbb{N} \to \mathbb{N} \cup \{\omega\}$. We define an upper bound structure as $\Delta := \bigcup_{(i,j) \in \mathcal{A}^2} \{\delta_{i \mapsto j}\}$.

Since, as we soon show, the time-bounded safety property is downward closed, we implement it by restriction of the set of environment protocols.

Definition A.46. For an upper-bound structure Δ , we define the set of timebounded communication environment protocols as

$$\mathscr{C}_{\epsilon}^{TC_{\Delta}} := \{ P_{\epsilon} \in \mathscr{C}_{\epsilon} \mid (\forall t \in \mathbb{N}) (\forall X_{\epsilon} \in P_{\epsilon}(t)) \\ greev(j, i, \mu, id(i, j, \mu, k, t')) \in X_{\epsilon} \rightarrow t' + \delta_{i \mapsto j} (\mu, t') \geq t \}.$$
(A.17)

Definition A.47. For an upper-bound structure Δ

 $\mathscr{E}^{TC_{\Delta}} := (\mathscr{C}^{TC_{\Delta}}_{\epsilon} \times \mathscr{C}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{N}, R)$

denotes the time-bounded communication extension.

Lemma A.48. $S^{TC_{\Delta}}$ is downward closed.

Proof. Suppose that by contradiction $S^{TC_{\Delta}}$ is not downward closed. This implies $X' \notin S^{TC_{\Delta}}(h)$ for some $h \in \mathscr{G}$, $X \in S^{TC_{\Delta}}$, and $X' \subseteq X$. It immediately follows that $X' \subset X$. Since in τ^N the neutral (event and action) filters are used we further get that there are $P_{\epsilon} \in \mathscr{C}_{\epsilon}^{TC_{\Delta}}$ and $P \in \mathscr{C}$, $X_{\epsilon} \in P_{\epsilon}(|h|)$, $X_i \in P_i(h_i)$

for all $i \in \mathcal{A}$, and $X = X_{\epsilon} \sqcup X_1 \sqcup \cdots \sqcup X_n$. Since the set of joint protocols is unrestricted there exists some joint protocol P' ensuring that together with P_{ϵ} , $X_{\epsilon} \sqcup X'_1 \sqcup \cdots \sqcup X'_n \in S^{TC_{\Delta}}(h)$ for all $X'_i \subseteq X_i$, $i \in \mathcal{A}$. Therefore, we conclude that the violation X' has to be caused by some $X'_{\epsilon} \subset X_{\epsilon} = X \sqcup GEvents$.

From $X \in S^{TC_{\Delta}}(h)$ we conclude that

$$grecv(j, i, \mu, id(i, j, \mu, k, t')) \in X_{\epsilon} \rightarrow t' + \delta_{i \mapsto j}(\mu, t') \ge |h|.$$
(A.18)

By semantics of " \rightarrow " and since $X'_{\epsilon} \subseteq X_{\epsilon}$ we get

$$\left(grecv(j,i,\mu,id(i,j,\mu,k,t')) \in X'_{\epsilon}\right) \to \left(grecv(j,i,\mu,id(i,j,\mu,k,t')) \in X_{\epsilon}\right).$$
(A.19)

Using (A.19) in (A.18) we get

$$(grecv(j, i, \mu, id(i, j, \mu, k, t')) \in X'_{\epsilon}) \to (grecv(j, i, \mu, id(i, j, \mu, k, t')) \in X_{\epsilon}) \to (t' + \delta_{i \mapsto j}(\mu, t') \ge |h|).$$

$$(A.20)$$

Finally from (A.20) by transitivity of " \rightarrow " we get that

$$\left(grecv(j,i,\mu,id(i,j,\mu,k,t'))\in X'_{\epsilon}\right)\to \left(t'+\delta_{i\mapsto j}\left(\mu,t'\right)\geq |h|\right).$$
(A.21)

Hence, we conclude that $X' \in S^{TC_{\Delta}}(h)$ and we are done.

Corollary A.49. $\mathscr{E}^{TC_{\Delta}} \in \text{EnvJP}_{DC}$.

Proof. Follows from Def. A.47 and Lemma A.48.

A.4 Synchronous Communication

The **Synchronous Communication** extension guarantees for a set of synchronous communication channels $C \subseteq \mathcal{A}^2$ that whenever a message is correctly received, it has been sent during the same round. This means that it is a special case of the time-bounded communication extension.

Definition A.50 (Synchronous Communication Environment Protocols). We define the synchronous message delay as

$$\delta^{SC_C}_{i \mapsto j}(\mu, t) := \begin{cases} 0 & \textit{if } (i, j) \in C \\ \omega & \textit{otherwise} \end{cases}$$

We define the synchronous communication upper bound structure Δ^{SC_C} as $\Delta^{SC_C} := \bigcup_{(i,j)\in\mathcal{A}^2} \{\delta_{i\mapsto j}^{SC_C}\}.$

$$\mathscr{E}_{\epsilon}^{SC_C} := \mathscr{E}_{\epsilon}^{TC_{\Delta^{SC_C}}} \tag{A.22}$$

Definition A.51. We denote by $\mathscr{E}^{SC_C} := (\mathscr{C}^{SC_C}_{\epsilon} \times \mathscr{C}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^N, R)$ the synchronous communication extension.

Lemma A.52. S^{SC_C} is downward closed.

Proof. Follows from Lemma A.48, as the synchronous communication extension is just an instance of the time-bounded communication extension (A.22).

Corollary A.53. $\mathscr{E}^{SC_C} \in \operatorname{EnvJP}_{DC}$.

Proof. Follows from Def. A.51 and Lemma A.52.

A.5 Multicast Communication

In the **multicast communication** paradigm, each agent has several multicast channels at its disposal and is restricted to sending messages using these particular channels. In this section, we provide a software based multicast, meaning that only correct agents have to adhere to this behavior (further along we provide a hardware based multicast as well, where also byzantine agents are forced to exhibit this multicast behavior).

First, we define a **multicast communication problem**. For each $i \in \mathcal{A}$ we define a collection Mc_i of groups of agents it can send messages to.

Definition A.54. For each $i \in A$ the set of available multicast channels is $Mc_i \subseteq 2^A \setminus \{\emptyset\}$. The **multicast communication problem** is the tuple of these collections of communication channels $Ch = (Mc_1, \ldots, Mc_n)$.

We denote the set of recipients for the copy μ_k of a message μ that has been sent according to some set $X \subseteq Actions$ by $Rec_X(\mu_k) = \{j \mid send(j, \mu_k) \in X\}$.

Since we implement a software based multicast (and since we want our extensions to be modular) we use a restriction of the joint protocol to do so.

Definition A.55. For a multicast communication problem Ch, we define the set of multicast joint protocols as

$$\mathscr{C}^{MC_{Ch}} = \{ (P_1, \dots, P_n) \in \mathscr{C} \mid (\forall i \in \mathcal{A}) (\forall h_i \in \mathscr{L}_i) (\forall X \in P_i(h_i)) (\forall \mu \in Msgs) (\forall k \in \mathbb{N}) \\ Rec_X(\mu_k) \neq \varnothing \rightarrow Rec_X(\mu_k) \in Mc_i \}.$$
(A.23)

Definition A.56. For a multicast communication problem Ch, we set $\mathscr{E}^{MC_{Ch}} := (\mathscr{C}_{\epsilon} \times \mathscr{C}^{MC_{Ch}}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{N,B}, R)$ to be the **multicast communication** extension where in $\tau^{N,B}$ the neutral event and the byzantine action filters are used (for all $i \in \mathcal{A}$).

An important special case of the multicast communication problem is the broadcast communication problem, where each agent must broadcast each message to all the agents:

Definition A.57. The broadcast communication extension \mathscr{E}^{BC} is a multicast communication extension $\mathscr{E}^{MC_{BCh}}$ for

$$BCh = (\underbrace{\{\mathcal{A}\}, \dots, \{\mathcal{A}\}}_{n})$$
(A.24)

 $\mathscr{E}^{BC} = (\mathscr{C}_{\epsilon} \times \mathscr{C}^{MC_{BCh}}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{N,B}, R), \text{ where in } \tau^{N,B} \text{ the neutral event filter and the byzantine action filters (for all <math>i \in \mathcal{A}$) are used.

Corollary A.58. $\mathscr{E}^{MC_{Ch}} \in \mathbf{JP} - \mathbf{AFB}$.

Proof. Follows from Def. A.56.

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Table A.1. filter dependencies in the lock-step synchronous agents extension

filter	dependency	removal
$filter^{S}_{\epsilon}$	go(i), sleep(i), hibernate(i)	go(i)
$filter^B_{\epsilon}$	$go(i), gsend(i, j, \mu, id), fake(i, gsend(i, j, \mu, id) \mapsto A)$	$grecv(j,i,\mu,id)$



Fig. 2. Dependence graph for $filter_{\epsilon}^{B}$ and $filter_{\epsilon}^{S}$

A.6 Lock-step Synchronous Agents

Table A.1 reveals that $filter_{\epsilon}^{B}$ depends on go(i) events, which $filter_{\epsilon}^{S}$ removes. Thus, we have a dependence relation from $filter_{\epsilon}^{B}$ to $filter_{\epsilon}^{S}$. $filter_{\epsilon}^{B}$ removes only correct receive events $grecv(j, i, \mu, id)$. $filter_{\epsilon}^{S}$ is independent of such events, hence, there is no dependence relation from $filter_{\epsilon}^{S}$ to $filter_{\epsilon}^{B}$.

Figure 2 shows the final dependence graph. Since there is no circular dependence, we can directly use the composition order given by the graph. This gives us $\mathscr{E}^{B \circ S} = (\mathscr{C}_{\epsilon} \times \mathscr{C}^{S}, 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{B \circ S, B}, R)$, where in $\tau^{B \circ S, B}$ the event filter is $filter_{\epsilon}^{B \circ S}$ and the action filters result in $filter_{i}^{B}$ for all $i \in \mathcal{A}$ (by idempotence of the byzantine action filter function). Following the rest of the extension combination guide finally leads to

$$\mathscr{E}^{B\circ S\circ BC\circ SC}{}_{\mathcal{A}^2}\circ RC_{\mathcal{A}^2} = \left(\mathscr{C}^{SC}{}_{\mathcal{A}^2} \times (\mathscr{C}^{MC}{}_{BCh} \cap \mathscr{C}^S), 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{B\circ S, B}, EDel_{\mathcal{A}^2}\right). \quad (A.25)$$

Lemma A.59. The extensions \mathscr{E}^B , \mathscr{E}^S , $\mathscr{E}^{SC}_{\mathcal{A}^2}$, $\mathscr{E}^{RC}_{\mathcal{A}^2}$, and \mathscr{E}^{BC} are compatible (w.r.t. the composition $B \circ S \circ BC \circ SC_{\mathcal{A}^2} \circ RC_{\mathcal{A}^2}$).

Proof. The only condition from Def. A.12 that does not trivially follow from the definition of the extensions in question is whether there exists an agent context χ , such that $\chi \in \mathscr{E}^{B \circ S \circ B C \circ S C}{}_{\mathcal{A}^2} \circ R C_{\mathcal{A}^2}$. Such a χ however can be easily constructed. Let $\chi = ((P'_{\epsilon}, \mathscr{G}(0), \tau^{B \circ S, B}, EDel_{\mathcal{A}^2}), P')$, where P'_{ϵ} only produces the set containing the empty set and P' for every agent produces the set containing the set that only contains the action $\mathring{\otimes}$, i.e., $P'_{\epsilon}(t) = \{\varnothing\}$ for all $t \in \mathbb{N}$ and $P'(h) = (\{\{\mathring{\otimes}\}\}, \ldots, \{\{\mathring{\otimes}\}\})$ for all $h \in \mathscr{G}$. Note that this agent context is part of the extension $\mathscr{E}^{B \circ S \circ B C \circ S C}{}_{\mathcal{A}^2} \circ R C_{\mathcal{A}^2}$, as $P'_{\epsilon} \in \mathscr{C}^{S C}{}_{\epsilon}{}^2$ and $P' \in (\mathscr{C}^{M C_{BCh}} \cap \mathscr{C}^S)$.

Lemma A.60. The extension $\mathcal{E}^{B \circ S \circ BC \circ SC_{A^2} \circ RC_{A^2}}$ satisfies all safety properties of its constituent extensions.

Proof. Follows from Table 2.

Finally, after having proved that the resulting extension $\mathscr{E}^{B \circ S \circ BC \circ SC}_{\mathcal{A}^2} \circ RC_{\mathcal{A}^2}$ satisfies all desired properties, we can define it as \mathscr{E}^{LSS} .

Definition A.61. We define the lock-step synchronous agents extension to be $\mathscr{E}^{LSS} = (\mathscr{C}^{SC_{\mathcal{A}^2}}_{\epsilon} \times (\mathscr{C}^{MC_{BCh}} \cap \mathscr{C}^S), 2^{\mathscr{G}(0)} \setminus \{\varnothing\}, \tau^{B \circ S, B}, EDel_{\mathcal{A}^2}).$

We will now add a few lemmas about properties, which the lock-step synchronous agents extension inherits from the synchronous agents extension.

Lemma A.62. An agent i in a lock-step synchronous agents context executes its protocol only during synced rounds, i.e., $go(i) \in \beta_{q_i}^t(r)$ iff t.5 is a synced round.

Proof. Lemma A.36 for the synchronous agents extension describes a property of S^S that by Lemma A.60, \mathscr{E}^{LSS} satisfies.

Lemma A.63. For a correct agent *i*, a $\tau_{P_{\epsilon}^{SC}A^2, P^{SMC}BCh}^{B \circ S, B}$ -transitional run *r* (where $P_{\epsilon}^{SC_{\mathcal{A}^2}} \in \mathscr{C}_{\epsilon}^{SC_{\mathcal{A}^2}} \text{ and } P^{SMC_{BCh}} \in \mathscr{C}^S \cap \mathscr{C}^{MC_{BCh}}), \text{ some timestamp } t' \geq 1,$ agent i's local history $r_i(t') = h_i = [\lambda_m, \ldots, \lambda_1, \lambda_0]$ (given the global history $h = r(t') \in \mathscr{G}$) and some round $(t-1)\frac{1}{2}$ ($t' \geq t \geq 1$), there exists some $a \in Actions_i \text{ such that } a \in \lambda_{k_t} \text{ where } \lambda_{k_t} = \sigma\left(\beta_{\epsilon_i}^{t-1}(r) \sqcup \beta_i^{t-1}(r)\right) \text{ if and only if}$ $(t-1)\frac{1}{2}$ is a synced round.

Proof. This again follows from Lemma A.37 for the synchronous agents extension, as the statement of this lemma is a safety property of \mathscr{E}^S and by Lemma A.60, \mathscr{E}^{LSS} satisfies S^S .

Lemma A.64. For any agent $i \in A$, any run $r \in R^{\chi}$, where $\chi \in \mathscr{E}^{LSS}$ and any timestamp $t \in \mathbb{N}$ it holds that $go(i) \in \beta_{q_i}^t(r) \iff (\exists A \in \overline{GActions_i}) A \in \beta_i^t(r)$.

Proof. Analogous to the proof of Lemma A.38 for synchronous agents.

Lemma A.65 (Lock-step Synchronous Brain-in-the-Vat Lemma). Let $\mathcal{A} = \llbracket 1; n \rrbracket$ be a set of agents with joint protocol $P^{SMC_{BCh}} = (P_1, \ldots, P_n) \in$ $(\mathscr{C}^{MC_{BCh}} \cap \mathscr{C}^S)$, let $P_{\epsilon}^{SC_{\mathcal{A}^2}} \in \mathscr{C}_{\epsilon}^{SC_{\mathcal{A}^2}}$ be the protocol of the environment, for $\chi \in \mathscr{E}^{LSS}$, where $\chi = ((P_{\epsilon}^{SC_{\mathcal{A}^2}}, \mathscr{G}(0), \tau^{B \circ S, B}, EDel_C), P^{SMC_{BCh}})$, let $r \in \mathbb{R}^{\chi}$, let $i \in \mathcal{A}$ be an agent, let t > 0 be a timestamp and let $adj = [B_{t-1}; \ldots; B_0]$ be an adjustment of extent t-1 satisfying $B_m = (\rho_1^m, \ldots, \rho_n^m)$ for all $0 \le m \le t-1$ with $\rho_i^m = BPFake_i^m$ and for all $j \neq i \ \rho_j^m \in \{CFreeze, BFreeze_j\}$. If the protocol $P_{\epsilon}^{SC_{\mathcal{A}^2}}$ makes

- agent i gullible,
- every agent $j \neq i$ delayable and fallible if $\rho_i^m = BFreeze_j$ for some m,

- all remaining agents delayable, then each run $r' \in R(\tau_{P_{e}^{SC}A^{2}, P^{SMC}BCh}^{B\circ S, B}, r, adj)$ satisfies the following properties: 1. $r' \in R^{\chi}$.

2. $(\forall m \leq t) r'_i(m) = r_i(m);$

3. $(\forall m \le t)(\forall j \ne i) \ r'_j(m) = r'_j(0);$

- 4. $(i, 1) \in Failed(r', 1)$ and thus $(i, m) \in Failed(r', m')$ for all $m' \ge m > 0$;
- 5. $\mathcal{A}(Failed(r'(t))) = \{i\} \cup \{j \neq i \mid (\exists m \leq t-1) \ \rho_j^m = BFreeze_j\};$ 6. $(\forall m < t) \ (\forall j \neq i) \ \beta_{\epsilon_j}^m(r') \subseteq \{fail(j)\}.$ More precisely, $\beta_{\epsilon_j}^m(r') = \emptyset$ iff $\rho_j^m = CFreeze \ and \ \beta_{\epsilon_j}^m(r') = \{fail(j)\} \ iff \ \rho_j^m = BFreeze_j;$

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7.
$$(\forall m < t) \beta_{\epsilon_i}^m(r') \setminus \beta_{f_i}^m(r') = \emptyset;$$

8. $(\forall m < t)(\forall j \in \mathcal{A}) \beta_i^m(r') = \emptyset.$

Proof. The proof is (similar to Lemma 8) analogous to the original Brain-inthe-Vat Lemma [12], since by Def. 7 of the interventions CFreeze, $BFreeze_i$, and $BPFake_i^t$ it holds that

$$\left(\forall r' \in R \left(\tau_{P_{\epsilon}^{SC} \mathcal{A}^{2}, P^{SMC_{BCh}}}^{B \circ S, B}, r, adj \right) \right) (\forall j \in \mathcal{A}) (\forall m \in \mathbb{N} \text{ s.t. } 0 \le m < t)$$

$$go(j) \notin \beta_{\epsilon_{j}}^{m}(r') \land \overline{\beta}_{\epsilon_{i}}^{m}(r') = \emptyset.$$
(A.26)

By Def. A.61 of the lock-step synchronous agents extension both its set of environment protocols and its admissibility condition from Def. A.43 only restrict runs (respectively environment protocols) w.r.t. correct receive events (see (A.22) and (A.17)). By (A.26) however correct events do not event occur in any such runs r'. Furthermore the synchronous agents event filter function by Def. 3 only additionally removes go events, which by (A.26) also are irrelevant for such runs r'. Additionally (A.26) makes the set of joint protocols superfluous for this lemma, hence the proof from [12] applies for the lock-step synchronous agents extension as well.

Here are some new properties unique to the lock-step synchronous extension.

Lemma A.66. Whenever a correct agent $i \in A$ in an agent context $\chi \in \mathscr{E}^{LSS}$ sends a message μ in round t, it sends μ to all agents and μ is received by all agents in the same round t.

Proof. When a correct agent *i* sends a message, this is done by executing its protocol (as a fake send initiated by the environment protocol would immediately make this agent faulty). From the definition of the joint protocol (Def. 4, (A.23) with (A.24)), an agent can only send a message to all agents or no one. From the admissibility condition $EDel_{\mathcal{A}^2}$ (A.16) and the synchronous communication environment protocol (A.22), it follows that a sent message has to be delivered to the receiving agent during the same round *t* it was sent. Suppose by contradiction that a message, sent in round *t*, is not received by some agent in round *t*. By (A.16), it follows that this message has to be correctly received at some later point in time t' > t. However by (A.22), a correct receive event can only happen during the same round of its corresponding send event, thus leading to a contradiction.

Theorem 67. A correct agent *i* with local history h_i in a lock-step synchronous agents context can infer from h_i the number of synced rounds that have passed. Formally, for an agent context $\chi \in \mathscr{E}^{LSS}$, a χ -based interpreted system $\mathcal{I} = (R^{\chi}, \pi)$, a run $r \in R^{\chi}$ and timestamp $t \in \mathbb{N}$, $(\mathcal{I}, r, t) \models H_i nsr_{NSR(r(t))}$.

Proof. Analogous to the proof of Theorem 13 from the synchronous agents extension as $\mathscr{C}^{MC_{BCh}} \cap \mathscr{C}^S \subseteq \mathscr{C}^S$.

Theorem 68. (Copy of Theorem 27)

There exists an agent context $\chi \in \mathscr{E}^{LSS}$, where $\chi = \left((P_{\epsilon}^{SC_{\mathcal{A}^2}}, \mathscr{G}(0), \tau^{B \circ S}, EDel_{\mathcal{A}^2}), \tilde{P}^{SMC_{BCh}} \right)$, and a run $r \in \mathbb{R}^{\chi}$, such that for agents $i, j \in \mathcal{A}$, where $i \neq j$, some timestamp $t \in \mathbb{N}$, and a χ -based interpreted system $\mathcal{I} = (\mathbb{R}^{\chi}, \pi)$

$$(\mathcal{I}, r, t) \models H_i fault y_j.$$

Proof. Suppose the joint protocol is such that for all global histories $h \in \mathscr{G}$

$$\dot{P}^{SMC_{BCh}}(h) = \{(S_1, \dots, S_n) \mid \\ (\forall i \in \mathcal{A})(\forall D \in S_i)(\exists \mu \in Msgs) \ \{send(j,\mu) \mid (\forall j \in \mathcal{A})\} \cup \{\mathring{o}\} \subseteq D\}.$$
(A.27)

meaning that every agent has to perform at least one broadcast in case it gets the opportunity to act. By Lemma A.62 (agents execute their protocols only during synced rounds), Lemma A.66 (whenever a message is sent by a correct agent, all agents receive it during the same round) and (A.27), it follows that every agent receives at least one message from every correct agent during a synced round. Thus in all states, where i is correct, it received a message from itself, but not from some agent j, j has to be faulty.